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PERIODIC SOLUTIONS OF A RESONANT HIGHER ORDER EQUATION

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Abstract: We study the existence of periodic solutions for an ordinary differential equation of resonant type. Under suitable conditions we prove the existence of at least one periodic solution of the problem applying Mawhin Coincidence degree theory.

1 – Introduction

In the last years there has been an increasing interest in higher order problems which have applications in many fields, such as beam theory [4], [5] and multi-ion electrodiffusion problems [8].

In this work, we consider the problem:

$$Lx + g(x, x', ..., x^{(N-2)}) \;=\; p(t) \tag{1}$$

where

$$Lx \;=\; x^{(N)} + a_{N-1}x^{(N-1)} + ... + a_0 x \tag{2}$$

under periodic conditions

$$\begin{aligned} x(0) &= x(2\pi) \\ x'(0) &= x'(2\pi) \\ &\;\;... \\ x^{(N-1)}(0) &= x^{(N-1)}(2\pi) \end{aligned}$$

for continuous and bounded g.

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We assume that L is a resonant operator, i.e. that the homogeneous problem $Lx = 0$ admits nontrivial periodic solutions. Namely, we assume that the polynomial

$$P(\lambda) = \lambda^N + a_{N-1}\lambda^{N-1} + ... + a_0 \tag{3}$$

admits imaginary roots $\pm im$ ($m \in \mathbb{Z}$).

For notational convenience, we introduce the n-dimensional symbolic vectors $V_{\pm\pm}$ given by

$$V_{++} = (+\infty, +\infty, -\infty, -\infty, ...)$$

$$V_{+-} = (+\infty, -\infty, -\infty, +\infty, ...)$$

$$V_{-+} = (-\infty, +\infty, +\infty, -\infty, ...)$$

$$V_{--} = (-\infty, -\infty, +\infty, +\infty, ...)$$

where the sequences of signs are 4-periodic.

Our main result is:

Theorem 1.1. *Let us assume that*

1. *The polynomial (3) has exactly two roots $\pm im$ in $i\mathbb{Z}$, which are simple.*
2. *$g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a continuous bounded function such that the four limits*

$$\lim_{s \to V_{\pm\pm}} g(s) := g_{\pm\pm}$$

exist.

Let $p \in L^2(0, 2\pi)$ and consider the m-th Fourier coefficients of p, namely:

$$a_m(p) = \frac{1}{\pi}\int_0^{2\pi} p(t)\cos(mt)\, dt \ ,$$

$$b_m(p) = \frac{1}{\pi}\int_0^{2\pi} p(t)\sin(mt)\, dt \ .$$

If furthermore, we assume that

$$a_m^2(p) + b_m^2(p) < \frac{2}{\pi^2}\left[(g_{+-} - g_{-+})^2 + (g_{++} - g_{--})^2\right] \tag{4}$$

then equation (1) has at least one 2π-periodic solution in $H^N(0, 2\pi)$.

Remark 1.1. The particular case $N=3$ with $g=g(x')$ has been considered in [1]. □

Remark 1.2. In the same way as in [1], Proposition 3.1, it is easy to see that condition (4) is "almost" necessary: namely, if (1) admits a 2π-periodic solution then

$$a_m^2(p)+b_m^2(p) \;\le\; \frac{16}{\pi^2}\,\|g\|_{L^\infty}^2 \;. \tag{5}$$

In particular, if g verifies

$$|g_{+-}-g_{-+}| \;=\; |g_{++}-g_{--}| \;=\; 2\,\|g\|_{L^\infty} \;,$$

and inequality (5) holds stricly, then (4) holds. □

Remark 1.3. We observe that (4) is a Landesman-Lazer type condition (see e.g. [7],[10]). Thus, Theorem 1.1 can be regarded as a Nth-order analogue of Theorem 1 in [11]. □

2 – Auxiliary results

2.1. Mawhin coincidence degree theory

Let us briefly summarize some aspects of Mawhin theory. This technique has been applied to many problems, see e.g. [2] and [6]. For further details see [9], [3].

