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CRUISING IN A CENTRAL FORCE FIELD

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Recommended by Carlos Rocha

Abstract: We study a particle in a central force field which has a *cruise motion*, namely which is constrained to keep a constant kinetic energy. It is an integrable dynamics. We describe the global geometry of the problem by introducing special variables and a new time. This permits us to prove some general facts such as the existence and the orbital stability of circular motions. As an application a Bertrand-like problem is solved. Moreover, some noteworthy potential functions are dealt with as the Newton gravity of a single celestial body.

1 – Introduction

A natural Lagrangian function and the nonholonomic constraint to keep the kinetic energy constant, define a *natural thermostatted system*, see [Z2] and [DM]. These system have nonlinear constraints on the velocities (see [Z1] Section 4, [GZ], and the references therein).

Our main aim is to study the thermostatted motion of a (test) particle in a central force field. We ask the kinetic energy, so the speed, of the particle to be constant, something which seems related to *"cruise controls"*, at least loosely. In this case of a single particle, we prefer to speak of *cruise motion* then of thermostatted motion.

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We find the global dynamics by introducing a new time and new spatial variables. We get a separate two-dimensional Hamiltonian system. The Hamiltonian function simply gives the full 3-dimensional dynamics. Then we can prove some general facts such as the existence and the orbital stability of the circular motions. Then, we deal with the special Hooke and Newton cases and with the logarithmic potential which is also important to this theory. Moreover we solve a Bertrand-like problem for this dynamics.

The celebrated Bertrand theorem says that only Hooke and Newton potentials have the property that all bounded orbits are closed (for a complete proof see, for instance [F]). Here it is worth mentioning that the authors of [KH] obtained an interesting generalization of the Bertrand problem for manifolds of constant curvature. We are going to see that for our cruising particle no potential function has a similar property in a whole interval of speeds.

Finally, we consider the thermostatted two-body problem. The center of mass remains at rest if its initial speed vanishes, and the relative motion coincides with the cruise motion of a single particle endowed with the reduced mass.

The papers Morriss & Dettmann [MD] and Benettin & Rondoni [BR] contain some relevant discussion on physical applications of thermostatted systems. Moreover, the review Ruelle [R] is basic for other relevant mathematical and physical results for these systems.

2 – Cruise motions: equations and conservation laws

Our system is a material point of mass $m > 0$ in 3-space under the central field

$$\vec{F}\colon \mathbb{R}^3\backslash\{(0,0,0)\} \to \mathbb{R}^3\,, \qquad \vec{x} \mapsto \vec{F}(\vec{x}) = -\nabla U(\vec{x}) = -\frac{\mathcal{U}'(|\vec{x}|)}{|\vec{x}|}\,\vec{x}\,, \tag{2.1}$$

where the *potential energy* $U(\vec{x}) = \mathcal{U}(|\vec{x}|)$ and $\mathcal{U}$ is a class $C^2(\mathbb{R}^*_+)$ function on the strictly positive real numbers. Moreover, we assume the *kinetic energy* keeps the constant value $k > 0$ by a device which mathematically has for us the precise meaning of a non-holonomic constraint. So d'Alembert–Lagrange principle says that the motion $\vec{q}\colon I \to \mathbb{R}^3\backslash\{(0,0,0)\}$ (defined on the interval $I \subseteq \mathbb{R}$) is dynamically possible if there exists $\lambda\colon I \to \mathbb{R}$, the "multiplier", such that for all $t \in I$

$$\begin{cases} m\,\ddot{\vec{q}}(t) + \nabla U(\vec{q}(t)) \;=\; \lambda(t)\; m\,\dot{\vec{q}}(t)\,, \\ \dfrac{1}{2}\; m\,|\dot{\vec{q}}(t)|^2 = k\;. \end{cases} \tag{2.2}$$

Such $\vec{q}$ could be called *thermostatted motions*, however, for a single point we prefer to say *cruise motions*. By replacing the last equation with its time derivative we arrive at once at the following equation where we get rid of the multiplier λ, the central dot means the standard scalar product, and we omit the time dependence

$$m\,\ddot{\vec{q}} + \nabla U(\vec{q}) \;=\; \frac{\nabla U(\vec{q})\cdot\dot{\vec{q}}}{|\dot{\vec{q}}|^2}\,\dot{\vec{q}}\,. \tag{2.3}$$

The last equation has $|\dot{\vec{q}}|^2$ as first integral, if we just consider the solutions which correspond to the value $2\,k/m$ of this integral, we then have all cruise motions.

Now, let us introduce the new time variable which is $\sqrt{2k/m}$ the old one. Then we have the following equation which gives all cruise motions for $|\dot{\vec{q}}| = 1$

$$2\,k\,\ddot{\vec{q}} + \nabla U(\vec{q}) \;=\; \frac{\nabla U(\vec{q})\cdot\dot{\vec{q}}}{|\dot{\vec{q}}|^2}\,\dot{\vec{q}}\,. \tag{2.4}$$

Next, let us consider $U_k = U/2k$ and the equation

$$\ddot{\vec{q}} + \nabla U_k(\vec{q}) \;=\; \Big(\nabla U_k(\vec{q})\cdot\dot{\vec{q}}\Big)\,\dot{\vec{q}}\,. \tag{2.5}$$

If at some t_0 we have $|\dot{\vec{q}}(t_0)| = 1$, then $|\dot{\vec{q}}(t)| = 1$ at all t where the solution of the last equation is defined and this function is also a solution to (2.4), hence the typical cruise motion (parametrized by the new time variable).

Since we are dealing with central forces, it is natural to study how the angular momentum $\vec{q}\times\dot{\vec{q}}$ changes

$$\frac{d}{dt}\,(\vec{q}\times\dot{\vec{q}}) \;=\; \vec{q}\times\ddot{\vec{q}} = \Big(\nabla U_k(\vec{q})\cdot\dot{\vec{q}}\Big)\,(\vec{q}\times\dot{\vec{q}})\;. \tag{2.6}$$

This formula gives the following first integral that we call *cruise angular momentum*

$$e^{-U_k(\vec{q})}\,\vec{q}\times\dot{\vec{q}}\,. \tag{2.7}$$

In particular, this shows that *the cruise motions are planar motions*.

