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WEIGHTED BOUNDEDNESS OF MULTILINEAR LITTLEWOOD–PALEY OPERATORS FOR THE EXTREME CASES OF p

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Abstract: In this paper, we prove the boundedness of multilinear Littlewood–Paley operators for the extreme cases of p.

1 – Preliminaries and results

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integral function f on R^n, denote that $f(Q) = \int_Q f(x)dx$, $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q} |Q|^{-1}\int_Q |f(y)-f_Q|dy$. For a weight functions $w \in A_1$(see[10]), f is said to belong to $BMO(w)$ if $f^{\#} \in L^\infty(w)$ and define that $\|f\|_{BMO(w)} = \|f^{\#}\|_{L^\infty(w)}$; If $w = 1$, we denote that $BMO(R^n) = BMO(w)$. Also, we give the concepts of atom and weighted H^1 space. A function a is called a $H^1(w)$ atom if there exists a cube Q such that a is supported on Q, $\|a\|_{L^\infty(w)} \le w(Q)^{-1}$ and $\int a(x)dx = 0$. It is well known that, for $w \in A_1$, the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization (see[2]).

In this paper, we will consider a class of multilinear operators related to Littlewood–Paley operators, whose definition are the following.

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Let ψ be a fixed function which satisfies the following properties:
(1) $\int \psi(x)\,dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y)-\psi(x)| \le C|y|(1+|x|)^{-(n+2)}$ when $2|y|<|x|$.

Let m be a positive integer and A be a function on R^n. We denote that $\Gamma(x)=\{(y,t)\in R_+^{n+1}\colon |x-y|<t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Littlewood–Paley operator is defined by

$$S_\psi^A(f)(x) \;=\; \left[\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2\,\frac{dy\,dt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) \;=\; \int_{R^n} \frac{R_{m+1}(A;x,z)}{|x-z|^m}\, f(z)\,\psi_t(y-z)\;dz\;,$$

$$R_{m+1}(A;x,y) \;=\; A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!}\, D^\alpha A(y)\,(x-y)^\alpha\;,$$

the derivatives $D^\alpha A$ are understood in the distributional sense and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t>0$. We denote that $F_t(f) = f*\psi_t$. We also define

$$S_\psi(f)(x) \;=\; \left(\int_{\Gamma(x)} |F_t(f)(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood–Paley operator (see [18]).

Let H be the Hilbert space $H=\left\{h\colon \|h\|=\left(\int\int_{R_+^{n+1}} |h(t)|^2\,dy\,dt/t^{n+1}\right)^{1/2}<\infty\right\}$. Then for each fixed $x\in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$S_\psi^A(f)(x) = \left\|\chi_{\Gamma(x)}\, F_t^A(f)(x,y)\right\| \quad \text{and} \quad S_\psi(f)(x) = \left\|\chi_{\Gamma(x)}\, F_t(f)(y)\right\|\;.$$

We also consider the variant of S_ψ^A, which is defined by

$$\tilde S_\psi^A(f)(x) \;=\; \left[\int_{\Gamma(x)} |\tilde F_t^A(f)(x,y)|^2\,\frac{dy\,dt}{t^{n+1}}\right]^{1/2},$$

where

$$\tilde F_t^A(f)(x,y) \;=\; \int_{R^n} \frac{Q_{m+1}(A;x,z)}{|x-z|^m}\;\psi_t(y-z)\,f(z)\;dz$$

and

$$Q_{m+1}(A;x,z) \;=\; R_m(A;x,z) - \sum_{|\alpha|=m} D^\alpha A(x)\,(x-z)^\alpha\;.$$

Note that when $m=0$, S_ψ^A is just the commutator of Littlewood–Paley operator (see [1],[15],[16]). It is well known that multilinear operators, as an extension of commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see [4–9],[12–14]). In [11],[17], the endpoint boundedness properties of commutators generated by the Calderon–Zygmund operator or fractional integral operator with BMO functions are obtained. The main purpose of this paper is to study the boundedness of the multilinear Littlewood–Paley operators for the extreme cases of p. We shall prove the following theorems in Section 3.

Theorem 1. *Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then S_ψ^A maps $L^\infty(w)$ continuously into $BMO(w)$.*

Theorem 2. *Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then $\tilde{S}_\psi^A$ maps $H^1(w)$ continuously into $L^1(w)$.*

Theorem 3. *Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then S_ψ^A maps $H^1(w)$ continuously into weak $L^1(w)$.*

Remark. In general, S_ψ^A is not $(H^1(w), L^1(w))$ bounded. □

2 – Some Lemmas

We begin with some preliminary lemmas.

