PORTUGALIAE MATHEMATICA
Vol. 62 Fasc. 3 – 2005
Nova Série

ON BI-LIPSCHITZ EMBEDDINGS

H. Movahedi-Lankarani and R. Wells

Abstract: Let μ be a finite Borel regular measure on a compact metric space (X,ρ), nontrivial on nonempty open sets. It is shown that whenever the map $\iota_\rho\colon X \longrightarrow L^p(\mu)$ given by $\iota_\rho(x) = \rho(x,\cdot)$ is lower Lipschitz for some $1 < p < \infty$, then there is a bi-Lipschitz embedding of (X,ρ) into some $\mathbb{R}^N$.

1 – Introduction

When does a metric space admit a bi-Lipschitz embedding into some Euclidean space? This problem has been studied in [3, 8, 10, 13, 20, 24, 27], to produce sufficient conditions for such embeddability, and in [1, 2, 4, 9, 11, 12, 16, 18, 21, 23, 24], to produce necessary conditions and counterexamples. Here, we use compactness of certain maps between Banach spaces to derive a sufficient condition for such embeddability of compact metric-measure spaces.

A compact metric-measure space (X,ρ,μ) consists of a compact metric space (X,ρ) equipped with a finite Borel regular measure μ, nontrivial on nonempty open sets. For such a space, we define following [20, 21], also cf. [5, 6, 7], the *canonical map* $\iota_\rho: X \longrightarrow L^p(\mu)$, $1 \le p \le \infty$, by setting $\iota_\rho(x) = \rho(x,\cdot)$. This map is always (upper) Lipschitz and sometimes, but certainly not always, lower Lipschitz.

Received: October 28, 2003; *Revised*: September 23, 2004.
AMS Subject Classification: Primary 54E40, 58C20; Secondary 54C25, 54F45, 54F50, 58C25, 57R35, 57R40, 26B05.
Keywords and Phrases: Lipschitz; bilipschitz; bi-Lipschitz; embedding; spherically compact; canonical map; evaluation map.

Main Theorem. *Let (X, ρ, μ) be a compact metric-measure space. If the canonical map $\iota_\rho \colon X \longrightarrow L^p(\mu)$ is bi-Lipschitz for some $1 < p < \infty$, then there exists a bi-Lipschitz embedding of (X, ρ) into some $\mathbb{R}^N$.*

Moreover, if $\iota_\rho \colon X \longrightarrow L^2(\mu)$ is bi-Lipschitz and $\varepsilon > 0$, then there is a bi-Lipschitz embedding $f \colon X \longrightarrow L^2(\mu)$ with image in some finite dimensional linear subspace of $L^2(\mu)$ and $\|\iota_\rho - f\|_{\mathrm{Lip}} < \varepsilon$.

See Section 2 for the definition of Lipschitz norm. The above theorem is an application of the following observation: Let $f \colon X \longrightarrow \mathbb{B}$ be a bi-Lipschitz embedding of a compact metric space into a Banach space. Then there is a bi-Lipschitz embedding of X into some $\mathbb{R}^N$ provided that f may be factored $f = \Phi \circ \hat{f}$, with $\hat{f}$ Lipschitz, through a suitable compact map Φ. (See Section 2 for details.)

Section 3 is entirely devoted to the proof of Main Theorem. In Section 4, we reformulate the condition that the canonical map is bi-Lipschitz. It turns out (Theorem 4.1) that the canonical map is bi-Lipschitz exactly when the measure μ satisfies a certain point separation property: The measure μ *separates points uniformly* with respect to the metric ρ if there exist constants $c > 0$ and $\varepsilon > 0$ such that for each $x, y \in X$ with $x \neq y$ we have

$$\mu\Big(\Big\{z \colon\, |\rho(x,z) - \rho(y,z)| \geq \varepsilon\, \rho(x,y)\Big\}\Big) \;\geq\; c\ .$$

Of course this condition is nontrivial, even for a doubling measure, as follows from the example of the Heisenberg group modulo its integer lattice. (See the last section for details.) In contrast, it follows from Theorem 4.1 below and [19] that if μ if a finite Borel regular measure on Euclidean space with compact support X, then μ separates points uniformly with respect to the Euclidean metric on X. More generally, a large class of metric measure spaces (X, ρ, μ) is given by compact subsets X of a Hilbert space $\mathbb{H}$, ρ the inherited metric, and μ any allowable measure. For these spaces, the canonical map ι_ρ is bi-Lipschitz exactly for the ones which are weak* (equivalently, weakly) spherically compact; see Section 2 for definitions, Theorem 4.2, and the last section. In particular, for X weakly spherically compact in $\mathbb{H}$, *any* allowable measure separates points uniformly.

Finally, we would like to thank Urs Lang for allowing us to use his unpublished argument in Lemma 2.4. We would also like to thank Gunnar Steffanson for careful reading of the manuscript.

2 – The Set Up

We recall from [15, 17, 19, 20] that a subset X of a normed linear space $\mathbb{B}$ is *spherically compact* if the set of unit vectors

$$U(X) \;=\; \left\{\frac{x-y}{\|x-y\|_{\mathbb{B}}} \;:\; x,y\in X \;\text{ with }\; x\neq y\right\}$$

has a compact closure in the norm topology of $\mathbb{B}$. For compact subsets of a Banach space, spherical compactness is an invariant under C^1-diffeomorphism but not under bi-Lipschitz equivalence [17] and [15, Examples 8.1, 8.2, 8.3]. We say that a subset $X\subset\mathbb{B}$ is *weakly spherically compact* if the weak closure of $U(X)$ does not contain zero. Clearly, spherically compact implies weakly spherically compact. However, it is shown in Example 5.1 that there is a compact subset of a Hilbert space which is weakly spherically compact but not spherically compact. Analogously, if $\mathbb{B}$ is a dual Banach space, we say that a subset $X\subset\mathbb{B}$ is *weak* spherically compact* if the weak* closure of $U(X)$ does not contain zero.

Now let (X,ρ) be a compact metric space and let $\mathrm{Lip}(X)$ denote the ring of all real-valued Lipschitz functions on X. It is well-known $\mathrm{Lip}(X)$ is a Banach space under the norm $\|\cdot\|_{\mathrm{Lip}}$ defined by setting

$$\|f\|_{\mathrm{Lip}} \;=\; \sup_{x\in X}|f(x)| + \mathrm{Lip}(f)\;,$$

where

$$\mathrm{Lip}(f) \;=\; \sup\left\{\frac{|f(x)-f(y)|}{\rho(x,y)} :\; x,y\in X \text{ with } x\neq y\right\}\;.$$

Let $\mathrm{Lip}(X)^*$ denote the dual Banach space with norm $\|\cdot\|_{\mathrm{Lip}(X)^*}$ and let $\langle\cdot,\cdot\rangle$ denote the dual pairing. Then the evaluation map $\mathrm{ev}\colon X\longrightarrow \mathrm{Lip}(X)^*$, defined by setting $\langle \mathrm{ev}(x), f\rangle = f(x)$, is a bi-Lipschitz embedding.(1) This gives the following reformulation of bi-Lipschitz embeddability.

Theorem 2.1. *Let (X,ρ) be a compact metric space. Then there exists a bi-Lipschitz embedding of (X,ρ) into some $\mathbb{R}^N$ if and only if the set* $\mathrm{ev}(X)$ *is weak* spherically compact in* $\mathrm{Lip}(X)^*$.

The proof follows from the definitions and Lemma 2.4 below. We note that the condition that $\mathrm{ev}(X)$ be weak* spherically compact in $\mathrm{Lip}(X)^*$ is itself a bi-Lipschitz invariant because, up to Banach space equivalence, $\mathrm{Lip}(X)$ depends

(1) In fact, Pestov [22] shows that in a slightly altered situation, the map ev is an isometric embedding.

only on the bi-Lipschitz equivalence class of the metric. Furthermore, it is easy to see that this set $\mathrm{ev}(X)$ is weak* spherically compact if and only if it is weakly spherically compact.

Next, let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Banach spaces and let $T\colon \mathbb{B}_1 \longrightarrow \mathbb{B}_2$ be a bounded linear transformation. By saying that T is *compact* we mean that T carries any bounded subset of $\mathbb{B}_1$ to a precompact subset of $\mathbb{B}_2$. We say that a map $\Phi: \mathbb{B}_1 \longrightarrow \mathbb{B}_2$ is C^1 if the Gâteaux derivative $d\Phi(x)$ exists, for all $x \in \mathbb{B}_1$, as a bounded linear operator on $\mathbb{B}_1$ and the map $x \mapsto d\Phi(x)$ is continuous with respect to the operator norm on its range. Then $d\Phi(x)$ is the Fréchet derivative. For a subset $X \subset \mathbb{B}_1$, a C^1 map $f: X \longrightarrow \mathbb{B}_2$ is simply the restriction of a C^1 map $\Phi\colon \mathbb{B}_1 \longrightarrow \mathbb{B}_2$. For simplicity, we say that a (possibly nonlinear) map $\Phi\colon \mathbb{B}_1 \longrightarrow \mathbb{B}_2$ is *compact* if and only if it is C^1 and $d\Phi(x)$ is compact for every $x \in \mathbb{B}_1$.

