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EXACT TRAVELING WAVE SOLUTIONS FOR DISCRETE CONSERVATION LAWS

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Abstract: In this paper, sine, cosine, hyperbolic sine and hyperbolic cosine travelling wave solutions for a class of linear partial difference equations modeling discrete conservation laws are obtained.

1 – Introduction

Consider a chain of chambers which interact through exchange of material. Assume the chain can be modeled by a doubly infinite sequence of identical chambers and that our material can, in a specific time period t, only flow from the $(n-1)$-th chamber to the n-th chamber. Let $u_n^{(t)}$ be the size of the material in the n-th chamber and in the time period t. Then a dynamical model describing the interaction as time evolves may take the form

$$u_n^{(t+1)} - u_n^{(t)} = F\left(u_{n-1}^{(t)} - u_n^{(t)}\right) ,$$

which roughly says that the increase or decrease of the size of the material in the n-th chamber in one time period is 'balanced' by the decrease or increase of the size of the material in the neighboring chamber.

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In particular, when our interaction assumes that $u_n^{(t+1)} - u_n^{(t)}$ is proportional to $u_{n-1}^{(t)} - u_n^{(t)}$, say, $r\left(u_{n-1}^{(t)} - u_n^{(t)}\right)$, then we have the following dynamic model

$$u_n^{(t+1)} - u_n^{(t)} = r\left(u_{n-1}^{(t)} - u_n^{(t)}\right), \tag{1}$$

where r is a proportionality constant.

Clearly, the above equation is a special case of the following more general conservation law

$$u_n^{(t+1)} = au_n^{(t)} + bu_{n-1}^{(t)}, \quad ab \neq 0, \tag{2}$$

where $n \in Z = \{..., -2, -1, 0, +1, ...\}$ and $t \in N = \{0, 1, 2, ...\}$.

We remark that when either a or b is 0, the equation in (2) is quite simple. For this reason, we have assumed that $ab \neq 0$.

For equation (2), the existence and uniqueness of solutions is easy to see. Indeed, if the initial distribution $\{u_n^{(0)}\}_{n\in Z}$ is known, then we may calculate successively the sequence

$$u_{-1}^{(1)}, u_0^{(1)}, u_1^{(1)}; u_{-2}^{(1)}, u_{-1}^{(2)}, u_0^{(2)}, u_1^{(2)}, u_2^{(1)}, ...$$

in a unique manner, which will give rise to a unique solution of (2).

An interesting question arises as to whether there is a solution $\left\{u_n^{(t)}\right\}$ of (2) such that $u_n^{(t+1)} = u_{n-m}^{(t)}$ for some integer m and all n and all t. If such a solution exist, it is naturally called a traveling wave since in one period of time, the initial distribution is shifted m units to the right if m is positive, or m units to the left if m is negative. In the particular case when m is 0, there is no shift and the corresponding solution is also called a stationary wave solution. For instance, the equation (1) has a traveling wave solution $\left\{u_n^{(t)}\right\}$ defined by $u_n^{(t)} = 1$ for $t \in N$ and $n \in Z$.

Traveling wave solutions are the subject of many investigations, see e.g. [1]. In particular, in [2], positive traveling wave solutions of the form

$$u_n^{(t)} = \lambda^{n-mt}, \quad m \in Z, \ \ \lambda > 0, \ \ n \in Z, \ \ t \in N .$$

have been found for the equation

$$u_n^{(t+1)} = au_{n-1}^{(t)} + bu_n^{(t)} + cu_{n+1}^{(t)}, \quad n \in Z, \ \ t \in N, \tag{3}$$

where the coefficients a, b and c are real numbers. In this paper, we will be interested in finding additional traveling wave solutions for the more special equation (2). As we will see later, these solutions are related to the sine, cosine, hyperbolic sine and hyperbolic cosine functions.

For background materials involving equations such as (1) and (3), the book by Cheng [3] can be consulted. An example illustrating the use of traveling wave solutions is also included in the last section for additional illustration.

As in [1,2], we first observe that a traveling wave solution is of the form

$$u_n^{(t)} = \varphi(n - mt), \quad n \in Z, \ t \in N \ . \tag{4}$$

Indeed, if $u_n^{(t)} = \varphi(n - mt)$ for some function $\varphi : Z \to R$, then $u_n^{(t+1)} = u_{n-m}^{(t)}$ for all $n \in Z$ and $t \in N$. Conversely, if we let $\varphi(k) = u_k^{(0)}$ for $k \in Z$, then

$$u_n^{(t)} = u_{n-m}^{(t-1)} = u_{n-2m}^{(t-2)} = ... = u_{n-mt}^{(0)} = \varphi(n - mt)$$

as required.

Before we discuss the main results, we first consider the stationary solution of (2). Note that if $m = 0$, then

$$u_n^{(t)} = \varphi(n) = u_n^{(0)}, \quad t \in N, \ n \in Z \ ,$$

and

$$(1 - a)\,\varphi(n) = b\,\varphi(n - 1), \quad n \in Z \ .$$

Thus

$$\varphi(n) = \frac{1-a}{b}\,\varphi(n + 1), \quad n \in Z \ . \tag{5}$$

The converse also holds as can be verified easily.

Theorem 1. *Let $\{\varphi(n)\}_{n\in Z}$ be a real sequence defined by (5). Then the initial distribution $\left\{u_n^{(0)}\right\} = \{\varphi(n)\}_{n\in Z}$ determines a stationary solution of (2). Conversely, if $\left\{u_n^{(t)}\right\}$ is a stationary solution of (2), then $u_n^{(0)} = b^{-1}(1 - a)u_{n+1}^{(0)}$ for all $n \in Z$.* ∎

We remark that in case $a = 1$, the real sequence $\{\varphi(n)\}_{n\in Z}$ that satisfies (5) is the trivial sequence, and in case $a \neq 1$, the real sequence $\{\varphi(n)\}_{n\in Z}$ defined by (5) is of the form

$$\varphi(n) = \left(\frac{1-a}{b}\right)^{-n} \varphi(0), \quad n \in Z \ .$$

Next we discuss non-stationary traveling wave solutions. Substituting $\varphi\,(n - mt)$ into the equations (2), we obtain

$$\varphi\,(n - mt - m) \ = \ a\varphi\,(n - mt) + b\varphi\,(n - mt - 1) \ . \tag{6}$$

