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QUADRATIC CONTROL OF AFFINE DISCRETE-TIME, PERIODIC SYSTEMS WITH INDEPENDENT RANDOM PERTURBATIONS

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Abstract: In this paper we consider the affine discrete-time, periodic systems with independent random perturbations and we solve, under stabilizability and uniform observability or detectability conditions, the discrete time version of the quadratic control problem introduced in [1].

1 – Introduction

We consider the quadratic control problem for the affine discrete-time, periodic systems with independent random perturbations in Hilbert spaces (see [1] for continuous time case). The existence of an optimal control is connected with the behavior of the discrete-time Riccati equation associated with this problem. We study the asymptotic behavior of the solutions of the Riccati equation and we find an optimal control. In 1974 J. Zabczyk [10] treated a similar problem for time homogeneous systems and proved that, under stabilizability and detectability conditions, the Riccati equation (14) has a unique nonnegative bounded solution. In connection with this problem, he also introduced the notion of stochastic observability (which is equivalent, in the finite dimensional case, with the one

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considered in this paper). The case of time varying systems in finite dimensions has been investigated by T.Morozan in [6]. He proved that, under uniform observability and stabilizability conditions, the discrete-time Riccati equation has a unique, uniformly positive, bounded on $\mathbb{N}$ solution. In this paper we generalize the results of T. Morozan. We also establish that, in the stochastic case, the uniform observability does not imply the detectability and, consequently, our result is different from that obtained by J. Zabczyk in the time invariant case. In [1] G. Da Prato and I. Ichikawa proposed a quadratic control problem for affine periodic systems (for both deterministic and stochastic cases), which is a generalization of the average cost criterion, usually considered for time-invariant systems. They proved that, under stabilizability and detectability conditions, the optimal control is given by a periodic feedback, which involves the periodic solution of the Riccati equation associated to this problem. In [9] we consider differential linear stochastic equations. We replace the detectability condition with the uniform observability property and, under stabilizability condition, we prove that the Riccati equation has a unique, uniformly positive, bounded on $\mathbb{R}_+$ solution, which is stabilizing for the controlled system. This result can be used to find the optimal control and the optimal cost for the quadratic control problem. We also proved in [9] that, in the stochastic case, uniform observability does not imply detectability, as in the deterministic case, and our result is different from the one of G. Da Prato. On the other hand, we note that the observability property is easier to verify than the detectability condition, both in the continuous and deterministic cases. So, the results of this paper are (in a certain sense) the discrete-time versions of those obtained in [9] and [1] for the continuous case. They are not obtained by a simple discretization of the results mentioned above (for example the algebraic Riccati equation, involved in the time invariant quadratic control problem, is not the same in the discrete-time (see (32)) and continuous cases (see [1])). There are many technical differences between the discrete time and the continuous cases. For example, in the discrete time case, we used the induction to prove the existence of the solution of the Riccati equation with final condition, while in the continuous case we work with specific properties of the functions, which are continuously time dependent.

2 – Notations and statement of the problem

Let H, V, U be separable real Hilbert spaces and let us denote by $L(H,V)$ the Banach space of all bounded linear operators which transform H into V.

If $H = V$ we put $L(H,V) = L(H)$. We write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements and operators. If $A \in L(H)$ then A^* is the adjoint operator of A. The operator $A \in L(H)$ is said to be nonnegative and we write $A \geq 0$, if A is self-adjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We denote by $\mathcal{H}$ the Banach subspace of $L(H)$ formed by all self-adjoint operators, by $\mathcal{K}$ the cone of all nonnegative operators of $\mathcal{H}$ and by I the identity operator on H. We also consider the Banach space $C_b(H) = \{\varphi : H \to \mathbb{R},\ \varphi \text{ is bounded and continuous}\}$. Let $\tau \in \mathbb{N}, \tau > 1$. The sequence $L_n \in L(H,V), n \in \mathbb{N}$ is *bounded* on $\mathbb{N}$ if $\sup\limits_{n\in\mathbb{N}} \|L_n\| < \infty$ and is *τ-periodic* if $L_n = L_{n+\tau}$ for all $n \in \mathbb{N}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and ξ be a real or H -valued random variable on Ω. We write $E(\xi)$ for mean value (expectation) of ξ. We will use the notation $\mathcal{B}(H)$ for the Borel σ-field of H.