Lemma 1 (see [7]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \;\le\; C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q \, dz \right)^{1/q} ,$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}\,|x-y|$. ∎

Lemma 2 (see [3]). *Let T_b be the commutator defined by*

$$T_b f(x) \;=\; \int \frac{b(x)-b(y)}{|x-y|^n} \, f(y) \, dy \;.$$

If $w \in A_1$, $1 < p < \infty$ and $b \in BMO(R^n)$. Then T_b is bounded on $L^p(w)$. ∎

Lemma 3 (see [8],[9]). *Let T_A be the multilinear operator defined by*

$$T_A f(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^{m+n}} f(y)\,dy\ .$$

If $w \in A_1$, $1 < p < \infty$, $1 < r \le \infty$, $1/q = 1/p + 1/r$ and $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$. Then T_A is bounded from $L^p(w)$ to $L^q(w)$, that is

$$\|T_A(f)\|_{L^q(w)} \le C\|f\|_{L^p(w)}\ . \blacksquare$$

Lemma 4. *Let $w \in A_1$, $1 < p < \infty$, $1 < r \le \infty$, $1/q = 1/p + 1/r$ and $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$. Then S_ψ^A is bounded from $L^p(w)$ to $L^q(w)$, that is*

$$\|S_\psi^A(f)\|_{L^q(w)} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}\, \|f\|_{L^p(w)}\ .$$

Proof: By Minkowski inequality and the condition of ψ, we have

$$\begin{aligned} S_\psi^A(f)(x) &\le \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_{\Gamma(x)} |\psi_t(y-z)|^2 \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\ &\le C \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \int_{|x-y|\le t} \frac{t^{-2n}}{(1+|y-z|/t)^{2n+2}} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\ &\le C \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \int_{|x-y|\le t} \frac{2^{2n+2}\cdot t^{1-n}}{(2t+|y-z|)^{2n+2}}\, dydt \right)^{1/2} dz\ , \end{aligned}$$

noting that $2t + |y-z| \ge 2t + |x-z| - |x-y| \ge t + |x-z|$ when $|x-y| \le t$ and

$$\int_0^\infty \frac{t\,dt}{(t+|x-z|)^{2n+2}} = C\,|x-z|^{-2n}\ ,$$

we obtain

$$\begin{aligned} S_\psi^A(f)(x) &\le C \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \frac{t\,dt}{(t+|x-z|)^{2n+2}} \right)^{1/2} dz \\ &= C \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^{m+n}}\, dz\ , \end{aligned}$$

thus, the lemma follows from Lemma 3. $\blacksquare$

3 – Proofs of Theorems

Proof of Theorem 1: We have only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)}\int_Q |S_\psi^A(f)(x)-C_Q|\,w(x)\,dx \;\le\; C\,\|f\|_{L^\infty(w)}$$

holds for any cube Q. Fix a cube $Q=Q(x_0,l)$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha$, then $R_m(A;x,y)=R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A}=D^\alpha A-(D^\alpha A)_{\tilde{Q}}$ for $|\alpha|=m$. We write, for $f_1=f\chi_{\tilde{Q}}$ and $f_2=f\chi_{R^n\setminus\tilde{Q}}$,

$$F_t^A(f)(x) \;=\; F_t^A(f_1)(x)+F_t^A(f_2)(x)\;,$$

then

$$\begin{aligned}
\frac{1}{w(Q)}\int_Q &|S_\psi^A(f)(x)-S_\psi^A(f_2)(x_0)|\,w(x)\,dx \;=\\
&= \frac{1}{w(Q)}\int_Q \Big|\|\chi_{\Gamma(x)}F_t^A(f)(x,y)\|-\|\chi_{\Gamma(x)}F_t^A(f_2)(x_0,y)\|\Big|\,w(x)\,dx\\
&\le \frac{1}{w(Q)}\int_Q S_\psi^A(f_1)(x)\,w(x)\,dx\\
&\quad+\frac{1}{w(Q)}\int_Q \Big\|\chi_{\Gamma(x)}F_t^A(f_2)(x,y)-\chi_{\Gamma(x)}F_t^A(f_2)(x_0,y)\Big\|\,w(x)\,dx\\
&:= I+II\;.
\end{aligned}$$