Let X and Y be real normed spaces, $L:\mathrm{dom}(L)\to Y$ be a linear mapping, and $N:X\to Y$ be a continuous mapping. The mapping L is called a Fredholm mapping of index 0 if $\mathrm{Im}(L)$ is a closed subspace of Y and

$$\dim(\mathrm{Ker}(L)) \;=\; \mathrm{codim}(\mathrm{Im}(L)) \;<\; \infty \;.$$

If L is a Fredholm mapping of index 0, then there exists continuous projectors $P:X\to X$ and $Q:Y\to Y$ such that $\mathrm{Im}(P)=\mathrm{Ker}(L)$ and $\mathrm{Ker}(Q)=\mathrm{Im}(L)$. It follows that

$$L_P \;=\; L|_{\mathrm{dom}(L)\cap\mathrm{Ker}(P)}\colon\;\; \mathrm{dom}(L)\cap\mathrm{Ker}(P) \;\to\; \mathrm{Im}(L)=\mathrm{Ker}(Q)$$

is one-to-one and onto $\operatorname{Im}(L)$. We denote its inverse by K_P. If Ω is a bounded open subset of X, N is called L-compact on Ω if $QN(\Omega)$ is bounded and $K_P(I-Q)N : \Omega \to X$ is compact. Since $\operatorname{Im}(Q)$ is isomorphic to $\operatorname{Ker}(L)$, there exists and isomorphism $J : \operatorname{Im}(Q) \to \operatorname{Ker}(L)$.

The following continuation theorem is due to Mawhin [9]:

Theorem 2.1. *Let L be a Fredholm mapping of index zero and N be L-compact on Ω. Suppose*

1. *For each $\lambda \in (0,1]$, $x \in \partial\Omega$ we have that $Lx \neq \lambda Nx$;*
2. *$QNx \neq 0$ for each $x \in \operatorname{Ker}(L) \cap \partial\Omega$;*
3. *The Brouwer degree $d_B(JQN, \Omega \cap \operatorname{Ker}(L), 0) \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom}(L) \cap \Omega$. ∎

In our case, we shall work in the usual Sobolev space of periodic functions, namely

$$H^K_{per}(0,2\pi) = \left\{ x \in H^K(0,2\pi) : \ x^{(j)}(0) = x^{(j)}(2\pi) \ \ \forall\, j=0,...,K-1 \right\}.$$

More precisely, we shall consider $X = H^{N-1}_{per}(0,2\pi)$, $Y = L^2(0,2\pi)$ and L the linear differential operator given by (2), with $\operatorname{dom}(L) = H^N_{per}(0,2\pi)$. Under the assumption 1 of Theorem 1.1, it is immediate to see that $Ker(L) = \mathcal{E}_m$ is the subspace generated by $\sin(mt)$ and $\cos(mt)$, and $\operatorname{Im}(L) = \mathcal{E}_m^{\perp}$; thus L is a Fredholm mapping of index zero. Moreover, we may take Q as the orthogonal projection P_m onto $\mathcal{E}_m$ in $L^2(0,2\pi)$ and P as the restriction of P_m to $H^{N-1}_{per}(0,2\pi)$. Ω will be an appropriate open bounded subset of $H^{N-1}_{per}(0,2\pi)$. Then, if $Nx = p(t) - g(x,...,x^{(N-2)})$, it will follow from the estimates in section 2.2 that N is L-compact on Ω.

2.2. Estimates for the linear operator L

We assume throughout this section that assumption 1 of Theorem 1.1 holds. We recall the following lemma from [1]:

Lemma 2.2. *There exists a constant c such that for any $x \in H^2(0,2\pi)$ we have*

$$\|x - P_m(x)\|_{H^1} \leq c\,\|x'' + m^2 x\|_{L^2}$$

where P_m is the orthogonal projection on $\mathcal{E}_m$ in $L^2(0,2\pi)$. ∎

Then we have:

Lemma 2.3. *There exists a constant c such that*

$$\|x - P_m(x)\|_{H^{N-1}} \leq c\,\|Lx\|_{L^2} \qquad \forall\, x \in H^N_{per}(0,2\pi)\ .$$