3 – Global 3-dimensional dynamics

As we have shown before, the cruise motion in a central field are planar. So, let us consider a plane with the origin at the centre of the force field referred to cartesian coordinates. Thus the cruise motion will be $t \mapsto (x(t), y(t))$ and

$t \mapsto (r(t), \theta(t))$ in polar coordinates. To simplify notations, in this section we assume $k = 1/2$, this choice is not restrictive to the present purposes; later the parameter k will have a crucial role dealing with the Bertrand-like problem. By taking into account (2.1), the equation (2.5) with $k = 1/2$ becomes

$$\begin{cases} \ddot{x}(t) + \dfrac{\mathcal{U}'(r(t))}{r(t)}\, x(t) \;=\; \mathcal{U}'(r(t))\, \dot{r}(t)\; \dot{x}(t) \\[2ex] \ddot{y}(t) + \dfrac{\mathcal{U}'(r(t))}{r(t)}\, y(t) \;=\; \mathcal{U}'(r(t))\, \dot{r}(t)\; \dot{y}(t) \end{cases} \quad \text{where } \; r(t) = \sqrt{x^2(t) + y^2(t)}\,. \tag{3.1}$$

This system has many solutions which are not of interest to us, namely those with $\dot{x}(t)^2 + \dot{y}(t)^2 \not\equiv 1$. To get rid of them, let us introduce a new coordinate ψ besides the polar ones, so

$$x = r\,\cos\theta\,, \quad y = r\,\sin\theta\,, \quad \dot{x} = \cos\psi\,, \quad \dot{y} = \sin\psi\;. \tag{3.2}$$

Then easy computations yield the following system

$$\begin{cases} \dot{r} \;=\; \cos(\psi - \theta) \\[1ex] \dot{\theta} \;=\; \dfrac{1}{r}\, \sin(\psi - \theta) \\[1ex] \dot{\psi} \;=\; \mathcal{U}'(r)\, \sin(\psi - \theta)\;. \end{cases} \tag{3.3}$$

The scalar angular momentum in polar coordinates is $r^2\, \dot{\theta}$, so the scalar cruise angular momentum is $e^{-\mathcal{U}(r)}\, r^2\, \dot{\theta}$ (see (2.7)), and the second equation in (3.3) gives the following first integral of the system (3.3)

$$e^{-\mathcal{U}(r)}\, r\, \sin(\psi - \theta)\;. \tag{3.4}$$

Moreover, the system (3.3) has an invariant measure with density $\mu = r\, e^{-\mathcal{U}(r)}$. Indeed, the divergence of μ times the vector field in (3.3) vanishes:

(3.5)
$$\frac{\partial}{\partial r}\Big(re^{-\mathcal{U}(r)}\cos(\psi{-}\theta)\Big) + \frac{\partial}{\partial\theta}\Big(e^{-\mathcal{U}(r)}\sin(\psi{-}\theta)\Big) + \frac{\partial}{\partial\psi}\Big(re^{-\mathcal{U}(r)}\mathcal{U}'(r)\sin(\psi{-}\theta)\Big) = 0\,.$$

The first integral (3.4), and the invariant measure, permit us to say that system (3.3) is *integrable*, see Theorem 13 in [A2], Ch. 4, and they give a well-known general technique to study our problem. However, we can also introduce new variables and a new time that make the global geometry of the cruise motions in a central field amazingly simple as we are going to see.

By introducing the new variables

$$\xi = \log r\,, \qquad \phi = \psi - \theta\,, \tag{3.6}$$

the system (3.3) becomes

$$\begin{cases} \dot{\xi} \;=\; e^{-\xi}\,\cos\phi \\ \dot{\phi} \;=\; e^{-\xi}\,\Big(\mathcal{V}'(\xi) - 1\Big)\,\sin\phi \qquad\quad \text{where} \quad \mathcal{V}(\xi) = \mathcal{U}(e^{\xi})\;. \\ \dot{\theta} \;=\; e^{-\xi}\,\sin\phi \end{cases} \tag{3.7}$$

Remark that the first two equations constitute a separate 2-dimensional system which can be written in Hamiltonian form if one multiplies the right hand side by $e^{2\xi-\mathcal{V}(\xi)}$ which never vanishes. So, let us do it and consider the new system whose solutions are reparametrizations of the old ones. So the new time variable defined by $d\tau/dt = e^{-2\xi+\mathcal{V}(\xi)}$ is considered, and, to remind this fact, we change the notation for time-derivatives

$$\begin{cases} \xi' \;=\; -\dfrac{\partial H}{\partial\phi}(\xi,\phi) \\ \phi' \;=\; \dfrac{\partial H}{\partial\xi}(\xi,\phi) \qquad\qquad \text{where} \quad H(\xi,\phi) = -e^{\xi-\mathcal{V}(\xi)}\,\sin\phi\;. \\ \theta' \;=\; -H(\xi,\phi) \end{cases} \tag{3.8}$$

By the 2π-periodicity, the first two equations can be viewed as an Hamiltonian system on the cylinder $\mathbb{R}\times\mathbb{S}^1$, so identifying the line $\phi = 2\pi$ with $\phi = 0$, and formula (3.8) is a system of differential equations on $\mathbb{R}\times\mathbb{T}^2$.

Consider the system of the first two equations

$$\begin{cases} \xi' \;=\; e^{\xi-\mathcal{V}(\xi)}\,\cos\phi \\ \phi' \;=\; e^{\xi-\mathcal{V}(\xi)}\Big(\mathcal{V}'(\xi) - 1\Big)\,\sin\phi \end{cases} \qquad\qquad \phi \bmod 2\pi\,, \tag{3.9}$$

and a level set of the Hamiltonian $H(\xi,\phi) = -c$. If $c = 0$ then the level set is the union of the lines $\phi = 0$ and $\phi = \pi$ and they are orbits of (3.9). Along the first line ξ strictly increases while it decreases along the second one. For $c < 0$, the level set is contained in the half cylinder $\pi < \phi < 2\pi$ and it is symmetric with respect to the line $\phi = 3\pi/2$, while for $c > 0$ we have $0 < \phi < \pi$ and symmetry with respect to the line $\phi = \pi/2$. In both cases the variable ξ satisfies the following two equivalent inequalities

$$e^{\xi-\mathcal{V}(\xi)} \geq |c| \quad\Longleftrightarrow\quad \xi - \mathcal{V}(\xi) \geq \log|c|\;. \tag{3.10}$$

So we have domains where the feasible orbits are localized.

Now, let us consider the critical points of H, namely the equilibria of (3.9). Those equilibria correspond to circular orbits of the problem under consideration. They exist whenever the equation $\mathcal{V}'(\xi) = 1$ has some solutions and, in this case, they are

$$(\xi, \phi) = (\hat{\xi}, \tfrac{1}{2}\pi)\,, \quad (\xi, \phi) = (\hat{\xi}, \tfrac{3}{2}\pi)\,, \qquad \text{where} \quad \mathcal{V}'(\hat{\xi}) = 1\ . \tag{3.11}$$

The Hessian matrix at the critical points is

$$\begin{aligned} H''(\hat{\xi}, \tfrac{1}{2}\pi) &= e^{\hat{\xi}-\mathcal{V}(\hat{\xi})} \begin{pmatrix} \mathcal{V}''(\hat{\xi}) & 0 \\ 0 & 1 \end{pmatrix}\,, \\ H''(\hat{\xi}, \tfrac{3}{2}\pi) &= e^{\hat{\xi}-\mathcal{V}(\hat{\xi})} \begin{pmatrix} -\mathcal{V}''(\hat{\xi}) & 0 \\ 0 & -1 \end{pmatrix}\ . \end{aligned} \tag{3.12}$$

So, if $\mathcal{V}''(\hat{\xi}) > 0$, then the critical point is a strict extremum and, in a neighborhood, the orbits are closed and contain the equilibrium in their interior, so we have a *center*. If $\mathcal{V}''(\hat{\xi}) < 0$, then the critical point is a saddle for the function H, and it is also a *saddle* for the system (3.9), namely it is an equilibrium where the Jacobian matrix of the vector field have eigenvalues with opposite signs

$$\mathcal{V}''(\hat{\xi}) > 0 \implies \textit{center}\,, \qquad \mathcal{V}''(\hat{\xi}) < 0 \implies \textit{saddle}\ . \tag{3.13}$$

