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BROWN–McCOY SEMISIMPLICITY OF CERTAIN BANACH ALGEBRAS

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Abstract: It is shown that if S is a free group, a free semigroup, or a free inverse semigroup then the Brown–McCoy radical of the Banach algebra $l^1(S)$ is zero.

Let S be a semigroup. We denote by $l^1(S)$ the Banach algebra consisting of all functions $a\colon S \to \mathbb{C}$ (the complex field), with finite or countably infinite support and such that $\sum_{x\in S} |a(x)| < \infty$, where addition and scalar multiplication are defined pointwise, multiplication is convolution and the norm of the element a is $\sum_{x\in S} |a(x)|$ ([1]). The semigroup algebra $\mathbb{C}[S]$ consists of all functions $a\colon S \to \mathbb{C}$ of finite support: this is clearly a subalgebra of $l^1(S)$. It is convenient to identify the elements of S with the corresponding characteristic functions: thus, if S is infinite, we can write $a \in l^1(S)$ in the form $\sum_{n=1}^{\infty} \alpha_n x_n$, where (α_n) is a sequence of complex numbers with $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ and (x_n) is a sequence of distinct elements of S.

The Brown–McCoy radical of an algebra A is denoted by $B(A)$. A survey of the basic properties of this radical may be found in [7, Ch. 7, §37]. In particular, $B(A)$ contains the Jacobson radical of A; and, assuming that A is nontrivial, $B(A) = \{0\}$ if and only if A is a subdirect product of simple algebras with unity. The purpose of the present paper is to show that $B(A) = \{0\}$ if $A = l^1(S)$, where S is a free group, a free semigroup or a free inverse semigroup.

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We note, in passing, that if S is a free group of rank at least two or a free semigroup of rank at least two then $l^1(S)$ is also primitive [10, 3; 9, 2].

The set of all congruences on a semigroup S is denoted by $\Lambda(S)$ and, for $\rho \in \Lambda(S)$, the ρ-class containing $x \in S$ is denoted by $x\rho$. We write $\Lambda_f(S) := \{\rho \in \Lambda(S) \colon\ S/\rho \text{ is finite}\}$. Observe that $\Lambda_f(S)$ is closed under finite intersections. Recall that S is termed *residually finite* if and only if, for every pair $(x,y) \in S \times S$ with $x \neq y$ there exists $\rho \in \Lambda_f(S)$ such that $(x,y) \notin \rho$. Thus S is residually finite if and only if $\bigcap\{\rho\colon\ \rho \in \Lambda_f(S)\} = \iota_S$, the identity relation on S. It is convenient here to introduce a further concept. We say that S is *residually* M-*finite* if and only if there exists a nonempty subset M of $\Lambda_f(S)$ such that (i) M is closed under finite intersections and (ii) $\bigcap\{\rho : \rho \in \mathrm{M}\} = \iota_S$. Note that if S is residually M-finite then it is residually finite and that if S is residually finite then it is residually $\Lambda_f(S)$-finite.

Lemma. *Let S be an infinite residually* M-*finite semigroup such that, for all $\rho \in \mathrm{M}$, the finite-dimensional algebra $\mathbb{C}[S/\rho]$ is semisimple. Then $B(l^1(S)) = \{0\}$.*

Proof: Let $\rho \in \mathrm{M}$. Define a surjective homomorphism $\theta_\rho \colon l^1(S) \to \mathbb{C}[S/\rho]$ by the rule that

$$\theta_\rho\Big(\sum_n \alpha_n\, x_n\Big) \;:=\; \sum_n \alpha_n (x_n\, \rho)\ ,$$

where (α_n) is a sequence of complex numbers with $\sum_n |\alpha_n| < \infty$ and (x_n) is a sequence of distinct elements of S. By hypothesis, $\mathbb{C}[S/\rho]$ is semisimple and so, for some positive integer k_ρ, $\mathbb{C}[S/\rho]$ is a direct sum of ideals $A_{\rho,i}$ $(i=1,2,...,k_\rho)$, each of which is a finite-dimensional simple algebra with unity. Hence, for each $i \in \{1,2,...,k_\rho\}$, there exists a surjective homomorphism $\phi_{\rho,i} \colon \mathbb{C}[S/\rho] \to A_{\rho,i}$; further,

$$\bigcap_{i=1}^{k_\rho} \ker \phi_{\rho,i} \;=\; \{0\}\ . \tag{1}$$

We now show that $l^1(S)$ is a subdirect product of the algebras $A_{\rho,i}$ $(\rho \in \Lambda_f(S);\ i = 1,2,...,k_\rho)$. Since each $A_{\rho,i}$ is a simple algebra with unity, this will establish the result.