Lemma 2.2. *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Banach spaces and let $\Phi\colon \mathbb{B}_1 \longrightarrow \mathbb{B}_2$ be a compact (possibly nonlinear) map. Let $X \subset \mathbb{B}_1$ be a compact subset. Then for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ so that the following conditions*

(**1**) $x, y, z \in X$ *with* $y \neq z$

(**2**) $\|x-y\|_{\mathbb{B}_1} < \delta(\varepsilon)$ *and* $\|x-z\|_{\mathbb{B}_1} < \delta(\varepsilon)$

imply that

$$\left\| \frac{\Phi(y)-\Phi(z)}{\|y-z\|_{\mathbb{B}_1}} - d\Phi(x)\, \frac{y-z}{\|y-z\|_{\mathbb{B}_1}} \right\|_{\mathbb{B}_2} < \varepsilon\,.$$

Lemma 2.3. *Let $\Phi\colon \mathbb{B}_1 \longrightarrow \mathbb{B}_2$ be a map of a normed linear space into a Banach space satisfying either of the following two hypotheses:*

(**1**) *$\mathbb{B}_1$ is a Banach space and Φ is a compact (possibly nonlinear) map.*

(**2**) *Φ is a compact linear map.*

Let X be a compact subset of $\mathbb{B}_1$ such that $\Phi|_X$ is bi-Lipschitz. Then $\Phi(X)$ is spherically compact.

In addition, if $\mathbb{B}_1$ is a linear subspace of a dual Banach space, then X itself is weak spherically compact.*

Lemma 2.4. *Let $\mathbb{B}$ be a normed linear space and let X be a compact subset of $\mathbb{B}$. Then there exist $N \in \mathbb{N}$ and a bounded linear map $\varphi\colon \mathbb{B} \longrightarrow \mathbb{R}^N$ such that $\varphi|_X$ is a bi-Lipschitz embedding if and only if X is weakly spherically compact.*

Proof: First, let X be weakly spherically compact. Following U. Lang([2]), there exist $\varepsilon > 0$ and a finite number of continuous linear functionals $\varphi_1, \cdots, \varphi_N$ on $\mathbb{B}$ such that for each $x, y \in X$ with $x \neq y$ we have $|\varphi_j(x-y)|/\|x-y\|_{\mathbb{B}} > \varepsilon$ for at least one index $1 \leq j \leq N$. Then $\varphi = (\varphi_1, \cdots, \varphi_N)$ is the desired map.

The converse follows from Case (2) of Lemma 2.3. ∎

As a corollary to the above proof we see that a compact weak* spherically compact subset of a dual Banach space admits a bi-Lipschitz embedding into $\mathbb{R}^N$ for some N. The following result is a corollary of Lemma 2.3 and Lemma 2.4.

Theorem 2.5. *Let X be a compact metric space. Suppose that there is a bi-Lipschitz embedding $f\colon X \longrightarrow \mathbb{B}$ which factors $f = \Phi \circ \hat{f}$*

$$\begin{tikzcd} & \mathbb{B}_1 \arrow[d, "\Phi"] \\ X \arrow[ur, "\hat{f}"] \arrow[r, "f"] & \mathbb{B} \end{tikzcd}$$

with $\hat{f}$ Lipschitz and the map $\Phi\colon \mathbb{B}_1 \longrightarrow \mathbb{B}$ satisfying either of the following two hypotheses:

(1) *$\mathbb{B}_1$ is a Banach space and Φ is a compact (possibly nonlinear) map.*

(2) *$\mathbb{B}_1$ is a normed linear space and Φ is a compact linear map.*

Then there exists a bi-Lipschitz embedding of X into $\mathbb{R}^N$ for some N.

Finally, because of the role played by spherical compactness and weak spherical compactness in the above, we include the following theorem. By a quotient map in the category of Banach apaces we mean a bounded surjective linear map $\pi\colon \mathbb{A} \longrightarrow \mathbb{B}$ which satisfies the following universal property: If $f\colon \mathbb{A} \longrightarrow \mathbb{E}$ is a bounded linear map which factors through $\mathbb{B}$ to produce a (unique) map $\hat{f}$,

$$\begin{tikzcd} \mathbb{A} \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ \mathbb{B} \arrow[r, "\hat{f}"] & \mathbb{E} \end{tikzcd}$$

then $|||\hat{f}||| = |||f|||$. Here, $|||\cdot|||$ denotes the operator norm. Of course, a map π

([2]) Private Communication.

is a quotient if and only if $\|b\|_{\mathbb{B}} = \inf\{\|a\|_{\mathbb{A}} :\ \pi(a) = b\}$. Given bounded linear maps

$$\mathbb{A}_1 \xrightarrow{\pi_1} \mathbb{A}_2 \xrightarrow{\pi_2} \mathbb{A}_3 \ ,$$

it is clear that if π_1 and $\pi_2 \circ \pi_1$ are quotients, then so is π_2.

Theorem 2.6. *Let X be a (norm) compact subset of a Banach space $\mathbb{B}$ with the inherited metric. Consider the following statements:*

(**1**) *The set X is spherically compact.*

(**2**) *For any $\varepsilon > 0$ there is a finite dimensional quotient $\mathbb{F}$ of $\mathbb{B}$ such that the quotient map $\pi_{\mathbb{F}} :\ \mathbb{B} \longrightarrow \mathbb{F}$ satisfies*

$$(1-\varepsilon)\,\|x-y\|_{\mathbb{B}} \ \le\ \|\pi_{\mathbb{F}}(x) - \pi_{\mathbb{F}}(y)\|_{\mathbb{F}} \ \le\ \|x-y\|_{\mathbb{B}} \tag{2.6.1}$$

for every $x, y \in X$.

(**3**) *The weak closure of $U(X)$ is contained in the unit sphere of $\mathbb{B}$.*

(**4**) *The set X is weakly spherically compact.*

(**5**) *There is a finite dimensional quotient $\mathbb{F}$ of $\mathbb{B}$ such that the quotient map $\pi_{\mathbb{F}}|_X :\ X \longrightarrow \mathbb{F}$ is a bi-Lipschitz embedding.*

(**6**) *There exists a C^1 map $g :\ X \longrightarrow \mathbb{R}^N$, for some N, which is a bi-Lipschitz embedding.*

(**7**) *There exist a finite dimensional linear subspace $\mathbb{F} \subset \mathbb{B}$, a compact set $Y \subset \mathbb{F}$, and a Lipschitz map $f :\ Y \longrightarrow \mathbb{G}$, where $\mathbb{B} = \mathbb{F} \oplus \mathbb{G}$, such that $X = \mathrm{graph}(f) = \{y + f(y) :\ y \in Y\}$.*

Then we have the following implications:

$$\begin{tikzcd}
(1) \arrow[d] & (3) \arrow[l, "\mathbb{B}=L^p,\ 1<p<\infty"'] \arrow[r] \arrow[dl, shift left, "\mathbb{B}=\mathbb{B}^{**}"] & (4) \arrow[dr] & (6) \arrow[l] \\
(2) \arrow[ur] & & (7) \arrow[u] & (5) \arrow[u] \arrow[l]
\end{tikzcd}$$

Example 5.1. below shows that (4) $\not\longrightarrow$ (3) even in a Hilbert space.

Proof: The implications (3) $\longrightarrow$ (4) and (5) $\longrightarrow$ (6) are immediate. Also, (4) $\longrightarrow$ (5) is Lemma 2.4, (6) $\longrightarrow$ (4) is Lemma 2.3, and (3) $\xrightarrow{\mathbb{B}=L^p,\,1<p<\infty}$ (1) is a standard exercise.