Letting $k = n - mt$, we obtain the difference equation

$$\varphi(k-m) = a\varphi(k) + b\varphi(k-1), \quad k \in Z . \tag{7}$$

In principle, if we can find an integer m and a corresponding solution $\{\varphi(k)\}_{k\in Z}$ of (7), then (4) defines a traveling wave solution of (2). To this end, we apply the well known result that the unknown solution is a linear combination of solutions of the form $\{\lambda^k\}$. Substituting $\varphi(k) = \lambda^k$ into (7), we obtain the characteristic equation

$$a\lambda^m + b\lambda^{m-1} - 1 = 0 . \tag{8}$$

For each integer m, we may then try to solve for the corresponding roots λ. As an example, let us consider the equation

$$\varphi(k-3) = 4\varphi(k) + 2\varphi(k-1) .$$

Solving the characteristic equation $4\lambda^3 + 2\lambda^2 - 1 = 0$, we obtain roots $\frac{1}{2}, -\frac{1}{2} - \frac{1}{2}i, -\frac{1}{2} + \frac{1}{2}i$. Hence the equation

$$u_n^{(t+1)} = 4u_n^{(t)} + 2u_{n-1}^{(t)}, \quad n \in Z, \ t \in N ,$$

has the traveling solutions $\left\{u_n^{(t)}\right\}$ defined by

$$u_n^{(t)} = \left(\frac{1}{2}\right)^{n-3t}, \quad \left(-\frac{1}{\sqrt{2}}\right)^{n-3t} \cos\frac{(n-3t)\pi}{4} \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}\right)^{n-3t} \sin\frac{(n-3t)\pi}{4} .$$

Next, the characteristic equation corresponding to

$$\varphi(k-3) = 2\varphi(k) + 3\varphi(k-1)$$

has roots $\frac{1}{2}$ and -1. Hence the equation

$$u_n^{(t+1)} = 2u_n^{(t)} + 3u_{n-1}^{(t)}, \quad n \in Z, \ t \in N ,$$

has the traveling solutions $\left\{u_n^{(t)}\right\}$ defined by

$$u_n^{(t)} = (1/2)^{n-3t} \ \text{ and } \ (-1)^{n-3t} .$$

Last, the characteristic equation corresponding to

$$\varphi(k-3) = -\frac{1}{4}\varphi(k) - \frac{1}{4}\varphi(k-1)$$

has the multiple root $\lambda = -2$. Hence the equation

$$u_n^{(t+1)} = -\frac{1}{4} u_n^{(t)} - \frac{1}{4} u_{n-1}^{(t)}, \qquad n \in Z, \quad t \in N ,$$

has the traveling solution $\left\{u_n^{(t)}\right\}$ defined by

$$u_n^{(t)} = (-2)^{n-3t} .$$

Although we can solve in an explicit manner some of the characteristic equations as seen above, in the general case, it is difficult to find the exact roots. We may turn to numerical methods of course. However, 'explicit' traveling wave solutions are of theoretical interest and may provide insights to the qualitative behavior of discrete conservation laws such as those described here and elsewhere. For this reason, in section 2, we will seek sine and cosine traveling wave solutions, and in section 3, we will seek hyperbolic sine and cosine traveling wave solutions.

In the following sections, for the sake of convenience, we set

$$\xi = \frac{1-a^2-b^2}{2ab}, \qquad \eta = \frac{1+a^2-b^2}{2a} \quad \text{and} \quad \zeta = \frac{1+b^2-a^2}{2b} . \tag{9}$$

Note that ξ, η and ζ are well defined when $ab \neq 0$, and

$$a + b\xi = \eta, \qquad b + a\xi = \zeta .$$

We will also take $v = \cos^{-1} u$ as the inverse function of $y = \cos x$ defined for $x \in [0, \pi]$.

2 – Sine and cosine traveling wave solutions

We seek explicit solutions of (8) in special forms. Among these is one that satisfies $\lambda = e^{i\theta}$ where $\theta \in [0, \pi]$. In other words, we will seek (complex valued) traveling wave solutions of the form $\{(e^{i\theta})^{n-mt}\}$ for (2). Note that such a solution then leads to real traveling wave solutions $\left\{u_n^{(t)}\right\}$ and $\left\{v_n^{(t)}\right\}$ defined by

$$u_n^{(t)} = \sin(n-mt)\theta, \qquad n \in Z, \quad t \in N ,$$

and

$$v_n^{(t)} = \cos(n-mt)\theta, \qquad n \in Z, \quad t \in N$$

respectively. Since (2) is a linear equation, linear combinations of these solution are also traveling wave solutions. In particular, $\{-\sin(n-mt)\theta\}$ is also a traveling wave solution. Therefore $\left\{e^{-i\theta(n-mt)}\right\}$ is a (complex valued) traveling wave and $-\theta$ now belongs to $[\pi, 2\pi]$. This is the reason why we have restricted our attention to $\theta \in [0, \pi]$.

It turns out such traveling solutions can be found when the pair (a, b) is inside the following region of the plane:

$$\Omega \equiv \left\{(x, y) \mid -1 \leq |x| - |y| \leq 1 \leq |x| + |y|\right\} . \tag{10}$$

Lemma 1. *Suppose $ab \neq 0$ and let ξ, η and ζ be defined by (9). Then*

$$|\xi| \leq 1 \iff |\eta| \leq 1 \iff |\zeta| \leq 1 \iff -1 \leq |a| - |b| \leq 1 \leq |a| + |b| . \tag{11}$$

Proof: First,

$$\begin{aligned} |\eta| \leq 1 \;&\iff\; |1 + a^2 - b^2| \leq 2|a| \\ &\iff\; -2|a| \leq 1 + a^2 - b^2 \leq 2|a| \\ &\iff\; 1 + a^2 - 2|a| \leq b^2 \leq 1 + a^2 + 2|a| \\ &\iff\; (1 - |a|)^2 \leq b^2 \leq (1 + |a|)^2 \\ &\iff\; -|b| \leq 1 - |a| \leq |b| \leq 1 + |a| \\ &\iff\; -1 \leq |a| - |b| \leq 1 \leq |a| + |b| . \end{aligned}$$

Similarly, $|\zeta| \leq 1 \iff -1 \leq |a| - |b| \leq 1 \leq |a| + |b|$.