The following observation is also of great importance. Stable circular orbits $\xi = \hat{\xi}$ are local maxima of the function $\exp(\xi - \mathcal{V}(\xi))$. We will use it when solving the Bertrand-like problem for cruising motions. We have just seen much of the dynamics of (3.9), so of system (3.8) which is easily related to (3.7). Going back to the original problem, we check at once that the straight line motions with $c = 0$ ($\phi = 0, \pi$) correspond to rectilinear motions also in the original coordinates. Moreover, we see that the equilibria of (3.9) correspond to circular motions of the original problem with the radius $\hat{r} = e^{\hat{\xi}}$ which satisfies the following equation

$$\mathcal{U}'(\hat{r}) = \frac{1}{\hat{r}}\ . \tag{3.14}$$

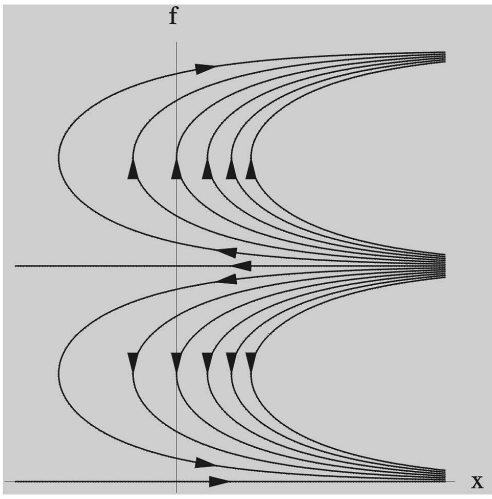
We may say that a circular cruise motion is *orbitally stable* whenever the corresponding equilibrium is Liapunov stable for (3.9). The condition (3.13) so gives

$$\begin{aligned} \mathcal{U}''(\hat{r}) > -\frac{1}{\hat{r}^2} &\implies \textit{orbital stability}\,, \\ \mathcal{U}''(\hat{r}) < -\frac{1}{\hat{r}^2} &\implies \textit{orbital instability}\ . \end{aligned} \tag{3.15}$$

4 – Remarkable potentials

Let us start from the trivial case of the lack of the potential function. So the function in (3.7) $\mathcal{V}(\xi) = \mathcal{U}(e^\xi) = 0$ and the phase portrait of the Hamiltonian system in (3.9) is

Free motion $\mathcal{U}(r) = 0$



Remark that the bottom line is $\phi = 0$ and the central one is $\phi = \pi$, moreover the line $\phi = 2\pi$, which is not shown in the picture, must be identified with $\phi = 0$ to get a cylinder.

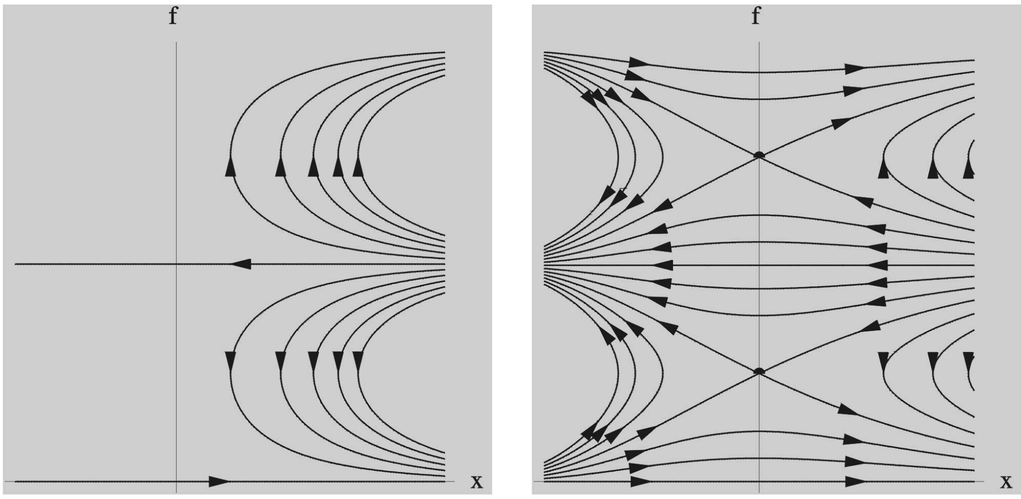
Next, let us consider the following potential

(4.1) *Coulomb* $(Q < 0)$ *and Newton* $(Q > 0)$ *potentials:* $\mathcal{U}(r) = -\dfrac{Q}{r}$,

which we have called Coulomb in the repulsive case and Newton in the attractive one (it could be Coulomb for opposite charges instead of Newton). The phase portraits of (3.9) are shown below. Notice that for the repulsive potential it looks like for the free motion above, while the attractive potential gives a new interesting dynamics.

The function $\mathcal{V}(\xi) = \mathcal{U}(e^\xi)$, see (3.7), for the Coulomb–Newton potential is $\mathcal{V}(\xi) = -Qe^{-\xi}$, so the condition (3.11) for the equilibria of the Hamiltonian 2-dimensional system (3.9) gives $Qe^{-\hat{\xi}} = 1$ and no equilibria for the repulsive case $Q < 0$, while for the Newton potential we have the two equilibria

$$(\xi, \phi) = (\log Q, \tfrac{1}{2}\pi)\,, \qquad (\xi, \phi) = (\log Q, \tfrac{3}{2}\pi)\,, \qquad Q > 0\ . \tag{4.2}$$



These equilibria are *saddles* as shown in the previous picture and proved by (remind (3.13))

$$\mathcal{V}''(\log Q) \;=\; -Q\, e^{-\log Q} \;=\; -1 \;<\; 0\ . \tag{4.3}$$

As we said in Section 3 these saddles correspond to orbitally unstable circular motions of radius $\hat{r} = e^{\hat{\xi}} = Q$. Let us consider the cruise motions which correspond to the orbits with $\xi \to -\infty$ in the future or in the past. The Hamiltonian

$$H(\xi, \phi) \;=\; -e^{\xi + Q e^{-\xi}}\, \sin\phi \tag{4.4}$$

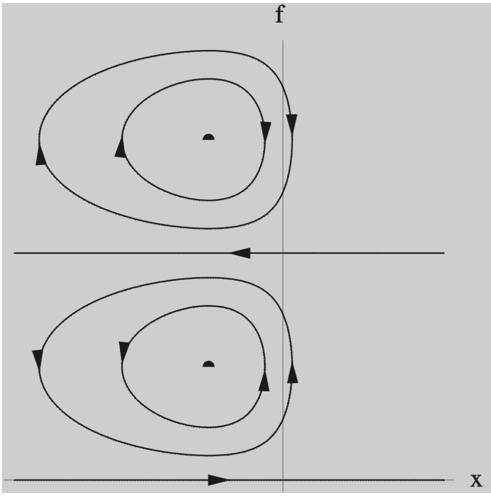
being a constant, we have that $\phi \to 0$ or $\phi \to \pi$ and, going back to (3.3) (and (3.6)) we have that $\dot{r} \to 1$ or $\dot{r} \to -1$, so the cruise motions collide with the particle which generates the field in finite time in the future or in the past (possibly both).