(1) $\longrightarrow$ (2): For any $\xi \in \overline{U(X)}$ there exists $\lambda_\xi \in \mathbb{B}^*$ with $\|\lambda_\xi\|_{\mathbb{B}^*} = 1$ and $\lambda_\xi(\xi) = 1$. Let $\mathbb{F}_\xi = \mathbb{B}/\ker\lambda_\xi$ and let $\pi_\xi\colon \mathbb{B} \longrightarrow \mathbb{F}_\xi$ be the quotient map. Then for the factorization $\lambda_\xi = \hat{\lambda}_\xi \circ \pi_\xi$ we have $\|\hat{\lambda}_\xi\| = \|\lambda_\xi\|_{\mathbb{B}^*} = 1$ implying that $\|\pi_\xi(\xi)\|_{\mathbb{F}} = 1$. Now let $\varepsilon > 0$ and let $N_\varepsilon(\xi_1), \cdots, N_\varepsilon(\xi_n)$ be a cover of $\overline{U(X)}$ by ε-balls. Let $M = \ker\lambda_{\xi_1} \cap \cdots \cap \ker\lambda_{\xi_n}$ and let $\mathbb{F}$ be the quotient $\mathbb{B}/M$ with quotient map $\pi_{\mathbb{F}}$. For each $\eta \in \overline{U(X)}$ we have $\|\eta - \xi_i\|_{\mathbb{B}} < \varepsilon$ for some index $1 \le i \le n$. Then

$$1 - \varepsilon \;\le\; \|\pi_{\xi_i}(\eta)\|_{\mathbb{F}_{\xi_i}} \le\; \|\pi_{\mathbb{F}}(\eta)\|_{\mathbb{F}} \;\le\; 1$$

so that (2.6.1) holds for any $x, y \in X$ with $x \neq y$.

(2) $\longrightarrow$ (3): Suppose that a net $\{\xi_i = (x_i - y_i)/\|x_i - y_i\|_{\mathbb{B}}\}_{i\in I}$ converges weakly to some ξ. For $\varepsilon > 0$, let $\mathbb{F}$ be a finite dimensional quotient of $\mathbb{B}$, with quotient map $\pi_{\mathbb{F}}$, so that $(1-\varepsilon)\|\eta\|_{\mathbb{B}} \le \|\pi_{\mathbb{F}}(\eta)\|_{\mathbb{F}} \le \|\eta\|_{\mathbb{B}}$ for all $\eta \in U(X)$. Then the net $\{\pi_{\mathbb{F}}(\xi_i)\}_{i\in I}$ converges in norm to $\pi_{\mathbb{F}}(\xi)$. Thus, $(1-\varepsilon) \le \|\pi_{\mathbb{F}}(\xi)\|_{\mathbb{F}} \le \|\xi\|_{\mathbb{B}} \le 1$ implying that $\|\xi\|_{\mathbb{B}} = 1$.

(3) $\xrightarrow{\mathbb{B}=\mathbb{B}^{**}}$ (2): We may argue as in the case (1) $\longrightarrow$ (2) except that we use a finite cover of $\overline{U(X)}^{\mathrm{weak}}$ by weakly open neighborhoods $U_1, \cdots, U_n$ of $\xi_1, \cdots, \xi_n$, respectively, so that for $\eta \in U_i$ we have $\|\pi_{\xi_i}(\eta) - \pi_{\xi_i}(\xi_i)\|_{\mathbb{F}_{\xi_i}} < \varepsilon$. Then, as before, we have $(1-\varepsilon) \le \|\pi_{\mathbb{F}}(\eta)\|_{\mathbb{F}} \le 1$ implying (2.6.1).

(5) $\longrightarrow$ (7): Let $\mathbb{F}$ be a finite dimensional quotient of $\mathbb{B}$ so that the quotient map $\pi_{\mathbb{F}}|_X\colon X \longrightarrow \mathbb{F}$ is bi-Lipschitz. Let $\mathbb{F}' \subset \mathbb{B}$ be a linear subspace so that $\pi_{\mathbb{F}}|_{\mathbb{F}'}$ is a linear isomorphism and let $Y = (\pi_{\mathbb{F}}|_{\mathbb{F}'})^{-1}\,\pi_{\mathbb{F}}(X)$. Let $\mathbb{G} = \ker\pi_{\mathbb{F}}$ and define $f\colon Y \longrightarrow \mathbb{G}$ by requiring

$$x \;=\; (\pi_{\mathbb{F}}|_{\mathbb{F}'})^{-1}\,\pi_{\mathbb{F}}(x) + f\Big((\pi_{\mathbb{F}}|_{\mathbb{F}'})^{-1}\,\pi_{\mathbb{F}}(x)\Big)\ .$$

(7) $\longrightarrow$ (4): Assume for contradiction that there exist nets $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$, both converging to some $y \in Y$, with $x_i \neq y_i$ and

$$\xi_i \;=\; \frac{x_i + f(x_i) - y_i - f(y_i)}{\|x_i + f(x_i) - y_i - f(y_i)\|_{\mathbb{B}}} \;\xrightarrow{\text{weak}}\; 0\ .$$

We may assume that $\{(x_i - y_i)/\|x_i - y_i\|_{\mathbb{F}}\}_{i\in I}$ converges in norm to some $\eta \in \mathbb{F}$ with $\|\eta\|_{\mathbb{F}} = 1$. Let $\eta^* \in \mathbb{F}^*$ with $\langle\eta^*, \eta\rangle = 1$ and let $\pi\colon \mathbb{B} \longrightarrow \mathbb{F}$ be a continuous projection. Then we have

$$0 \;=\; \lim_{i\in I}|\langle\pi^*\eta^*, \xi_i\rangle| \;=\; \lim_{i\in I}|\langle\eta^*, \pi\xi_i\rangle| \;\ge\; \frac{1}{1 + \mathrm{Lip}(f)}$$

which is a contradiction. ∎

Hence, we arrive at the following:

Complement to Theorem 2.5. *In Theorem 2.5, if* $\mathbb{B} = \mathbb{H}$ *is a Hilbert space, then for any* $\varepsilon > 0$ *there exists a bi-Lipschitz embedding* $\varphi\colon X \longrightarrow \mathbb{R}^N$ *for some* N*, such that* $\|\varphi - f\|_{\mathrm{Lip}} < \varepsilon$.

3 – Proof of Main Theorem

Let (X, ρ) be a compact metric space. Using the metric ρ on X, we define a lift $\lambda\colon X \longrightarrow \mathrm{Lip}(X)$ of the canonical map by setting $\lambda(x) = \rho(x, \cdot)$. Unfortunately, λ is totally discontinuous, with $\lambda(X)$ metrically discrete; see the last section for some related issues. However, there is a topology on $\mathrm{Lip}(X)$ with respect to which λ is continuous. Specifically, let $E(X)$ denote the closed linear span of $\mathrm{ev}(X) \subset \mathrm{Lip}(X)^*$. We call the weak topology induced by $E(X)$ on $\mathrm{Lip}(X)$ the $E(X)$-topology on $\mathrm{Lip}(X)$.

Lemma 3.1. *The map* $\lambda\colon X \longrightarrow \mathrm{Lip}(X)$ *is continuous with respect to the* $E(X)$*-topology on* $\mathrm{Lip}(X)$.

Proof: Suppose that a sequence $x_n \longrightarrow x$ in X. Then, for any $y \in X$, we have

$$\langle \lambda(x_n), \mathrm{ev}(y)\rangle = \rho(x_n, y) \longrightarrow \rho(x, y) = \langle \lambda(x), \mathrm{ev}(y)\rangle \ .$$

Consequently, $\langle \lambda(x_n), \xi\rangle \longrightarrow \langle \lambda(x), \xi\rangle$ for any finite linear combination ξ of elements of $\mathrm{ev}(X)$. Finally, for any $\varepsilon > 0$ and any $\eta \in E(X)$ there is ξ as above with $\|\xi - \eta\|_{\mathrm{Lip}(X)^*} < \varepsilon$. Then

$$\begin{aligned}
\lim_{n\to\infty} \langle \lambda(x_n), \xi\rangle - \varepsilon(1 + \operatorname{diam} X) \ &\leq \ \liminf_{n\to\infty} \langle \lambda(x_n), \eta\rangle \\
&\leq \ \limsup_{n\to\infty} \langle \lambda(x_n), \eta\rangle \\
&\leq \ \lim_{n\to\infty} \langle \lambda(x_n), \xi\rangle + \varepsilon(1 + \operatorname{diam} X)
\end{aligned}$$

as desired. ∎

Now, let μ be a finite Borel regular measure supported on X; we may as well assume that $\mu(X) = 1$. We define a map

$$J\colon \ \mathrm{Lip}(X)^* \longrightarrow \mathbb{R}^X$$

by setting $J\xi(x) = \langle \xi, \lambda(x)\rangle$ for any $\xi \in \mathrm{Lip}(X)^*$. Of course, the function $J\xi$ is bounded by $(1+\operatorname{diam} X)\,\|\xi\|_{\mathrm{Lip}(X)^*}$ but it need not be μ-measurable. However, for $\xi \in E(X)$ it is continuous. Then we have

Lemma 3.2. *For any $\xi \in E(X)$, the function $J\xi$ represents an element $J_p\xi \in L^p(\mu)$, $1 \le p \le \infty$, with $\|J_p\xi\|_{L^p(\mu)} \le (1+\mathit{diam}\, X)\,\|\xi\|_{\mathrm{Lip}(X)^*}$.*

We have thus defined, for each $1 \le p \le \infty$, a bounded linear map

$$J_p \colon E(X) \longrightarrow L^p(\mu)\ .$$

We next define $J_p^* \colon L^p(\mu) \longrightarrow \mathrm{Lip}(X)$ by setting (The adjoint notation will be justified below.)

$$J_p^* f(x) \,=\, \int_X f(y)\,\rho(x,y)\,d\mu(y)\ .$$

Then $\|J_p^* f\|_{\mathrm{Lip}(X)} \le (1+\operatorname{diam} X)\,\|f\|_{L^p(\mu)}$. We may identify $\mathrm{Lip}(X)$ with a subset of $E(X)^*$ and write $\mathrm{Lip}(X) \subset E(X)^*$ by setting $\langle f,\xi\rangle = \langle \xi, f\rangle$ for any $\xi \in E(X)$. That $\mathrm{Lip}(X)$ is actually a closed subset of $E(X)^*$, and that the above correspondence is one-one are consequences of the following lemma.

