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CONGRUENCES OF LINES WITH ONE-DIMENSIONAL FOCAL LOCUS *

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Abstract: In this article we obtain the classification of the congruences of lines with one-dimensional focal locus. It turns out that one can restrict to study the case of $\mathbb{P}^3$.

Introduction

A congruence B of lines in $\mathbb{P}^n$ is a family of lines of dimension $n-1$. The order of B is the number of lines of B passing through a general point in $\mathbb{P}^n$ and its class is the number of lines of the congruence contained in a general hyperplane and meeting a general line contained in it.

The study of congruences of lines in $\mathbb{P}^3$ was started by E. Kummer in [Kum66], in which he gave a classification of those of order one. More recently, congruences of lines in $\mathbb{P}^3$ were studied in [Gol85] by N. Goldstein, who tried to classify these from the point of view of their focal locus. The focal locus is, roughly speaking, the set of critical values of the natural map between the total space Λ of the family B and $\mathbb{P}^n$. Successively, Z. Ran in [Ran86] studied the surfaces of order one in the general Grassmannian $\mathbb{G}(r,n)$ *i.e.* families of r-planes in $\mathbb{P}^n$ for which the general $(n-r-2)$-plane meets only one element of the family. He gave a classification of such surfaces, in particular, in the case of $\mathbb{P}^3$, obtaining a modern and more correct proof of Kummer's classification of first order congruences of lines. Moreover, he proved that if the class of these congruences in $\mathbb{P}^3$ is greater

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than three, these are not smooth, as conjectured by I. Sols. Another proof of Kummer's classification is given by F. L. Zak and others in [ZILO]. Moreover, E. Arrondo and I. Sols, in [AS92] classified all the smooth congruences with small invariants. Successively, E. Arrondo and M. Gross in [AG93] classified all the smooth congruences in $\mathbb{P}^3$ with a fundamental curve and this classification was extended to all smooth congruences with a fundamental curve by E. Arrondo, M. Bertolini e C. Turrini in [ABT94].

In this paper the classification of congruences with focal locus of dimension one (*i.e.* if the focal locus reduces to be only a — possibly reducible or non-reduced — fundamental curve) is obtained. In Section 1 we fix the notation and we recall some known results. In Section 2 we obtain some general results on the possible minimal dimension of the focal locus of a congruence in $\mathbb{P}^n$: first of all, we prove in Theorem 2.1 that if the focal locus has codimension at least two, then the congruence has order either zero or one. This result has been suggested to us by F. Catanese. In Theorem 2.2 we prove that the dimension of the focal locus is at least $\frac{n-1}{2}$ — if the congruence is not a star of lines, *i.e.* the set of lines passing through a point $P \in \mathbb{P}^n$ (in which case the focal locus has support in P). Moreover, we give also some results concerning the case of dimension $\frac{n-1}{2}$.

In Section 3 we prove the main result of this paper, namely Theorem 0.1; in order to state it, we need the following notation: first of all, let ℓ be a fixed line in $\mathbb{P}^3$ and $\mathbb{P}^1_\ell$ the set of the planes containing ℓ. Let ϕ be a general nonconstant morphism from $\mathbb{P}^1_\ell$ to ℓ and let Π be a general element in $\mathbb{P}^1_\ell$. We define $\mathbb{P}^1_{\phi(\Pi),\Pi}$ as the pencil of lines passing through the point $\phi(\Pi)$ and contained in Π.

Theorem 0.1. *If the focal locus F of a congruence of lines B in $\mathbb{P}^n$ has dimension one, then $n = 2, 3$. If $n = 2$ F is an irreducible curve of degree greater than one and B is its dual; if $n = 3$ B is a first order congruence, and we have the following possibilities:*

(1) *F is an irreducible curve, which is either:*

- **(a)** *a rational normal curve C_3 in $\mathbb{P}^3$. In this case B is the family of secant lines of C_3 and has bidegree $(1, 3)$ in the Grassmannian;*
- **(b)** *or a non-reduced scheme whose support is a line ℓ, and the congruence is — using the above notation — $\bigcup_{\Pi\in\mathbb{P}^1_\ell} \mathbb{P}^1_{\phi(\Pi),\Pi}$; if we set $d := \deg(\phi)$, the congruence has bidegree $(1, d)$; or*