Next, let us consider Hooke's potential

$$\mathcal{U}(r) = \frac{K^2\, r^2}{2}\,, \qquad K > 0\ , \tag{4.5}$$

which gives the following phase portrait for the system (3.9) as we are going to prove.

Hooke's potential :



Indeed,

(4.6) $$\mathcal{V}(\xi) = \frac{K^2}{2}\, e^{2\xi}$$

so the equation $\mathcal{V}'(\xi) = 1$ gives the two equilibria

(4.7) $$(\xi, \phi) = (-\log K, \tfrac{1}{2}\pi)\,, \qquad (\xi, \phi) = (-\log K, \tfrac{3}{2}\pi)\ .$$

Moreover

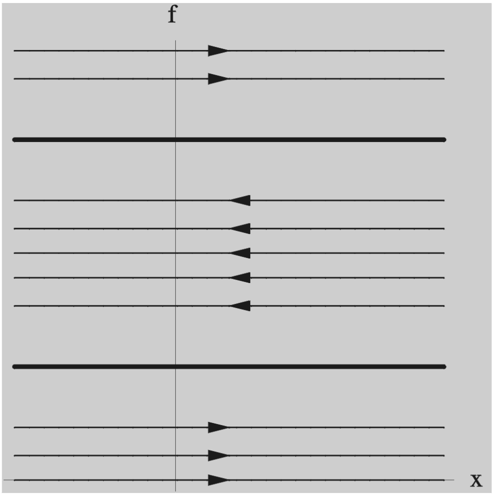
(4.8) $$\mathcal{V}''(-\log K) = 2\,K^2\, e^{-2\log K} = 2 > 0$$

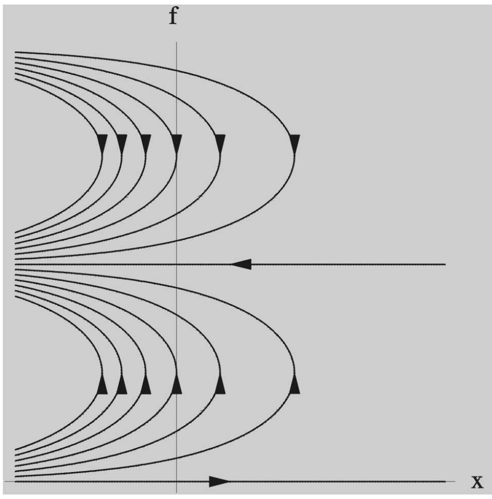
shows that they are both centers. A more careful argument on the level sets of the function H shows that all orbits, but the lines $\phi = 0$ and $\phi = \pi$, are actually closed. Indeed, the function $\mathcal{V}(\xi) - \xi$ is strictly convex (unlike $H(\xi, \phi)$), and has a minimum at $-\log K$ so the second inequality in (3.10) says that for $|c| > -\log K$ the level set $H(\xi, \phi) = -c$ is empty, for $|c| = -\log K$ we have the critical point and for $|c| < -\log K$ we have a compact set since ξ ranges in a compact interval $[\xi_1(c), \xi_2(c)]$. Being a regular curve (with no critical points) the conclusion follows. Remark that we can estimate $[\xi_1(c), \xi_2(c)]$. Back to the original 3-dimensional problem, we have that the circular orbits are orbitally stable. Of course we do not expect the other orbits to be generally closed, however the radial coordinate ranges in $[e^{\xi_1(c)}, e^{\xi_2(c)}]$.

Finally, the vector field (3.9) suggests to study the

(4.9) *log potential:* $\mathcal{U}(r) = a\,\log r\,,\quad H(\xi,\phi) = -e^{(1-a)\xi}\,\sin\phi\,,\qquad a\in\mathbb{R}\ .$

For $a<1$ the 2-dimensional phase Hamiltonian flow looks like the one of the free motion which is included as the particular case $a=0$. Below we can see on the left the special case $a=1$, with the two lines $\phi=\pi/2$ and $\phi=3\pi/2$ made of equilibria, and the case $a>1$ on the right. For the log potential, the system (3.8) can be explicitly integrated by means of elementary functions as we are going to see.





Let us consider first the exceptional case $a=1$. The system (3.8) becomes

$$(4.10)\qquad \begin{cases} \xi' \;=\; -\dfrac{\partial H}{\partial\phi}(\xi,\phi) \;=\; \cos\phi \\[2ex] \phi' \;=\; \dfrac{\partial H}{\partial\xi}(\xi,\phi) \;=\; 0 \\[2ex] \theta' \;=\; -H(\xi,\phi) = \sin\phi\ . \end{cases}$$

The solution is

$$(4.11)\qquad \phi(\tau)=\phi_0\,,\qquad \xi(\tau)=\xi_0+\tau\cos\phi_0\,,\qquad \theta(\tau)=\theta_0+\tau\sin\phi_0\ .$$

Back to the polar coordinates

$$(4.12)\qquad r(\tau)=r_0\,e^{\tau\cos\phi_0}\,,\qquad \theta(\tau)=\theta_0+\tau\sin\phi_0\ .$$

Remind that

$$\frac{d\tau}{dt} = e^{-2\xi+\mathcal{V}(\xi)} = e^{-2\log r+\mathcal{U}(r)} = \frac{1}{r} = \frac{1}{r_0}\, e^{-\tau\cos\phi_0} \; . \tag{4.13}$$

So we have

$$r(t) = r_0\,, \qquad \theta(t) = \theta_0 \pm \frac{1}{r_0}(t-t_0) \quad \text{if} \;\; \cos\phi_0 = 0 \tag{4.14}$$

(of course, the equilibria of the two straight lines in the left picture above correspond to uniform circular motions). Otherwise,

$$r(t) = r_0 + (t-t_0)\cos\phi_0\,, \quad \theta(t) = \theta_0 + (\tan\phi_0)\,\log\Big(1+(t-t_0)\frac{\cos\phi_0}{r_0}\Big)\,. \tag{4.15}$$

We have just integrated the case $a = 1$. Now, let us study the generic case $a \neq 1$. As we know, the straight lines $\phi = 0$ and $\phi = \pi$ give straight lines also in dimension 3. Along them $\dot{r} = \pm 1$, so we have uniform straight line motions. Otherwise we can write system (3.8) in the following form where we fix $-c$ as the value of the Hamiltonian function

$$\begin{cases} \xi' = c\cot\phi \\ \phi' = (a-1)\,c \\ \theta' = c\;. \end{cases} \tag{4.16}$$

This easily gives

$$\xi(\tau) = \xi_0 + \frac{1}{a-1}\,\log\frac{\sin\Big(\phi_0 + (a-1)\,c\,\tau\Big)}{\sin\phi_0}\,, \qquad \theta(\tau) = \theta_0 + (a-1)\,c\,\tau\,, \tag{4.17}$$

$$r(\tau) = r_0\left(\frac{\sin\Big(\phi_0 + (a-1)\,c\,\tau\Big)}{\sin\phi_0}\right)^{\frac{1}{a-1}}, \qquad \theta(\tau) = \theta_0 + (a-1)\,c\,\tau\;. \tag{4.18}$$

Back to the old time variable

$$\frac{d\tau}{dt} = e^{-2\xi+\mathcal{V}(\xi)} = e^{(a-2)\xi} = r^{a-2} = r_0^{a-2}\left(\frac{\sin\Big(\phi_0 + (a-1)\,c\,\tau\Big)}{\sin\phi_0}\right)^{\frac{a-2}{a-1}}\,. \tag{4.19}$$

We guess that this equation is not elementarily integrable for generic a. We end our discussion with the special case $a = 2$:

$$r(t) = r_0\,\frac{\sin\Big(\phi_0 + c\,(t-t_0)\Big)}{\sin\phi_0}\,, \qquad \theta(t) = \theta_0 + c\,(t-t_0)\;. \tag{4.20}$$

All orbits, but the straight lines discussed above, are circles passing through the origin (which is not included in the configuration space).