Second,

$$\begin{aligned} |\xi| \leq 1 \;&\iff\; |1 - a^2 - b^2| \leq 2|ab| \\ &\iff\; -2|ab| \leq 1 - a^2 - b^2 \leq 2|ab| \\ &\iff\; -2|ab| \leq 1 - a^2 - b^2 \leq 2|ab| \\ &\iff\; (|a| - |b|)^2 \leq 1 \leq (|a| + |b|)^2 \\ &\iff\; -1 \leq |a| - |b| \leq 1 \leq |a| + |b| . \end{aligned}$$

The proof is complete. ∎

Theorem 2. *Suppose $ab \neq 0$ and $(a, b) \in \Omega$ where Ω is defined by (10). If $\left\{e^{i\theta(n-mt)}\right\}$, where $\theta \in [0, \pi]$ and $m \in Z$, is a (complex valued) traveling wave solution of (2), then θ and m must satisfy the system of equations:*

$$\begin{cases} \cos\theta = \xi , \\ \cos(m-1)\theta = \zeta , \\ \cos m\theta = \eta . \end{cases} \tag{12}$$

Conversely, if $\theta \in [0,\pi]$ and $m \in Z$ satisfy (12), then $\left\{e^{i\theta(n-mt)}\right\}$ is a (complex valued) traveling wave solution of (2).

Proof: If $\left\{e^{i\theta(n-mt)}\right\}$, where $\theta \in [0,\pi]$ and $m \in Z$, is a (complex valued) traveling wave solution of (2), then $e^{i\theta}$ will satisfy (8)

$$ae^{im\theta} + be^{i(m-1)\theta} = 1 \ ,$$

that is, θ and m form a solution pair of

$$\left\{\begin{aligned} a\cos m\theta + b\cos(m-1)\theta \ &= \ 1\,, \\ a\sin m\theta + b\sin(m-1)\theta \ &= \ 0 \ . \end{aligned}\right. \tag{13}$$

Thus

$$\Big[a\cos m\theta + b\cos(m-1)\theta\Big]^2 + \Big[a\sin m\theta + b\sin(m-1)\theta\Big]^2 = 1 \ ,$$

so that

$$\cos\theta = \frac{1-a^2-b^2}{2ab} \ . \tag{14}$$

Rewriting (13) as,

$$\left\{\begin{aligned} a\cos m\theta &= 1 - b\cos(m-1)\theta\,, \\ a\sin m\theta &= -b\sin(m-1)\theta \ , \end{aligned}\right.$$

we see also that

$$a^2\cos^2 m\theta + a^2\sin^2 m\theta \ = \ \Big(1-b\cos(m-1)\theta\Big)^2 + \Big(-b\sin(m-1)\theta\Big)^2 \ ,$$

and

$$\cos(m-1)\theta \ = \ \frac{1+b^2-a^2}{2b} \ . \tag{15}$$

Similarly rewriting (13) as

$$\left\{\begin{aligned} b\cos(m-1)\theta \ &= \ 1 - a\cos m\theta\,, \\ b\sin(m-1)\theta \ &= \ a\sin m\theta \ , \end{aligned}\right.$$

we may obtain

$$\cos m\theta = \frac{1+a^2-b^2}{2a} \ . \tag{16}$$

Conversely, assume (12) holds, we need to show that (13) holds. Indeed,

$$a\cos m\theta + b\cos(m-1)\theta \ = \ a\frac{1+a^2-b^2}{2a} + b\frac{1+b^2-a^2}{2b} \ = \ 1 \ .$$

Furthermore, note that if $\theta = 0$ or π, then the second equation in (13) is obviously true. If $\theta \in (0, \pi)$, then $\sin\theta \neq 0$, so that

$$\begin{aligned}
&\sin\theta\Big[a\sin m\theta + b\sin(m-1)\theta\Big] \;= \\
&\qquad = a\sin\theta\sin m\theta + b\sin\theta(\sin m\theta\cos\theta - \sin\theta\cos m\theta) \\
&\qquad = a\sin\theta\sin m\theta + b\sin\theta\sin m\theta\cos\theta - b\sin^2\theta\cos m\theta \\
&\qquad = (a+b\cos\theta)\sin\theta\sin m\theta - b(1-\cos^2\theta)\cos m\theta \\
&\qquad = (a+b\cos\theta)\Big[\cos(m-1)\theta - \cos m\theta\cos\theta\Big] - b(1-\cos^2\theta)\cos m\theta \\
&\qquad = \left(a + b\frac{1-a^2-b^2}{2ab}\right)\left(\frac{1+b^2-a^2}{2b} - \frac{1+a^2-b^2}{2a}\frac{1-a^2-b^2}{2ab}\right) \\
&\qquad\quad -b\left(1-\left(\frac{1-a^2-b^2}{2ab}\right)^2\right)\frac{1+a^2-b^2}{2a} \\
&\qquad = 0
\end{aligned}$$

implies

$$a\sin m\theta + b\sin(m-1)\theta \;=\; 0\;.$$

The proof is complete. ∎

Suppose $ab \neq 0$ and $(a,b) \in \Omega$ where Ω is defined by (10). Further assume that there exist θ and m such that $\cos\theta = \xi$ and $\cos m\theta = \eta$. We remark that we cannot conclude that $\left\{e^{i\theta(n-mt)}\right\}$ is a (complex valued) traveling wave solution of (2). Consider the following example

$$-\lambda^m + \lambda^{m-1} = 1\;. \tag{17}$$

Here $a = -1$ and $b = 1$. Thus, we have

$$\begin{cases} \cos\theta \;=\; \dfrac{1-a^2-b^2}{2ab} \;=\; \dfrac{1}{2}\,, \\[2ex] \cos m\theta \;=\; \dfrac{1+a^2-b^2}{2a} \;=\; -\dfrac{1}{2}\;. \end{cases} \tag{18}$$

Clearly, $\theta = \frac{\pi}{3}$ and $m = 4$ satisfy (18). However, $\lambda = e^{i\pi/3}$ is not the root of equation (17). In fact,

$$\begin{aligned}
-\lambda^m + \lambda^{m-1} \;&=\; -e^{4i\pi/3} + e^{3i\pi/3} \\
&=\; -\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3} + \cos\pi + i\sin\pi
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} + \frac{\sqrt{3}}{2}i - 1 \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \neq 1 \,. \end{aligned}$$

Hence, we know that $\cos\theta = \xi$ and $\cos m\theta = \eta$ are only necessary conditions, but not sufficient.