Lemma 3.3. *The space $\mathrm{Lip}(X)$ is a continuous linear retract of $E(X)^*$.*

Proof: We define $\alpha \colon E(X)^* \longrightarrow \mathrm{Lip}(X)$ by setting $\alpha(\zeta)(x) = \langle \zeta, \mathrm{ev}(x)\rangle$, for $\zeta \in E(X)^*$. Then, for $f \in \mathrm{Lip}(X) \subset E(X)^*$, we have $\alpha(f)(x) = \langle f, \mathrm{ev}(x)\rangle = \langle \mathrm{ev}(x), f\rangle = f(x)$. It is easily seen that $\|\alpha(\zeta)\|_{\mathrm{Lip}(X)} \le 2\|\zeta\|_{E(X)^*}$. ∎

Now, for $1 < p < \infty$ and $f \in L^p(\mu)$, we define an element $\int_X f(y)\,\lambda(y)\,d\mu(y) \in E(X)^*$ by setting

$$\left\langle \int_X f(y)\,\lambda(y)\,d\mu(y),\, \xi \right\rangle = \int_X f(y)\,\langle \lambda(y), \xi\rangle\,d\mu(y) \,=\, \int_X f(y)\,J_q\xi(y)\,d\mu(y)$$

for $1/p + 1/q = 1$ and any $\xi \in E(X)$.

Lemma 3.4. *The element $\int_X f(y)\,\lambda(y)\,d\mu(y) \in \mathrm{Lip}(X) \subset E(X)^*$ and for $1 < p < \infty$ we have*

$$J_p^* f \,=\, \int_X f(y)\,\lambda(y)\,d\mu(y)\ .$$

Proof: It suffices to show that $J_p^* f$ and $\int_X f(y)\,\lambda(y)\,d\mu(y)$ agree as bounded linear functionals on $E(X)$. First, they agree on $\mathrm{ev}(X)$ since

$$\left\langle \int_X f(y)\,\lambda(y)\,d\mu(y),\, \mathrm{ev}(x) \right\rangle = \int_X f(y)\,\rho(x,y)\,d\mu(y) \,=\, J_p^* f(x) \,=\, \langle J_p^* f, \mathrm{ev}(x)\rangle\ .$$

Consequently, they agree on $\operatorname{span}\{\operatorname{ev}(X)\}$ by linearity and on $\overline{\operatorname{span}}\{\operatorname{ev}(X)\} = E(X)$ by continuity. ∎

In what follows, $\langle\cdot,\cdot\rangle_{p,q}$ denotes the pairing $L^p(\mu)\times L^q(\mu)\longrightarrow\mathbb{R}$ for $1/p+1/q=1$. We note that for $1<p<\infty$ and $1/p+1/q=1$, we have

$$\langle\xi, J_q^* f\rangle \;=\; \langle J_q^* f, \xi\rangle \;=\; \left\langle \int_X f(y)\,\lambda(y)\,d\mu(y),\; \xi \right\rangle$$
$$= \int_X f(y)\, J_p\xi(y)\,d\mu(y) \;=\; \langle J_p\xi, f\rangle_{p,q} \;,$$

where $\xi\in E(X)$ and $f\in L^q(\mu)$. This proves the following lemma.

Lemma 3.5. *For $1<p<\infty$ and $1/p+1/q=1$, we have*

$$\langle\xi, J_q^* f\rangle = \langle J_p\xi, f\rangle_{p,q} \;,$$

where $\xi\in E(X)$ and $f\in L^q(\mu)$.

Proposition 3.6. *For $1<p<\infty$, the map $J_p\colon\, E(X)\longrightarrow L^p(\mu)$ is compact.*

Proof: It suffices to show that for any bounded net $\{\xi_i\}_{i\in I}\subset E(X)$, the net $\{J_p\xi_i\}_{i\in I}\subset L^p(\mu)$ has a convergent subnet.

To this end, let $\{\xi_i\}_{i\in I}\subset E(X)$ be a net with $\|\xi_i\|_{\operatorname{Lip}(X)^*}<K<\infty$ for all $i\in I$. By Alaoglu's Theorem, we may assume, without loss of generality, that $\xi_i \xrightarrow{\text{weak}^*} \xi\in\operatorname{Lip}(X)^*$. Thus, for any $g\in L^q(\mu)$, we have

$$\lim_{i\in I}\langle J_p\xi_i, g\rangle_{p,q} \;=\; \lim_{i\in I}\langle\xi_i, J_q^* g\rangle \;=\; \langle\xi, J_q^* g\rangle \;.$$

Moreover, $\|J_p\xi_i\|_p\le K(1+\operatorname{diam}X)$ for each $i\in I$. By the Riesz Representation Theorem, there exists $f\in L^p(\mu)$ such that $\|f\|_p\le K(1+\operatorname{diam}X)$ and f is the weak limit of the net $\{J_p\xi_i\}_{i\in I}\subset L^p(\mu)$. That is, $\lim_{i\in I}\langle J_p\xi_i, g\rangle_{p,q}=\langle f,g\rangle_{p,q}$ for every $g\in L^q(\mu)$. Because (X,ρ) is compact metric and μ is Borel regular, the weak topology of $L^p(\mu)$ is metrizable. Therefore, there exists a sequence $i(1)<i(2)<\cdots$ in the directed set I so that

$$\text{weak-}\lim_{n\to\infty} J_p\xi_{i(n)} \;=\; f \;=\; \text{weak-}\lim_{i\in I} J_p\xi_i \;.$$

There are two cases to consider:

Case 1. If the sequence $\{i(n)\}_{n\ge1}$ is not cofinal in I, the directed set $I'=\{j\colon\, j>i(n),\; n=1,2,\cdots\}$ is nonempty and cofinal in I. Furthermore, for each

$j \in I'$, the element $J_p\xi_j$ lies in the intersection of a countable base for the weak topology at $f \in L^p(\mu)$. Hence, $J_p\xi_j = f$ and the net $\{J_p\xi_j\}_{j\in I'}$ is a subnet of $\{J_p\xi_i\}_{i\in I}$, constant at f and so converging, now in the norm topology of $L^p(\mu)$, to f.(3)

Case 2. If the sequence $\{i(n)\}_{n\geq 1}$ is cofinal in I, we have

$$\lim_{i\in I} J_p\xi_i(x) \;=\; \lim_{i\in I}\langle \xi_i, \lambda(x)\rangle \;=\; \langle \xi, \lambda(x)\rangle$$

for every $x \in X$. Thus, in the topology of pointwise convergence on $\mathbb{R}^X$, the net $\{J_p\xi_i\}_{i\in I}$ converges to the function $\gamma\colon\, x \mapsto \langle \xi, \lambda(x)\rangle$. Therefore, the cofinal subsequence $\{J_p\xi_{i(n)}\}_{n\geq 1}$ also converges pointwise to the function γ. Consequently, the function γ is measurable and bounded by $K(1 + \operatorname{diam} X)$ since each of the functions $J_p\xi_{i(n)}$, $n \geq 1$, is so. It follows from the Bounded Convergence Theorem that $\lim_{n\to\infty} \|J_p\xi_{i(n)}\|_{L^p(\mu)} = \|\gamma\|_{L^p(\mu)}$ and that

$$\lim_{n\to\infty} \int_X J_p\xi_{i(n)}(x)\, g(x)\, d\mu(x) \;=\; \int_X \gamma(x)\, g(x)\, d\mu(x)$$

for every $g \in L^q(\mu)$. Hence, $\lim_{n\to\infty} \|J_p\xi_{i(n)} - \gamma\|_{L^p(\mu)} = 0$ by the Banach–Steinhaus Theorem. ∎

Incidentaly, we find that $\gamma = f$ so that $J_p\xi = f$ in the second case. At this point we have the following commutative diagram

$$\begin{tikzcd} & E(X) \arrow[d, "J_p"] \\ X \arrow[ur, "\mathrm{ev}"] \arrow[r, "\iota_\rho"'] & L^p(\mu) \end{tikzcd}$$

where the map $\iota_\rho\colon\, X \longrightarrow L^p(\mu)$ is the canonical map $\iota_\rho(x) = \rho(x,\cdot)$ and J_p is a compact linear operator for $1 < p < \infty$.(4) Clearly, if ι_ρ is bi-Lipschitz, then so is $J_p \circ \mathrm{ev}$ and hence $J_p\colon\, \mathrm{ev}(X) \longrightarrow J_p\,(\mathrm{ev}(X))$ is a bi-Lipschitz equivalence.