For each pair (ρ, i), with $\rho \in \Lambda_f(S)$ and $i \in \{1,2,...,k_\rho\}$, write $\psi_{\rho,i} := \phi_{\rho,i} \circ \theta_\rho$. Then $\psi_{\rho,i} : l^1(S) \to A_{\rho,i}$ is a surjective homomorphism. Let $a \in l^1(S)$ be such that $\psi_{\rho,i}(a) = 0$ for all $\rho \in \Lambda_f(S)$ and all $i \in \{1,2,...,k_\rho\}$. Thus, from (1),

$$(\forall\ \rho \in \Lambda_f(S)) \quad \theta_\rho(a) = 0\ . \tag{2}$$

Suppose that $a \neq 0$. Then $a = \sum_n \alpha_n x_n$ for some sequence (α_n) of complex numbers, not all zero, such that $\sum_n |\alpha_n| < \infty$ and some sequence (x_n) of distinct elements of S. Choose a positive integer m such that $\alpha_m \neq 0$. Since $\sum_n |\alpha_n|$ converges, there exists a positive integer $p > m$ such that $\sum_{n>p} |\alpha_n| < \frac{1}{2}|\alpha_m|$. Now, since S is residually M-finite, for each pair (r,s) of positive integers with $r < s \leq p$, there exists $\rho_{rs} \in \mathrm{M}$ such that $(x_r, x_s) \notin \rho_{rs}$. Write $\rho := \bigcap_{(r,s)} \rho_{rs}$. Then $\rho \in \mathrm{M}$; also, for each pair (r,s) with $r < s \leq p$, we have that $(x_r, x_s) \notin \rho$, that is, $x_r\rho \neq x_s\rho$. In particular, $x_r\rho \neq x_m\rho$ for all r such that $r \neq m$ and $1 \leq r \leq p$. Let β denote the coefficient of $x_m\rho$ in $\theta_\rho(a)$ and let T be the set of all positive integers t such that $t > p$ and $x_t\rho = x_m\rho$. If $T = \emptyset$ then $\beta = \alpha_m \neq 0$. On the other hand, if $T \neq \emptyset$ then

$$|\beta| \;\geq\; |\alpha_m| - \left|\sum_{t\in T} \alpha_t\right| \;\geq\; |\alpha_m| - \sum_{t\in T} |\alpha_t| \;\geq\; |\alpha_m| - \sum_{n>p} |\alpha_n| \;>\; \frac{1}{2}|\alpha_m| \;.$$

Thus, in either case, $\beta \neq 0$. However, by (2), $\theta_\rho(a) = 0$, which implies that $\beta = 0$. From this contradiction we see that $a = 0$. Hence $l^1(S)$ is a subdirect product of the algebras $A_{\rho,i}$ ($\rho \in \Lambda_f(S)$; $i = 1, 2, ..., k_\rho$), as required. ∎

Theorem 1. *Let G_X and S_X denote, respectively, the free group and the free semigroup on a nonempty set X. Then $B(l^1(G_X)) = \{0\}$ and $B(l^1(S_X)) = \{0\}$.*

Proof: Note that S_X can be regarded as a subsemigroup of G_X. Let S be an infinite subsemigroup of G_X. It suffices to show that $B(l^1(S)) = \{0\}$.