Let us consider the sequence $\xi_n, n \in \mathbb{Z}$ of real independent random variables, which satisfy the conditions $E(\xi_n) = 0$ and $E|\xi_n|^2 = b_n < \infty$. If $\mathcal{F}_n$ is the σ-algebra generated by $\{\xi_i, i \leq n-1\}$, then we will denote by $L^2_n(H) = L^2(\Omega, \mathcal{F}_n, P, H)$ the space of all equivalence class of H-valued random variables η (i.e. η is a measurable mapping from $(\Omega, \mathcal{F}_n)$ into $(H, \mathcal{B}(H))$) such that $E\|\eta\|^2 < \infty$. Analogously we define $L^2(\Omega, \mathcal{F}, P, H)$ and we denote it L^2.

We introduce the controlled system

$$(1) \qquad \begin{cases} x_{n+1} = A_n x_n + \xi_n B_n x_n + D_n u_n + f_n \\ x_k = x \in H,\ k \in \mathbb{N} \end{cases}$$

where $A_n,\ B_n \in L(H), D_n \in L(U,H)$. The control $\{u_k, u_{k+1}, ...\}$ belongs to the class $\widetilde{U}_k$ defined by the property that $u_n,\ n \geq k$ is an U-valued random variable, $\mathcal{F}_n$-measurable and $\sup\limits_{n\geq k} E\|u_n\|^2 < \infty$. For every $x \in H$ and $k \in \mathbb{N}$, fixed, we will denote by $U_{k,x}$ the subset of admissible controls from $\widetilde{U}_k$ with the property that (1) has a bounded solution.

If $f_n = 0$ for all $n \in \mathbb{N}$ we use the notation $\{A : D, B\}$ for the system (1). In the sequel we need the hypotheses:

H_0: The sequences $A_n, B_n \in L(H), D_n \in L(U,H), C_n \in L(H,V),\ K_n \in L(U),\ f_n, b_n, n \in \mathbb{N}$ are bounded on $\mathbb{N}$ and

$$(2) \qquad K_n \geq \delta I, \delta > 0 \text{ for all } n \in \mathbb{N}.$$

H_1: The sequences $A_n, B_n, D_n, C_n, K_n, f_n, n \in \mathbb{N}, b_n, n \in \mathbb{Z}$ introduced above are τ-periodic.

If H_0 (respectively H_1) holds we will use the notation $\widetilde{Z} = \sup_{n\in\mathbb{N}} \|Z_n\|$ (respectively $\widetilde{Z} = \max_{n\ =\ 0,1,...,\tau-1} \|Z_n\|$) for $Z = A, B, D, C, F, K, b$.

Assuming the hypotheses H_0, H_1 we study the following problem:

For every $k \in \mathbb{N}$ and $x \in H$, we look for an optimal control $u = \{u_k, u_{k+1}, ...\}$, which belongs to the class $U_{k,x}$ and minimizes the following quadratic cost

$$I_k(x, u) = \overline{\lim_{q\to\infty}} \frac{1}{q-k} E \sum_{n=k}^{q-1} [\|C_n x_n\|^2 + < K_n u_n, u_n >], \tag{3}$$

where x_n is the solution of (1) for all $n \in \mathbb{N}$, $n \geq k$. (It is clear that if $u \in U_{k,x}$ then $I_k(x, u) < \infty$).

We will establish that under stabilizability and uniform observability (or detectability) conditions (see Theorem 26) the optimal cost, given by (28), is obtained for the optimal control (29).

3 – Preliminaries

3.1. Properties of the solutions of the linear discrete time systems

We associate to (1) the linear stochastic system $\{A, B\}$

$$\begin{cases} x_{n+1} = A_n x_n + \xi_n B_n x_n \\ x_k = x \in H, n, k \in \mathbb{N}. \end{cases} \tag{4}$$

The random evolution operator of (4) is the operator $X(n, k)$ $n \geq k \geq 0$, where $X(k, k) = I$ and $X(n, k) = (A_{n-1} + \xi_{n-1} B_{n-1}) ... (A_k + \xi_k B_k)$, for all $\dot{n} > k$.

Definition 1. A sequence $\{x_n\}, n \in \mathbb{Z}$ of H-valued random variables is τ-periodic ($\tau \in \mathbb{N}, \tau > 1$) if

$$P\{x_{n_1+\tau} \in A_1, ..., x_{n_m+\tau} \in A_m\} = P\{x_{n_1} \in A_1, ..., x_{n_m} \in A_m\}, \tag{5}$$

for all $n_1, n_2, ..., n_m \in \mathbb{Z}$ and all $A_p \in \mathcal{B}(H)$, $p = 1, .., m$. □

It is known that (5) is equivalent with

$$E\varphi(x_{n_1+\tau}, ..., x_{n_m+\tau}) = E\varphi(x_{n_1}, .., x_{n_m}),$$

for all $\varphi \in C_b(H^m)$.

