PORTUGALIAE MATHEMATICA
Vol. 62 Fasc. 2 – 2005
Nova Série

HELMHOLTZ'S VORTICITY TRANSPORT EQUATION WITH PARTIAL DISCRETIZATION IN BOUNDED 3-DIMENSIONAL DOMAINS

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Abstract: We prove the existence of a unique classical solution w to the initial value problem of Helmholtz's vorticity transport equation with partial discretization in a smoothly bounded 3-dimensional domain for each bounded interval of time. The solution w depends continuously on its initial value and, in addition, fulfills a discretized form of Cauchy's vorticity equation.

1 – Introduction

Let (v,p) denote any smooth solution of the initial boundary value problem of Euler's equation

$$\text{(1.1)} \qquad \frac{\partial}{\partial t}v + v\cdot\nabla v + \nabla p = 0\,, \qquad \nabla\cdot v = 0\,, \qquad (t,x)\in(0,a]\times\Omega\ , \\ v(0,x) = v_0(x)\,, \qquad x\in\overline{\Omega}\ ,$$

$$\text{(1.2)} \qquad n(x)\cdot v(t,x) = 0\,, \qquad (t,x)\in[0,a]\times\partial\Omega\ ,$$

at times $t\in J=[0,a]$ in a bounded connected open set $\Omega\subset\mathbb{R}^3$ with $C^{2+\alpha}$-smooth boundary $\partial\Omega=S$. We write $n(x)$ for the exterior normal of S in $x\in S$. The boundary S may be composed of several connected components, $S=S_0\cup\cdots\cup S_h$,

Received: October 16, 2003; *Revised*: February 13, 2004.
AMS Subject Classification: 35A05, 35A35, 34A40, 34A12, 76C05.
Keywords and Phrases: Helmholtz's and Cauchy's vorticity equations with a discretization in Hölder spaces; quasi-Lipschitz conditions.

where S_0 being an exterior boundary and $S_1, ..., S_h$ are inside S_0, outside one another, so that $\Omega = \Omega_0 \backslash (\overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_h)$, $\partial\Omega_k = S_k$ for $k = 0, ..., h$.

In the following, we consider domains $\Omega \subset \mathbb{R}^3$ which have the topological type of a ball B with m solid handles, $m \geq 0$, inside of B a number h of smaller balls $\overline{\Omega}_k$ being cut out. Such a domain is characterized (a) by the presence of a homotopy basis of m simply closed smooth curves $l_i \subset \Omega$ which inside Ω neither can be continuously deformed into each other nor can be continuously contracted into a single point, and (b) by the fact that making m cuts along smooth surfaces $\Sigma_i \subset \overline{\Omega}$ we can transform the domain Ω into a simply connected one. In each such $m+1$-times connected domain Ω with $C^{2+\alpha}$-boundary $\partial\Omega = S$ there exist precisely m linearly independent "Neumann vector fields" $u_i \in C^{1+\alpha}(\overline{\Omega})$ satisfying the conditions

$$\operatorname{rot} u_i(x) = 0, \quad \operatorname{div} u_i(x) = 0 \quad \text{in } \Omega\,, \qquad u_i \cdot n = 0 \quad \text{on } S\,,$$

u_i having the fluxes

$$\int_{\Sigma_k} u_i \cdot n\, dS \;=\; \delta_{ik}$$

across Σ_k, or the circulations

$$\int_{l_k} u_i \cdot dl \;=\; \delta_{ik}\,,$$

$i, k = 1, ..., m$, respectively. Here $n = n(x)$ denotes a unit normal vector in $x \in \Sigma_k$, $l = l(x)$ a tangential vector of l_k with any prescribed orientation, see [9, 11, 25].

Taking rot of (1.1), for the vorticity $w = \operatorname{rot} v$ we get Helmholtz's vorticity equation

$$\frac{\partial}{\partial t} w + v \cdot \nabla w \;=\; w \cdot \nabla v\,, \qquad (t, x) \in (0, a] \times \Omega\,. \tag{1.3}$$

For smooth v fulfilling (1.2), the particles' pathlines

$$x(t) = X(t, s, \widehat{x}) \,\in\, \overline{\Omega} \tag{1.4}$$

calculated from the initial value problem

$$\frac{d}{dt} x = v(t, x), \quad t \in J\,, \qquad x(s) = \widehat{x} \in \overline{\Omega}\,, \tag{1.5}$$

are unique and exist globally in $J \times J \times \overline{\Omega}$, [16]. Thus for fixed $s = 0$, we get the representation

$$\widehat{w}(t, \widehat{x}) = w\Big(t,\, X(t, 0, \widehat{x})\Big) \tag{1.6}$$

of w in dependence on the "Lagrangean" (or "material") coordinates $\widehat{x} \in \overline{\Omega}$. The solution of (1.3) is expressed in Cauchy's vorticity equation

$$\widehat{w}(t,\widehat{x}) \;=\; \widehat{w}_0(\widehat{x}) \cdot (\nabla X)(t,0,\widehat{x}) \;, \tag{1.7}$$

[22] (cp. Remark 7.2 below), where X has to be calculated from (1.5) in dependence on v.

On the other side, for all $t \in J$, by definition of w the vector function $v = v(t,\cdot)$ is solution of the boundary value problem

$$\operatorname{rot} v = w, \quad \nabla \cdot v = 0 \quad \text{in } \ \Omega\,, \qquad n(x) \cdot v(t,x) = 0, \quad x \in S \;, \tag{1.8}$$

$$\int_{l_k} v \cdot dl \;=\; \gamma_k\,, \qquad k = 1,...,m \;, \tag{1.9}$$

with arbitrarily given $\gamma_k \in \mathbb{R}$.

It is well known that problem (1.8), (1.9) admits a unique solution for arbitrary divergence free w satisfying natural compatibility conditions

$$\int_{S_k} w \cdot n\, dS \;=\; 0\,, \qquad k = 0,...,h \;, \tag{1.10}$$

and that the solution satisfies coercive estimates in Hölder and Sobolev norms [2, 4, 9, 11, 23, 25]. In [19] we have proved