(2) *F is a reducible curve, union of a line F_1 and a rational curve F_2 such that* $\mathrm{length}(F_1 \cap F_2) = \deg(F_2) - 1$*; B is the family of lines meeting F_1 and F_2 and it has bidegree $(1, \deg(F_2))$.*

1 – Notation and preliminaries

We will work with schemes and varieties over the complex field $\mathbb{C}$, with the standard definitions and notation as in [Har77]. For us a variety will always be projective. More information about general results and references about families of lines, focal diagrams and congruences can be found in [De 01b], [De 01a] and [De 00], or [De 99]. Besides, we refer to [GH78] for notation about Schubert cycles and to [Ful84] for the definitions and results of intersection theory. So we denote by σ_{a_0,a_1} the Schubert cycle of the lines in $\mathbb{P}^n$ contained in a fixed $(n-a_1)$-dimensional subspace $H \subset \mathbb{P}^n$ and which meet a fixed $(n-1-a_0)$-dimensional subspace $\Pi \subset H$. Here we recall that a congruence of lines in $\mathbb{P}^n$ is a (flat) family (Λ, B, p) of lines in $\mathbb{P}^n$ obtained as the pull-back of the universal family under the desingularization map of a subvariety B' of dimension $n-1$ of the Grassmannian $\mathbb{G}(1,n)$ of lines in $\mathbb{P}^n$. So $\Lambda \subset B \times \mathbb{P}^n$ and p is the restriction of the projection $p_1 : B \times \mathbb{P}^n \to B$ to Λ, while we will denote the restriction of $p_2 : B \times \mathbb{P}^n \to \mathbb{P}^n$ by f. $\Lambda_b := p^{-1}(b)$, $(b \in B)$ will be a line of the family and $f(\Lambda_b) =: \Lambda(b)$ is the corresponding line in $\mathbb{P}^n$. A point $y \in \mathbb{P}^n$ is called fundamental if its fibre $f^{-1}(y)$ has dimension greater than the dimension of the general one. The fundamental locus is the set of the fundamental points. It is denoted by Ψ. Moreover the locus of the points $y \in \mathbb{P}^n$ for which the fibre $f^{-1}(y)$ has positive dimension will be denoted by Φ.

Since Λ is smooth of dimension n, we define the focal divisor $R \subset \Lambda$ as the ramification divisor of f. The schematic image of the focal divisor R under f (see, for example, [Har77]) is the focal locus $F \subset \mathbb{P}^n$. Clearly, we have $\Psi \subset \Phi \subset (F)_{\mathrm{red}}$.

To a congruence is associated a sequence of degrees $(a_0, ..., a_\nu)$ since B is rationally equivalent to a linear combination of the Schubert cycles of dimension $n-1$ in the Grassmannian $\mathbb{G}(1,n)$:

$$[B] = \sum_{i=0}^{\nu} a_i \, \sigma_{n-1-i,i}$$

(where we set $\nu := [\frac{n-1}{2}]$); in particular, the order a_0 is the number of lines of B passing through a general point in $\mathbb{P}^n$, and the class a_1 is the number of lines intersecting a general line L and contained in a general hyperplane $H \supset L$ (*i.e.* as a Schubert cycle, $B \cdot \sigma_{n-2,1}$). a_0 is the degree of $f : \Lambda \to \mathbb{P}^n$: thus if $a_0 > 0$, $\Psi = \Phi$, while if $a_0 = 1$, set-theoretically, $\Psi = \Phi = F$.

An important result — independent of order and class — is the following:

Proposition 1.1 (C. Segre, [Seg88]). *On every line $\Lambda_b \subset \Lambda$ of the family, the focal divisor R either coincides with the whole Λ_b — in which case $\Lambda(b)$ is called a focal line — or is a zero dimensional subscheme of Λ_b of length $n-1$. Moreover, in the latter case, if Λ is a first order congruence, $\Psi = F$ and $F \cap \Lambda(b)$ has length $n-1$.*

This result was proven classically in [Seg88]; the first modern proof in the case of the congruences (*i.e.* families of dimension two) of planes in $\mathbb{P}^4$ is in [CS92]. See [De 01b], Proposition 1 for a proof of Proposition 1.1.