Remark 2. Assume that H_1 holds and the sequence $\{\xi_n\}, n \in \mathbb{Z}$ is τ-periodic. There exist the functions $F_{n,k} : \mathbb{R}^{n-k} \to H$ measurable $(\mathcal{B}(\mathbb{R}^{n-k}), \mathcal{B}(H))$ such that $X(n,k)x = F_{n,k}(\xi_{n-1}, ..., \xi_k), X(n+\tau, k+\tau)x = F_{n+\tau,k+\tau}(\xi_{n-1+\tau}, ..., \xi_{k+\tau})$ and $F_{n+\tau,k+\tau} = F_{n,k}$. Since the random variables $\xi_n, n \in \mathbb{Z}$ are independent and τ-periodic, then it follows that the random variables $X(n,k)x$ and $X(n+\tau, k+\tau)x$ have the same distribution function for all $n \geq k, n, k \in \mathbb{Z}$. □

If $x_n = x_n(k,x)$ is the solution of the system (4) then it is unique and $x_n(k,x) = X(n,k)x$.

It is not very difficult to see that we have the following lemma:

Lemma 3. *$X(n,k)$ is a bounded linear operator from $L_k^2(H)$ into $L_n^2(H)$ and we have*

$$E\left\|X(n,k)(\xi)\right\|^2 \leq (\|A_{n-1}\|^2 + b_{n-1}\|B_{n-1}\|^2)...(\|A_k\|^2 + b_k\|B_k\|^2)E\|\xi\|^2$$

for all $n > k$ and $\xi \in L_k^2(H)$.

From the above considerations it is clear that (1) has a unique solution $x_n(x,k;u)$. Using the induction it follows that $x_n(x,k;u)$ satisfies the relation

$$x_n(x,k;u) = X(n,k)x + \sum_{i=k}^{n-1} X(n,i+1)\left(D_i u_i + f_i\right) \tag{6}$$

for $n \geq k+1$. Moreover, $x_n(x,k;u)$ is $\mathcal{F}_n$-measurable and ξ_n-independent.

Now, we introduce the mappings $\mathcal{U}_n, T(n,k) : \mathcal{H} \to \mathcal{H}$

$$\begin{aligned} \mathcal{U}_n(S) &= A_n^* S A_n + b_n B_n^* S B_n, \\ T(n,k) &= U_k U_{k+1} ... U_{n-1}, \text{for all } n-1 \geq k \text{ and } T(k,k) = I, \end{aligned} \tag{7}$$

where $I \in L(\mathcal{H})$ is the identity operator. It is easy to see that $\mathcal{U}_n$ and $T(n,k)$ are linear and bounded operators.

Theorem 4 ([8])**.** *If $X(n,k)$ is the random evolution operator associated to (4), then $T(n,k)(\mathcal{K}) \subset \mathcal{K}$ and we have*

$$\langle T(n,k)(S)x, y\rangle = E\langle SX(n,k)x, X(n,k)y\rangle \tag{8}$$

for all $S \in \mathcal{H}$, $n \geq k \geq 0$ and $x, y \in H$. Moreover $\|T(n,k)(I)\| = \|T(n,k)\|$.

Remark 5. If H_1 holds then $T(n,k)$ is τ-periodic that means $T(n,k) = T(n+\tau, k+\tau)$ for all $n \geq k \geq 0$. □

3.2. Uniform exponential stability and uniform observability

Definition 6. We say that $\{A, B\}$ is uniformly exponentially stable iff there exist $\beta \geq 1$, $a \in (0,1)$ and $n_0 \in \mathbb{N}$ such that we have

$$E \left\| X(n,k)x \right\|^2 \leq \beta a^{n-k} \left\| x \right\|^2 \tag{9}$$

for all $n \geq k \geq n_0$ and $x \in H$. □

If $B_n = 0$ for all $n \in \mathbb{N}$, we obtain the definition of the uniform exponential stability of the deterministic system $x_{n+1} = A_n x_n, x_k = x \in H, n \geq k$ denoted $\{A\}$.