Proof of Main Theorem: First assume that the canonical map ι_ρ is bi-Lipschitz. Then $J_p \circ \mathrm{ev}\colon\, X \longrightarrow J_p\,(\mathrm{ev}(X))$ must be bi-Lipschitz. Moreover, the map J_p is compact and the map ev is Lipschitz. Accordingly, Theorem 2.5 implies that there is a bi-Lipschitz embedding of X into some $\mathbb{R}^N$. The approximation part of Main Theorem follows from the Complement to Theorem 2.5. ∎

(3) Of course, we cannot say that $J_p(\xi) \in L^p(\mu)$ or that $J_p(\xi) = f$.

(4) Recall that both λ and ι_ρ are the map $x \mapsto \rho(x,\cdot)$. The difference is the choice of target space.

4 – The canonical map and point separation

Let (X, ρ, μ) be a compact metric-measure space. For simplicity, in this section we restrict our attension to the case of $p = 2$ and consider the question of when the canonical map $\iota_\rho\colon X \longrightarrow L^2(\mu)$ is Lipschitz below.

Theorem 4.1. *Let (X, ρ, μ) be a compact metric-measure space. Then the canonical map $\iota_\rho\colon X \longrightarrow L^2(\mu)$ is bi-Lipschitz if and only if the measure μ separates points uniformly with respect to ρ.*

Proof: Clearly, if μ separates points uniformly with respect to ρ, then there exist $c > 0$ and $\varepsilon > 0$ with $\|\iota_\rho(x) - \iota_\rho(y)\|_{L^2(\mu)} \geq c\sqrt{\varepsilon}\,\rho(x, y)$ for all $x, y \in X$.

Conversely, suppose that $\|\iota_\rho(x) - \iota_\rho(y)\|_{L^2(\mu)} \geq K\rho(x, y)$ for some $K > 0$ and all $x, y \in X$; we may assume that $K < 1$. For $t \geq 0$, we define

$$f(t) \;=\; f(t\colon x, y) \;=\; \mu\left(\left\{z\colon\, |\rho(x, z) - \rho(y, z)| \geq \sqrt{t}\,\rho(x, y)\right\}\right) .$$

Then f is a monotone decreasing function with $f(t) \equiv 0$ for $t > 1$. Hence,

$$K^2 \;\leq\; \frac{\|\iota_\rho(x) - \iota_\rho(y)\|^2_{L^2(\mu)}}{\rho(x, y)^2} \;=\; -\int_0^2 t\,df(t) \;=\; \int_0^1 f(t)\,dt \;.$$

Now assume, for contradiction, that for all $r > 0$ the square on the t-axis with the lower left corner at the origin and side r fails to lie entirely under the graph of $s = f(t)$. Then the corner (r, r) lies above the graph so that the area under the graph of $s = f(t)$ from $t = 0$ to $t = 1$ lies in the union of rectangles $([0, 1]\times[0, r])\cup([0, r]\times[r, 1])$. Thus $\int_0^1 f(t)\,dt \leq r + r(1-r)$. Consequently, we must have $K^2 \leq r + r(1-r)$ for all $r \in [0, 1]$ which is a contradiction. In fact, we have that the square of side $r_\circ = 1 - \sqrt{1 - K^2}$ lies under the graph. This value of r is independent of the choice of x and y and hence we have

$$\mu\Big(\Big\{z\colon\, |\rho(x, z) - \rho(y, z)| \geq \sqrt{r_\circ}\,\rho(x, y)\Big\}\Big) \;\geq\; r_\circ$$

for all $x, y \in X$ with $x \neq y$. ∎

Theorem 4.2. *Let X be a compact subset of a Hilbert space $\mathbb{H}$ with the inherited metric and let μ be a finite Borel regular measure with closed support X.*

(1) *If X is weakly spherically compact, then the canonical map $\iota\colon X \longrightarrow L^2(\mu)$ is bi-Lipschitz.*

(**2**) *Assume that all atoms of μ are isolated. If the canonical map $\iota\colon X \longrightarrow L^2(\mu)$ is bi-Lipschitz, then X is weakly spherically compact.*

Here and in the proof, ι stands for $\iota_{\|\cdot\|_{\mathbb{H}}}$. We note that (2) follows from Lemma 2.3 only when ι is C^1; that is when the scaling dimension $\dim(\mu) > 2$, see [19, Theorem 2.1].

Proof: To prove (1), let ν be the Lebesgue measure on $\mathbb{R}^3$ normalized so that $\nu(D^3) = 1$, where D^3 is the closed unit ball in $\mathbb{R}^3$. Let $j\colon \mathbb{H}\times\mathbb{R}^3 \longrightarrow L^2(\mu\times\nu|_{D^3})$ be the canonical map defined by setting

$$j(u,v)(u',v') = \Big(\|u-u'\|_{\mathbb{H}}^2 + \|v-v'\|_{\mathbb{R}^3}^2\Big)^{1/2} = \|(u,v)-(u',v')\|_{\mathbb{H}\times\mathbb{R}^3} \ .$$

It is shown in [19, Theorem 2.1] that this map j is C^1 and bi-Lipschitz.(5) Thus, for $x,y \in X$, we may calculate

$$\begin{aligned} K\|x-y\|_{\mathbb{H}} &\le \|j(x,0)-j(y,0)\|_{L^2(\mu\times\nu|_{D^3})} \\ &= \int_{D^3}\int_X \Big|\|(x,0)-(u,v)\|_{\mathbb{H}\times\mathbb{R}^3} - \|(y,0)-(u,v)\|_{\mathbb{H}\times\mathbb{R}^3}\Big|^2 d\mu(u)\,d\nu(v) \\ &\le \int_{D^3}\int_X \frac{\Big|\|x-u\|_{\mathbb{H}}^2 - \|y-u\|_{\mathbb{H}}^2\Big|^2}{\Big(\|x-u\|_{\mathbb{H}}+\|y-u\|_{\mathbb{H}}\Big)^2}\ d\mu(u)\,d\nu(v) \\ &= \|\iota(x)-\iota(y)\|_{L^2(\mu)}^2 \ . \end{aligned}$$

Hence, ι is lower Lipschitz and so bi-Lipschitz.

For (2) it suffices, by Theorem 4.1, to show that if X is not weakly spherically compact, then μ does not separate points uniformly with respect to $\|\cdot\|_{\mathbb{H}}$. If X is not weakly spherically compact, there exists a pair of sequences $\{x_n\}_{n\ge1}$ and $\{y_n\}_{n\ge1}$ in X, with $x_n \ne y_n$ for all n, both converging to some $x \in X$ such that the sequence of unit vectors $(x_n-y_n)/\|x_n-y_n\|_{\mathbb{H}} \xrightarrow{\text{weak}} 0$; note that x cannot be isolated. Let $\varepsilon > 0$ be given and consider

$$X_n = \left\{ z \in X\colon \ \frac{\Big|\|x_n-z\|_{\mathbb{H}} - \|y_n-z\|_{\mathbb{H}}\Big|}{\|x_n-y_n\|_{\mathbb{H}}} \ge \varepsilon \right\} .$$

Then for $z \ne x$ we have

$$\frac{\Big|\|x_n-z\|_{\mathbb{H}} - \|y_n-z\|_{\mathbb{H}}\Big|}{\|x_n-y_n\|_{\mathbb{H}}} = \left\langle \frac{x_n-y_n}{\|x_n-y_n\|_{\mathbb{H}}}, \frac{x_n+y_n-2z}{\|x_n-z\|_{\mathbb{H}}+\|y_n-z\|_{\mathbb{H}}} \right\rangle \longrightarrow 0 \ .$$

(5) The map j is C^1 because the scaling dimension $\dim(\mu\times\nu|_{D^3}) > 2$.

Thus, the characteristic functions $\chi_{X_n-\{x\}}$ converges to zero pointwise. By the Bounded Convergence Theorem $\mu(X_n - \{x\}) \longrightarrow 0$ and so $\mu(X_n) \longrightarrow 0$. Consequently, for any $c > 0$, the inequality $\mu(X_n) \geq c$ is impossible. ∎

5 – Concluding Remarks

I. The following example shows that the two notions of spherical compactness and weak spherical compactness are not the same.