Theorem 1.1. *For arbitrary bounded continuous, weakly divergence free w satisfying (1.10) and prescribed $\gamma_k \in \mathbb{R}$, $k = 1,...,m$, the solution of problem (1.8), (1.9) has the form*

$$v(x) = -\mathit{rot}\, A(x) + \nabla\phi(x) + \sum_{k=1}^{m} (\gamma_k + \lambda_k) u_k\,, \qquad \lambda_k = -\int_{l_k} (\operatorname{rot} A) \cdot dl \;, \tag{1.11}$$

$$A(x) \;=\; \int_\Omega E(x-y)\, w(y)\, dy \;-\; \int_S E(x-y)\, n(y)\, \varphi(y)\, dS_y \;:=\; A_1 + A_2 \;,$$

where $E(z) = -\frac{1}{4\pi|z|}$ is a fundamental solution of the Laplace equation, and $\varphi(x), \phi(x)$ are solution to the following exterior and interior Neumann problems:

$$\Delta\varphi(x) = 0, \quad x \in \mathbb{R}^3 \backslash \overline{\Omega}\,, \qquad \frac{\partial\varphi}{\partial n}|_S \;=\; w \cdot n|_S \;, \tag{1.12}$$

$$\Delta\phi(x) = 0, \quad x \in \Omega\,, \qquad \frac{\partial\phi}{\partial n}|_S \;=\; n \cdot \mathit{rot}\, A|_S \;. \tag{1.13}$$

The necessary compatibility conditions for the boundary data in (1.12), (1.13) follow from (1.10) and from the Stokes formula

$$\int_{S_k} n \cdot \operatorname{rot} A \, dS = 0 \,, \qquad k = 0, .., h \ . \tag{1.14}$$

Notations: In the following, besides the norm $|f|_0 = \sup_{x \in \Omega} |f(x)|$ in $C^0(\overline{\Omega})$, the Hölder seminorms $[f]_\alpha = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(y)-f(x)|}{|y-x|^\alpha}$, and norms $|f|_\alpha = |f|_0 + [f]_\alpha$ of the Hölder spaces $C^\alpha(\overline{\Omega}) \subset C^0(\overline{\Omega})$, $0 < \alpha < 1$, we will decisively use the seminorms

$$[f]_\ell = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(y) - f(x)|}{\ell(|y-x|)} \tag{1.15}$$

with the function

$$\ell(r) = \begin{cases} 0, & r = 0 \\ -r ln(r), & r \in (0, e^{-1}) \\ r, & r \geq e^{-1}. \end{cases}$$

By means of the latter we define the subspace $C^\ell(\overline{\Omega}) = \{f \in C^0(\overline{\Omega}) \mid [f]_\ell < \infty\}$. Following [8], the requirement $[f]_\ell < \infty$ is refered to as quasi-Lipschitz condition for f. Below the mark $\overline{\Omega}$ will be omitted, if no confusion is possible. Finally by C^ℓ_ν we will denote the subspace of all vector functions $f \in C^\ell$, $f : \overline{\Omega} \to \mathbb{R}^3$ having normal component zero on S.

We do no longer require $w = \operatorname{rot} v$. In [19] we also have proved

Theorem 1.2. *For abritrary $w \in C^0(\overline{\Omega})$ and arbitrarily fixed $\gamma_k \in \mathbb{R}$, $k = 1, ..., m$, the function v represented by (1.11), with (1.13) and*

$$\begin{aligned} &\Delta \varphi(x) = 0, \quad x \in \mathbb{R}^3 \backslash \overline{\Omega} \,, \qquad \frac{\partial \varphi}{\partial n}|_S = b(x) \ , \\ &b(x) \ = \ n(x) \cdot w(x) - |S_k|^{-1} \cdot \int_{S_k} n \cdot w \, dS \,, \qquad x \in S_k \ , \end{aligned} \tag{1.12a}$$

satisfies

$$|v|_0 + [v]_\ell \ \leq \ c \, |w|_0 \tag{1.16}$$

with some constant c independent of w. Consequently for each $t \in J$, the formula (1.11), (1.12a), (1.13) together define a bounded linear map

$$K : C^0 \to C^\ell_\nu \,, \qquad v(t, \cdot) = K w(t, \cdot) \ , \tag{1.17}$$

[19, Remark 3.5, Theorem 6.2.]

Here the compatibility conditions (1.10) are unnecessary since we do not require $w = \operatorname{rot} v$.

However, for the fixpoint equation for w resulting from the system of the four equations (1.17) (here with (1.12) instead of (1.12a)), (1.4), (1.6), (1.7) together, as well as for Euler's initial boundary value problem (1.1), (1.2), the existence of a global unique solution remains an open problem until now. In 2 space dimensions, quasi-Lipschitz conditions have essentially been used in [7], [8], [28] for proving global existence of solutions to the nonstationary Euler equations. In 3 space dimensions, a lot of work has been done on solutions to (1.1), (1.2) which exist locally in time and on blow-up criteria, cp. [13, 14, 15] and the citations there. Therefore at least with regard to numerical approaches it seems to be remarkable that a discretization of the right hand side in (1.3) leads to initial value problems for approximations to w, which are globally solvable in a unique way.

We will prove

Theorem 1.3. *Let the prescribed initial value w_0 be one times continuously differentiable, the vector valued function $Z(x) = x + \epsilon\, w_0(x)$ for $x \in \overline{\Omega}$ taking its values in $\overline{\Omega}$, where $\epsilon \neq 0$ denotes a constant. Then the initial value problem*

$$(1.18) \qquad \frac{\partial}{\partial t} w + v \cdot \nabla w \;=\; \frac{1}{\epsilon}\Big\{ v\Big(t,\, x + \epsilon w(t,x)\Big) - v(t,x)\Big\}\,, \qquad (t,x) \in J \times \overline{\Omega}\,,$$

$$w(0,x) = w_0(x)\,,$$

with $v = Kw$ from (1.17) (the function v being somehow continuously extended to $J \times \mathbb{R}^3$), has a unique global solution $w \in C^1(J \times \overline{\Omega})$. The function w can be approximated by iteration of a contracting map T and depends continuously on w_0.

The construction of the map T is similar as in [17], where the Cauchy problem of (1.18) in $\mathbb{R}^3$ without boundary condition (1.2) for the velocity field v has been solved, the initial value w_0 having compact support. The fulfilling of (1.2) requires the new potential theoretical tools developed in [19].