Definition 7 ([4])**.** The deterministic system $\{A\}$ is uniformly exponentially stable iff there exist $\beta \geq 1$, $a \in (0,1)$ and $n_0 \in \mathbb{N}$ such that we have $\|A_{n-1}A_{n-2}...A_k\| \leq \beta a^{n-k}$ for all $n \geq k \geq n_0$. □

It is easy to see that if $\{A, B\}$ is uniformly exponentially stable then (9) holds for $n_0 = 0$. The following result is known [4] for the finite dimensional case but it is presented for the readers' convenience.

Proposition 8. *If H_1 holds and $\{A\}$ is uniformly exponentially stable then the system*

$$y_n = A_n^* y_{n+1} + f_n \tag{10}$$

has a unique τ-periodic solution.

Proof: Using the condition required by the τ-periodic sequences we can extend the sequences A_n, f_n for all $n \in \mathbb{Z}$. Let us denote $Y(n,k) = A_k^* A_{k+1}^* ... A_{n-1}^*$ if $n \neq k$ and $Y(k,k) = I$ (the identity operator). If $\{A\}$ is uniformly exponentially stable then it is easy to see that there exist $a \in (0,1)$ and $\beta > 1$ such as $\|Y(n,k)^*\| = \|Y(n,k)\| \leq \beta a^{n-k}$. Hence the series $\sum\limits_{p=n}^{\infty} Y(p,n) f_p$ converges in H.

It is not difficult to see that $y_n = \sum_{p=n}^{\infty} Y(p,n) f_p$ satisfies (10). From H_1 it follows $Y(p+\tau, n+\tau) = Y(p,n)$ for all $p \geq n$ and consequently

$$y_{n+\tau} = \sum_{p=n+\tau}^{\infty} Y(p, n+\tau) f_{p+\tau} = \sum_{p=n}^{\infty} Y(p+\tau, n+\tau) f_p = y_n.$$

Thus y_n is a τ-periodic solution of (10). If $\overline{y}_n$ is another τ-periodic solution of (10) we have $\|y_{n+1} - \overline{y}_{n+1}\| \leq \|Y(n,k)\| \max_{k\ =\ 0,..,\tau-1} \|y_k - \overline{y}_k\| \leq \beta a^{n-k} \max_{k\ =\ 0,..,\tau-1} \|y_k - \overline{y}_k\|$. As $k \to -\infty$ we get $y_{n+1} = \overline{y}_{n+1}$ for all $n \in \mathbb{Z}$ and the proof is complete. ∎

Now we consider the discrete time stochastic system $\{A, B; C\}$ formed by the system (4) and the observation relation $z_n = C_n x_n$, where $C_n \in L(H, V), n \in \mathbb{N}$.

Definition 9 (see Definition 6 in [6])**.** We say that $\{A, B; C\}$ is uniformly observable if there exist $n_0 \in \mathbb{N}$ and $\rho > 0$ such that

$$\sum_{n=k}^{k+n_0} E \left\| C_n X(n,k) x \right\|^2 \geq \rho \left\| x \right\|^2 \tag{11}$$

for all $k \in \mathbb{N}$ and $x \in H$. □

If the stochastic perturbation is missing, that is $B_n = 0$ for all $n \in \mathbb{N}$, we will use the notation $\{A, \_; C\}$ for the observed (deterministic) system. We have the following definition of the deterministic uniform observability (see [3] and [2]).

Definition 10. We say that $\{A, \_; C\}$ is uniformly observable iff there exist $n_0 \in \mathbb{N}$ and $\rho > 0$ such that $\sum_{n=k}^{k+n_0} \left\| C_n A_{n-1} A_{n-2} ... A_k x \right\|^2 \geq \rho \left\| x \right\|^2$ for all $k \in \mathbb{N}$ and $x \in H$. □

Remark 11. It is not difficult to see that, in the time-invariant, finite dimensional case, the deterministic system $\{A, \_; C\}$ is uniformly observable iff $rank(C^*, A^*C^*, ..., (A^*)^{n-1} C^*) = n$, where n is the dimension of H. □

The following proposition is a consequence of the Theorem 4.

Proposition 12 ([8]).

a) *The system (4) is uniformly exponentially stable if and only if there exist* $\beta \geq 1$, $a \in (0,1)$ *and* $n_0 \in \mathbb{N}$ *such that we have*

$$\|T(n,k)\| \leq \beta a^{n-k} \tag{12}$$

for all $n \geq k \geq n_0$.

b) *The system* $\{A,B;C\}$ *is uniformly observable if and only if there exist* $n_0 \in \mathbb{N}$ *and* $\rho > 0$ *such that*

$$\sum_{n=k}^{k+n_0} T(n,k)(C_n^* C_n) \geq \rho I \tag{13}$$

for all $k \in \mathbb{N}$.