5 – Action-angle variables and the Bertrand-like problem

System (3.8) is obviously integrable. Then the Liouville–Arnol'd theory, see Theorem 8 of Ch. 4 in [A2], gives

Proposition 5.1. *Let the component M of the level set $H(\xi,\phi) = -c$ in $\mathbb{R}\times\mathbb{S}^1$ be compact, and let M contain no critical points of H. Then there exist a neighborhood Ω of M and a canonical map $f\colon\ \Omega \to\ ]I_1, I_2[\ \times \mathbb{S}^1$ (with $\det f' \equiv 1$) such that, in the new variables $(I,\zeta) = f(\xi,\phi)$, the Hamiltonian function depends on I only, so that $(H\circ f^{-1})(I,\zeta) = \mathcal{H}(I)$, and, in the variables (I,ζ,θ), the system (3.8) becomes*

$$(5.1)\qquad \begin{cases} I' \,=\, 0 \\ \zeta' \,=\, \omega(I) \\ \theta' \,=\, -\mathcal{H}(I) \end{cases} \qquad \text{where} \quad \omega(I) = \frac{d\mathcal{H}}{dI}(I)\ .$$

The phase space of this system is $]I_1, I_2[\ \times\ \mathbb{T}^2$ and the solutions are standard windings on the torus with frequencies $\omega(I)$ and $-\mathcal{H}(I)$.

In a standard terminology, (5.1) is a two-frequency system. By using this framework, we are going to prove the following theorem whose hypothesis is satisfied for instance by Hooke's potential (4.5) as easily checked.

Theorem 5.2. *Bertrand-like theorem. Let a cruising particle in a central force field have a stable circular orbit for each value of the kinetic energy in some open interval. Then there do not exist potentials for which all the neighboring trajectories are closed.*

Proof: Let us fix a certain value of the constant c (for simplicity we assume that $c > 0$) and consider the following region in the 3-dimensional phase space of equation (3.8):

$$(5.2)\qquad \Big\{\xi \in [\xi_1(c), \xi_2(c)]\,,\ \ (\phi,\theta) \in \mathbb{T}^2\Big\}\ ,$$

where $\xi_1(c), \xi_2(c)$ are two neighboring roots of the equation $\exp\left(\xi - \mathcal{V}(\xi)\right) = c$ such that the value $\hat{\xi}$ corresponding to a stable circular orbit belongs to the interval $(\xi_1(c), \xi_2(c))$. Let us assume that all the trajectories in the above region are closed. This means that

$$\frac{\omega(I)}{\mathcal{H}(I)} = \frac{p}{q} \in \mathbb{Q}\,. \tag{5.3}$$

The left-hand side of (5.3), being a continuous function, may have only discrete values. Therefore,

$$\frac{\omega(I)}{\mathcal{H}(I)} = \frac{d}{dI} \log \mathcal{H}(I) = -\delta = \text{const} \tag{5.4}$$

for any value of the action variable I. Consequently,

$$\mathcal{H}(I) = -\hat{c}\, e^{-\delta I} \tag{5.5}$$

where $\hat{c}$ is a suitable real constant. Since I is an integral of (5.1) there is a one-to-one correspondence between the constants of motion I and c. By fixing $\mathcal{H}(I) = -c$, we get

$$I = -\frac{1}{\delta} \log \frac{c}{\hat{c}}\,. \tag{5.6}$$

The angle-action variables (I, ζ) can be constructed as follows (see [A2]). There exists a smooth generating function $S = S(I, \phi)$ such that

$$\xi = \frac{\partial S}{\partial \phi}(I, \phi)\,, \qquad \zeta = \frac{\partial S}{\partial I}(I, \phi) \tag{5.7}$$

and satisfying the Hamilton–Jacobi equation in the following form

$$H\left(\frac{\partial S}{\partial \phi}(I, \phi)\,,\, \phi\right) = \mathcal{H}(I) = -\hat{c}\, e^{-\delta I}\,. \tag{5.8}$$

Let M_c be the 2-dimensional domain surrounded by the curve $\gamma_c = \{H(\xi, \theta) = -c\}$. Then the action variable I can be constructed as follows

$$I = \frac{1}{2\pi} \int_{M_c} d\xi\, d\theta = \frac{S_c}{2\pi}\,, \tag{5.9}$$

where S_c is the area of M_c. Moreover, let us assume that there are no other critical points inside M_c besides $(\hat{\xi}, \frac{\pi}{2})$, $\hat{\xi} = \log \hat{r}$. Then

$$S_c = \int_{M_c} d\xi\, d\theta = 2 \int_{\xi_1(c)}^{\xi_2(c)} \arccos c\, e^{-\xi + \mathcal{V}(\xi)}\, d\xi\,. \tag{5.10}$$

Formula (5.9) shows that I must be non-negative. The value $I = 0$ ($c = \hat{c}$) corresponds to the circular orbit $\xi = \hat{\xi}$. Since the orbit is stable, $\hat{c}$ is the maximal value of the function $\exp(\xi - \mathcal{V}(\xi))$ on the segment $[\xi_1(c), \xi_2(c)]$ and, therefore, $c < \hat{c}$. Then it immediately follows from (5.6) that $\delta > 0$. Let us denote $I = \epsilon^2$ and calculate S_c for a small region surrounding the stable circular orbit ($c \approx \hat{c}$). Hence, $\epsilon > 0$ is a small parameter. The following identity is almost obvious

$$c\, e^{-\xi + \mathcal{V}(\xi)} = e^{\mathcal{V}(\xi) - \xi - (\mathcal{V}(\hat{\xi}) - \hat{\xi}) - \delta\, \epsilon^2} \; . \tag{5.11}$$

Let us denote $F(\xi) = \mathcal{V}(\xi) - \xi - (\mathcal{V}(\hat{\xi}) - \hat{\xi})$. This function has the following properties

$$F(\hat{\xi}) = 0\,, \qquad F'(\hat{\xi}) = \mathcal{V}'(\hat{\xi}) - 1 = 0\,, \qquad F''(\hat{\xi}) = \mathcal{V}''(\hat{\xi}) > 0\;. \tag{5.12}$$

To proceed further we need some auxiliary technical results.