If $\theta \in [0, \pi]$ and $m \in Z$ satisfy the system (12). Then

$$\begin{cases} \cos\theta = \xi \\ \cos(m-1)\theta = \zeta \\ \cos m\theta = \eta \end{cases} \Leftrightarrow \begin{cases} \theta = \cos^{-1}\xi \\ (m-1)\theta = \pm\cos^{-1}\zeta + 2l\pi \\ m\theta = \pm\cos^{-1}\eta + 2k\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} \theta = \cos^{-1}\xi \\ m-1 = (\pm\cos^{-1}\zeta + 2l\pi)/\cos^{-1}\xi \\ m = (\pm\cos^{-1}\eta + 2k\pi)/\cos^{-1}\xi \end{cases}$$

$$\Leftrightarrow \quad \theta = \cos^{-1}\xi \text{ and } 1 + (\pm\cos^{-1}\zeta + 2l\pi)/\cos^{-1}\xi = (\pm\cos^{-1}\eta + 2k\pi)/\cos^{-1}\xi \,.$$

Thus, we immediately obtain the following fact.

Corollary 1. *Suppose $ab \neq 0$ and $(a, b) \in \Omega$ where Ω is defined by (10). Then $\left\{e^{i\theta(n-mt)}\right\}$, where $\theta \in [0, \pi]$ and $m \in Z$, is a (complex valued) traveling wave solution of (2) if, and only if, there exist integral numbers l and k such that*

$$\theta = \cos^{-1}\xi$$

and

(19) $$m = 1 + (\pm\cos^{-1}\zeta + 2l\pi)/\cos^{-1}\xi = (\pm\cos^{-1}\eta + 2k\pi)/\cos^{-1}\xi \in Z \,. \blacksquare$$

As an example, consider the equation

$$u_n^{(t+1)} = -2u_n^{(t)} + \sqrt{3}u_{n-1}^{(t)}\,, \qquad n \in Z, \quad t \in N \,.$$

We note that here $(a, b) = (-2, \sqrt{3}) \in \Omega$,

$$\theta = \cos^{-1}\xi = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

and

$$\begin{aligned} m &= (\pm\cos^{-1}\eta + 2k\pi)/\cos^{-1}\xi \\ &= (\pm\cos^{-1}(-\frac{1}{2}) + 2k\pi)/\frac{\pi}{6} \\ &= (\pm\frac{2\pi}{3} + 2k\pi)/\frac{\pi}{6} \\ &= \pm 4 + 12k \end{aligned}$$

$k \in Z$. Now we check whether θ and m satisfy the second equation of (12). In fact,

$$\cos(m-1)\theta = \begin{cases} \cos(3+12k)\frac{\pi}{6}, & m = 4+12k \\ \cos(-5+12k)\frac{\pi}{6}, & m = -4+12k \end{cases}$$

$$= \begin{cases} 0, & m = 4+12k \\ -\frac{\sqrt{3}}{2}, & m = -4+12k\ . \end{cases}$$

Thus $m = 4+12k$ satisfies the equation. And then, we get all the sine and cosine traveling solutions of this equation,

$$u_n^{(t)} = e^{i\theta(n-mt)} = e^{i\frac{\pi}{6}(n-(4+12k)t)}\ .$$

For some cases, such as $k = 0$, we get $m = 4$ and the traveling wave solutions are $\{\sin\pi(n-4t)/6\}$ and $\{\cos\pi(n-4t)/6\}$; for $k = -1$, we get $m = -8$ and the traveling wave solutions $\{\sin\pi(n+8t)/6\}$ and $\{\cos\pi(n+8t)/6\}$.

As another immediate corollary of Theorem 2, under $ab \neq 0$ and $(a,b) \in \Omega$ where Ω is defined by (10), if

$$\xi = \eta \quad \text{and} \quad \zeta = 1\ , \tag{20}$$

then (2) has the traveling wave solution $\left\{e^{i\theta(n-t)}\right\}$ where $\theta = \cos^{-1}\xi$. Similarly, note that $\cos 2\theta = 2\cos^2\theta - 1$, therefore if

$$2\xi^2 - 1 = \eta \quad \text{and} \quad \xi = \eta\ , \tag{21}$$

then (2) has the traveling wave solution $\left\{e^{i\theta(n-2t)}\right\}$ where $\theta = \cos^{-1}\xi$. The same principle leads to the following result, which involves the m-th Tchebysheff polynomial $T_m : [-1,1] \to R$ defined by $T_0(x) = 1$, $T_1(x) = x$ and $T_m(\cos\theta) = \cos m\theta$ for $m = 2, 3, \ldots$.

Corollary 2. *Suppose $ab \neq 0$ and $(a,b) \in \Omega$ where Ω is defined by (10). If $T_m(\xi) = \eta$ and $T_{m-1}(\xi) = \zeta$ where $m \geq 1$, then (2) has the traveling wave solution $\left\{e^{i\theta(n-mt)}\right\}$ where $\theta = \cos^{-1}\xi$.* ∎

In particular, if

$$\begin{cases} T_3(\xi) = 4\xi^3 - 3\xi = \eta \\ T_2(\xi) = 2\xi^2 - 1 = \zeta\ , \end{cases} \tag{22}$$

or

$$\begin{cases} T_4(\xi) = 8\xi^4 - 8\xi^2 + 1 = \eta \\ T_3(\xi) = 4\xi^3 - 3\xi = \zeta \ , \end{cases} \tag{23}$$

then (2) has traveling solutions $\left\{e^{i\theta(n-3t)}\right\}$ or $\left\{e^{i\theta(n-4t)}\right\}$ respectively.