Let $\rho \in \Lambda_f(G_X)$ and let $T_\rho := \{w\rho\colon\ w \in S\}$. Then T_ρ is a subsemigroup of the finite group G_X/ρ and so is itself a finite group. Thus, by Maschke's theorem, $\mathbb{C}[T_\rho]$ is semisimple. Write $\rho_S := \rho \cap (S \times S)$. Then ρ_S is a congruence on S and $S/\rho_S \cong T_\rho$. Hence $\rho_S \in \Lambda_f(S)$ and $\mathbb{C}[S/\rho_S]$ is semisimple. Let $\mathrm{M} := \{\rho_S :\ \rho \in \Lambda_f(G_X)\}$. Since $\Lambda_f(G_X)$ is closed under finite intersections, so also is M. Further, by [8, Theorem 8.18], G_X is residually finite and so $\bigcap\{\rho_S\colon \rho \in \Lambda_f(G_X)\} = \bigcap\{\rho\colon \rho \in \Lambda_f(G_X)\} \cap (S \times S) = \iota_S$. Thus S is residually M-finite. Applying the lemma, we see that $B(l^1(S)) = \{0\}$. ∎

By an *inverse semigroup* we mean a semigroup S such that

$$(\forall\, x \in S)\;\; (\exists!\, x' \in S) \qquad x\,x'x = x \;\text{ and }\; x'x\,x' = x' \;.$$

A basic account of such semigroups is provided in [4, Chapter V]; for an extended discussion, see [6].

Theorem 2. *Let FI_X denote the free inverse semigroup on a nonempty set X. Then $B(l^1(FI_X)) = \{0\}$.*

Proof: Write $S := FI_X$. By [12, Theorem 3.6], S is residually finite. Further, by [4, Proposition V.1.6], for all $\rho \in \Lambda_f(S)$, S/ρ is a finite inverse semigroup and so $\mathbb{C}[S/\rho]$ is semisimple [14, Theorem 4; 11, Theorem 4.4]. The result now follows by the lemma. ∎

We conclude with some remarks about the semigroup algebra $F[S]$ of a semigroup S over an arbitrary field F [13]. Theorems analogous to those above hold for S a free group, a free semigroup or a free inverse semigroup, the analogue of Theorem 1 being deducible from more general results of Jespers and Puczylowski [5, Corollary 6 and Corollary 13].

To obtain these theorems, we may proceed as follows. First, note that the lemma still holds if we replace '$\mathbb{C}[S/\rho]$' by '$F[S/\rho]$' and '$l^1(S)$' by '$F[S]$'. As in the proof of Theorem 1, consider an infinite subsemigroup S of G_X. Choose a prime p different from the characteristic of F and let $\Pi := \{\rho \in \Lambda_f(G_X) : G_X/\rho$ is a p-group$\}$. For $\rho \in \Pi$, let $T_\rho := \{w\rho : w \in S\}$. Then T_ρ is a subsemigroup of the finite p-group G_X/ρ and so is itself a finite p-group. Thus, by Maschke's theorem, $F[T_\rho]$ is semisimple. Further, $S/\rho_S \cong T_\rho$, where $\rho_S := \rho \cap (S \times S)$. Now G_X is residually Π-finite [8, Chapter 8, Problem 16]. Hence, taking $\mathrm{M} := \{\rho_S : \rho \in \Pi\}$, we see that S is residually M-finite. It follows from the modified lemma that $B(F[S]) = \{0\}$. In particular, $B(F[G_X]) = \{0\}$ and $B(F[S_X]) = \{0\}$ ([5]).

Next, let $S := FI_X$, the free inverse semigroup on X. For a proper ideal T of S we define the *Rees congruence* ρ_T on S by

$$(x,y) \in \rho_T \iff x = y \ \text{ or } \ x, y \in T\ .$$

The proof of [12, Theorem 3.6] shows that S is residually M-finite, where $\mathrm{M} := \{\rho \in \Lambda_f(S) : \rho = \rho_T$ for a proper ideal T of $S\}$. Further, by [12, Theorem 3.2(iii)], S has only trivial subgroups. Hence, for all $\rho \in \mathrm{M}$, S/ρ has only trivial subgroups and so $F[S/\rho]$ is semisimple [14, 11]. The modified lemma now shows that $B(F[S]) = \{0\}$.

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