Proof: Writing $P(\lambda) = (\lambda^2 + m^2)\,\widetilde{P}(\lambda)$ we may decompose the operator L as

$$L \,=\, L_2\,\widetilde{L}$$

where $L_2 x = x'' + m^2 x$ and $\widetilde{L}$ is the differential operator associated to $\widetilde{P}$. For $x \in \mathcal{E}_m^\perp$ in $L^2(0,2\pi)$, let us write the equality $Lx = f$ as a system:

$$\begin{cases} L_2 y \,=\, f \\ \widetilde{L}x \,=\, y\ . \end{cases}$$

Since $y = \widetilde{L}x \in \mathcal{E}_m^\perp$, from Lemma 2.2, we have that

$$\|y\|_{H^1} \leq c\,\|f\|_{L^2}\ .$$

We claim that

$$\|x\|_{H^{N-1}} \leq c\,\|y\|_{H^1}\ . \tag{6}$$

Indeed, as $P(0) \neq 0$, from Lemma 4.3 in [1] we know that (6) holds for $N = 3$. Assume that (6) holds for $N-1$, and let a be any root of $\widetilde{P}$. Thus, if $\widetilde{P}(\lambda) = (\lambda - a)\widehat{P}(\lambda)$, by inductive hypothesis we have that

$$\|x'\|_{H^{N-2}} \leq c\,\|\widehat{L}(x')\|_{H^1} \leq c\,\|\widehat{L}x\|_{H^2}$$

and from Lemma 4.3 in [1] we know that

$$\|\widehat{L}x\|_{H^2} \,\leq\, c\,\|y\|_{H^1}\ .$$

It follows that

$$\|x'\|_{H^{N-2}} \,\leq\, c\,\|y\|_{H^1}\ .$$

On the other hand, it is easy to see that

$$\left|\int_0^{2\pi} z(t)\,dt\right| \,\leq\, c\,\|\widetilde{L}z\|_{L^2}$$

for any $z \in H^N_{per}(0,2\pi)$, and the claim follows from Wirtinger inequality.

For $x \notin \mathcal{E}_m^\perp$, it suffices to write

$$\|x - P_m(x)\|_{H^{N-1}} \leq c\,\|L(x - P_m(x))\|_{L^2} \,=\, c\,\|Lx\|_{L^2}\ . \ \blacksquare$$

Remark 2.1. It follows that the operator $Nx = p - g(x, ..., x^{(N-2)})$ defined in the previous section is L-compact on Ω for any open bounded $\Omega \subset H^{N-1}_{per}(0, 2\pi)$. □

3 – A priori estimates

Lemma 3.1. *Assume that $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a continuous bounded function, that the four limits $g_{\pm\pm}$ exist, and that*

$$a_m^2(p) + b_m^2(p) \neq \frac{2}{\pi^2} \Big[(g_{+-} - g_{-+})^2 + (g_{++} - g_{--})^2\Big] .$$

Then the solutions of

$$L(x) = \lambda\Big(p(t) - g(x, ..., x^{(N-2)})\Big)$$

with $\lambda \in (0, 1]$ are a priori bounded in $H^{N-1}(0, 2\pi)$.

Proof: Let $x_n(t)$ be a solution of

$$Lx_n = \lambda_n\Big(p(t) - g(x_n, ..., x_n^{(N-2)})\Big)$$

with $\lambda_n \in (0, 1]$ and assume that $\|x_n\|_{H^{N-1}} \to +\infty$. Let us write

$$x_n = y_n + z_n$$

where $y_n = P_m(x_n)$ and $z_n = x_n - P_m(x_n) \in \mathcal{E}_m^\perp$. We have that

$$\|z_n\|_{H^{N-1}} \leq c_1 \Big\|p(t) - g(x_n, ..., x_n^{(N-2)})\Big\|_{L^2} \leq c_2 .$$