We remark that conditions (20) and (21) can be written as implicit relations between a and b. For instance, (20) can be written as

$$\begin{cases} \dfrac{1-a^2-b^2}{2ab} = \dfrac{1+a^2-b^2}{2a} \\ \dfrac{1+b^2-a^2}{2b} = 1 \ , \end{cases}$$

which has solutions $(a,b) = (a,-a+1)$, $(a,a+1)$. In view of the assumptions $ab \neq 0$ and $(a,b) \in \Omega$ where Ω is defined by (10), we see that when

$$(a,b) \in \left\{(x,y) |\ y = -x+1,\ xy \neq 0\right\} , \tag{24}$$

or

$$(a,b) \in \left\{(x,y) |\ y = x+1,\ xy \neq 0\right\} , \tag{25}$$

(2) has traveling wave solutions of the form $\{e^{i\theta(n-t)}\}$ where $\theta = \cos^{-1}\xi \in [0,\pi]$.

Similarly,

$$\begin{cases} 2\left(\dfrac{1-a^2-b^2}{2ab}\right)^2 - 1 = \dfrac{1+a^2-b^2}{2a} \\ \dfrac{1-a^2-b^2}{2ab} = \dfrac{1+b^2-a^2}{2b} \end{cases}$$

has solutions $(a,b) = (a,a-1)$, $(a,-a+1)$. In view of the assumptions $ab \neq 0$ and $(a,b) \in \Omega$ where Ω is defined by (10), we see that when

$$(a,b) \in \left\{(x,y) |\ y = x-1,\ xy \neq 0\right\} , \tag{26}$$

or

$$(a,b) \in \left\{(x,y) |\ y = -x+1,\ xy \neq 0\right\} , \tag{27}$$

then (2) has traveling wave solutions of the form $\left\{e^{i\theta(n-2t)}\right\}$ where $\theta = \cos^{-1}\xi \in [0,\pi]$.

For the cases where $m = 3$ or 4, we can also find explicit conditions similar to those above for the existence of traveling wave solutions. For the case where $m > 4$, we can also find traveling wave solutions in theory, but the conditions become very complicated.

3 – Hyperbolic sine and cosine traveling wave solutions

In this section, we seek new explicit solutions of (8) in the form $\sinh(n-mt)\theta$ or $\cos(n-mt)\theta$. It turns out such traveling solutions can be found when the pair (a,b) is inside the following region of the plane:

$$\Gamma \equiv \left\{(x,y) \mid \frac{1-(x^2+y^2)}{2xy} > 1 \text{ and } \frac{1+x^2-y^2}{2x} > 1\right\} . \tag{28}$$

We remark that by symmetry considerations, we may show that Γ is also equal to

$$\left\{(x,y) \mid \frac{1-(x^2+y^2)}{2xy} > 1,\ \frac{1+x^2-y^2}{2x} > 1 \text{ and } \frac{1+y^2-x^2}{2y} > 1\right\} .$$

Theorem 3. *Suppose $ab \neq 0$ and $(a,b) \in \Gamma$ where Γ is defined by (28). If the double sequences $\{\cosh(n-mt)\theta\}$ and $\{\sinh(n-mt)\theta\}$, where $m \in Z$ and $\theta \in R$, are traveling wave solutions of (2), then $e^\theta = \xi \pm \sqrt{\xi^2-1}$ and $m \in Z$ must satisfy*

$$\left(\xi+\sqrt{\xi^2-1}\right)^m = \eta+\sqrt{\eta^2-1} \qquad \text{if } \ b>0 \tag{29}$$

or

$$\left(\xi+\sqrt{\xi^2-1}\right)^m = \eta-\sqrt{\eta^2-1} \qquad \text{if } \ b<0 \ . \tag{30}$$

Conversely, if $e^\theta = \xi \pm \sqrt{\xi^2-1}$ and $m \in Z$ satisfy (29) or (30), then the double sequences $\{\cos(n-mt)\theta\}$ and $\{\sinh(n-mt)\theta\}$ are traveling wave solutions of (2).

Proof: If $\{\cosh(n-mt)\theta\}$ and $\{\sinh(n-mt)\theta\}$ are traveling wave solutions of (2), then $\{\cosh k\theta\}$ and $\{\sinh k\theta\}$ are solutions of (7). Thus

$$\begin{cases} a+b\cosh\theta = \cosh m\theta \\ b\sinh\theta = \sinh m\theta \end{cases} \tag{31}$$

which implies

$$\begin{cases} a+b(\cosh\theta+\sinh\theta) = \cosh m\theta+\sinh m\theta \\ a+b(\cosh\theta-\sinh\theta) = \cosh m\theta-\sinh m\theta \end{cases} \tag{32}$$

and

$$\begin{cases} a+be^\theta = e^{m\theta} \ , \\ a+be^{-\theta} = e^{-m\theta} \ . \end{cases}$$

Let $t = e^\theta$, we get the following system

$$\begin{cases} a + bt = t^m \\ a + \dfrac{b}{t} = \dfrac{1}{t^m} \end{cases}$$

and

$$a^2 + b^2 + ab\left(t + \frac{1}{t}\right) = 1 \ .$$

Since $\xi > 1$, the above equation has solutions

$$t = \xi \pm \sqrt{\xi^2 - 1}\ , \tag{33}$$

so that $e^\theta = \xi \pm \sqrt{\xi^2 - 1}$.