Conclusion 13. *From the above proposition it follows that if the deterministic system* $\{A, \_; C\}$ *is uniformly observable then the stochastic system* $\{A,B;C\}$ *is uniformly observable.*

Proposition 14. *Assume that* H_1 *holds,* $D_n = 0$ *for all* $n \in \mathbb{N}$ *and the sequence* $\{\xi_n\}, n \in \mathbb{Z}$ *is* τ*-periodic. If* $\{A,B\}$ *is uniformly exponentially stable then the system (1) (without initial condition) has a unique* τ*-periodic solution in* L^2.

Proof: As in the proof of the Proposition 8 we consider the system (1) on $\mathbb{Z}$. Let us consider the series $\sum\limits_{p=-\infty}^{n-1} X(n,p+1)f_p$ in the Hilbert space L^2. We have

$$\begin{aligned}\left\|\sum_{p=-\infty}^{n-1} X(n,p+1)f_p\right\|_{L^2} &\leq \sum_{p=-\infty}^{n-1} \|X(n,p+1)f_p\|_{L^2} \\ &= \sum_{p=-\infty}^{n-1} \sqrt{E\,\|X(n,p+1)f_p\|^2}\end{aligned}$$

If $T(n,k)$ is the operator associated to the system $\{A,B\}$ according to the Theorem 4, we deduce by Remark 5 and Proposition 12 that (12) holds for all $n \geq k > -\infty$.

Using Theorem 4 and the above considerations we get

$$\left\| \sum_{p=-\infty}^{n-1} X(n,p+1) f_p \right\|_{L^2} \leq \sum_{p=-\infty}^{n-1} \sqrt{\left\langle T(n,p+1) f_p, f_p \right\rangle}$$
$$\leq \sum_{p=-\infty}^{n-1} \beta^{1/2} a^{(n-p-1)/2} \widetilde{f} < \infty.$$

Consequently the series converges in L^2. We denote $y_n = \sum_{p=-\infty}^{n-1} X(n,p+1) f_p$. It is a simple exercise to verify that y_n satisfies (1). Now we will prove that it is a τ-periodic solution of (1). We consider the random variables $y_{n,m} = \sum_{p=m}^{n-1} X(n,p+1) f_p$ and $y_{n+\tau,m+\tau} = \sum_{p=m}^{n-1} X(n+\tau, p+\tau+1) f_p$.

Since $X(n,p+1)f_p$ and $X(n+\tau,p+\tau+1)f_p$ have the same distribution functions for all $n \geq p+1 > m$ it is clear that the distributions of $y_{n,m}$ and $y_{n+\tau,m}$ coincide.

Thus $E\varphi(y_{n,m}) = E\varphi(y_{n+\tau,m+\tau})$ for all $\varphi \in C_b(H)$. Since $\|y_{n,m} - y_n\|_{L^2} \underset{m \longrightarrow -\infty}{\longrightarrow} 0$ we deduce that there exists a subsequence y_{n,m_k} such that y_{n,m_k} converges to y_n $P.a.s$, as $k \longrightarrow \infty$.

Analogously, from $\|y_{n+\tau,m_k+\tau} - y_{n+\tau}\|_{L^2} \underset{m_k \longrightarrow -\infty}{\longrightarrow} 0$ it follows that there exists a subsequence $y_{n+\tau,m_{k_h}+\tau}$ such that $y_{n+\tau,m_{k_h}+\tau} \underset{h \longrightarrow \infty}{\longrightarrow} y_{n+\tau}$ $P.a.s.$ We consider now the last subsequence and we denote $y_{n+\tau,m_{k_h}+\tau} = y_{n+\tau,\widetilde{m}_h+\tau}$. It is clear that both sequences $y_{n+\tau,\widetilde{m}_h+\tau}, y_{n,\widetilde{m}_h}$ converges to their limit $P.a.s$ and we deduce that $\varphi(y_{n,\widetilde{m}_h}) \underset{h \longrightarrow \infty}{\longrightarrow} \varphi(y_n)$ (respectively $\varphi(y_{n+\tau,\widetilde{m}_h+\tau}) \underset{h \longrightarrow \infty}{\longrightarrow} \varphi(y_{n+\tau})$) $P.a.s$ for all $\varphi \in C_b(H)$. Using the Bounded Convergence Theorem it follows that $E\varphi(y_{n,\widetilde{m}_h}) \underset{h \longrightarrow \infty}{\longrightarrow} E\varphi(y_n)$ (respectively $E\varphi(y_{n+\tau,\widetilde{m}_h+\tau}) \underset{h \longrightarrow \infty}{\longrightarrow} E\varphi(y_{n+\tau})$). From Remark 2 we deduce that $y_{n,\widetilde{m}_h}$ and $y_{n+\tau,\widetilde{m}_h+\tau}$ have the same distribution function and $E\varphi(y_{n,\widetilde{m}_h}) = E\varphi(y_{n+\tau,\widetilde{m}_h+\tau})$. Hence $E\varphi(y_n) = E\varphi(y_{n+\tau})$ for all $\varphi \in C_b(H)$ and y_n, $y_{n+\tau}$ have the same distribution function. Using the same way of proof it can be shown that $y_{n_1}, y_{n_2}, ..., y_{n_m}$ and $y_{n_1+\tau}, y_{n_2+\tau}, ..., y_{n_m+\tau}$ for all $n_1, n_2, ..., n_m \in \mathbb{Z}$ have the same joint distribution functions and it follows that y_n is τ-periodic.