From the compactness of the imbedding $H^{N-1} \hookrightarrow C^{N-2}[0, 2\pi]$ there exists a convergent subsequence, still denoted z_n, with

$$z_n \to z \quad \text{in} \;\; C^{N-2}[0, 2\pi] .$$

We have that $\|y_n\|_{H^{N-1}} \to \infty$. Since $P_m(x_n) \in \mathcal{E}_m$,

$$y_n(t) = \alpha_n \cos(mt - \beta_n)$$

where $\alpha_n \geq 0$, $\alpha_n \to +\infty$ and $\beta_n \in [0, 2\pi]$. Taking a subsequence, we may assume that $\beta_n \to \beta$. On the other hand, if $\varphi \in \mathcal{E}_m$ and $L^*x = (-1)^N x^{(N)} + (-1)^{N-1} a_{N-1}\, x^{(N-1)} + ... + a_0 x$, integrating by parts we have:

$$0 \;=\int_0^{2\pi} x_n L^*(\varphi)\, dt \;=\; \lambda_n \int_0^{2\pi} \Big[p(t) - g(x_n, ..., x_n^{(N-2)})\Big]\varphi\ dt\ .$$

In particular, choosing $\varphi = \cos(mt)$

$$a_m(p) \;=\; \frac{1}{\pi}\int_0^{2\pi} p(t)\cos(mt)\ dt \;=\; \frac{1}{\pi}\int_0^{2\pi} g(x_n, ..., x_n^{(N-2)})\cos(mt)\ dt$$

and choosing $\varphi = \sin(mt)$

$$b_m(p) \;=\; \frac{1}{\pi}\int_0^{2\pi} p(t)\sin(mt)\ dt \;=\; \frac{1}{\pi}\int_0^{2\pi} g(x_n, ..., x_n^{(N-2)})\sin(mt)\ dt\ .$$

From the identities

$$\begin{aligned} x_n(t) &= z_n'(t) + \alpha_n \cos(mt - \beta_n) \\ x_n'(t) &= z_n'(t) - \alpha_n m \sin(mt - \beta_n) \\ x_n''(t) &= z_n''(t) - \alpha_n m^2 \cos(mt - \beta_n) \\ &\ \ \cdots \end{aligned}$$

we obtain:

$$g\Big(x_n(t), ..., x_n^{(N-2)}(t)\Big) \;\to\; \begin{cases} g_{+-} & \text{if } \ t \in C_\beta^{++} \\ g_{--} & \text{if } \ t \in C_\beta^{-+} \\ g_{-+} & \text{if } \ t \in C_\beta^{--} \\ g_{++} & \text{if } \ t \in C_\beta^{+-} \end{cases}$$

where

$$\begin{aligned} C_\beta^{++} &= \Big\{t \in [0, 2\pi]:\ \cos(mt - \beta) > 0,\ \ \sin(mt - \beta) > 0\Big\} \\ C_\beta^{-+} &= \Big\{t \in [0, 2\pi]:\ \cos(mt - \beta) < 0,\ \ \sin(mt - \beta) > 0\Big\} \\ C_\beta^{--} &= \Big\{t \in [0, 2\pi]:\ \cos(mt - \beta) < 0,\ \ \sin(mt - \beta) < 0\Big\} \\ C_\beta^{+-} &= \Big\{t \in [0, 2\pi]:\ \cos(mt - \beta) > 0,\ \ \sin(mt - \beta) < 0\Big\}\ . \end{aligned}$$

As $n \to \infty$, by dominated convergence we obtain:

$$\begin{aligned} a_m(p) \;=\; \frac{1}{\pi}\bigg[& g_{+-}\int_{C_\beta^{++}} \cos(mt)\, dt \;+\; g_{--}\int_{C_\beta^{-+}} \cos(mt)\, dt \\ & +\; g_{-+}\int_{C_\beta^{--}} \cos(mt)\, dt \;+\; g_{++}\int_{C_\beta^{+-}} \cos(mt)\, dt \bigg] \end{aligned}$$

and

$$b_m(p) \;=\; \frac{1}{\pi}\Bigg[g_{+-}\int_{C_\beta^{++}}\sin(mt)\,dt \;+\; g_{--}\int_{C_\beta^{-+}}\sin(mt)\,dt$$
$$+\; g_{-+}\int_{C_\beta^{--}}\sin(mt)\,dt \;+\; g_{++}\int_{C_\beta^{+-}}\sin(mt)\,dt\Bigg]\;.$$