Proof of Theorem 1.3:

2 – Notations

In order to prove Theorem 1.3 we have to consider (vector valued) functions on $J\times\overline{\Omega}$ which may have different smoothness properties with respect to t and x. Therefore, denoting by ∂_x^p, $p=(p_1,p_2,p_3)$, an arbitrary x-derivative of order $|p|=p_1+p_2+p_3$, $p_j=0,1,2,...$, we introduce the linear spaces $C^{0,k}$ of all (vector valued) functions f for which all derivatives $\partial_x^p f$ exist and are continuous on $J\times\overline{\Omega}$, $0\le|p|\le k$. $C^{0,k+\alpha}$ is the subspace of $C^{0,k}$ of all functions f which together with all of their derivatives $\partial_x^p f$, $0\le|p|\le k$, are uniformly Hölder continuous in $x\in\overline{\Omega}$ with exponent $\alpha\in(0,1)$. As usual, by $C^k=C^k(A)$ or $C^{k+\alpha}=C^{k+\alpha}(A)$ we denote the class of all functions f which together with all of their partial derivatives of all orders $j\le k=0,1,2,...$ are continuous or uniformly Hölder continuous with exponent α, respectively, on their domain of definition A. We will write $C^{0,0}_\nu$, or $C^{0,\ell}_\nu$, or $C^{0,\ell}_{\nu,M}$ for the subclass of $C^{0,0}=C^0(J\times\overline{\Omega})$ of all vector valued functions $f\in C^{0,0}$ having zero normal component on S or which in addition uniformly in $t\in J$ fulfill the quasi-Lipschitz condition $[f(t,\cdot)]_\ell<\infty$ or $[f(t,\cdot)]_\ell\le M$, respectively. Moreover we set $C^{0,1+\gamma}_\nu=C^{0,0}_\nu\cap C^{0,1+\gamma}$, $\gamma\in[0,1)$. Since each $f\in C^{0,1+\gamma}$ is uniformly Lipschitz continuous in $x\in\overline{\Omega}$, we find $C^{0,1+\gamma}_\nu\subset C^{0,\ell}_\nu$. The norms needed below are

$$|f|_0=\sup_{t\in J}|f(t,\cdot)|_0\,,\qquad [f]_\alpha=\sup_{t\in J}[f(t,\cdot)]_\alpha\,,\qquad [f(t,\cdot)]_1=|\nabla f(t,\cdot)|_0\qquad\text{and}$$

$$|f|_k=\sum_{j=0}^{k}\sum_{|p|=j}|\partial_x^p f|_0\,,\qquad |f|_{k+\alpha}=|f|_k+\sum_{|p|=k}[\partial_x^p f]_\alpha\,.$$

By $C^{0,k+\gamma}_M$ we will denote the ball of radius M in $C^{0,k+\gamma}$, and we will write $C^{0,k+\gamma}_{\nu,M}=C^{0,0}_\nu\cap C^{0,k+\gamma}_M$, $\gamma\in[0,1)$.

From the estimates (1.16) of $v=Kw$ above for each $t\in J$, and from the estimates in Proposition 3.1 below we see that proving continuity of $v=v(t,\cdot)$ or of $\nabla v(t,\cdot)$ with respect to t requires continuity of $|w(t,\cdot)|_0$ or, for some $\alpha\in(0,1)$, of $|w(t,\cdot)|_\alpha$ in t, respectively, which will be ensured by the following

Remark 2.1.

(**a**) Since $J=[0,a]$ and $\overline{\Omega}\subset\mathbb{R}^3$ are compact, for any continuous function $f:J\times\overline{\Omega}\to\mathbb{R}^n$ the function $F(t)=\sup_{x\in\overline{\Omega}}|f(t,x)|$ is (uniformly) continuous.

(**b**) In case $f \in C^{0,\alpha}$, for any $\alpha' \in (0,\alpha)$ the Hölder quotients

$$H_{\alpha'} f(t,x,y) = \begin{cases} \dfrac{|f(t,y)-f(t,x)|}{|y-x|^{\alpha'}} & \text{for} \quad x \neq y\,, \\ 0 & \text{for} \quad x = y \end{cases}$$

are continuous in $(t,x,y) \in J \times \overline{\Omega} \times \overline{\Omega}$. Therefore by (a) (with $\overline{\Omega} \times \overline{\Omega}$ instead of $\overline{\Omega}$) the Hölder seminorm $[f(t,\cdot)]_{\alpha'} = \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} H_{\alpha'} f(t,x,y)$ is (uniformly) continuous for each fixed $\alpha' \in (0,\alpha)$. □

3 – Construction of the velocity v from the approximate vorticity w

Proposition 3.1. *The linear map in (1.17), $K : w \to v = Kw$ defined by equations (1.11), (1.12a), (1.13) in Theorem 1.2 has the following properties:*

(**i**) *K: $C^{0,0} \to C^{0,0}_{\nu}$ is a bounded linear operator.*

(**ii**) *For all $w \in C^{0,0}_M$ we have $Kw \in C^{0,\ell}_{\nu,cM}$ with constant c independent of w.*

(**iii**) *For all $w \in C^{0,\alpha}_M$ we have $Kw \in C^{0,1+\alpha}_{\nu,cM}$.*

Proof: As shown in [19] , the linear map K from Theorem 1.2 is well defined for each $w(t,\cdot) \in C^0(\overline{\Omega})$, K being bounded with respect to the norm $|\cdot|_0$, and $(Kw)(t,\cdot)$ fulfilling the quasi-Lipschitz condition $[(Kw)(t,\cdot)]_\ell \le c|w(t,\cdot)|_0$ as well as having zero normal component in $x \in S$. From this, recalling the continuity of $|w(t,\cdot)|_0$, we see (i) and (ii).

From the well known regularity theorems and Hölder norm estimates of potential theory [5, 21], assuming $w \in C^{0,\alpha}$, thus $w(t,\cdot) \in C^\alpha(\overline{\Omega})$, we conclude $\varphi(t,\cdot) \in C^{1+\alpha}(\mathbb{R}^3 \backslash \Omega)$ for the solution $\varphi(t,\cdot)$ of the exterior Neumann boundary value problem (1.12a), consequently we find $A_2(t,\cdot) \in C^{2+\alpha}(\overline{\Omega})$ for the single layer $A_2(t,\cdot)$ having the density φ on S, and we get $A_1(t,\cdot) \in C^{2+\alpha}(\overline{\Omega})$ for the Newton potential A_1 in (1.11) with density $w(t,\cdot)$. Therefore the solution $\phi(t,\cdot)$ of the interior Neumann boundary value problem (1.13) with boundary value $n \cdot \mathrm{rot}(A_1 + A_2)$ on S fulfills $\phi(t,\cdot) \in C^{2+\alpha}(\overline{\Omega})$, too. The related Hölder estimates include $|Kw(t,\cdot)|_{1+\alpha} \le c \cdot |w(t,\cdot)|_\alpha$ which shows the boundedness of the linear operator $K : C^\alpha(\overline{\Omega}) \to C^{1+\alpha}(\overline{\Omega})$. Recalling Remark 2.1 (b), from the continuity in $t \in J$ of the Hölder norm $|w|(t,\cdot)|_{\alpha'}$ for $\alpha' \in (0,\alpha)$ we find the continuity of $(\nabla Kw)(t,x)$, thus $Kw \in C^{0\cdot,1+\alpha}_{\nu,cM}$. ■

4 – Particles' paths following v

Proposition 4.1. *Assume $v \in C^{0,\ell}_\nu$. Then*

(i) *the initial value problem*

$$\frac{d}{dt}x = v(t,x), \quad x(s) = x_s \in \overline{\Omega}, \quad t, s \in J \tag{4.1}$$

has a unique global solution $x(t) = X(t,s,x_s)$ being continuous on $J{\times}J{\times}\overline{\Omega}$. We will write $L : C^{0,\ell}_\nu \to C^0(J{\times}J{\times}\overline{\Omega})$ for the map defined by $Lv = X$.

