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OSCILLATION OF THIRD-ORDER DIFFERENCE EQUATIONS

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Abstract: In this paper we will study the oscillatory properties of third order difference equations. By means of the Reccati transformation techniques we will establish some sufficient conditions which are sufficient for all solutions to be oscillatory or tend to zero.

1 – Introduction

In recent years, the oscillation theory and the asymptotic behavior of difference equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs [1, 2, 4]. Compared to the second order difference equations, the study of third order difference equations has received considerably less attention in the literature, even though such equations arise in the study of Economics, Mathematical Biology, and other areas of mathematics where discrete models are used (see for example [3]). Some recent results on third order difference equations can be found in [5-10].

In this paper we shall consider the third order difference equation

$$\Delta^3 V_n + P_n V_{n+1} = 0\,, \qquad n \geq n_0\ , \tag{1.1}$$

where $P_n > 0$ for $n \geq n_0$ and Δ denotes the forward difference operator $\Delta V_n = V_{n+1} - V_n$ for any sequence $\{V_n\}$ of real numbers.

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By a solution of (1.1) we mean a nontrivial real sequence $\{V_n\}$ that is defined for $n \geqslant n_0$ and satisfies equation (1.1) for n$\geq n_0$. A solution $\{\mathrm{V}_n\}$ of (1.1) is said to be oscillatory, if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

A number of dynamical behavior of solutions of difference equations are possible; here we will only be concerned with conditions which are sufficient for all solutions of (1.1) to be oscillatory or tend to zero as $n \to \infty$.

2 – Main results

In this section, by using the Reccati transformation techniques we establish some new conditions which are sufficient for all solutions of (1.1) to be oscillatory or tend to zero as $n \to \infty$.

Theorem 2.1. *Assume that*

$$\sum_{l=n_3}^{\infty} \left[\sum_{t=n_3}^{l-1} \sum_{s=n_2}^{t-1} P_s \right] = \infty \,, \tag{2.1}$$

and there exists a positive sequence $\{\rho_n\}_{n=n_0}^{\infty}$ *such that,*

$$\lim_{n\to\infty} \sup \sum_{l=n_2}^{n} \left[\rho_l P_l - \frac{(\Delta\rho_l)^2}{4\rho_l(l-n_1)} \right] = \infty, \qquad \textit{for} \;\; n_2 > n_1 \,. \tag{2.2}$$

Then every solution $\{V_n\}$ *of Eq.(1.1) oscillates or* $\lim_{n\to\infty} V_n = 0$.

Proof: Let $\{V_n\}$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $V_n > 0$ for $n \geqslant n_1$ where $n_1 \geq n_0$ is chosen so large. From (1.1) we have $\Delta^3 V_n \leq 0$ for $n \geqslant n_1$. Then $\{V_n\}$, $\{\Delta V_n\}$ and $\{\Delta^2 V_n\}$ are monotone and eventually of one sign. We claim $\Delta^2 V_n > 0$. Suppose to the contrary that $\Delta^2 V_n \leq 0$ for $n \geqslant n_2$ for $n_2 \geqslant n_1$. Since $\Delta^2 V_n$ is nonincreasing there exists a negative constant C and $n_3 \geqslant n_2$ such that $\Delta^2 V_n \leq C$ for $n \geqslant n_3$. Summing from n_3 to $n-1$, we obtain

$$\Delta V_n \,\leq\, \Delta V_{n_3} + C(n-1-n_3) \,.$$

Letting n$\to \infty$, then $\Delta V_n \to -\infty$. Thus, there is an integer $n_4 \geqslant n_3$ such that for $n \geqslant n_4$, $\Delta V_n \leq \Delta V_{n_4} < 0$. Summing from n_4 to $n-1$ we obtain

$$V_n - V_{n_4} \,\leq\, C(n-1-n_4) \,,$$

this implies that $V_n \to -\infty$ as n$\to \infty$ which is a contradiction with the fact that V_n is positive. Then $\Delta^2 V_n > 0$. Therefore, there are only the following two cases for $n \geqslant n_1$ sufficiently large:

(**I**) $V_n > 0,\ \ \Delta V_n > 0,\ \ \Delta^2 V_n > 0.$

(**II**) $V_n > 0,\ \ \Delta V_n < 0,\ \ \Delta^2 V_n > 0.$

First we consider the Case (I): Define w_n by the *Reccati substitution*

$$w_n = \rho_n \frac{\Delta^2 V_n}{V_{n+1}}, \qquad n \geqslant n_1 \tag{2.3}$$

we have $w_n > 0$ and

$$\Delta w_n = \Delta^2 V_{n+1}\, \Delta\left[\frac{\rho_n}{V_{n+1}}\right] + \frac{\rho_n\, \Delta^3 V_n}{V_{n+1}} ,$$

this and (1.1), imply that

$$\Delta w_n \le -\rho_n P_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n\, \Delta^2 V_n\, \Delta(V_{n+1})}{V_{n+1} V_{n+2}} , \tag{2.4}$$