Lemma 5.3. *There exists a smooth change of variable* $\eta = \eta(\xi)$, *a bijection from* $]\xi_1(c), \xi_2(c)[$ *onto* $]-\sqrt{\delta}\,\epsilon, \sqrt{\delta}\,\epsilon[$, *such that in the new variable*

$$F(\xi(\eta)) = \eta^2 \tag{5.13}$$

and

$$\xi(0) = \hat{\xi}, \quad \eta(\hat{\xi}) = 0\;. \tag{5.14}$$

Proof: The construction is very simple. Indeed, $F(\xi) \geq 0$ on $]\xi_1(c), \xi_2(c)[$ and $\hat{\xi}$ is the only point in the above interval, for which $F(\xi) = F'(\xi) = 0$. Hence,

$$\eta(\xi) = \begin{cases} \sqrt{F(\xi)}\,, & \xi \geq \hat{\xi} \\ -\sqrt{F(\xi)}\,, & \xi \leq \hat{\xi}\;. \end{cases} \tag{5.15}$$

For every value of $\xi \neq \hat{\xi}$

$$\eta'(\xi) = \operatorname{sgn}(\xi - \hat{\xi})\, \frac{F'(\xi)}{2\sqrt{F(\xi)}} > 0\;. \tag{5.15}$$

By using asymptotic expansions of the numerator and the denominator of the fracture in (5.15), it is also possible to calculate the limit

$$\eta'(\hat{\xi}) = \lim_{\xi \to \hat{\xi}} \eta'(\xi) = \frac{F''(\hat{\xi})}{2} > 0\;. \tag{5.16}$$

Hence, the function $\eta(\xi)$ is increasing. This means that there is a smooth inverse function $\xi = \xi(\eta)$ and Lemma 5.3 is proved. ∎

Lemma 5.4. *The derivatives of the inverse function $\xi(\eta)$ can be computed as follows*

$$\xi'(0) \;=\; \sqrt{2}\left(F''(\hat{\xi})\right)^{-\frac{1}{2}} \;, \tag{5.17}$$

$$\xi''(0) \;=\; -\frac{2}{3}\left(F''(\hat{\xi})\right)^{-2} F'''(\hat{\xi}) \;, \tag{5.18}$$

$$\xi'''(0) \;=\; \frac{\sqrt{2}}{6}\left(F''(\hat{\xi})\right)^{-\frac{7}{2}}\left(5\left(F'''(\hat{\xi})\right)^2 - 3\,F^{(4)}(\hat{\xi})\,F''(\hat{\xi})\right) \;. \tag{5.19}$$

Formulas (5.17–5.19) can be obtained by substituting a formal Maclaurin expansion

$$\xi(\eta) \;=\; \hat{\xi} \;+\; \xi'(0)\eta \;+\; \frac{\xi''(0)}{2}\eta^2 \;+\; \frac{\xi'''(0)}{6}\eta^3 \;+\; o(\eta^3) \tag{5.20}$$

into the following expression

$$\begin{aligned} F(\xi) \;&=\; \frac{F''(\hat{\xi})}{2}(\xi-\hat{\xi})^2 + \frac{F'''(\hat{\xi})}{6}(\xi-\hat{\xi})^3 + \frac{F^{(4)}(\hat{\xi})}{24}(\xi-\hat{\xi})^4 + o(\xi-\hat{\xi})^4 \\ &=\; \eta^2 \end{aligned} \tag{5.21}$$

and equating terms at different powers of η.

Lemma 5.5. *The following asymptotic formula holds*

$$\arccos e^{-x^2} \;=\; \sqrt{2}\,|x|\left(1 \;-\; \frac{1}{6}x^2 \;+\; o(x^2)\right) \qquad \textit{as} \quad x \to 0 \;. \tag{5.22}$$

This result can be proved by standard methods of asymptotic analysis.

By using the new variable η, formula (5.10) reads

$$S_c \;=\; 2\int_{-\sqrt{\delta}\epsilon}^{\sqrt{\delta}\epsilon} \xi'(\eta)\ \arccos\ e^{\eta^2-\delta\epsilon^2}\,d\eta \tag{5.23}$$

or, by carrying out another change of variables $\eta = \sqrt{\delta}\,\epsilon\sin\zeta$,

$$S_c \;=\; 2\sqrt{\delta}\,\epsilon\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi'(\epsilon\sqrt{\delta}\sin\zeta)\ \cos\zeta\ \arccos\ e^{-\delta\epsilon^2\cos^2\zeta}\ d\zeta \;. \tag{5.24}$$

Since $I = \epsilon^2$, by exploiting expansion (5.20) and the formula for the area S_c (5.24), we can get the following asymptotic formula

$$\epsilon^2 = \epsilon^2 \frac{\delta}{\sqrt{2}} \xi'(0) + \epsilon^4 \frac{\delta^2\sqrt{2}}{16}\Big(\xi'''(0) - \xi'(0)\Big) + o(\epsilon^4) \ . \tag{5.25}$$

Formula (5.25) results in two important equalities

$$\xi'(0) \frac{\delta}{\sqrt{2}} = 1 \tag{5.26}$$

and

$$\xi'''(0) - \xi'(0) = 0 \ . \tag{5.27}$$

Let us now consider a certain interval of kinetic energy values and a 'representative' k from this interval. Then it follows immediately from (5.26) that

$$F''(\hat{\xi}) = \delta^2 \tag{5.28}$$

and does not depend on k. Now we should take into account that

$$F(\xi) = F_k(\xi) = \mathcal{V}_k(\xi) - \xi - \Big(\mathcal{V}_k(\xi) - \hat{\xi}\Big) = \frac{1}{2k}\Big(\mathcal{V}(\xi) - \mathcal{V}(\hat{\xi})\Big) - (\xi - \hat{\xi}) \, . \tag{5.29}$$

Since $\mathcal{V}_k'(\hat{\xi}) = 1$ ($\hat{\xi}$ corresponds to stable circular orbits),

$$\mathcal{V}'(\hat{\xi}) = 2k \ . \tag{5.30}$$

Then (5.28) gives us that

$$F_k''(\hat{\xi}) = \frac{1}{2k}\mathcal{V}''(\hat{\xi}) = \frac{\mathcal{V}''(\hat{\xi})}{\mathcal{V}'(\hat{\xi})} = C_1 = \text{const} \ , \tag{5.31}$$

where C_1 does not depend on k. If k varies, $\hat{\xi}$ also varies, and (5.31) may be considered as an ordinary differential equation

$$\mathcal{V}''(\xi) = C_1 \mathcal{V}'(\xi) \ , \tag{5.32}$$

having evident general solutions

$$\mathcal{V}_{(1)}(\xi) = C_2\,\xi + C_3 \quad \Big(\mathcal{U}_{(1)}(r) = C_2 \log r + C_3\Big) \ , \qquad \text{if } C_1 = 0 \ , \tag{5.33}$$

and

$$\mathcal{V}_{(2)}(\xi) = C_2\, e^{C_1 \xi} + C_3 \quad \Big(\mathcal{U}_{(2)}(r) = C_2\, r^{C_1} + C_3\Big) \ , \qquad \text{if } C_1 \neq 0 \ . \tag{5.34}$$