2 – Congruences of lines in $\mathbb{P}^n$

We start with the following result, suggested to us by F. Catanese:

Theorem 2.1. *Let B be a congruence whose focal locus F has codimension at least two. Then, B has order either zero or one.*

Proof: Let us consider the restriction of the map $f : \Lambda \to \mathbb{P}^n$ to the set $\Lambda \setminus f^{-1}(F)$. Then, either $f^{-1}(F) = \Lambda$, in which case B is a congruence of order zero, or the map $f\,|_{\Lambda\setminus f^{-1}(F)}$ defines an unramified covering of the set $\mathbb{P}^n \setminus F$. But it is a well-known fact that — by dimensional reasons — $\mathbb{P}^n \setminus F$ is simply connected and $\Lambda \setminus f^{-1}(F)$ is connected. Therefore, $f\,|_{\Lambda\setminus f^{-1}(F)}$ is a homeomorphism, hence f is a birational map and B is a first order congruence. ∎

A central definition, introduced in [De 01b], is the notion, $\forall d$ such that $0 \le d \le n-2$, of fundamental d-locus. This locus $\mathcal{F}^d$, is so defined: let $R_1, ..., R_s$ be the horizontal components (*i.e.* $p_{|R_i} : R_i \to B$ is dominant). Now, let $F_i := f(R_i)$, (as before, we take the schematic image); then, $\forall d$ such that $0 \le d \le n-2$ we define $\mathcal{F}^d := \bigcup_{\dim(F_i)=d} F_i$. By dimensional reasons, the lines of the congruence passing through $P_d \in \mathcal{F}^d$ form a family of dimension (at least) $n-1-d$ (so, set-theoretically, $\mathcal{F}^d \subset \Phi$).

From this, we infer for example that if the focal locus has dimension zero, then the congruence is a star of lines (see Corollary 1 of [De 01b]). In this article, we will be interested in the subsequent case, *i.e.* the case in which $\dim(F) = 1$.

We first note that if our congruence has order one, then there exists a d such that $\mathcal{F}^d \neq \emptyset$.

The first important results of this paper are the following

Theorem 2.2. *Let B be a congruence of lines in $\mathbb{P}^n$. If* $\dim(F) := i > 0$, *then* $\frac{n-1}{2} \leq i \leq n-1$. *Besides, if* $i = \frac{n-1}{2}$, $(F)_{\text{red}}$ *is irreducible and the general line of the congruence meets F — set-theoretically — in only one point, then F is — set-theoretically — an i-plane.*

Proof: Clearly, we have only to prove that $\dim(F) \geq \frac{n-1}{2}$. Moreover, by Theorem 2.1, we can reduce to study the cases of order zero and one.

If the congruence has order zero, $F = f(\Lambda)$ and F contains a family of lines of dimension $n-1$. Then the the observation that among varieties of fixed dimension i, the projective space is the unique one which contains a family of lines of dimension greater than or equal to $2(i-1)$.

Let now suppose that the order is one. Let $\mathcal{F}^i$ be the fundamental i-locus of maximal dimension $i(>0)$. Given $i+1$ general hyperplanes $H_0, ..., H_i$ in $\mathbb{P}^n$ and the corresponding hyperplane sections of $\mathcal{F}^i$, $D_0, ..., D_i$, the lines of Λ which meet D_j form a family of dimension $n-2$, $j = 0, ..., i$ which will fill a hypersurface M_{D_j} in $\mathbb{P}^n$. Since $D_0 \cap \cdots \cap D_i = \emptyset$ and $M := M_{D_0} \cap \cdots \cap M_{D_i} \subset \mathbb{P}^n$ is such that $\dim(M) \geq n-1-i \geq 0$, take $Q \in M$. By definition, there exist $\ell_i \in D_i$ with $Q \in \ell_i$. If the ℓ_i's are different, then $Q \in F$. If all ℓ_i's are equal, it is absurd. It follows that $M \subset F$.

Now $M \subset F$; if $\dim(F) < \frac{n-1}{2}$, then, since $\dim(M) \geq n-1-i$, we have $2i > n-1$, which cannot be, since $\mathcal{F}^i \subset F$.