From the Case (I) we have $V_{n+2} \ge V_{n+1}$, then from (2.4) we obtain

$$\Delta w_n \le -\rho_n P_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n\, \Delta^2 V_{n+1}\, \Delta(V_{n+1})}{V_{n+2}^2} . \tag{2.5}$$

Also From the Case (I) and Eq.(1.1) we have

$$\Delta V_n = \Delta V_{n_1} + \sum_{s=n_1}^{n-1} \Delta^2 V_s \geqslant (n-1-n_1)\, \Delta^2 V_n , \qquad n \geqslant n_1 + 1 . \tag{2.6}$$

This implies that

$$\Delta V_{n+1} \geqslant (n - n_1) \Delta^2 V_{n+1} , \qquad n \geqslant n_2 = n_1 + 1 , \tag{2.7}$$

Substituting from (2.7) in (2.8), we obtain

$$\Delta w_n \le -\rho_n P_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n\, (n-n_1)\, (\Delta^2 V_{n+1})^2}{V_{n+2}^2} . \tag{2.8}$$

From (2.3) and (2.8) we obtain

$$\Delta w_n \le -\rho_n P_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n (n-n_1)}{\rho_{n+1}^2} w_{n+1}^2 . \tag{2.9}$$

By completing the square we have

$$\begin{aligned}\Delta w_n \;&\leq\; -\rho_n P_n + \frac{(\Delta\rho_n)^2}{4\rho_n(n-n_1)} - \left[\frac{\sqrt{\rho_n(n-n_1)}}{\rho_{n+1}} w_{n+1} - \frac{\Delta\rho_n}{2\sqrt{\rho_n(n-n_1)}}\right]^2 \\ &<\; -\left[\rho_n P_n - \frac{(\Delta\rho_n)^2}{4\rho_n(n-n_1)}\right].\end{aligned}$$

Then, we have

$$\Delta w_n \;<\; -\left[\rho_n P_n - \frac{(\Delta\rho_n)^2}{4\rho_n(n-n_1)}\right]. \tag{2.10}$$

Summing (3.11) from n_2 to n, we obtain

$$-w_{n_2} \;<\; w_{n+1} - w_{n_2} \;<\; -\sum_{l=n_2}^{n}\left[\rho_l P_l - \frac{(\Delta\rho_l)^2}{4\rho_l(l-n_1)}\right],$$

which yields

$$\sum_{l=n_2}^{n}\left[\rho_l P_l - \frac{(\Delta\rho_l)^2}{4\rho_l(l-n_1)}\right] \;<\; c_1\ , \tag{2.11}$$

for all large n, and this is contrary to (2.2). Next we assume that the Case (II) holds. Since $\{V_n\}$ is positive and decreasing it follows that $\lim_{n\to\infty} V_n = b \geqslant 0$. Now we claim that $b = 0$. If not then $V_n \to b > 0$ as n$\to\infty$, and hence there exists $n_2 \geq n_1$ such that $V_{n+1} \geq b$. Therefore from (1.1) we have

$$\Delta^3 V_n + P_n b \leq 0\,, \qquad n \geq n_2\ , \tag{2.12}$$

Define the sequence $u_n = \Delta^2 V_n$ for n$\geq n_2$. Then we have

$$\Delta u_n \leq -P_n b\ .$$

Summing the last inequality from n_2 to $n-1$, we have

$$u_n \;\leq\; u_{n_2} - b\sum_{s=n_2}^{n-1} P_s\ , \tag{2.13}$$

From (2.2), by choosing $\rho_n = 1$ we have $\sum\limits_{n=n_0}^{\infty} P_n = \infty$, and then from (2.13) it is possible to choose an integer n_3 sufficiently large such that for all $n \geq n_3$

$$u_n \;\leq\; -\frac{b}{2}\sum_{s=n_2}^{n-1} P_s\ ,$$

and hence

$$\Delta^2 V_n \le -\frac{b}{2}\sum_{s=n_2}^{n-1} P_s \,.$$

Summing the last inequality from n_3 to $n-1$ we obtain

$$\Delta V_n \le \Delta V_{n_3} - \frac{b}{2}\sum_{t=n_3}^{n-1}\left(\sum_{s=n_2}^{t-1} P_s\right).$$