Thermostatted systems with potentials of the (5.33) type do not have circular orbits at all. To examine potentials $\mathcal{V}_{(2)}$, let us exploit condition (5.27). In terms of the function $F_k(\xi)$ it becomes

$$5\left(F_k'''(\hat{\xi})\right)^2 - 3\,F_k^{(4)}(\hat{\xi})\,F_k''(\hat{\xi}) - 6\left(F_k''(\hat{\xi})\right)^3 = 0 \tag{5.35}$$

or, by passing to original potentials,

$$\mathcal{V}'(\hat{\xi})\left(5\left(\mathcal{V}'''(\hat{\xi})\right)^2 - 3\,\mathcal{V}^{(4)}(\hat{\xi})\,\mathcal{V}''(\hat{\xi})\right) - 6\left(\mathcal{V}''(\hat{\xi})\right)^3 = 0\,. \tag{5.36}$$

After substituting the potential $\mathcal{V}_{(2)}$ (see (5.34)) into (5.36), we arrive at the following equation

$$2\,C_2^3\,C_1^6\,e^{3\,C_1\,\hat{\xi}}\,(C_1 - 3) = 0\,, \tag{5.37}$$

which results immediately in the equality

$$C_1 = 3\,, \tag{5.38}$$

since the parameter $\hat{\xi}$ may vary. So only cubic potentials are suspect to satisfy the Bertrand conditions. Nevertheless, it is not the case. We will show that thermostatted systems with cubic potentials do not have periodic orbits in a neighborhood of the stable circular one and explain why the method used before fails. Indeed, let us consider the cruising motion with a cubic potential $\mathcal{U}(r) = a\,r^3/3$ (a free constant does not play an important role). Evidently, $a > 0$. Otherwise, the system under consideration does not have circular orbits. In polar coordinates the equations of motion are

$$\begin{cases} \ddot{r}(t) - r\,\dot{\theta}^2 + a\,r^2 = \dfrac{a}{2k}\,r^2\,\dot{r}^2 \\[2ex] 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} = \dfrac{a}{2k}\,r^3\,\dot{r}\,\dot{\theta}\,, \end{cases} \tag{5.39}$$

which have two first integrals

$$\dot{r}^2 + r^2\,\dot{\theta}^2 = 2k \tag{5.40}$$

and

$$r^2\,\dot{\theta} = c\,\exp\left(\tfrac{a}{6k}\,r^3\right)\,. \tag{5.41}$$

By combining the first equation of (5.39) and integrals (5.40) and (5.41), we obtain the following reduced equation

$$\ddot{r} + \frac{d}{dr}\left(c^2\,\frac{\exp\left(\frac{a}{3k}\,r^3\right)}{2r^2}\right) = 0\,, \tag{5.42}$$

which, in turn, has the following first integral

$$\dot{r}^2 + \left(c^2\,\frac{\exp\left(\frac{a}{3k}\,r^3\right)}{r^2}\right) = h^2\,. \tag{5.43}$$

The first integral (5.43) is analog to the total energy. By combining integrals (5.41) and (5.43) and carrying out the change of variables

$$r = \left(\frac{6\,k}{a}\right)^{\frac{1}{3}} u\,, \tag{5.44}$$

we arrive at the following differential equation

$$\pm\,\frac{u^2\,\exp\left(\frac{u^3}{2}\right)\,du}{\sqrt{\alpha\,u^2 - \exp(u^3)}} = d\theta\,, \tag{5.45}$$

where

$$\alpha = \frac{h^2}{c^2}\left(\frac{6\,k}{a}\right)^{\frac{1}{3}} \tag{5.46}$$

is a new parameter of the problem. Now we can see that for the exceptional case of the cubic potentials the two independent constants of integration are reduced to only one. This is the reason why the previous method detected the cubic case as suspect. Equation (5.45) can be slightly simplified by means of the substitution

$$v = \exp\left(\frac{u^3}{2}\right). \tag{5.47}$$

If the constant α varies, the variable v varies between two roots $v_1(\alpha)$ and $v_2(\alpha)$ of the following transcendent equation

$$\alpha\,\log^{\frac{2}{3}} v^2 - v^2 = 0\,. \tag{5.48}$$

The roots are obviously simple. Let $\hat{\alpha}$ be the minimal value of the parameter α, for which equation (5.48) has real solutions. It is not difficult to compute the value of $\hat{\alpha}$ and the corresponding value of the variable $v = \hat{v}$

$$\hat{\alpha} = \left(\frac{3\,e}{2}\right)^{\frac{2}{3}}, \quad \hat{v} = e^{\frac{1}{3}}\,. \tag{5.49}$$

These values clearly correspond to stable circular orbits of the system under consideration. Consider the so-called apsis angle

$$\theta_\star(\alpha) \;=\; \frac{2}{3}\int_{v_1(\alpha)}^{v_2(\alpha)} \frac{dv}{\sqrt{\alpha\,\log^{\frac{2}{3}} v^2 \;-\; v^2}}\;. \tag{5.50}$$

It is the minimal angle between two directions from the center to the nearest point and the most distant one of the trajectory. The trajectory is closed if and only if the apsis angle is rationally commensurable with 2π. The apsis angle is a continuous function of α. Consequently, for systems satisfying the Bertrand condition, $\theta_\star$ must be constant for a certain region of values of α. Let us compute $\lim_{\alpha\to\hat\alpha}\theta_\star(\alpha)$. We need to introduce a new small parameter $\mu = \sqrt{\alpha-\hat\alpha}$. Further let us compute the roots $v_1(\alpha)$, $v_2(\alpha)$ of equation (5.48) in a form of power series with respect to μ. Evidently, if μ is rather small, then $v_2(\alpha)-v_1(\alpha)\sim\mu$. By substituting expansions $v_{1,2}(\alpha) = \hat v + v^{(1)}_{1,2}\,\mu + o(\mu)$, where $v^{(1)}_{1,2}$ are certain real numbers to be computed, and $\alpha = \hat\alpha+\mu^2$ into (5.48), we can find that

$$v^{(1)}_{1,2} \;=\; \mp\, 2^{\frac{1}{3}}\, 3^{-\frac{5}{6}}\;. \tag{5.51}$$

Let us perform a linear change of variables mapping magnitudes $v_1(\alpha)$ and $v_2(\alpha)$ to -1 and 1 respectively. An appropriate formula reads

$$v \;=\; v_2 + \frac{1}{2}\,(v_2-v_1)\,(w-1)\;. \tag{5.52}$$

Then the function under the radical in (5.50) can be rewritten as follows

$$\alpha\,\log^{\frac{2}{3}} v^2 - v^2 \;=\; \mu^2\,(1-w^2)\,\Phi(w,\mu)\;, \tag{5.53}$$

where $\Phi(w,\mu)$ is strictly positive for $w\in[-1,1]$ and $\mu>0$. It is not difficult to show that

$$\Phi(w,\mu) \;=\; \left(\frac{2}{3}\right)^{\frac{2}{3}} + o(1) \qquad \text{as}\quad \mu\to 0\;. \tag{5.54}$$

Therefore,

$$\theta_\star(\alpha) \;=\; \frac{1}{3\mu}\int_{-1}^{1}\frac{(v_2-v_1)\,dw}{\sqrt{(1-w^2)\,\Phi(w,\mu)}} \;=\; \frac{2\pi\sqrt{3}}{9} + o(1), \qquad \text{as}\quad \mu\to 0\;. \tag{5.55}$$

So the apsis angle is not rationally commensurable with 2π even if $\theta_\star(\alpha)$ is constant. Thus the thermostatted systems with cubic potentials do not satisfy the Bertrand conditions. Theorem 5.2 is proved. ∎