(ii) *$X(t,s,\cdot)\colon \overline{\Omega}\to\overline{\Omega}$ being topological, $X^{-1}(t,s,\cdot) = X(s,t,\cdot)$, $X(s,s,\cdot) = id$. $v \in C^{0,1}_\nu$ includes $X \in C^1 = C^1(J{\times}J{\times}\overline{\Omega})$.*

(iii) *Ω and $\partial\Omega$ are flow invariant for (4.1).*

(iv) *In case $v \in C^{0,\ell}_{\nu,M}$ the estimate*

$$[X(t,s,\cdot)]_\alpha \le \max\{eD, 1\} \quad \text{with } \alpha = e^{-Ma} \tag{4.2}$$

holds, D denoting the diameter of Ω. In case v having normal component zero on $\partial\Omega$, $v \in C^{0,1}$ or $v \in C^{0,1+\alpha}$, we have

$$|\nabla X(t,s,\cdot)|_0 \;\le\; c \cdot e^{c|\nabla v|_0 |t-s|} \tag{4.3}$$

or additionally

$$[\nabla X(t,s\cdot)]_\alpha \;\le\; c[\nabla v]_\alpha \, |t-s| \, e^{c|\nabla v|_0|t-s|} \;, \tag{4.4}$$

respectively.

(v) *In case $v_{[m]} \in C^{0,1}_\nu$, $|\nabla v_{[m]}|_0 \le M$, $m = 1,2$, the solutions $X_{[m]}$ of (4.1) with $v = v_{[m]}$, respectively, fulfil*

$$\begin{aligned} \Big|X_{[2]}(t,s,\cdot) - X_{[1]}(t,s,\cdot)\Big|_0 \;&\le \\ &\le\; c\, e^{cM|t-s|} \left| \int_s^t e^{-cM|t'-s|} \Big|v_{[2]}(t',\cdot) - v_{[1]}(t',\cdot)\Big|_0 \, dt' \right| \;, \end{aligned} \tag{4.5}$$

each constant c above being independent of $v, v_{[m]}$.

Proof: Statements (i), (iii) have been proved in [16], for general results concerning flow invariance cp. [1], [20].

(ii) is well known in differential equation theory, cp. [3]. In a slightly different formulation, the estimates (iv) and (v) have been proved in [8], [18]. For completeness we give the proofs in the Appendix Section 7 below. ∎

5 – Construction of the approximate vorticity from the particles' paths: A discretized form of Cauchy's vorticity equation

Proposition 5.1.

(**i**) *For any given* $v \in C^{0,1}_\nu$, *each solution* $w \in C^1$ *of (1.18) in Theorem 1.3 with initial value* $w_0 \in C^1(\overline{\Omega})$ *fulfilling*

$$Z(x) = x + \epsilon w_0(x) \in \overline{\Omega} \quad \text{for all } \; x \in \overline{\Omega}\ , \tag{5.1}$$

has the representation

$$w(t,x) \;=\; \frac{1}{\epsilon}\Big\{X(t,0,Z(\cdot)) - X(t,0,\cdot)\Big\} \circ X(0,t,x) \;=\; HX \tag{5.2}$$

for all $(t,x) \in J{\times}\overline{\Omega}$, $\quad X = Lv$ *denoting the solution of (4.1).*

(**ii**) *Conversely in case* $w_0 \in C^1(\overline{\Omega})$ *fulfilling (5.1),* $v = Kw \in C^{0,1}_\nu$, *the function* $w = HX$ *in (5.2) belongs to* C^1 *and solves (1.18).*

Proof: Recalling Proposition 4.1, the representation of w in Lagrange coordinates $\widehat{x}$, $\widehat{w}(t,\widehat{x}) = w(t, X(t,0,\widehat{x}))$, is well defined. Because of

$$\Big(\frac{\partial}{\partial t}\widehat{w}(t,\cdot)\Big) \circ X^{-1}(t,0,x) \;=\; \frac{\partial}{\partial t} w + v \cdot \nabla w\ , \tag{5.3}$$

initial value problem (1.18) is equivalent to

$$\frac{\partial}{\partial t}\widehat{w} \;=\; \frac{1}{\epsilon}\Big\{v(t, X{+}\epsilon w(t,X)) - v(t,X)\Big\} \quad \text{or} \tag{5.4}$$

$$\frac{\partial}{\partial t}\{X + \epsilon\widehat{w}\} = v(t, X{+}\,\epsilon\widehat{w})\,, \quad t \in [0,a], \quad \text{with} \tag{5.5}$$

$$X + \epsilon\widehat{w} = \widehat{x} + \epsilon w_0(\widehat{x})\,, \quad t = 0\ ,$$

where $v(t,X) = \frac{\partial}{\partial t}X$, $X = X(t,0,\widehat{x})$.

Equation (5.5) represents the initial value problem (4.1) with direction field v for the function $X + \epsilon\widehat{w}$ having the initial value $\widehat{x} + \epsilon w_0(\widehat{x}) \in \overline{\Omega}$. Thus recalling again Proposition 4.1 we find

$$(X + \epsilon\widehat{w})(t,\widehat{x}) \;=\; X(t,0,\widehat{x} + \epsilon w_0(\widehat{x})) \quad \text{or}$$

$$\widehat{w}(t,\widehat{x}) \;=\; \frac{1}{\epsilon}\Big\{X(t,0,Z(\widehat{x})) - X(t,0,\widehat{x})\Big\}\ , \tag{5.6}$$

which shows (5.2) because of $\widehat{x} = X(0,t,x)$.

Finally, recalling Propositions 3.1 (iii), 4.1 (ii), our requirement in (ii) gives $w = HX \in C^1(J\times\overline{\Omega})$ in (5.2), the latter equation being equivalent to (5.6). Differentiating with respect to t and using the representation above of $\frac{\partial}{\partial t}\widehat{w}(t,\widehat{x})$ with $\widehat{x} = X^{-1}(t,0,x)$, we get (1.18). ∎

Corollary 5.2. *Under the assumptions of Proposition 5.1 we have* $x + \epsilon w(t,x) \in \overline{\Omega}$ *for all* $x \in \overline{\Omega}$.