Moreover,

$$\int_{C_\beta^{++}} e^{imt}\,dt \;=\; e^{i\beta}\int_{C_0^{++}} e^{ims}\,ds \;=\; e^{i\beta}m\int_0^{\pi/2m} e^{ims}\,ds \;=\; (1+i)\,e^{i\beta}$$

and in a similar way we compute:

$$\int_{C_\beta^{-+}} e^{imt}\,dt \;=\; e^{i\beta}\int_{C_0^{-+}} e^{ims}\,ds \;=\; e^{i\beta}\,m\int_{\pi/2m}^{\pi/m} e^{ims}\,ds \;=\; (-1+i)\,e^{i\beta}$$

$$\int_{C_\beta^{--}} e^{imt}\,dt \;=\; e^{i\beta}\int_{C_0^{--}} e^{ims}\,ds \;=\; e^{i\beta}\,m\int_{\pi/m}^{3\pi/2m} e^{ims}\,ds \;=\; (-1-i)\,e^{i\beta}$$

$$\int_{C_\beta^{-+}} e^{imt}\,dt \;=\; e^{i\beta}\int_{C_0^{-+}} e^{ims}\,ds \;=\; e^{i\beta}\,m\int_{3\pi/2m}^{2\pi/m} e^{ims}\,ds \;=\; (1-i)\,e^{i\beta}\;.$$

Hence we have that

$$a_m(p) \;=\; \frac{1}{\pi}\Big[(g_{+-}-g_{-+})\,(cos\beta-\sin\beta)+(g_{++}-g_{--})\,(cos\beta+\sin\beta)\Big]$$

and

$$b_m(p) \;=\; \frac{1}{\pi}\Big[(g_{+-}-g_{-+})\,(cos\beta+\sin\beta)+(g_{++}-g_{--})\,(cos\beta-\sin\beta)\Big]\;.$$

Then

$$a_m^2(p)+b_m^2(p) = \frac{1}{\pi^2}\Big[(cos\beta-\sin\beta)^2+(cos\beta+\sin\beta)^2\Big]\Big[(g_{+-}-g_{-+})^2+(g_{++}-g_{--})^2\Big]$$
$$= \frac{2}{\pi^2}\Big[(g_{+-}-g_{-+})^2+(g_{++}-g_{--})^2\Big]\;,$$

a contradiction. ∎

4 – A degree computation

In this section we compute the degree

$$d\Big(QN,\,\Omega\cap\mathrm{Ker}(L),\,0\Big)$$

where

$$\Omega \;=\; \Big\{u\in H^{N-1}_{per}(0,2\pi)\colon\; \|u\|_{H^{N-1}}<R\Big\}$$

for some R to be chosen. With this aim, let us consider the isomorphism $\psi:\mathbb{R}^2\to\mathcal{E}_m=\mathrm{Ker}(L)$ given by

$$\psi(a,b) \;=\; a\cos(mt)+b\sin(mt)$$

and define

$$h \;=\; \psi^{-1}QN\psi\;.$$

Let us call $\tilde\Omega=\psi^{-1}(\Omega\cap\mathcal{E}_m)\subset\mathbb{R}^2$. As we shall prove, if R is large enough, then h does not vanish on $\partial\tilde\Omega$, and hence

$$d\Big(QN,\,\Omega\cap\mathrm{Ker}(L),\,0\Big) \;=\; d(h,\tilde\Omega,0)\;. \tag{7}$$

If we introduce polar coordinates $a=r\cos\omega$, $b=r\sin\omega$, a simple computation shows that the components of $h=(h_1,h_2)$ are

$$h_1(r\cos\omega,\,r\sin\omega) \;=$$
$$=\; a_m(p)-\frac{1}{\pi}\int_0^{2\pi} g\Big(r\cos(mt-\omega),-mr\sin(mt-\omega),-m^2r\cos(mt-\omega),...\Big)\cos(mt)\,dt$$

and

$$h_2(r\cos\omega,\,r\sin\omega) \;=$$
$$=\; b_m(p)-\frac{1}{\pi}\int_0^{2\pi} g\Big(r\cos(mt-\omega),-mr\sin(mt-\omega),-m^2r\cos(mt-\omega),...\Big)\sin(mt)\,dt\,.$$