Now, let us estimate I and II . First, by the L^∞ boundedness of S_ψ^A (see Lemma 4), we get

$$I \;\le\; \|S_\psi^A(f_1)\|_{L^\infty(w)} \;\le\; C\,\|f\|_{L^\infty(w)}\;.$$

To estimate II, we write

$$\begin{aligned}
\chi_{\Gamma(x)}&F_t^A(f_2)(x,y)-\chi_{\Gamma(x_0)}F_t^A(f_2)(x_0,y) \;=\\
&= \int\left[\frac{1}{|x-z|^m}-\frac{1}{|x_0-z|^m}\right]\chi_{\Gamma(x)}\,\psi_t(y-z)\,R_m(\tilde{A};x,z)\,f_2(z)\,dz\\
&\quad+\int\frac{\chi_{\Gamma(x)}\,\psi_t(y-z)\,f_2(z)}{|x_0-z|^m}\Big[R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)\Big]\,dz\\
&\quad+\int(\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)})\,\frac{\psi_t(y-z)\,R_m(\tilde{A};x_0,z)\,f_2(z)}{|x_0-z|^m}\,dz\\
&\quad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int\left[\frac{\chi_{\Gamma(x)}(x-z)^\alpha}{|x-z|^m}-\frac{\chi_{\Gamma(x_0)}(x_0-z)^\alpha}{|x_0-z|^m}\right]\psi_t(y-z)\,D^\alpha\tilde{A}(z)\,f_2(z)\,dz\\
&:= II_1^t(x)+II_2^t(x)+II_3^t(x)+II_4^t(x)\;.
\end{aligned}$$

Note that $|x-z| \sim |x_0-z|$ for $x \in Q$ and $z \in R^n \setminus Q$, similarly to the proof of Lemma 4 and by Lemma 1, we have

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q \|II_1^t(x)\|\, w(x)\ dx \ \le \\
&\qquad \le \frac{C}{w(Q)}\int_Q \left(\int_{R^n\setminus\tilde{Q}} \frac{|x-x_0|\,|f(z)|}{|x-z|^{n+m+1}}\,|R_m(\tilde{A};x,z)|\,dz\right) w(x)\ dx \\
&\qquad \le \frac{C}{w(Q)}\int_Q \left(\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|\,|f(z)|}{|x-z|^{n+m+1}}\,|R_m(\tilde{A};x,z)|\,dz\right) w(x)\ dx \\
&\qquad \le C\sum_{k=0}^{\infty}\frac{k\,l\,(2^kl)^m}{(2^kl)^{n+m+1}}\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\left(\int_{2^k\tilde{Q}}|f(z)|\,dz\right) \\
&\qquad \le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\sum_{k=0}^{\infty}k\,2^{-k} \\
&\qquad \le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\ ;
\end{aligned}$$

For $II_2^t(x)$, by the formula (see [7]):

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z) = R_m(\tilde{A};x,x_0)+\sum_{0<|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x_0,z)(x-x_0)^\beta$$

and Lemma 1, we get

$$\begin{aligned}
&|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)| \ \le \\
&\qquad \le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\left(|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}\,|x-x_0|^{|\beta|}\right),
\end{aligned}$$

thus, for $x \in Q$,

$$\begin{aligned}
\|II_2^t(x)\| &\le C\int_{R^n}\frac{|f_2(z)|}{|x-z|^{m+n}}\,|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|\ dz \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\int_{R^n}\frac{|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}\,|x-x_0|^{|\beta|}}{|x_0-z|^{m+n}}\,|f_2(z)|\,dz \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\sum_{k=0}^{\infty}\frac{kl^m}{(2^kl)^{m+n}}\int_{2^k\tilde{Q}}|f(z)|\,dz \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\sum_{k=1}^{\infty}k2^{-km} \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\ ;
\end{aligned}$$

For $II_3^t(x)$, note that $|x+y-z| \sim |x_0+y-z|$ for $x \in Q$ and $z \in R^n \setminus Q$, we obtain, similarly to the estimate of $II_1^t(x)$,

$$
\begin{aligned}
&\|II_3^t(x)\| \le \\
&\le C\int_{R^n}\left(\iint_{R_+^{n+1}}\left[\frac{|\psi_t(y-z)||f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}|\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t)|\right]^2\frac{dydt}{t^{n+1}}\right)^{1/2}dz \\
&\le C\int_{R^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\left|\iint_{\Gamma(x)}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2}}-\iint_{\Gamma(x_0)}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2}}\right|^{1/2}dz \\
&\le C\int_{R^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\qquad\cdot\left(\iint_{|y|\le t}\left|\frac{1}{(t+|x+y-z|)^{2n+2}}-\frac{1}{(t+|x_0+y-z|)^{2n+2}}\right|\frac{dydt}{t^{n-1}}\right)^{1/2}dz \\
&\le C\int_{R^n}\frac{|f_2(z)|\,|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\left(\iint_{|y|\le t}\frac{|x-x_0|\,t^{1-n}\,dydt}{(t+|x+y-z|)^{2n+3}}\right)^{1/2}dz \\
&\le C\int_{R^n}\frac{|f_2(z)|\,|x-x_0|^{1/2}\,|R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n+1/2}}\,dz \\
&\le C\sum_{k=0}^{\infty}\frac{kl^{1/2}(2^kl)^m}{(2^kl)^{n+m+1/2}}\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\left(\int_{2^k\tilde{Q}}|f(z)|\,dz\right) \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\sum_{k=0}^{\infty}k2^{-k/2} \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\ ;
\end{aligned}
$$