Proof: From (5.2) recalling $x = X(t,0,\widehat{x})$ and (1.6) we find $X + \epsilon\widehat{w} = X + X(t,0,Z(\widehat{x})) - X = X(t,0,Z(\widehat{x})) \in \overline{\Omega}$ because of (5.1) and Proposition 4.1 (ii). ∎

Remark 5.3. Due to the smoothness we have required for $\partial\Omega$, each vector valued function $v \in C^{0,1+\alpha}_{\nu,M}(J\times\overline{\Omega})$ can be extended to $\tilde{v} \in C^{0,1+\alpha}_{c\cdot M}(J\times\mathbb{R}^3)$. The uniform Lipschitz continuity of $\tilde{v}$ ensures the existence of the unique global solution $\tilde{X}$ of (4.1) with $\tilde{v}$ instead of v and $x_s \in \mathbb{R}^3$. Rewriting (5.4) and recalling $x = \tilde{X}(t,0,\widehat{x}) \in \overline{\Omega}$ if $\widehat{x} \in \overline{\Omega}$ we see

$$\frac{\partial}{\partial t}\widehat{w} \;=\; \frac{1}{\epsilon}\int_0^\epsilon \frac{\partial}{\partial\sigma}\tilde{v}(t,\tilde{X}+\sigma w(t,\tilde{X}))d\,\sigma \;=\; w(t,\tilde{X})\cdot(\nabla v)(t,\tilde{X}) + E\;, \tag{5.7}$$

where

$$\begin{aligned} |E| \;&=\; \left|\frac{1}{\epsilon}\int_0^\epsilon w(t,\tilde{X})\cdot\left\{(\nabla\tilde{v})\,(t,\tilde{X}+\sigma w(t,\tilde{X})) - (\nabla v)(t,\tilde{X})\right\}d\sigma\right| \\ &\le\; [\nabla\tilde{v}]_\alpha\cdot|w|_0^{1+\alpha}\cdot|\epsilon|^\alpha\;. \end{aligned} \tag{5.8}$$

Because of (5.3) and $\tilde{X}|_{J\times J\times\overline{\Omega}} = X$, equation (5.7) and the latter inequality show that in the special case where the semi-norms in question of $\tilde{v}$ and w are bounded independently of $|\epsilon|$, a solution w of (1.18) solves Helmholtz's vorticity equation (1.3) modulo $c|\epsilon|^\alpha$. Moreover, in (5.2) with $\tilde{X}$ instead of X taking the limit $\epsilon\to 0$, we again get Cauchy's vorticity equation (1.7). □

We will further on write D for the diameter of $\overline{\Omega}$.

Proposition 5.4. *Assume*

(**a**) $X \in L(C^{0,\ell}_{\nu,M})$, *thus* $X(t,s,\cdot) \in C^\alpha(\overline{\Omega})$ *by (4.2),* $t,s \in J$, *and*

(**b**) $w_0 \in C^1(\overline{\Omega})$ *with (5.1).*

Then the function HX in (5.2) is continuous on $J\times\overline{\Omega}$ and obeys the estimates

(**i**) $|HX|_0 \leq \frac{D}{|\epsilon|}$,

(**ii**) $[(HX)(t,\cdot)]_{\alpha^2} \leq \frac{1}{|\epsilon|}\Big\{c\cdot(1+|\epsilon|[w_0]_1)^\alpha\cdot[X(t,0,\cdot)]_\alpha\cdot[X(0,t,\cdot)]_\alpha^\alpha + D^{1-\alpha^2}\Big\}$.
Particularly $X\in L(C_\nu^{0,1})$, thus $\alpha=1$ includes $(HX)(t,\cdot)\in C^1(\overline{\Omega})$.

In case

(**c**) $X_m\in L(C_\nu^{0,1})$, *thus* $X_m(t,s,\cdot)\in C^1(\overline{\Omega})$ *by (4.3),* $t,s\in J$, *and*

(**d**) $w_{m0}\in C^1(\overline{\Omega})$ *with* $Z_m(x)=x+\epsilon w_{m0}(x)\in\overline{\Omega}$ *if* $x\in\overline{\Omega}$, $m=1,2$,

the Lipschitz estimate

(**iii**) $|\epsilon|\,|(HX_2)(t,\cdot)-(HX_1)(t,\cdot)|_0 \;\leq$
$$\leq \left\{\begin{array}{l} |\epsilon|\,[X_1(t,0,\cdot)]_1\,|w_{20}-w_{10}|_0 + 3\,|X_2(t,0,\cdot)-X_1(t,0,\cdot)|_0\;+ \\ \qquad\qquad +\; c\cdot[X_1(t,0,\cdot)]_1\cdot[Z_1]_1\cdot|X_2(0,t,\cdot)-X_1(0,t,\cdot)|_0 \end{array}\right.$$
holds true.

Remark 5.5. Here and in the following we write $[f]_1=|\nabla f|_0$ for any function $f\in C^1(\overline{\Omega})$. ▫

Proof: From (a) and (b) recalling Proposition 4.1 (i) we see that HX in (5.2) must be continuous, since it is composition of continuous functions. Inequality (i) holds because of $X(t,s,x)\in\overline{\Omega}$ for $x\in\overline{\Omega}, t,s\in J$. In order to prove (ii), writing $X\circ Z = X(t,0,Z)$, we observe that $HX=\frac{1}{\epsilon}\{X\circ Z\circ X^{-1}-\mathrm{id}\}$ holds because of Proposition 4.1 (ii). Thus taking

$$\widehat{x}_m\in\overline{\Omega}\,,\qquad X_m=X(t,0,\widehat{x}_m)\,,\qquad \widehat{x}_m=X_m^{-1}=X(0,t,x_m)\,,\qquad x_1\neq x_2\ ,$$

first of all in case $Z\circ X_2^{-1}\neq Z\circ X_1^{-1}$ we see

$$|\epsilon|\cdot|HX_2-HX_1| \;=\; \left|X\circ Z\circ X_2^{-1}-x_2-(X\circ Z\circ X_1^{-1}-x_1)\right| \;\leq$$
$$\leq\; \frac{|X\circ Z\circ X_2^{-1}-X\circ Z\circ X_1^{-1}|}{|Z\circ X_2^{-1}-Z\circ X_1^{-1}|^\alpha}\cdot\frac{|Z\circ X_2^{-1}-Z\circ X_1^{-1}|^\alpha}{|X_2^{-1}-X_1^{-1}|^\alpha}\cdot|X_2^{-1}-X_1^{-1}|^\alpha+|x_2-x_1|$$
$$\leq\; \Big\{[X(t,0,\cdot)]_\alpha\cdot([Z]_1\cdot[X(0,t,\cdot)]_\alpha)^\alpha+|x_2-x_1|^{1-\alpha^2}\Big\}\,|x_2-x_1|^{\alpha^2}\ .$$

Since the last bound obviously holds in case

$Z\circ X_2^{-1} = Z\circ X_1^{-1}$, too, noting $[Z]_1 = |\nabla(x+\epsilon w_0(x))|_0 \le c\cdot\{1+|\epsilon|\,[w_0]_1\}$,

we find (ii). Namely, the last statement in (ii) is obvious by Proposition 4.1 (ii).