Example 5.1. Let C be the Cantor ternary subset of $[0,1]$. According to [17], there is an infinite dimensional Hilbert space $\mathbb{H}$ and a bi-Lipschitz map $f\colon C \longrightarrow \mathbb{H}$ such that any weakly convergent sequence of normalized secants in $f(C)$ has weak limit zero. That is, $f(C)$ fails to be weakly spherically compact. Let

$$\Gamma = \Big\{(x, f(x))\colon\, x \in C\Big\} \subset \mathbb{R}{\times}\mathbb{H} .$$

This set Γ is weakly spherically compact but not spherically compact, as the following argument shows.

For some $0 < a < A$ we have

$$a\,|x-y| \;\leq\; \|f(x)-f(y)\|_{\mathbb{H}} \;\leq\; A\,|x-y| .$$

Let $\{(x_n, f(x_n))\}_{n\geq 1}$ and $\{(y_n, f(y_n))\}_{n\geq 1}$ be sequences in Γ, both converging to some $(x, f(x)) \in \Gamma$, such that the sequence

$$\xi_n = \frac{(x_n - y_n,\, f(x_n) - f(y_n))}{\sqrt{|x_n-y_n|^2 + \|f(x_n)-f(y_n)\|^2_{\mathbb{H}}}} \xrightarrow{\text{weak}} \xi .$$

We have that

$$\frac{1}{\sqrt{1+A^2}} \;\leq\; |\langle(1,0),\xi\rangle| \;\leq\; \frac{1}{\sqrt{1+a^2}} ,$$

implying that $\xi \neq 0$ and so Γ is weakly spherically compact. But, we also have that $\langle(0,\eta),\xi\rangle = 0$ for any $\eta \in \mathbb{H}$. Thus, $\xi \in \mathbb{R}$ and $|\xi| = |\langle(1,0),\xi\rangle| \leq (1+a^2)^{-1/2} < 1$, implying that Γ is not spherically compact. □

II. Of course if (X,ρ,μ) is as in the Main Theorem and ι_ρ fails to be Lipschitz below, but $\iota_\rho|_Y$ is Lipschitz below for some $Y \subset X$, then there is a bi-Lipschitz embedding of Y into some $\mathbb{R}^N$. However, such Y may be uninteresting as the following example shows: For $X = f(C)$ as in **I**, ρ the metric on X inherited from $\mathbb{H}$, and μ the standard probability measure from C, we see that for any $1 < p < \infty$, the map $\iota_\rho\colon X \longrightarrow L^p(\mu)$ is bi-Lipschitz only on finite subsets of X. This fact follows from an argument similar to the proof of Proposition 3.6.

Thus, $X = f(C)$ gives an example of a space which admits a bi-Lipschitz embedding into some $\mathbb{R}^N$ even though ι_ρ fails, in a strong sense, to be lower Lipschitz.

III. In [25, 26] it is shown that $\mathrm{Lip}([0,1])$ is not separable. The map $\lambda\colon X \longrightarrow \mathrm{Lip}(X)$ in Section 3 provides a quick and easy proof that $\mathrm{Lip}(X)$ is not separable unless the compact space X is at most countable. Specifically, for $x \neq y$ in X we have

$$\|\lambda(x) - \lambda(y)\|_{\mathrm{Lip}(X)} \;=\; \rho(x,y) + 2$$

which proves the following proposition.

Proposition 5.2. *Let X be a compact uncountable metric space. Then the set $\lambda(X)$ is closed, uncountable, and discrete in the norm topology of* $\mathrm{Lip}(X)$. *In particular,* $\mathrm{Lip}(X)$ *is not separable.*

In contrast, the map $\lambda\colon X \longrightarrow \mathrm{Lip}(X)$ is a topological embedding if $\mathrm{Lip}(X)$ is equipped with the $E(X)$-topology.

IV. The choice of the map λ in Section 3 may be broadened considerably. Suppose that $\hat{\lambda}\colon X \times X \longrightarrow \mathbb{R}$ is symmetric and satisfies only

$$\left|\hat{\lambda}(x,y) - \hat{\lambda}(x,y')\right| \;\leq\; K\rho(y,y')$$

for some $0 < K < \infty$. Then the maps $\lambda\colon X \longrightarrow \mathrm{Lip}(X)$ and $\iota\colon X \longrightarrow L^p(\mu)$ are defined in the obvious way:

$$\iota(x) = \lambda(x) = \hat{\lambda}(x,\cdot)\ .$$

Then the argument proving the Main Theorem actually proves the following result.

Theorem 5.3. *If the map $\iota\colon X \longrightarrow L^p(\mu)$ is bi-Lipschitz for some $1<p<\infty$, then there is a bi-Lipschitz embedding of (X,ρ) into some $\mathbb{R}^N$.*

V. We note that the retraction $\alpha\colon E(X)^* \longrightarrow \mathrm{Lip}(X)$ in Lemma 3.3 extends to a retraction $\alpha\colon \mathbb{B}^* \longrightarrow \mathrm{Lip}(X)$, where $\mathbb{B}$ is any closed linear subspace of $\mathrm{Lip}(X)^*$ containing $E(X)$.

VI. The results of this paper do not assume the doubling condition on the metric (equivalently, finiteness of Assouad dimension $\dim_A$; see [2, 3, 12]). Rather, the finiteness of Assouad dimension here appears as consequence. In fact, by combining with [14], one arrives at the following result.

Proposition 5.4. *If a compact metric-measure space (X, ρ, μ) has a lower Lipschitz canonical map $\iota_\rho \colon X \longrightarrow L^p(\mu)$, for some $1 < p < \infty$, then*

(**1**) $\dim_A(X) < \infty$, *and*

(**2**) (X, ρ) *supports a doubling measure ν (not necessarily $= \mu$).*

VII. Tomi Laakso(6) has informed us that his non-embeddability theorem [9] can be generalized in such a way that the non-embeddability theorem of Cheeger [4] no longer implies it. In a forthcoming paper, he gives a sufficient condition for a metric space to admit *no* bi-Lipschitz embedding into uniformly convex Banach spaces. Furthermore, he shows that there exists a metric space which does not admit a bi-Lipschitz embedding into any uniformly convex Banach space even though it admits a David-Semmes regular mapping onto $\mathbb{R}^2$.

VIII. Let X be the Heisenberg group modulo its integer lattice and let ρ be any metric on X, bi-Lipschitz equivalent to the Carnot–Carathéodory metric. If the canonical measure μ on X should separate points uniformly with respect to ρ, then by Theorem 4.1, the canonical map $\iota_\rho \colon X \longrightarrow L^2(\mu)$ would be lower Lipschitz and hence there would be a bi-Lipschitz embedding of (X, ρ) into some $\mathbb{R}^N$. However, it is well known that no such embedding exists [12, 21].

IX. Unfortunately, the canonical map is not a bi-Lipschitz invariant of the metric. It may happen that ι_ρ is not bi-Lipschitz, but ι_σ is bi-Lipschitz for some metric σ, bi-Lipschitz equivalent to ρ. However, we do have the following:

Complement to Main Theorem. *If there exists a bi-Lipschitz embedding of (X, ρ) into some $\mathbb{R}^N$, then the metric σ induced on X by the embedding has a bi-Lipschitz canonical map $\iota_\sigma \colon X \longrightarrow L^p(\mu)$ for all $1 < p < \infty$.*

Proof: Let $\varphi \colon X \longrightarrow \mathbb{R}^N$ be a bi-Lipschitz embedding. Let σ be the pull-back of the Euclidean metric e on $\mathbb{R}^N$ restricted to $\varphi(X)$. Then (X, σ) is bi-Lipschitz equivalent to (X, ρ) and the map $\varphi \colon (X, \sigma) \longrightarrow (\varphi(X), e)$ is an isometry. Replacing X with $X \times S^2$ if necessary, we may assume that the scaling

(6) Private Communication.

dimension $\dim(X) > 1$. It then follows from [19, Corollary 2.2] that the map $\iota_e\colon \varphi(X) \longrightarrow L^p(\varphi_*\mu)$ is a C^1 embedding for $1<p<\dim X$, and so a bi-Lipschitz embedding. Then the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota_\sigma} & L^p(\mu) \\ {\scriptstyle\varphi}\downarrow & & \uparrow{\scriptstyle\varphi^*} \\ \varphi(X) & \xrightarrow{\iota_e} & L^p(\varphi_*\mu) \end{array}$$

with φ^* an isometry, shows that $\iota_\sigma\colon X \longrightarrow L^p(\mu)$ is bi-Lipschitz.