On the other hand, we also get

$$\begin{cases} a - t^m = bt\ , \\ a - \dfrac{1}{t^m} = \dfrac{b}{t}\ . \end{cases}$$

From this, we get

$$t^m = \eta \pm \sqrt{\eta^2 - 1}\ . \tag{34}$$

If $b > 0$, then from (31) we know $\sinh m\theta$ and $\sinh\theta$ have the same sign. Hence $m > 0$. Thus $t = \xi + \sqrt{\xi^2 - 1}$ and $t^m = \eta + \sqrt{\eta^2 - 1}$, or $t = \xi - \sqrt{\xi^2 - 1}$ and $t^m = \eta - \sqrt{\eta^2 - 1}$. Therefore, we have

$$\left(\xi + \sqrt{\xi^2 - 1}\right)^m = \eta + \sqrt{\eta^2 - 1}\ .$$

If $b < 0$, then $\sinh m\theta$ and $\sinh\theta$ have different signs. Hence $m < 0$. Thus $t = \xi + \sqrt{\xi^2 - 1}$ and $t^m = \eta - \sqrt{\eta^2 - 1}$, or $t = \xi - \sqrt{\xi^2 - 1}$ and $t^m = \eta + \sqrt{\eta^2 - 1}$. Therefore, we have

$$\left(\xi - \sqrt{\xi^2 - 1}\right)^m = \eta - \sqrt{\eta^2 - 1}\ .$$

The proof of necessity is complete.

Now we prove the sufficiency of the condition. Assume first that $e^\theta = \xi + \sqrt{\xi^2 - 1}$, then

$$\cosh(n - mt)\theta = \frac{1}{2}\left\{\left(\xi + \sqrt{\xi^2 - 1}\right)^{n-mt} + \left(\xi - \sqrt{\xi^2 - 1}\right)^{n-mt}\right\}, \quad n \in Z,\ \ t \in N\,,$$

and

$$\sinh(n-mt)\theta = \frac{1}{2}\left\{\left(\xi+\sqrt{\xi^2-1}\right)^{n-mt} - \left(\xi-\sqrt{\xi^2-1}\right)^{n-mt}\right\}, \quad n \in Z, \quad t \in N\,.$$

Suppose $b > 0$ and $\left(\xi+\sqrt{\xi^2-1}\right)^m = \eta+\sqrt{\eta^2-1}$. Then the double sequence $\left\{u_n^{(t)}\right\} = \{\cosh(n-mt)\theta\}$ satisfies

$$\begin{aligned}
u_n^{(t+1)} &= \frac{1}{2}\left[\left(\xi+\sqrt{\xi^2-1}\right)^{n-m(t+1)} + \left(\xi-\sqrt{\xi^2-1}\right)^{n-m(t+1)}\right] \\
&= \frac{1}{2}\left[\left(\xi+\sqrt{\xi^2-1}\right)^{n-mt}\left(\xi+\sqrt{\xi^2-1}\right)^{-m} + \left(\xi-\sqrt{\xi^2-1}\right)^{n-mt}\left(\xi-\sqrt{\xi^2-1}\right)^{-m}\right] \\
&= \frac{1}{2}\left[\left(\xi+\sqrt{\xi^2-1}\right)^{n-mt}\left(\xi-\sqrt{\xi^2-1}\right)^{m} + \left(\xi-\sqrt{\xi^2-1}\right)^{n-mt}\left(\xi+\sqrt{\xi^2-1}\right)^{m}\right] \\
&= \frac{1}{2}\left[\left(\xi+\sqrt{\xi^2-1}\right)^{n-mt}\left(\eta-\sqrt{\eta^2-1}\right) + \left(\xi-\sqrt{\xi^2-1}\right)^{n-mt}\left(\eta+\sqrt{\eta^2-1}\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
u_{n-1}^{(t)} &= \frac{1}{2}\left[\left(\xi+\sqrt{\xi^2-1}\right)^{n-1-mt} + \left(\xi-\sqrt{\xi^2-1}\right)^{n-1-mt}\right] \\
&= \frac{1}{2}\left[\left(\xi+\sqrt{\xi^2-1}\right)^{n-mt}\left(\xi+\sqrt{\xi^2-1}\right)^{-1} + \left(\xi-\sqrt{\xi^2-1}\right)^{n-mt}\left(\xi-\sqrt{\xi^2-1}\right)^{-1}\right] \\
&= \frac{1}{2}\left[\left(\xi+\sqrt{\xi^2-1}\right)^{n-mt}\left(\xi-\sqrt{\xi^2-1}\right) + \left(\xi-\sqrt{\xi^2-1}\right)^{n-mt}\left(\xi+\sqrt{\xi^2-1}\right)\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
au_n^{(t)} + bu_{n-1}^{(t)} = \frac{1}{2}\Bigg\{&\left(\xi+\sqrt{\xi^2-1}\right)^{n-mt}\left[a+b\left(\xi-\sqrt{\xi^2-1}\right)\right] \\
&+ \left(\xi-\sqrt{\xi^2-1}\right)^{n-mt}\left[a+b\left(\xi+\sqrt{\xi^2-1}\right)\right]\Bigg\}.
\end{aligned}$$

To prove

$$u_n^{(t+1)} = au_n^{(t)} + bu_{n-1}^{(t)}\,,$$

we need to show that

$$\begin{cases} \eta-\sqrt{\eta^2-1} = a+b\left(\xi-\sqrt{\xi^2-1}\right) \\ \eta+\sqrt{\eta^2-1} = a+b\left(\xi+\sqrt{\xi^2-1}\right) \end{cases}$$

which is equivalent to

$$\begin{cases} \eta = a + b\xi\,, \\ \sqrt{\eta^2-1} = b\sqrt{\xi^2-1}\ . \end{cases}$$