For proving (iii), writing $X_m = X_m(t,0,\cdot)$, $X_m^{-1} = X_m(0,t,\cdot)$, $X_m\circ Z_n = X_m(t,0,Z_n)$, $n=1,2$, we find

$$|\epsilon|\cdot|HX_2 - HX_1| \;=\; \Big|(X_2\circ Z_2 - X_2)\circ X_2^{-1} - (X_1\circ Z_1 - X_1)\circ X_1^{-1}\Big| \;\le\; \delta_1+\delta_2\ ,$$

where

$$\begin{aligned}\delta_1 \;&=\; \Big|\Big\{(X_2\circ Z_2 - X_2) - (X_1\circ Z_1 - X_1)\Big\}\circ X_2^{-1}\Big|\\ &\le\; |X_2\circ Z_2 - X_1\circ Z_2|_0 + |X_1\circ Z_2 - X_1\circ Z_1|_0 + |X_2-X_1|_0\ ,\\ \delta_2 \;&=\; \Big|(X_1\circ Z_1 - X_1)\circ X_2^{-1} - (X_1\circ Z_1 - X_1)\circ X_1^{-1}\Big|\\ &\le\; c\cdot[X_1]_1\,[Z_1]_1\cdot|X_2^{-1}-X_1^{-1}|_0 + |X_2-X_1|_0\end{aligned}$$

since $|\mathrm{id} - X_1\circ X_2^{-1}|_0 = |X_2-X_1|_0$ because of Proposition 4.1 (ii). Noting $|X_2\circ Z_2 - X_1\circ Z_2|_0 \le |X_2-X_1|_0$ and $|X_1\circ Z_2 - X_1\circ Z_1|_0 \le [X_1]_1\,|\epsilon|\,|w_{20}-w_{10}|_0$, we see that the right hand side in (iii) is upper bound of $\delta_1+\delta_2$. The constant $c>0$ depends only on the special norm we use for 3×3-matrices, c being independent of X_m, Z_m. ∎

6 – Fixpoint equation $w = Tw$ with the contracting map $T = HLK$

For solving the fixpoint equation non-locally in time, separate estimates of the norm $|w|_0$ (which, as we will see, determines the Hölder seminorm $[Tw]_\alpha$) and of $[w]_\alpha$ are decisive. Therefore we introduce the bounded subsets

$$C^{0,\alpha}_{M_0,M_1} \;=\; \Big\{f\in C^{0,\alpha}(J\times\overline{\Omega})\ \big|\ |f|_0\le M_0,\ [f]_\alpha\le M_1\Big\}\ .$$

Combining Propositions 3.1–5.4 we find

Proposition 6.1. *Assume $w_0\in C^1(\overline{\Omega})$ and that (5.1) holds. Then the composed map $T = HLK$ fulfills*

(i) $T\colon C^{0,0}_M \to C^{0,\beta}_{M_0,M_1}$, *where $M>0$ arbitrary, $\alpha = e^{-cMa}$, $\beta=\alpha^2$, $M_0 = \frac{D}{|\epsilon|}$, $M_1 = \frac{1}{|\epsilon|}\Big\{c(1+|\epsilon|\,[w_0]_1)^\alpha\cdot(\max\{eD,1\})^{1+\alpha} + D^{1-\beta}\Big\}$,* (**ii**) $T\colon C^{0,\gamma}_{M_0,M_1} \to C^1 \cap C^{0,\beta}_{M_0,M_1}$, *where* $\gamma \in (0,1)$ *arbitrary,* $\beta = e^{-2cM_0 a}$,

and in case of two vector valued functions $w_m \in C^{0,\beta}_{M_0,M_1}$ *having the initial values* $w_m(0,\cdot) = w_{m0} \in C^1(\overline{\Omega})$ *which both fulfill (5.1),* $m = 1,2$, *the inequality*

(**iii**) $|Tw_2 - Tw_1|_* \le c \cdot |w_{20} - w_{10}|_0 + \frac{c_1}{b}|w_2 - w_1|_*$

holds in the norm

$$|f|_* = \sup_{t\in J} e^{-(b+cM)t}\,|f(t,\cdot)|_0 \tag{6.1}$$

which is equivalent to the norm $|f|_0$ *for* $f \in C^0(J\times\overline{\Omega})$, *with* $M = c(M_0 + M_1)$ *and arbitrary* $b \in (0,\infty)$.

Proof: For any $w \in C^{0,0}_M$, Proposition 3.1 (ii) shows $v = Kw \in C^{0,\ell}_{\nu,cM}$, thus $X = (Lv) \in C^0(J\times J\times\overline{\Omega})$ and $[X(t,s,\cdot)]_\alpha \le \max\{eD, 1\}$, $\alpha = e^{-cMa}$ by Proposition 4.1 (i), (iv). Recalling Proposition 5.4 (i), (ii) we find statement (i) above.

To prove (ii) in case $w \in C^{0,\gamma}_{M_0,M_1}$, Proposition 3.1 (iii) gives $v = Kw \in C^{0,1+\gamma}_{\nu,M}$, $M = c(M_0 + M_1)$, therefore we have $X \in C^1(J\times J\times\overline{\Omega})$ by Proposition 4.1 (ii) which includes $Tw = HLKw \in C^1$ due to Proposition 5.1 (ii).

Finally in case $w_m \in C^{0,\beta}_{M_0,M_1}$ writing the bounds for $[X_m(t,s\cdot)]_1$ and $|X_2(t,s,\cdot) - X_1(t,s,\cdot)|_0$, which we find from Proposition 3.1 and Proposition 4.1 (iv), (v), in the Lipschitz estimate (iii) of Proposition 5.4, with $Tw_m = HX_m$, $M = c\cdot(M_0 + M_1)$ we get

$$\begin{aligned} |\epsilon|\cdot\Big|(Tw_2)(t,\cdot) - (Tw_1)(t,\cdot)\Big|_0 \ \le \\ &\le\ c\,|\epsilon|\,e^{cMt}\,|w_{20} - w_{10}|_0 \ +\ c\Big(3 + c\,e^{cMt}(1 + |\epsilon|\,[w_{10}]_1)\Big)\,e^{cMt} \\ &\quad\cdot\ \int_0^t e^{-cMt'}\cdot\Big|w_2(t',\cdot) - w_1(t',\cdot)\Big|_0\,dt'\ . \end{aligned}$$