Since $\Delta V_n < 0$ for $n \geqslant n_0$, the last inequality implies that

$$\Delta V_n \le -\frac{b}{2}\sum_{t=n_3}^{n-1}\left(\sum_{s=n_2}^{t-1} P_s\right).$$

Summing from n_3 to $n-1$ we have

$$V_n \le V_{n_3} - \frac{b}{2}\sum_{l=n_3}^{n-1}\left[\sum_{t=n_3}^{l-1}\sum_{s=n_2}^{t-1} P_s\right].$$

Condition (2.1) implies that $V_n \to -\infty$ as $n \to \infty$ which is a contradiction with the fact that V_n is positive. Then $b = 0$ and this completes the proof. ∎

Remark 2.1. From Theorem 2.1, we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of $\{\rho_n\}$. Let $\rho_n = n^\lambda$, $n \ge n_0$ and $\lambda \ge 1$ is a constant. Hence we have the following results. □

Corollary 2.1. *Assume that all the assumptions of Theorem 2.1 hold, except that the condition (2.2) is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_2}^{n}\left[s^\lambda P_s - \frac{\left((s+1)^\lambda - s^\lambda\right)^2}{4\, s^\lambda (s-n_1)}\right] = \infty, \qquad \textit{for} \quad n_2 > n_1 \,.$$

Then, every solution $\{V_n\}$ of Eq.(1.1) oscillates or $\lim_{n\to\infty} V_n = 0$.

Theorem 2.2. *Assume that (2.1) holds. Let $\{\rho_n\}_{n=n_0}^{\infty}$ be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H_{m,n}\colon\, m \ge n \ge 0\}$ such that*

(i) $H_{m,m} = 0$ *for* $m \ge 0$,

(ii) $H_{m,n} > 0$ *for* $m > n \ge 0$,

(iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \le 0$ *for* $m \ge n \ge 0$.

If

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,n_2}}\sum_{n=n_2}^{m-1}\left[H_{m,n}\rho_nP_n-\frac{(\rho_{n+1})^2}{4\rho_n(n-n_1)}\left(h_{m,n}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right]=\infty\,. \tag{2.14}$$

for $n_2>n_1$, *where*

$$h_{m,n}=-\frac{\Delta_2H_{m,n}}{\sqrt{H_{m,n}}}\,,\qquad m>n\geq 0\ .$$

Then, every solution $\{V_n\}$ *of Eq.(1.1) oscillates or* $\lim_{n\to\infty}V_n=0$.

Proof: Proceeding as in Theorem 2.1, we assume that Eq.(1.1) has a nonoscillatory solution, say $V_n>0$ for all $n\geq n_1$. From the proof of Theorem 2.1 there are two possible cases. First, we consider the case Case (I). Defining again $\{w_n\}$ by (2.3), then from Theorem 2.1, we have $w_n>0$ and (2.9) holds. From (2.9) we have for $n\geq n_2$

$$\rho_nP_n\ \leq\ -\Delta w_n+\frac{\Delta\rho_n}{\rho_{n+1}}\,w_{n+1}-\frac{\bar{\rho}_n}{(\rho_{n+1})^2}\,w_{n+1}^2\ . \tag{2.15}$$

Therefore, we have

$$\begin{aligned}\sum_{n=n_2}^{m-1}H_{m,n}\rho_nP_n\ \leq\ &-\sum_{n=n_2}^{m-1}H_{m,n}\,\Delta w_n\ +\sum_{n=n_2}^{m-1}H_{m,n}\,\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}\\&-\sum_{n=n_2}^{m-1}H_{m,n}\,\frac{\bar{\rho}_n}{(\rho_{n+1})^2}\,w_{n+1}^2\ .\end{aligned} \tag{2.16}$$

which yields after summing by parts

$$\begin{aligned}\sum_{n=n_2}^{m-1}H_{m,n}\,\rho_nP_n\ \leq\ &H_{m,n_2}\,w_{n_2}\ +\sum_{n=n_2}^{m-1}w_{n+1}\,\Delta_2H_{m,n}\\&+\sum_{n=n_2}^{m-1}H_{m,n}\,\frac{\Delta\rho_n}{\rho_{n+1}}\,w_{n+1}\ -\sum_{n=n_2}^{m-1}H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2}\,w_{n+1}^2\end{aligned}$$

hence

$$\sum_{n=n_2}^{m-1} H_{m,n}\,\rho_n P_n \;=$$

$$= \; H_{m,n_2} w_{n_2} - \sum_{n=n_2}^{m-1} h_{m,n}\sqrt{H_{m,n}}\,w_{n+1} + \sum_{n=n_2}^{m-1} H_{m,n}\,\frac{\Delta\rho_n}{\rho_{n+1}}\,w_{n+1} - \sum_{n=n_2}^{m-1} H_{m,n}\,\frac{\bar{\rho}_n}{(\rho_{n+1})^2}\,w_{n+1}^2$$

$$= \; H_{m,n_2}\,w_{n_2} - \sum_{n=n_2}^{m-1}\left[\frac{\sqrt{H_{m,n}\;\bar{\rho}_n}}{\rho_{n+1}}\,w_{n+1} + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}\;\bar{\rho}_n}}\left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}}H_{m,n}\right)\right]^2 + \frac{1}{4}\sum_{n=n_2}^{m-1}\frac{(\rho_{n+1})^2}{\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2 .$$