In the following, we verify this result:

$$a+b\xi \ = \ a + b\cdot\frac{1-a^2-b^2}{2ab} \ = \ a + \frac{1-a^2-b^2}{2a} \ = \ \frac{1+a^2-b^2}{2a} \ = \ \eta\ ,$$

and

$$\begin{aligned}
\sqrt{\eta^2-1} &= \sqrt{\left(\frac{1+a^2-b^2}{2a}\right)^2-1} \\
&= \frac{1}{|2a|}\sqrt{(1+a^2-b^2)^2-4a^2} \\
&= \frac{1}{|2a|}\sqrt{\Big[(1+a^2-b^2)-2a\Big]\Big[(1+a^2-b^2)+2a\Big]} \\
&= \frac{1}{|2a|}\sqrt{\Big[(1-a)^2-b^2\Big]\Big[(1+a)^2-b^2\Big]} \\
&= \frac{1}{|2a|}\sqrt{(1-a-b)\,(1-a+b)\,(1+a-b)\,(1+a+b)}\ ,
\end{aligned}$$

$$\begin{aligned}
b\sqrt{\xi^2-1} \ &= \ b\sqrt{\left(\frac{1-a^2-b^2}{2ab}\right)^2-1} \\
&= b\sqrt{\frac{(1-a^2-b^2)^2}{4a^2b^2}-1} \\
&= \frac{b}{2|ab|}\sqrt{(1-a^2-b^2)^2-4a^2b^2} \\
&= \frac{1}{2|a|}\sqrt{(1-a^2-b^2-2ab)\,(1-a^2-b^2+2ab)} \\
&= \frac{1}{2|a|}\sqrt{\Big[1-(a+b)^2\Big]\Big[1-(a-b)^2\Big]} \\
&= \frac{1}{2|a|}\sqrt{\Big[1-(a+b)\Big]\Big[1+(a+b)\Big]\Big[1-(a-b)\Big]\Big[1+(a-b)\Big]} \\
&= \frac{1}{2|a|}\sqrt{(1-a-b)\,(1+a+b)\,(1-a+b)\,(1+a-b)} \\
&= \sqrt{\eta^2-1}\ .
\end{aligned}$$

Hence, we prove $\left\{u_n^{(t)}\right\}$ is a traveling wave solution of (2) under the conditions $e^\theta = \xi + \sqrt{\xi^2 - 1}$ and $b > 0$ as well as $\left(\xi + \sqrt{\xi^2 - 1}\right)^m = \eta + \sqrt{\eta^2 - 1}$. The other cases can be proved in a similar manner. The proof is complete. ∎

Corollary 3. *Assume that $b = -1$ and $a > 2$. Then equation (2) has traveling solutions $\left\{u_n^{(t)}\right\}$ and $\left\{v_n^{(t)}\right\}$ with velocity $m = -1$ defined by*

$$u_n^{(t)} = \frac{1}{2}\left\{\left(\frac{a + \sqrt{a^2 - 4}}{2}\right)^{n+t} + \left(\frac{a - \sqrt{a^2 - 4}}{2}\right)^{n+t}\right\}, \quad n \in Z, \quad t \in N .$$

and

$$u_n^{(t)} = \frac{1}{2}\left\{\left(\frac{a + \sqrt{a^2 - 4}}{2}\right)^{n+t} - \left(\frac{a - \sqrt{a^2 - 4}}{2}\right)^{n+t}\right\}, \quad n \in Z, \quad t \in N . \blacksquare$$

Corollary 4. *Assume that $a = -1$ and $b > 2$. Then equation (2) has traveling solutions $\left\{u_n^{(t)}\right\}$ and $\left\{v_n^{(t)}\right\}$ with velocity $m = 2$ defined by*

$$u_n^{(t)} = \frac{1}{2}\left\{\left(\frac{b + \sqrt{b^2 - 4}}{2}\right)^{n-2t} + \left(\frac{b - \sqrt{b^2 - 4}}{2}\right)^{n-2t}\right\}, \quad n \in Z, \quad t \in N ,$$

and

$$u_n^{(t)} = \frac{1}{2}\left\{\left(\frac{b + \sqrt{b^2 - 4}}{2}\right)^{n-2t} - \left(\frac{b - \sqrt{b^2 - 4}}{2}\right)^{n-2t}\right\}, \quad n \in Z, \quad t \in N . \blacksquare$$

As an example, consider the equation

$$u_n^{(t+1)} = 3\sqrt{5}u_n^{(t)} - 4u_{n-1}^{(t)}, \quad n \in Z, \quad t \in N .$$

Simple calculation shows that $m = -3$ and $e^\theta = \left(\sqrt{5} \pm 1\right)/2$. Hence this equation has traveling wave solutions $\left\{u_n^{(t)}\right\}$ and $\left\{v_n^{(t)}\right\}$ defined by

$$u_n^{(t)} = \frac{1}{2}\left\{\left(\frac{\sqrt{5} - 1}{2}\right)^{n+3t} + \left(\frac{\sqrt{5} + 1}{2}\right)^{n+3t}\right\}, \quad n \in Z, \quad t \in N ,$$

and

$$u_n^{(t)} = \frac{1}{2}\left\{\left(\frac{\sqrt{5} - 1}{2}\right)^{n+3t} - \left(\frac{\sqrt{5} + 1}{2}\right)^{n+3t}\right\}, \quad n \in Z, \quad t \in N .$$

4 – Applications

As applications of our results, we first consider the following partial difference equation

$$u_n^{(t+1)} = au_n^{(t)} + bu_{n-1}^{(t)}, \qquad ab \neq 0 , \tag{35}$$

defined on the 'discrete cylinder': $(n,t) \in \{1,2,...,M\} \times N$. Let us seek its solutions of the form $\left\{u_n^{(t)}\right\}$ defined for

$$(n,t) \in \Psi = \{0,1,...,M\} \times N$$

under the periodic boundary condition

$$u_0^{(t)} = u_M^{(t)}, \qquad t \in N . \tag{36}$$

Note that the equations in (35) and (36) can be written as

$$\begin{cases} u_1^{(t+1)} = au_1^{(t)} + bu_M^{(t)} \\ u_2^{(t+1)} = au_2^{(t)} + bu_1^{(t)} \\ \cdots \\ u_n^{(t+1)} = au_n^{(t)} + bu_{n-1}^{(t)} \\ \cdots \\ u_M^{(t+1)} = au_M^{(t)} + bu_{M-1}^{(t)} \end{cases} \tag{37}$$

for each $t \in N$. If $\left\{u_n^{(t)}\right\}_{(n,t)\in Z\times N} = \{e^{i(n-mt)\theta}\}$ is a traveling solution of (2), then it is easy to see that $\left\{u_n^{(t)}\right\}_{(n,t)\in\Psi} = \{e^{i(n-mt)\theta}\}_{(n,t)\in\Psi}$ satisfies all the equations of (37) except the first one. In order that the first equation is also satisfied, it suffices to require

$$e^{-imt\theta} = e^{i(M-mt)\theta} ,$$

or equivalently,

$$e^{iM\theta} = 1 .$$

Thus, we have the following result in view of Theorem 2.