Under the integral sign introducing firstly the factor $e^{bt'}\cdot e^{-bt'}$, then the norm (6.1), by integration we get

$$\Big|(Tw_2)(t,\cdot) - (Tw_1)(t,\cdot)\Big|_0\ e^{-(b+cM)t}\ \le\ c\cdot|w_{20} - w_{10}|_0\ +\ \frac{c_1}{b}|w_2 - w_1|_*\ ,$$

with $c_1 = \frac{c}{|\epsilon|}(3 + c\,e^{cMa}(1 + |\epsilon|\,[w_{10}]_1))$, for all $t \in [0,a]$. Taking sup on the left hand side we find (iii). ∎

Proposition 6.2. *Assume $w_0 \in C^1(\overline{\Omega})$ fulfills (5.1). We set*

$$(6.2)\qquad \begin{aligned} M_0 &= \frac{D}{|\epsilon|}\,, \qquad \alpha = e^{-cM_0 a}\,, \qquad \beta = \alpha^2\,, \\ M_1 &= \frac{1}{|\epsilon|}\Big\{c\cdot(1+|\epsilon|\,[w_0]_1)^\alpha\,(\max\{eD,1\})^{1+\alpha} + D^{1-\beta}\Big\}\,. \end{aligned}$$

Then

(**i**) *the class C_{w_0} of all functions $f \in C^{0,\beta}_{M_0,M_1}$ having the initial value $f(0,\cdot) = w_0$ constitutes a closed subset in $C^{0,\beta}_{M_0,M_1}$ with respect to the norm $|\cdot|_*$.*

(**ii**) *There holds $TC_{w_0} \subset C_{w_0}$, T being in case $b > c_1$ a contraction of C_{w_0} with respect to $|\cdot|_*$.*

(**iii**) *The fixpoint equation $w = Tw$ has a unique solution $w \in C_{w_0}$. The fixpoint w belongs even to C^1, w being there the unique solution of the initial value problem (1.18) with $v = Kw$.*

(**iv**) *In the norm $|\cdot|_0$, the solution $w = Tw \in C^{0,0}_{M_0,M_1}$ depends continuously on its initial value $w_0 \in C^1(\overline{\Omega})$, w_0 fulfilling (5.1).*

Proof: The norms $|\cdot|_*$ and $|\cdot|_0$ being equivalent, the closeness of C_{w_0} with respect to $|\cdot|_*$ in the closed bounded set $C^{0,\beta}_{M_0,M_1}$ of the Hölder space $C^{0,\beta}$ results from the closeness of $C^{0,0}_{M_0}$ with respect to $|\cdot|_0$, since in case of uniform convergence $|f_k - f|_0 \to 0$ in $C^0(J\times\overline{\Omega})$ with $k\to\infty$ any uniform Hölder estimate $[f_k(t,\cdot)]_\alpha \le M_1$ remains valid for the limit f, too.

The first statement in (ii) follows from Proposition 6.1 (i) because of $C_{w_0} \subset C^{0,0}_{M_0}$ and $(Tw)(0,\cdot) = w_0$ for all $w \in C_{w_0}$. If we take $b > c_1$, the contractivity of T on C_{w_0} is seen from Proposition 6.1 (iii), since there the first term on the right hand side vanishes.

Because of (i) and (ii), the contracting mapping principle [27] ensures the existence of a unique fixpoint $w = Tw \in C_{w_0}$, which can be approximated (with respect to the norm $|\cdot|_*$) by iteration of T.

For any fixpoint $w = Tw \in C_{w_0}$, from Proposition 3.1 (iii) and 5.1 (ii) we see $X = LKw \in C^1(J\times J\times\overline{\Omega})$, thus $w \in C^1$ fulfills (1.18). Conversely for each solution $w \in C^1 \subset C^{0,\alpha}$ of the initial value problem (1.18), we find $v = Kw \in C^{0,1}_\nu$ from Proposition 3.1 (iii). Therefore, as stated in Proposition 5.1 (i), w has the representation $w = HX$ in (5.2). Thus $w = Tw$ holds because of $v = Kw$, $X = Lv$, and from Proposition 6.1 we find $|w|_0 \le \frac{D}{|\epsilon|} = M_0$, consequently $w \in C^{0,\beta}_{M_0,M_1} \cap C_{w(0,\cdot)} = C_{w_0}$.

Finally, if $w_m = Tw_m \in C^{0,\beta}_{M_0,M_1}$ with initial values $w_{m,0} \in C^1$ fulfilling (5.1) are fixpoints in $C_{w_{m,0}}$, $m = 1, 2$, respectively, and if we take any $b > c_1$, setting $q = \frac{c_1}{b} < 1$, Proposition 6.1 (iii) becomes

$$|w_2 - w_1|_* \le \frac{c}{1-q} \cdot |w_{20} - w_{10}|_0 ,$$

which shows the continuous dependence of w on its initial value even in the norm $|\cdot|_0$ of $C^0(J\times\Omega)$. ∎

7 – Appendix

We will prove the estimates (iv), (v) in Proposition 4.1 by the methods of differential inequalities: Assume $v_m \in C^{0,\ell}_{\nu,M}$, $m=1,2$ and let $X_m = X_m(t, s_m, x_m) = X_m(t)$ denote the unique solution of $\frac{d}{dt}x = v_m(t,x)$, $x(s) = x_m \in \overline{\Omega}$, $t, s \in J$. The Dini derivative $(D^-_t f)(t) = \limsup_{\tau\to t-0} \frac{f(t)-f(\tau)}{t-\tau} \in [-\infty, \infty]$ is defined for all continuous functions f. In case of a continuously differentiable function φ, the inequalitiy

$$D^-_t |\varphi(t)| \le \left|\frac{d}{dt}\varphi(t)\right| \tag{7.1}$$

holds, [26]. Setting $\varphi(t) = |X_2 - X_1|$ we get

$$D^-_t \varphi \le |v_2(t, X_2) - v_1(t, X_1)| \le \delta_1 + M\ell(\varphi) , \tag{7.2}$$

$$\varphi(s_1) = |X_2(s_2) - X_1(s_1)| \le |v_2|_0 \cdot |s_2 - s_1| + |x_2 - x_1| = \delta_0 , \tag{7.3}$$

with a constant $\delta_1 \ge |v_2(t,\cdot) - v_1(t,\cdot)|_0$.