For $II_4^t(x)$, similarly to the estimates of $II_1^t(x)$ and $II_3^t(x)$, we get

$$
\begin{aligned}
\|II_4^t(x)\| &\le C\int_{R^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x-z|^{n+1/2}}\right)\sum_{|\alpha|=m}|D^\alpha\tilde{A}(z)|\,|f(z)|\,dz \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\sum_{k=0}^{\infty}k(2^{-k}+2^{-k/2}) \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^\infty(w)}\ .
\end{aligned}
$$

Combining these estimates, we complete the proof of Theorem 1. ∎

Proof of Theorem 2: It suffices to show that there exists a constant $C>0$ such that for every $H^1(w)$-atom a, the following holds:

$$\|\tilde{S}_\psi^A(a)\|_{L^1(w)} \le C\ .$$

We write

$$\int_{R^n} \tilde{S}_\psi^A(a)(x)\, w(x)\ dx\ =\ \left[\int_{2Q} + \int_{(2Q)^c}\right] \tilde{S}_\psi^A(a)(x)\, w(x)\ dx\ :=\ J + JJ\ .$$

For J, by the following equality

$$Q_{m+1}(A;x,y)\ =\ R_{m+1}(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha \left(D^\alpha A(x) - D^\alpha A(y)\right)\ ,$$

we get, similarly to the proof of Lemma 4,

$$\tilde{S}_\psi^A(a)(x)\ \le\ S_\psi^A(a)(x)\ +\ C\sum_{|\alpha|=m}\int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n}\, |a(y)|\ dy\ ,$$

thus, $\tilde{S}_\psi^A$ is L^∞-bounded by Lemma 2 and Lemma 4. We see that

$$J\ \le\ C\|\tilde{S}_\psi^A(a)\|_{L^\infty(w)}\, w(2Q)\ \le\ C\|a\|_{L^\infty(w)}\, w(Q)\ \le\ C\ .$$

To obtain the estimate of JJ, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2Q} x^\alpha$. Then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$. We write, by the vanishing moment of a and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for $x\in(2Q)^c$,

$$\begin{aligned}
\tilde{F}_t^A(a)(x,y) &= \int \frac{\psi_t(y-z)\, R_m(\tilde{A};x,z)}{|x-z|^m}\ a(z)\ dz \\
&\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int \frac{\psi_t(y-z)\, D^\alpha\tilde{A}(z)\,(x-z)^\alpha}{|x-z|^m}\ a(z)\ dz \\
&= \int \left[\frac{\psi_t(y-z)\, R_m(\tilde{A};x,z)}{|x-z|^m} - \frac{\psi_t(y-x_0)\, R_m(\tilde{A};x,x_0)}{|x-x_0|^m}\right] a(z)\ dz \\
&\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int\left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-x_0)(x-x_0)^\alpha}{|x-x_0|^m}\right] D^\alpha\tilde{A}(x)\, a(z)\, dz\,,
\end{aligned}$$

thus, similarly to the proof of II in Theorem 1, we obtain

$$\|\tilde{F}_t^A(a)(x,y)\| \le C\frac{|Q|^{1+1/n}}{w(Q)}\left(\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}|x-x_0|^{-n-1} + |x-x_0|^{-n-1}|D^\alpha\tilde{A}(x)|\right);$$

Note that if $w \in A_1$, then $\frac{w(Q_2)}{|Q_2|}\frac{|Q_1|}{w(Q_1)} \le C$ for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$. Thus, by Holder'inequality and the reverse of Holder' inequality for $w \in A_1$, choose $p > 1$ and $1/p + 1/p' = 1$, we obtain

$$\begin{aligned} JJ \;&\le\; C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=1}^{\infty} 2^{-k} \left(\frac{|Q|}{w(Q)}\, \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right) \\ &\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} 2^{-k} \frac{|Q|}{w(Q)} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^\alpha \tilde{A}(x)|^p \, dx \right)^{1/p} \\ &\quad \cdot \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^{p'} \, dx \right)^{1/p'} \\ &\le\; C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=1}^{\infty} k\, 2^{-k} \left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\, \frac{|Q|}{w(Q)} \right) \;\le\; C \;, \end{aligned}$$

which together with the estimate of J yields the desired result. This finishes the proof of Theorem 2. ∎