In the same way as before we obtain:

$$\lim_{r\to\infty} h(r\cos\omega,r\sin\omega) \;=\; (a_m(p),b_m(p))-C(\omega)$$

where $C(\omega)\in\mathbb{R}^2$ is given by

$$C(\omega)=\frac{1}{\pi}\Big(A(cos\omega-\sin\omega)+B(cos\omega+\sin\omega),\;A(cos\omega+\sin\omega)+B(cos\omega-\sin\omega)\Big)$$

with $A=g_{+-}-g_{-+}$ and $B=g_{++}-g_{--}$.

Remark 4.1. In the same way as in [1] it can be proved that the previous limits are uniform in $\omega \in [0, 2\pi)$. □

Lemma 4.1. *Assume that* $g : \mathbb{R}^{N-1} \to \mathbb{R}$ *is continuous and bounded, that the four limits* $g_{\pm\pm}$ *exist, and that*

$$a_m^2(p) + b_m^2(p) \;<\; \frac{2}{\pi^2}\Big[(g_{+-} - g_{-+})^2 + (g_{++} - g_{--})^2\Big] \ .$$

Then for R *large enough*

$$d\Big(QN,\, \Omega \cap \mathrm{Ker}(L),\, 0\Big) = -1 \ .$$

Proof: Let us introduce the function

$$\tilde{h}(x,y) \;=\; \begin{cases} \Big(a_m(p), b_m(p)\Big) - T\Big(\dfrac{x}{\|(x,y)\|}, \dfrac{y}{\|(x,y)\|}\Big) & \text{if} \ \ \|(x,y)\| \ge 1 \\ \Big(a_m(p), b_m(p)\Big) - T(x,y) & \text{if} \ \ \|(x,y)\| < 1 \end{cases}$$

where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear operator given by

$$T(x,y) \;=\; \frac{1}{\pi}\Big(A(x-y) + B(x+y),\; A(x+y) - B(x-y)\Big)$$

with A and B as above. Next, define the homotopy

$$H(x,y,\lambda) \;=\; \lambda\, h(x,y) + (1-\lambda)\,\tilde{h}(x,y) \ .$$

From the previous computations, for $\lambda \neq 0$ it is clear that $H(x,y,\lambda) \neq 0$ for $\|(x,y)\|$ large. On the other hand, for $\lambda = 0$, $H = \tilde{h}$. If $\tilde{h}(x_0,y_0) = 0$ for some (x_0,y_0) with $\|(x_0,y_0)\| \ge 1$, then

$$a_m^2(p) + b_m^2(p) \;=\; \frac{2}{\pi^2}(A^2 + B^2) \ ,$$

a contradiction. Note that $\mathrm{dist}(\partial\tilde{\Omega}, 0) \to +\infty$ for $R \to +\infty$. Hence, if R is large enough, by the homotopy invariance and the excision property of the degree:

$$d(h, \tilde{\Omega}, 0) \;=\; d(\tilde{h}, \tilde{\Omega}, 0) \;=\; d(\tilde{h}, B_1, 0) \ .$$

But $\tilde{h}|_{B_1}$ has a unique zero at

$$(x_0,y_0) = \frac{\pi}{2(A^2+B^2)}\Big((A+B)a_m(p) + (A-B)b_m(p),\, -(A-B)a_m(p) + (A+B)b_m(p)\Big) \ .$$

Moreover,

$$D\tilde{h}(x_0, y_0) \;=\; -\frac{1}{\pi}\begin{pmatrix} A{+}B & -(A{-}B) \\ A{-}B & A{+}B \end{pmatrix} .$$

Hence $\operatorname{sgn}(D\tilde{h}(x_0, y_0)) = -1$, and we conclude that $d(h, B_R, 0) = -1$. ∎

Proof of Theorem 1.1: From Lemma 3.1 and Lemma 4.1, if we choose R large enough, all the conditions of Theorem 2.1 are fulfilled, and the proof is complete. ∎

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