Remark 7.1. The function $\ell(r)$ obeys the inequality

$$0 \le \ell(r_2) - \ell(r_1) \le \ell(r_2 - r_1) \quad \text{for} \;\; 0 \le r_1 \le r_2 . \tag{7.4}$$

Namely, the statement is clear for both arguments $r_j \ge 1/e$ as well as for $r_1 = 0$. In case $0 < r_1 \le r_2 < 1/e$ observing the strict decrease of the derivative $\ell'(r)$ in $r > 0$ and $\ell(0) = 0$ we find $\int_{r_1}^{r_2} \ell'(\rho)\, d\rho \le \int_0^{r_2-r_1} \ell'(\rho)\, d\rho$ which gives (7.4). □

The function ℓ being Osgood function and fulfilling Wintner's condition [6, 26], too, because of (7.4) there is a unique global solution $\psi(t) = \Psi(t - s_1, \delta_0)$ of the autonomous differential equation

$$\frac{d}{dt}\psi = \delta_1 + M\ell(\psi) , \qquad t \in [s_1, a], \quad \psi(s_1) = \delta_0 , \tag{7.5}$$

and the well known comparison theorem gives

$$\varphi(t) \le \psi(t) = \Psi(t - s_1, \delta_0) \quad \text{for} \quad s_1 \le t \le a \,, \tag{7.6}$$

[26].

Similarly for $0 \le t \le s_1$ introducing the variable $\tau = -t$ firstly we get

$$D^-_\tau \varphi \le \delta_1 + M\ell\varphi \,, \qquad \varphi(-s_1) = \delta_0 \;. \tag{7.7}$$

Then the comparison of $\varphi(\tau)$ with the unique solution $\psi(\tau) = \Psi(\tau + s_1, \delta_0)$ of

$$\frac{d}{d\tau}\psi = \delta_1 + M\ell(\psi) \,, \qquad \tau \in [-s_1, 0], \quad \psi(-s_1) = \delta_0 \tag{7.8}$$

leads to

$$\varphi(\tau) \le \psi(\tau) = \Psi(s_1 - t, \delta_0) \quad \text{for} \quad 0 \le t = -\tau \le s_1 \;,$$

which together with (7.6) shows

$$\varphi(t) \le \Psi(|t - s_1|, \delta_0) \quad \text{for} \quad t \in [0, a] \;. \tag{7.9}$$

In case $\delta_1 = 0$ and under the restriction

$$\psi(t) \le e^{-1} \tag{7.10}$$

the solution of (7.5), (7.8) is

$$\Psi(|t - s_1|, \delta_0) = \delta_0^{\alpha(t)} \quad \text{with} \quad \alpha(t) = e^{-M|t-s_1|} \;. \tag{7.11}$$

Therefore taking $s_2 = s_1$ and $\alpha = e^{-Ma} \le \alpha(t)$, the restriction (7.10) surely holds if $|x_2 - x_1| \le e^{-\alpha^{-1}}$, therefore from (7.11) we find $\frac{|X_2 - X_1|}{|x_2 - x_1|^\alpha} \le 1$. Otherwise, if $|x_2 - x_1| \ge e^{-\alpha^{-1}}$, we get $\frac{|X_2 - X_1|}{|x_2 - x_1|^\alpha} \le eD$ because of $|X_2 - X_1| \le D$. This shows the first inequality (iv).

Quite similarly we estimate $[X(t,s,\cdot)]_1$ or $[\nabla X(t,s,\cdot)]_\alpha$ in case $v \in C^{0,1}_\nu$ or $v \in C^{0,1+\alpha}_\nu$, respectively: Each derivative $\left(\frac{\partial}{\partial x_j} X\right)(t,s,x) = X_{(j)}$ is solution of the differential equation

$$\frac{\partial}{\partial t} X_{(j)} = (\nabla v) \cdot X_{(j)} \,, \qquad t \in J, \quad X_{(j)} = (\delta_{ij}), \quad t = s, \quad j = 1,2,3 \;, \tag{7.12}$$

where (∇v) denotes the Jacobian matrix of $v(t,X)$. As above by means of the comparison theorem, for the continuous functions $\varphi(t) = |X_{(j)}|$, the inequality $D^-_t \varphi \le |\frac{\partial}{\partial t} X_{(i)}| \le c \cdot |\nabla v|_0 \cdot \varphi$ with $\varphi(s) = 1$ leads to $|X_{(i)}(t,s,\cdot)|_0 \le \varphi(t) \le$

$e^{c|\nabla v|_0|t-s|}$ which includes $|\nabla X(t,s,\cdot)|_0 \le c\, e^{c|\nabla v|_0|t-s|}$. The constant c depends only on the special norm we use for 3×3-matrices, c being independent of X and v.

In case $v \in C_\nu^{0,1+\alpha}$ we consider $X_{(j)m} = (\frac{\partial}{\partial x_j}X)(t,s,\widehat{x}_m)$, $X_m = X(t,s,\widehat{x}_m)$, $\widehat{x}_m \in \overline{\Omega}$, $t,s \in J$, $m=1,2$. For the continuous function $\varphi(t) = |X_{(j)2} - X_{(j)1}|$, from (7.12) we find the inequality

$$D_t^- \varphi \le c \cdot [\nabla v]_\alpha \, |\nabla X|_0^{1+\alpha} \cdot |\widehat{x}_2 - \widehat{x}_1|^\alpha + c\,|\nabla v|_0 \cdot \varphi \,, \tag{7.13}$$

which leads to the third inequality in (iv) again by means of the comparison theorem.

Finally the proof of estimate (4.5) in Proposition 4.1 (v) results from the observation that the right hand side in (4.5) represents the solution of (7.5) for $t \in [s_1, a]$ or of (7.8) for $-\tau = t \in [0, s_1]$, respectively, if in the right hand sides of both equations we replace $\ell(\psi)$ by ψ, the constant δ_1 by the continuous function $\delta_1(t) = |v_2(t,\cdot) - v_1(t,\cdot)|_0$ and require $\delta_0 = 0$.

Remark 7.2. Assume the functions $v \in C_\nu^{0,1}$ and $X \in C^1(J\times J\times\overline{\Omega})$ from (4.1) are given. Then from (7.12) we see that $(X_{(j)})$, $j=1,2,3$, represents a fundamental system of the initial value problem (7.12) which is linear in $X_{(j)}$. Using the identity (5.3) and the transformation $X = X(t,0,\widehat{x})$ we get from Helmholtz's vorticity equation (1.3)

$$\frac{\partial}{\partial t}\widehat{w} = (\widehat{w}\cdot\nabla)v \equiv (\nabla v)\cdot\widehat{w}\,, \qquad t \in J, \quad \widehat{w} = \widehat{w}_0 = (\widehat{w}_{0j}), \quad t = 0 \,, \tag{7.14}$$

where $\nabla v = (\nabla v)(t,X)$. Thus by a well known theorem on linear systems of ordinary differential equations, $\widehat{w}$ must be linear combination of the solutions $X_{(j)}$ of (7.12) with the constant coefficients $\widehat{w}_{0j}$, which proves Cauchy's vorticity equation (1.7), [22, p. 151–152]. ▫

ACKNOWLEDGEMENTS – The authors thank the Deutsche Forschungsgemeinschaft for support to the first author, and we thank the Alexander von Humboldt-Foundation for supporting the stay of the second author at the University of Paderborn.

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