Let us now suppose that the equality holds, $D := (F)_{\text{red}}$ is irreducible, and the general line of the congruence meets D in only one point. If $P, Q \in D$ are two general points, the set of lines of Λ passing through them generate two cones $M_P, M_Q \subset \mathbb{P}^n$ of dimension $i+1$. Therefore, by our hypotheses $\dim(M_P \cap M_Q) \geq 1$ and $M_P \cap M_Q$ contains the line joining P and Q. So, D is a linear space. ∎

Proposition 2.3. *Let Λ be a congruence in $\mathbb{P}^n$ and D the component of the focal locus of maximal dimension $i > 0$; if $\mathcal{F}^k$ is a fundamental k-locus contained in D, then $n-1-i \leq k \leq i$. In particular, if $i = \frac{n-1}{2}$ we have only the i-fundamental locus $\mathcal{F}^i$.*

Proof: With notation as in the proof of the preceding theorem, we consider the hyperplane sections of F^k, $D_0, ..., D_k$, for which we have $D_0 \cap \cdots \cap D_k = \emptyset$, and so $M_{D_0} \cap \cdots \cap M_{D_k} \subset F$. Therefore our thesis easily follows. ∎

Corollary 2.4. *Let Λ be a congruence in $\mathbb{P}^n$; if $\dim(F) = 1$, then $n \leq 3$.*

Remark. The preceding results are new in the literature, although the method of proof of Theorem 2.2 and Proposition 2.3 is very similar to the respective proofs of Theorem 6 and Corollary 2 of [De 01b]. Actually, in Theorem 6 of [De 01b] we forgot the hypothesis — in the border case $i = \frac{n-1}{2}$ — that the general line should meet the focal locus only once. □

3 – Congruences in $\mathbb{P}^3$

Our aim in this article is to classify the congruences with $\dim(F) = 1$. In view of Corollary 2.4, it is sufficient to consider congruences in $\mathbb{P}^2$ and $\mathbb{P}^3$. The case of $\mathbb{P}^2$ is well known: all the congruences but the ones of order one (*i.e.* the pencils) have a focal curve, and the congruence is given by the tangents of the focal curve. Therefore, we will study congruences in $\mathbb{P}^3$ with $\dim(F) = 1$. By Theorem 2.1, these are the first order congruences (but the star of lines) of $\mathbb{P}^3$, since the order cannot be zero; this can be deduced easily from the fact that the only surface containing a family of dimension two of lines is the plane. Actually, the first order congruences of $\mathbb{P}^3$ are classified in [Ran86] and [ZILO]. Here we give another easy and — as far as we know — new proof of this classification.

In what follows, we will also call the focal locus fundamental curve.

3.1. Congruences with irreducible fundamental curve

Proposition 3.1. *If the fundamental curve F of a congruence of lines in $\mathbb{P}^3$ is irreducible, then we have the following possibilities:*

(**1**) *if F is reduced, we are in case (1)(a) of Theorem 0.1;*

(**2**) *If F is non-reduced, we are in case (1)(b) of Theorem 0.1.*

Proof: First of all, we consider the case of the secant lines of a curve, and we know that we have a first order congruence. We could apply Theorem 2.16 of [De 00] to obtain that $\deg(F) = 3$, but for the sake of completeness, we will prove this result here. Clearly we have that $\deg F \geq 3$ for degree reasons. Then, let us denote by a the class of our congruence, and by r a general line in $\mathbb{P}^3$. Let us also denote by V_r the scroll of the lines of B meeting r. An easy application of the Schubert calculus shows that $\deg(V_r) = a + 1$.

Let V_r and $V_{r'}$ be two such surfaces; we denote by k the (algebraic) multiplicity of F in V_r (and $V_{r'}$). First of all, we note that the complete intersection of two general surfaces V_r and $V_{r'}$ is a (reducible) curve Γ whose components are the focal locus F and $(a+1)$ lines, *i.e.* the lines of B meeting r and r': in fact, a point of Γ not belonging to the lines meeting both r and r' is a focal point, since through it pass infinitely many lines of B, *i.e.* a line meeting r and another one meeting r', and r and r' are general. The lines of B meeting r and r' are $a+1$ since this is the degree of V_r.

Then, by Bézout, we have that

$$\begin{aligned}\deg(V_r \cap V_{r'}) &= (a+1)^2 \\ &= a+1+k^2 d\end{aligned}$$

— where $d := \deg(F)$ — and we obtain

$$k^2 d = (a+1)\,a \; . \tag{1}$$

Besides, since a line of the congruence not belonging to the $(a+1)$ lines meeting r and r' must intersect the curve F in (at least) two points, we deduce

$$a+1 = 2\,k \; . \tag{2}$$

From formulas (1) and (2) we conclude

$$k^2(4-d) - 2\,k = 0 \; ,$$

from which we get the result, *i.e.* $d = 3$.