Theorem 4. *Suppose $ab \neq 0$ and $(a,b) \in \Omega$ where Ω is defined by (10). Suppose $\theta \in [0,\pi]$ and $m \in Z$ satisfy (12). Suppose further that $e^{iM\theta} = 1$. Then $\left\{e^{i\theta(n-mt)}\right\}_{(n,t)\in\Psi}$ is a (complex valued) solution of the dynamical system (37).* ∎

For example, dynamical system

$$\left\{\begin{array}{l} u_1^{(t+1)} = -2u_1^{(t)} + \sqrt{3}u_{12}^{(t)} \\ u_2^{(t+1)} = -2u_2^{(t)} + \sqrt{3}u_1^{(t)} \\ \qquad\quad ...... \\ u_{12}^{(t+1)} = -2u_{12}^{(t)} + \sqrt{3}u_{11}^{(t)} \end{array}\right. , \quad t \in N , \tag{38}$$

has the solution

$$u_n^{(t)} = e^{i\theta(n-mt)} = e^{i\frac{\pi}{6}(n-(4+12k)t)}, \quad n \in \{1, 2, ..., 12\}, \quad t \in N . \tag{39}$$

On the other hand, (35)–(36) or (37) can also be expressed as the dynamical system

$$u^{(t+1)} = au^{(t)} + b\Lambda_M u^{(t)}, \quad t \in N , \tag{40}$$

where $u^{(t)} = \left(u_1^{(t)}, ..., u_M^{(t)}\right)^T$ and Λ_M is the circulant matrix

$$\Lambda_M = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & .. & ... \\ 0 & 0 & ... & 1 & 0 \end{pmatrix}_{M\times M} .$$

In terms of vectors, a solution of (40) takes the form $\left\{u^{(t)}\right\}_{t\in N}$. Let us now seek a solution of (40) which is periodic in time, where a vector sequence $\left\{u^{(t)}\right\}$ is said to be ω-periodic if ω is a positive integer such that $u^{(t+\omega)} = u^{(t)}$ for $t \in N$. Clearly, if $\left\{u_n^{(t)}\right\}_{(n,t)\in\Psi}$ is a solution of (37), then

$$\left\{\left(u_1^{(t)}, ..., u_M^{(t)}\right)^T\right\}_{t\in N}$$

will be a ω-periodic solution of (40) provided

$$u_n^{(t+\omega)} = u_n^{(t)}, \quad n = 1, ..., M; \quad t \in N . \tag{41}$$

Corollary 5. *Suppose* $\left\{e^{i(n-mt)\theta}\right\}_{(n,t)\in\Psi}$ *is a solution of (35) such that* $\frac{2l\pi}{m\theta}$ *is a positive integer for certain* $l \in Z$. *Then*

$$\left\{\left(e^{i(1-mt)\theta}, e^{i(2-mt)\theta}, ..., e^{i(M-mt)\theta}\right)^T\right\}_{t\in N}$$

is a periodic solution of (40) with period $\omega = \frac{2l\pi}{m\theta}$. ∎

Indeed, this follows from

$$u_n^{(t+\omega)} = e^{i(n-m(t+\omega))\theta} = e^{i(n-mt)\theta - im\omega\theta} = e^{i(n-mt)\theta - i2l\pi} = u_n^{(t)}$$

for $n \in \{1, ..., M\}$ and $t \in N$.

As an example, consider the following discrete time dynamical system

$$u^{(t+1)} = -2u^{(t)} + \sqrt{3}\Lambda_{12}u^{(t)}, \quad t \in N . \tag{42}$$

As we know

$$\{e^{i\theta(n-mt)}\} = \{e^{i\frac{\pi}{6}(n-(4+12k)t)}\} \tag{43}$$

is a solution of (42) where $\theta = \frac{\pi}{6}$and $m = 4 + 12k$. Since

$$\frac{2l\pi}{m\theta} = \frac{12l}{(4+12k)},$$

which is equal to 3 when $k = 0$ and $l = 1$, hence

$$\left\{\left(e^{i\frac{\pi}{6}(1-(4+12k)t)}, ..., e^{i\frac{\pi}{6}(12-(4+12k)t)}\right)^T\right\}_{t\in N}$$

is a periodic solution of (42) with period $\omega = 3$.

As a final example, note that if we set

$$W_M = aI + b\Lambda_M = \begin{pmatrix} a & 0 & ... & 0 & b \\ b & a & ... & 0 & 0 \\ 0 & b & ... & 0 & 0 \\ ... & ... & ... & .. & ... \\ 0 & 0 & ... & b & a \end{pmatrix}_{M\times M},$$

from (40), we have

$$u^{(t+1)} = W_M u^{(t)} . \tag{44}$$

Thus we get $u^{(1)} = W_M u^{(0)}$, $u^{(2)} = W_M u^{(1)} = W_M^2 u^{(0)}$, and in general $u^{(t)} = W_M^t u^{(0)}$ for $t \geq 1$. If $\left\{u^{(t)}\right\}_{t\in N}$ is a nontrivial ω-periodic solution of (44), then

$$W_M^{\omega} u^{(0)} = u^{(0)} , \tag{45}$$

that is, 1 is an eigenvalue of the matrix W_M^{ω}. For example, consider the previous example (42), where

$$W_{12} = \begin{pmatrix} -2 & 0 & ... & 0 & \sqrt{3} \\ \sqrt{3} & -2 & ... & 0 & 0 \\ 0 & \sqrt{3} & ... & 0 & 0 \\ ... & ... & ... & .. & ... \\ 0 & 0 & ... & \sqrt{3} & -2 \end{pmatrix}_{12\times 12} .$$

Since we have a nontrivial 3-periodic solution of (44) in this case, 1 is an eigenvalue of W_{12}^3.

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