Proof of Theorem 3: By the equality

$$R_{m+1}(A;x,y) \;=\; Q_{m+1}(A;x,y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha \Big(D^\alpha A(x) - D^\alpha A(y) \Big)$$

and similarly to the proof of Lemma 4, we get

$$S_\psi^A(f)(x) \;\le\; \tilde{S}_\psi^A(f)(x) + C \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} \, |f(y)| \, dy \;,$$

by Theorem 1,2 and Lemma 2, we obtain

$$\begin{aligned} & w\Big(\Big\{ x \in R^n \colon\, S_\psi^A(f)(x) > \lambda \Big\} \Big) \;\le \\ &\qquad \le\; w\Big(\Big\{ x \in R^n \colon\, \tilde{S}_\psi^A(f)(x) > \lambda/2 \Big\} \Big) \\ &\qquad\quad + w\left(\left\{ x \in R^n : \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} \, |f(y)| \, dy > C\lambda \right\} \right) \\ &\qquad \le\; C \, \|f\|_{H^1(w)} / \lambda \;. \end{aligned}$$

This completes the proof of Theorem 3. ∎

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REFERENCES

[1] ALVAREZ, J.; BABGY, R.J.; KURTZ, D.S. and PEREZ, C. – Weighted estimates for commutators of linear operators, *Studia Math.,* 104 (1993), 195–209.

[2] BUI HUY QUI – Weighted Hardy spaces, *Math. Nachr.,* 103 (1981), 45–62.

[3] CHANILLO, S. – A note on commutators, *Indiana Univ. Math. J.,* 31 (1982), 7–16.

[4] CHEN, W. and HU, G. – Weak type (H^1, L^1) estimate for multilinear singular integral operator, *Adv. in Math. (China),* 30 (2001), 63–69.

[5] COHEN, J. – A sharp estimate for a multilinear singular integral on R^n, *Indiana Univ. Math. J.,* 30 (1981), 693–702.

[6] COHEN, J. and GOSSELIN, J. – On multilinear singular integral operators on R^n, *Studia Math.,* 72 (1982), 199–223.

[7] COHEN, J. and GOSSELIN, J. – A BMO estimate for multilinear singular integral operators, *Illinois J. Math.,* 30 (1986), 445–465.

[8] DING, Y. – A note on multilinear fractional integrals with rough kernel, *Adv. in Math. (China),* 30 (2001), 238–246.

[9] DING, Y. and LU, S.Z. – Weighted boundedness for a class rough multilinear operators, *Acta Math. Sinica,* 3 (2001), 517–526.

[10] GARCIA-CUERVA, J. and RUBIO DE FRANCIA, J.L. – *Weighted norm inequalities and related topics*, North-Holland Math. 16, Amsterdam, 1985.

[11] HARBOURE, E.; SEGOVIA, C. and TORREA, J.L. – Boundedness of commutators of fractional and singular integrals for the extreme values of p, *Illinois J. Math.,* 41 (1997), 676–700.

[12] HU, G. and YANG, D.C. – A variant sharp estimate for multilinear singular integral operators, *Studia Math.,* 141 (2000), 25–42.

[13] HU, G. and YANG, D.C. – Multilinear oscillatory singular integral operators on Hardy spaces, *Chinese J. of Contemporary Math.,* 18 (1997), 403–413.

[14] KOMORI, Y. – Weighted endpoint estimates for multilinear singular integrals, *Far East J. Math. Sci.,* 4(2) (2002), 189–207.

[15] LIU LANZHE – Weighted weak type estimates for commutators of Littlewood–Paley operator, *Japanese J. of Math.,* 29(1) (2003), 1–13.

[16] LIU LANZHE – Weighted weak type (H^1,L^1) estimates for commutators of Littlewood–Paley operator, *Indian J. of Math.,* 45(1) (2003), 71–78.

[17] PEREZ, C. – Endpoint estimate for commutators of singular integral operators, *J. Func. Anal.,* 128 (1995), 163–185.

[18] TORCHINSKY, A. – *The real variable methods in harmonic analysis*, Pure and Applied Math. 123, Academic Press, New York, 1986.

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