So the only possibility is to have the twisted cubic C^3, which has, in fact, an apparent double point. Since this curve has degree three, the bidegree is $(1,3)$.

Then we pass at the other case: by Theorem 2.2, $(F)_{\mathrm{red}} =: \ell$ is a line; so, if we restrict the congruence to a (general) plane Π through ℓ, we get a congruence of lines in $\mathbb{P}^2$ such that its focal locus is contained in ℓ, and therefore it is a pencil of lines with its fundamental point P_Π in ℓ. Then φ of Theorem 0.1, (1)(b) is the map defined by $\varphi(\Pi) = P_\Pi$. This congruence has bidegree $(1,d)$, as it is easily seen if we consider the lines of the congruence contained in a general plane. ∎

3.2. Congruences with reducible fundamental curve

Proposition 3.2. *If the fundamental curve F of a first order congruence of lines in $\mathbb{P}^3$ is reducible, then the congruence is as in Theorem 0.1, case (2).*

Proof: Let us denote by Z the 0-dimensional scheme given by $F_1 \cap F_2$ and we set $u = \mathrm{length}(Z)$.

Let P be a general point in $\mathbb{P}^3$; then, if we set $\deg(F_1) := m_1 \geq m_2 =: \deg(F_2)$, the cone given by the join PF_j has degree m_j; as usual, by Bézout

$$\deg(PF_1 \cap PF_2) \;=\; m_1\, m_2 \;.$$

Since we have a congruence of order one, we obtain:

$$u \;=\; m_1\, m_2 - 1 \;; \tag{3}$$

this is due to the fact that only one of the lines of $PF_1 \cap PF_2$ is a line of the congruence; therefore the others must be the u lines of the join PZ.

If Q is a general point of F_1, there will pass $m_2(m_1 - 1)$ secant lines of F_1 through Q meeting F_2 also; but these lines will pass through the points of Z, since if one of these lines did not intersect Z, this line would be a focal line, and varying these lines when we vary Q on F_1, we would obtain a focal surface. Besides, since Q is general, we can suppose that it does not belong to the tangent cones of the two curves at the points of Z. Then we have that

$$u \;=\; (m_1 - 1)\, m_2 \;, \tag{4}$$

and by equation (3) we obtain $m_2 = 1$, $u = m_1 - 1$. To see that F_1 is a rational curve, we can simply project it from a general point of the line F_2 onto a plane: by the Clebsh formula, the projected curve has geometric genus zero.

The class is — as usual — calculated intersecting with a general plane. ∎

3.3. Final remarks about first order congruences

We finish this article analysing which of these congruences are smooth as surfaces in the Grassmannian $\mathbb{G}(1,3)$.

Proposition 3.3. *The smooth first order congruences of lines B in $\mathbb{P}^3$ are*

(**1**) *the secants of the twisted cubic, in which case B is the Veronese surface;*

(**2**) *the join of a line and a smooth conic meeting in a point, in which case B is a rational normal cubic scroll;*

(**3**) *the join of two — non-meeting — lines in which case B is a quadric.*

Proof: We see from [AG93] and [ABT94] that the only possible bidegrees for a smooth congruence in $\mathbb{P}^3$ with a fundamental curve are $(1,1)$, $(1,2)$, $(1,3)$ and $(3,6)$. The last case can be excluded because has order three, by Theorem 0.1. From the description of these congruence given in [AS92] we get the proposition. ∎

Remark. Observe that, if the fundamental curve is non-reduced, its support is a line ℓ. Its image in $\mathbb{G}(1,3)$ is a singular point L for B: more precisely, B is a 2-dimensional linear section, tangent to $\mathbb{G}(1,3)$ at L.

We observe finally that the case of two lines is the one in which B is the intersection of the two tangent hyperplane sections of the Grassmannian corresponding to the two lines, *i.e.* B is a general 2-dimensional linear section of $\mathbb{G}(1,3)$. From this description we see that the case in which the fundamental curve is a line only is a limit case of this one, when the two lines coincide. □

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