6 – Thermostatted two-body problem

Consider two material points with masses m_1, m_2 and with radius vectors $\vec{x}_1, \vec{x}_2$ in a fixed coordinate frame. Let $\vec{F}$ be the force acting on the first particle and $-\vec{F}$ the one acting on the second particle. Furthermore, let us assume that the system obeys the nonholonomic constraint

$$m_1\,|\dot{\vec{x}}_1|^2 + m_2\,|\dot{\vec{x}}_2|^2 = 2\,k\ . \tag{6.1}$$

The motions which are dynamically possible can be called *thermostatted motions* since the kinetic energy (the "temperature") is constant. By the d'Alembert–Lagrange principle, the equations of the thermostatted motions are (6.1) and

$$m_1\,\ddot{\vec{x}}_1 = \vec{F} + \lambda\,m_1\,\dot{\vec{x}}_1\,, \qquad m_2\,\ddot{\vec{x}}_2 = -\vec{F} + \lambda\,m_2\,\dot{\vec{x}}_2\ . \tag{6.2}$$

We take the time derivative of (6.1) and use (6.2) to determine the multiplier λ. In this way we get the following equations without λ

$$\ddot{\vec{x}}_1 = \frac{\vec{F}}{m_1} + \frac{\vec{F}\cdot(\dot{\vec{x}}_2 - \dot{\vec{x}}_1)}{m_1|\dot{\vec{x}}_1|^2 + m_2|\dot{\vec{x}}_2|^2}\,\dot{\vec{x}}_1\,, \qquad \ddot{\vec{x}}_2 = -\frac{\vec{F}}{m_2} + \frac{\vec{F}\cdot(\dot{\vec{x}}_2 - \dot{\vec{x}}_1)}{m_1|\dot{\vec{x}}_1|^2 + m_2|\dot{\vec{x}}_2|^2}\,\dot{\vec{x}}_2\ . \tag{6.3}$$

The left hand side of (6.1) is a first integral, we just have to consider the solutions for which (6.1) is true. Equivalently, we may consider the solutions of a simpler system

$$\ddot{\vec{x}}_1 = \frac{\vec{F}}{m_1} + \frac{1}{2k}\,\vec{F}\cdot(\dot{\vec{x}}_2 - \dot{\vec{x}}_1)\,\dot{\vec{x}}_1\,, \qquad \ddot{\vec{x}}_2 = -\frac{\vec{F}}{m_2} + \frac{1}{2k}\,\vec{F}\cdot(\dot{\vec{x}}_2 - \dot{\vec{x}}_1)\,\dot{\vec{x}}_2\ , \tag{6.4}$$

for which (6.1) is true at some time t_0, then is always true. Now, let

$$\vec{x} = \vec{x}_1 - \vec{x}_2\,, \quad M = m_1 + m_2\,, \quad \vec{X} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{M}\,, \quad m = \frac{m_1 m_2}{M}\ , \tag{6.5}$$

the relative position of the two points, the total mass, the center of mass radius vector, and the reduced mass respectively. Then (6.4) give

$$\ddot{\vec{x}} = \frac{\vec{F}}{m} - \frac{1}{2k}\,(\vec{F}\cdot\dot{\vec{x}})\,\dot{\vec{x}}\,, \qquad \ddot{\vec{X}} = -\frac{1}{2k}\,(\vec{F}\cdot\dot{\vec{x}})\,\dot{\vec{X}}\ . \tag{6.6}$$

Moreover (6.1) is equivalent to

$$m\,|\dot{\vec{x}}|^2 + M\,|\dot{\vec{X}}|^2 = 2\,k\ . \tag{6.7}$$

Now, let $\vec{F}$ depend only on $\vec{x}$ and be as in (2.1), namely a central force field. Then we have the following equations

$$\ddot{\vec{x}} = -\frac{\nabla U(\vec{x})}{m} + \frac{1}{2k}\left(\nabla U(\vec{x}) \cdot \dot{\vec{x}}\right)\dot{\vec{x}}\,, \qquad \ddot{\vec{X}} = \frac{1}{2k}\left(\nabla U(\vec{x}) \cdot \dot{\vec{x}}\right)\dot{\vec{X}}\;. \tag{6.8}$$

We have the following vector first integral which shows that the center of mass motion is on a straight line (though generally non uniform)

$$e^{-\mathcal{U}(r)/2k}\,\dot{\vec{X}} = \vec{L}\;. \tag{6.9}$$

If the initial speed of the center of mass vanishes, then it stays at rest and the first equation in (6.8) with the new time variable which is $\sqrt{2k/m}$ the old one, becomes

$$\ddot{\vec{x}} + \nabla U_k(\vec{x}) = \left(\nabla U_k(\vec{x}) \cdot \dot{\vec{x}}\right)\dot{\vec{x}}\,, \qquad \text{with} \quad U_k = \frac{U}{2k}\;. \tag{6.10}$$

We must take into account (6.7), which can be now written as $|\dot{\vec{x}}| = 1$. This last equation holds whenever it is satisfied at the initial time. If we recall (2.5) and the sentence which follows it, we then have

Proposition 6.1. *The thermostatted two-bodies which interact with a force (2.1) where $\vec{x}$ is the relative position, have the center of mass at rest if its initial speed vanishes. In this case, their relative motion coincides with the cruise motion of a single particle which has the reduced mass.*

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REFERENCES

[A1] Arnol'd, V.I. – *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1978.

[A2] Arnol'd, V.I. (Ed.) – *Dynamical Systems III*, Encyclopaedia of Mathematical Sciences, Vol. 3, Springer-Verlag, 1988.

[BR] Benettin, G. and Rondoni, L. – A new model for the transport of particles in a thermostatted system, *Math. Phys. Electron. J.,* 7 (2001), Paper 3, 22 pp. (electronic).

[DM] Dettmann, C.P. and Morriss, G.P. – Hamiltonian formulation of the Gaussian isokinetic thermostat, *Phys. Rev. E,* 54 (1996), 2495–2500.

[F] Furta, S. – *Two-Body Problem from Different Points of View* (in Russian), ElfLtd, Moscow, 1997.

[GZ] Gorni, G. and Zampieri, G. – Time reversibility and energy conservation for Lagrangian systems with nonlinear nonholonomic constraints, *Rep. Math. Phys.,* 45 (2000), 217–227.

[KH] Kozlov, V.V. and Harin, A. – Kepler's problem in constant curvature spaces, *Celestial Mech. Dynam. Astronom.,* 54 (1992), 393–399.

[MD] Morriss, G.P. and Dettmann, C.P. – Thermostats: analysis and application, *Chaos,* 8 (1998), 321–336.

[O] Oliva, W.M. – *Geometric Mechanics*, Lecture Notes in Mathematics 1798, Springer-Verlag, 2002.

[R] Ruelle, D. – Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics, *J. Stat. Phys.,* 95 (1999), 393–468.

[Z1] Zampieri, G. – Nonholonomic versus vakonomic dynamics, *J. Diff. Eqns.,* 163 (2000), 335–347.

[Z2] Zampieri, G. – Dynamic convexity for natural thermostatted systems, *J. Diff. Eqns.,* 191 (2003), 55–66.

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