Then,

$$\sum_{n=n_2}^{m-1}\left[H_{m,n}\,\rho_n P_n - \frac{(\rho_{n+1})^2}{4\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < \; H_{m,n_2} w_{n_2} \; .$$

which implies that

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,n_2}}\sum_{n=n_2}^{m-1}\left[H_{m,n}\,\rho_n P_n - \frac{(\rho_{n+1})^2}{4\;\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < w_{n_2} < \infty\,,$$

which contradicts (2.14). If the Case (II) holds we are then back to the proof of the second case of Theorem 2.1 to prove that $\lim_{n\to\infty} V_n = 0$. The proof is complete. ∎

Remark 2.2. By choosing the sequence $\{H_{m,n}\}$ in appropriate ways, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$H_{m,n} = (m-n)^\lambda\,, \qquad \lambda \ge 1, \quad m \ge n \ge 0 \;,$$

or

$$H_{m,n} = \left(\log\frac{m+1}{n+1}\right)^\lambda\,, \qquad \lambda \ge 1, \quad m \ge n \ge 0 \;,$$

or

$$H_{m,n} = (m-n)^{(\lambda)}, \qquad \lambda > 2, \quad m \geq n \geq 0 .$$

where $(m-n)^{(\lambda)} = (m-n)(m-n+1)\cdots(m-n+\lambda-1)$ and

$$\Delta_2 (m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda (m-n)^{(\lambda-1)} .$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Hence we have the following results. □

Corollary 2.2. *Assume that all the assumptions of Theorem 2.2 hold, except that the condition (2.14) is replaced by*

$$\lim_{m\to\infty}\sup \frac{1}{m^{\lambda}} \sum_{n=n_2}^{m-1} \left[(m-n)^{\lambda} \rho_n P_n - \frac{\rho_{n+1}^2}{4\rho_n (n-n_1)} \left(\lambda (m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{(m-n)^{\lambda}} \right)^2 \right] = \infty.$$

Then, every solution $\{V_n\}$ of Eq.(1.1) oscillates or $\lim_{n\to\infty} V_n = 0$.

Corollary 2.3. *Assume that all the assumptions of Theorem 2.2 hold, except that the condition (2.14) is replaced by*

$$\lim_{m\to\infty}\sup \frac{1}{\Big(\log(m+1)\Big)^{\lambda}} \sum_{n=n_2}^{m-1} \left[\left(\log \frac{m+1}{n+1} \right)^{\lambda} \rho_n P_n - \right.$$
$$\left. - \frac{\rho_{n+1}^2}{4\rho_n (n-n_1)} \left(\frac{\lambda}{n+1} \left(\log \frac{m+1}{n+1} \right)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{\left(\log \frac{m+1}{n+1} \right)^{\lambda}} \right)^2 \right] = \infty .$$

Then, every solution $\{V_n\}$ of Eq.(1.1) oscillates or $\lim_{n\to\infty} V_n = 0$.

Corollary 2.4. *Assume that all the assumptions of Theorem 2.2 hold, except that the condition (2.14) is replaced by*

$$\lim_{m\to\infty}\sup \frac{1}{m^{(\lambda)}} \sum_{n=n_2}^{m-1} (m-n)^{(\lambda)} \left[\rho_n P_n - \frac{\rho_{n+1}^2}{4\rho_n (n-n_1)} \left(\frac{\lambda}{m-n+\lambda-1} - \frac{\Delta \rho_n}{\rho_{n+1}} \right)^2 \right] = \infty .$$

Then, every solution $\{V_n\}$ of Eq.(1.1) oscillates or $\lim_{n\to\infty} V_n = 0$.

Remark 2.3. Our results can be extended to nonlinear difference equations of the form

$$\Delta^3 V_n + P_n f(V_{n+1}) = 0 , \qquad n \geq n_0 .$$

where f: $\mathbf{R} \to \mathbf{R}$ is continuous such that $uf(u) > 0$ for $u \neq 0$ and $f(u)/u \geqslant K > 0$ except that the term P_n is replaced by KP_n. □

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