Finally, we note that if $\iota_\sigma\colon X \longrightarrow L^p(\mu)$ is bi-Lipschitz for some p, then $\iota_\sigma\colon X \longrightarrow L^{p'}(\mu)$ is bi-Lipschitz for all $p' \geq p$. ∎

Given a compact metrizable space X supporting a finite Borel regular measure μ which is nontrivial on nonempty open sets, we let $\mathfrak{M}(X)$ denote the set of all metrics on X which are continuous on $X \times X$. There is a natural but not Hausdorff topology on $\mathfrak{M}(X)$ with the property that the (possibly empty) subset

$$\mathfrak{I}(X) = \Big\{\text{all metircs } \rho \in \mathfrak{M}(X) \text{ with lower Lipschitz canonical map } \iota_\rho\Big\}$$

is open. This topology on $\mathfrak{M}(X)$ is determined by a pseudometric $\mathfrak{D}$, given by

$$\mathfrak{D}(\rho,\sigma) \;=\; \sup_{x\neq y}\left\|\frac{\iota_\rho(x)-\iota_\rho(y)}{\rho(x,y)} - \frac{\iota_\sigma(x)-\iota_\sigma(y)}{\sigma(x,y)}\right\|_{L^2(\mu)}.$$

That $\mathfrak{D}$ is a pseudometric follows from $\mathfrak{D}(\rho, c\rho) = 0$ for any $c > 0$. We note that the closely related number $\sup_{x\neq y}\|(\iota_\rho(x)-\iota_\rho(y))/\rho(x,y)\|_{L^2(\mu)}$ is simply $\mathrm{Lip}(\iota_\rho)$. Clearly if $\mathrm{Lip}(\iota_\rho) = 0$, then (X,ρ) is a finite space with all distances equal.

As mentioned above, two metrics in $\mathfrak{M}(X)$ may be bi-Lipschitz equivalent with one in $\mathfrak{I}(X)$ and the other not. An open problem is to determine whether every metric sufficiently near (with respect to $\mathfrak{D}$) to $\rho \in \mathfrak{I}(X)$ must be bi-Lipschitz equivalent to ρ. This problem is of interest in the following situation: X is a compact subset of some (finite dimensional) Euclidean space with the inherited metric ρ and suitable measure μ. Then $\rho \in \mathfrak{I}(X)$ and there is a bi-Lipschitz embedding of (X,σ) into some Euclidean space for any $\sigma \in \mathfrak{M}(X)$ with $\mathfrak{D}(\rho,\sigma)$ sufficiently small. This construction would yield new examples of spaces which admit bi-Lipschitz embeddings into Euclidean spaces, provided that one could arrange for σ not to be bi-Lipschitz equivalent to ρ.

X. The following is an immediate corollary of the Main Theorem and Theorem 4.1.

Corollary 5.5. *Let (X,ρ) be a compact metric space. Then there exists a bi-Lipschitz embedding of (X,ρ) into some $\mathbb{R}^N$ if and only if there is a finite Borel regular measure μ with closed support X which separates points uniformly with respect to a metric σ, bi-Lipschitz equivalent to ρ.*

Finally, by combining Theorem 4.1 and Theorem 4.2, we see that for a compact metric-measure space (X,ρ,μ), the condition that the canonical map $\iota_\rho\colon X \longrightarrow L^p(\mu)$ is bi-Lipschitz for some $1 < p < \infty$ may be viewed as a generalization of weak spherical compactness for a compact subset of a Hilbert space.

APPENDIX — L-bi-Lipschitz embeddings

It is obvious that any finite metric space admits bi-Lipschitz embeddings into $\mathbb{R}^N$ for any $N \geq 1$. However, the upper (or lower) Lipschitz constant may depend on N and be very large (or very small). Also, since any compact metric space may be "approximated" as closely as we please by finite subsets, the following question arises naturally: When does bi-Lipschitz embeddability of ε-dense subspaces imply bi-Lipschitz embeddability of the space itself? The purpose of this appendix is to provide an answer to this question.

We begin by recalling that a map $f\colon X \longrightarrow Y$ of metric spaces is an L-bi-Lipschitz embedding for some $L \geq 1$ if and only if

$$\rho_X(x,x')/L \;\leq\; \rho_Y\left(f(x),f(x')\right) \;\leq\; L\,\rho_X(x,x')\ .$$

for all $x,x' \in X$.

Now let (X,ρ) be a metric space, let $\varepsilon > 0$, and let $L \geq 1$. We set

$$\mathfrak{N}(X,\varepsilon,L) \;=\; \min\Big\{n \geq 0 \mid \text{ There exists an } \varepsilon\text{-dense closed subset } X_\varepsilon \subset X \\ \text{and an } L\text{-bi-Lipschitz embedding } f\colon X_\varepsilon \hookrightarrow \mathbb{R}^n\Big\}\ .$$

Theorem A.1. *Let (X,ρ) be a metric space with $\operatorname{diam}(X) < \infty$ and let $L \geq 1$. If $\liminf_{\varepsilon\to 0}\mathfrak{N}(X,\varepsilon,L) \leq N$ for some $N \in \mathbb{N}$, then there is an L-bi-Lipschitz embedding $f\colon X \hookrightarrow \mathbb{R}^N$.*

Proof: Let $\varepsilon_1 > \varepsilon_2 > \cdots \searrow 0$ be such that $\mathfrak{N}(X, \varepsilon_i, L) \leq N$ with $\sum_{i\geq 1} \varepsilon_i < \infty$ and let $\varepsilon > 0$. Let X_i be a closed ε_i-dense subset of X with $f_i\colon X_i \hookrightarrow \mathbb{R}^N$ an L-bi-Lipschitz embedding.

Claim 1. If $\liminf_{\varepsilon\to 0} \mathfrak{N}(X, \varepsilon, L) \leq N$, then $\lim_{\varepsilon\to 0} \mathfrak{N}(X, \varepsilon, L)$ exists and is $\leq N$.

To see this, we note that for any $\varepsilon > 0$ we have some $\varepsilon_i < \varepsilon$ with an ε_i-dense subset X_i and an L-bi-Lipschitz embedding $f_i\colon X_i \hookrightarrow \mathbb{R}^N$. The ε_i-dense subset X_i is ε-dense in X and hence $\mathfrak{N}(X, \varepsilon, L) \leq N$; the claim follows.

Now, since $\operatorname{diam}(X) < \infty$, we may assume that for each $i \geq 1$ the set $Y_i = f_i(X_i) \subset B(0, R) \subset \mathbb{R}^N$. Here $B(0, R)$ denotes the closed ball of radius R centered at 0. Moreover, since Y_i is closed for each $i \geq 1$, we may assume, by taking subsequences, that the sequence $\{Y_i\}_{i\geq 1}$ converges in the Hausdorff metric to some set $Y \subset B(0, R)$.

For each $i \geq 1$, we define a map $\psi_i\colon X_i \longrightarrow X_{i+1}$ by choosing $\rho(\psi_i(x), x) < \varepsilon_{i+1}$. We may then define $\varphi_i\colon Y_i \longrightarrow Y_{i+1}$ so that the diagram

$$\begin{tikzcd} X_i \arrow[r, "\psi_i"] \arrow[d, "f_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\ Y_i \arrow[r, "\varphi_i"] & Y_{i+1} \end{tikzcd}$$

commutes. For $j > i$, we set $\varphi_i^j = \varphi_{j-1} \circ \cdots \circ \varphi_i\colon Y_i \longrightarrow Y_j$. We choose a subsequence $1 = i(1,1) < i(1,2) < i(1,3) < \cdots$ so that the sequence $\left\{\varphi_{i(1,1)}^{i(1,k)}(y)\right\}_{k\geq 1}$ converges for each $y \in Y_{i(1,1)}$. Then we choose a subsequence $i(1,2) = i(2,2) < i(2,3) < i(2,4) < \cdots$ so that the sequence $\left\{\varphi_{i(2,2)}^{i(2,k)}(y)\right\}_{k\geq 1}$ converges for each $y \in Y_{i(2,2)}$. Inductively, we choose a subsequence

$$i(j-1, j) = i(j,j) < i(j, j+1) < i(j, j+2) < \cdots$$

so that the sequence $\left\{\varphi_{i(j,j)}^{i(j,k)}(y)\right\}_{k\geq 1}$ converges for each $y \in Y_{i(j,j)}$. The next step is Cantor diagonalization: We set

$$\hat{X}_n = X_{i(n,n)}, \quad \hat{f}_n = f_{i(n,n)} \quad \text{and} \quad \hat{Y}_n = Y_{i(n,n)}.$$

Then for $m > n$, the maps $\sigma_n^m\colon \hat{X}_n \longrightarrow \hat{X}_m$ and $\tau_n^m\colon \hat{Y}_n \longrightarrow \hat{Y}_m$ are defined the obvious way. Also the sequence $\hat{\varepsilon}_n = \varepsilon_{i(n,n)} \searrow 0$ as $n \to \infty$ with $\sum_{n\geq 1} \hat{\varepsilon}_n < \infty$.

Now let $x \in X$ and choose $\hat{x}_n(x) \in \hat{X}_n$ so that $\rho(x, \hat{x}_n(x)) < \hat{\varepsilon}_n$ and set $\hat{y}_n(x) = \hat{f}_n(\hat{x}_n(x))$.

Claim 2. The limit $\lim_{n\to\infty} \hat{y}_n(x)$ exists.

For proof of the claim, we note that $\rho\left(\hat{x}_n(x), \hat{x}_{n+1}(x)\right) < \hat{\varepsilon}_n + \hat{\varepsilon}_{n+1}$. Hence, $\|\hat{y}_n(x) - \hat{y}_{n+1}(x)\| < L(\hat{\varepsilon}_n + \hat{\varepsilon}_{n+1})$ implying that $\|\hat{y}_n(x) - \hat{y}_m(x)\| < 2\,L\sum_{j\geq n}\hat{\varepsilon}_j$ for $n \leq m$. Consequently, the sequence $\{\hat{y}_n(x)\}_{n\geq 1} \subset B(0,R)$ is a Cauchy sequence; this fact establishes the claim.

Finally, we define a map $f\colon\ X \longrightarrow \mathbb{R}^N$ by setting $f(x) = \lim_{n\to\infty} \hat{y}_n(x)$. Then $\|f(x) - f(x')\| = \lim_{n\to\infty} \|\hat{y}_n(x) - \hat{y}_n(x')\|$ for $x, x' \in X$. But $\lim_{n\to\infty} \hat{x}_n(x) = x$ for each $x \in X$, and

$$\rho\left(\hat{x}_n(x), \hat{x}_n(x')\right)/L \;\leq\; \|\hat{y}_n(x) - \hat{y}_n(x')\| \;\leq\; L\,\rho\left(\hat{x}_n(x), \hat{x}_n(x')\right) \;.$$

Hence, by taking the limit as $n \to \infty$, we have

$$\rho\left(x, x'\right)/L \;\leq\; \|f(x) - f(x')\| \;\leq\; L\,\rho\left(x, x'\right). \blacksquare$$

The following is an immediate corollary of the above theorem.

Corollary A.2. *Let (X,ρ) be a metric space with $\operatorname{diam}(X) < \infty$ and let $L \geq 1$. If $\liminf_{\varepsilon\to 0} \mathfrak{N}(X,\varepsilon,L) \leq N$, then the Assouad dimension $\dim_A(X) \leq N$.*

For the definition of Assouad dimension see [2, 3, 12].

Now, given $D > 0$ and a function $\eta\colon\,(0,\infty) \longrightarrow \mathbb{N}$ with the property that $\lim_{\varepsilon\to 0}\eta(\varepsilon)$ exists, we set

$$\begin{aligned}\mathfrak{C}\left(D, L, \eta(\cdot)\right) \;=\; & \text{The class of all compact metric spaces } X \text{ with} \\ & \operatorname{diam}(X) \leq D \;\text{ and }\; \mathfrak{N}(X,\varepsilon,L) \leq \eta(\varepsilon)\;.\end{aligned}$$

The following is another immediate corollary of Theorem A.1.

Corollary A.3. *The class $\mathfrak{C}\left(D, L, \eta(\varepsilon)\right)$ is Hausdorff–Gormov compact.*

This corollary is related to the following result of Gromov.

Theorem A.4 ([Gromov])**.** *A class $\mathcal{M}$ of compact metric spaces is Hausdorff–Gromov pre-compact if and only if there is a function $n\colon(0,\infty) \longrightarrow \mathbb{N}$ such that for any $\varepsilon > 0$ any $X \in \mathcal{M}$ can be covered by $n(\varepsilon)$-many ε-balls.*

By setting $n(\varepsilon) = (D/\varepsilon)^{\eta(\varepsilon)}$ we see that Theorem A.4 implies that $\mathfrak{C}(D, L, \eta(\varepsilon))$ is Hausdorff–Gromov pre-compact; it does not, however, imply that $\mathfrak{C}(D, L, \eta(\varepsilon))$ is Hausdorff–Gromov closed.

REFERENCES

[1] Assouad, P. – Espaces métriques, plongements, facteurs, *Publ. Math. Orsay,* (1977), 223–769.

[2] Assouad, P. – Étude d'une dimension métrique liée à la possibilité de plongements dans $\mathbb{R}^n$, *C.R. Acad. Sci. Paris Sér. A,* 288 (1979), 731–734.

[3] Assouad, P. – Plongements Lipschitzien dans $\mathbb{R}^n$, *Bull. Soc. Math. France,* 111 (1983), 429–448.

[4] Cheeger, J. – Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.,* 9 (1999), 428–517.

[5] Dugundji, J. – *Topology*, Allyn and Bacon, Boston, 1966.

[6] Gromov, M. – Filling Riemannian manifolds, *J. Differential Geom.,* 18 (1983), 1–147.

[7] Heinonen, J. – *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.

[8] Johnson, W.B.; Lindenstrauss, J. and Schechtman, G. – On Lipschitz embedding of finite metric spaces in low-dimensional normed spaces, *Lecture Notes in Math,* 1267, Springer, Berlin-NewYork (1987), 177–184.

[9] Laakso, T.J. – Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality, *Geom. Funct. Anal.,* 10 (2000), 111–123.

[10] Lang, U. and Plaut, C. – *Bilipschitz embeddings of metric spaces into space forms*, preprint, May 2000.

[11] Luosto, K. – Ultrametric spaces bi-Lipschitz embeddable in $\mathbb{R}^n$, *Fund. Math.,* 150 (1996), 25–42.

[12] Luukkainen, J. – Assouad dimension: Antifractal metrization, porous sets, and homogeneous measures, *J. Korean Math. Soc.,* 35 (1998), 23–76.

[13] Luukkainen, J. and Movahedi-Lankarani, H. – Minimal bi-Lipschitz embedding dimension of ultrametric spaces, *Fund. Math.,* 144 (1994), 181–193.

[14] Luukkainen, J. and Saksman, E. – Every complete doubling metric space carries a doubling measure, *Proc. Amer. Math. Soc.,* 126 (1998), 531–534.

[15] Mansfield, R.; Movahedi-Lankarani, H. and Wells, R. – Smooth finite dimensional embeddings, *Can. J. Math.,* 51 (1999), 585–615.

[16] Movahedi-Lankarani, H. – On the inverse of Mañé's projection, *Proc. Amer. Math. Soc.,* 116 (1992), 555–560.

[17] Movahedi-Lankarani, H. – On the theorem of Rademacher, *Real Anal. Exchange,* 17 (1992), 802–808.

[18] Movahedi-Lankarani, H. and Wells, R. – Ultrametrics and geometric measures, *Proc. Amer. Math. Soc.,* 123 (1995), 2579–2584.

[19] Movahedi-Lankarani, H. and Wells, R. – *Sets with Lie isometry groups*, in "Topics in Low Dimensional Topology" (A. Banyaga, *et al.*, Ed.), World Scientific Publishing Co., Singapore, 1999, pp. 114–125.

[20] Movahedi-Lankarani, H. and Wells, R. – *Bi-Lipschitz embeddings: A sufficient condition*, preprint, February 2001.

[21] Pansu, P. – Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un, *Ann. Math.,* 129 (1989), 1–60.

[22] Pestov, V. – Douady's conjecture on Banach analytic spaces, *C. R. Acad. Sci. Paris Sér. I Math.,* 319 (1994), 1043–1048.

[23] Semmes, S. – On the nonexistence of bilipschitz parametrizations and geometric problems about A_∞-weights, *Rev. Mat. Iberoamericana,* 12 (1996), 337–410.

[24] Semmes, S. – Bilipschitz embeddings of metric spaces into Euclidean spaces, *Publl. Math.,* 43 (1999), 571–653.

[25] Sherbert, D.R. – Banach algebras of Lipschitz functions, *Pacific. J. Math.,* 13 (1963), 1387–1399.

[26] Sherbert, D.R. – The structure of ideals and point derivations in Banach algebras of Lipschitz functions, *Trans. Amer. Math. Soc.,* 111 (1964), 240–272.

[27] Tukia, P. and Väisälä, J. – Quasisymmetric embeddings of metric spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.,* 5 (1980), 97–114.

H. Movahedi-Lankarani,
Department of Mathematics and Statistics, Penn State Altoona,
Altoona, PA 16601-3760 – USA
E-mail: hml@math.psu.edu

and

R. Wells,
Department of Mathematics, Penn State University,
University Park, PA 16802 – USA
E-mail: wells@math.psu.edu