If $z_n \in L^2, n \in \mathbb{Z}$ is another τ-periodic solution of (1) then we have

$$E\left\|y_{n+1} - z_{n+1}\right\|^2 = E\left\langle T(n,k)\,(I)\,(y_k - z_k)\,, y_k - z_k\right\rangle$$

$$\leq \|T(n,k)\| \max_{p=0,..,\tau-1} E\left\|y_p - z_p\right\|^2 \leq \beta^{\frac{1}{2}} a^{\frac{n-k}{2}} \max_{p=0,..,\tau-1} E\left\|y_p - z_p\right\|^2$$

for all $n \geq k$. As $k \to -\infty$ we get $y_{n+1} = z_{n+1}$ $P.a.s$ for all $n \in \mathbb{Z}$ and the proof is complete. ∎

The above proposition is the infinite dimensional version of the statement i) of Theorem 3 from [5].

4 – Optimal quadratic control for affine discrete-time systems

In this section we assume that the hypothesis H_0 holds.

4.1. The discrete-time Riccati equation of stochastic control and the uniform observability

We consider the transformation

$$\mathcal{G}_n : \mathcal{K} \to \mathcal{K}, \mathcal{G}_n(S) = A_n^* S D_n (K_n + D_n^* S D_n)^{-1} D_n^* S A_n,$$

which is well defined. Let $\mathcal{U}_n \in L(\mathcal{H})$ be the linear operator defined by (7). We consider the following Riccati equation

$$R_n = \mathcal{U}_n(R_{n+1}) + C_n^* C_n - \mathcal{G}_n(R_{n+1}) \tag{14}$$

on $\mathcal{K}$, connected with the quadratic cost (3).

Definition 15. A sequence $\{R_n\}_{n\in\mathbb{N}}$, $R_n \in \mathcal{K}$ such as (14) holds is said to be a solution of the Riccati equation (14). □

We need the following definitions (see D.3 from [6]).

Definition 16. A solution $R = (R_n)_{n\in\mathbb{N}}$ of (14) is said to be stabilizing for $\{A : D, B\}$ if $\{A + DF, B\}$ with

$$F_n = -(K_n + D_n^* R_{n+1} D_n)^{-1} D_n^* R_{n+1} A_n, n \in \mathbb{N} \tag{15}$$

is uniformly exponentially stable. □

Definition 17 ([6])**.** The system $\{A : D, B\}$ is stabilizable if there exists a bounded on $\mathbb{N}$ sequence $F = \{F_n\}_{n\in\mathbb{N}}, F_n \in L(H, U)$ such that $\{A + DF, B\}$ is uniformly exponentially stable. □

Proposition 18. *The Riccati equation (14) has at most one stabilizing and bounded on $\mathbb{N}$ solution.*

Proof: Let $R_{n,1}$ and $R_{n,2}$ be two stabilizing and bounded on $\mathbb{N}$ solutions of equation (14). We introduce the systems

$$\left\{\begin{array}{c} x_{n+1,i} = (A_n + D_n F_{n,i}) x_{n,i} + \xi_n B_n x_{n,i} \\ x_{k,i} = x \in H \end{array}\right. \tag{16}$$

for all $n \geq k$, $n, k \in \mathbb{N}$, where $F_{n,i} = -(K_n + D_n^* R_{n+1,i} D_n)^{-1} D_n^* R_{n+1,i} A_n$, $i = 1, 2$. If we denote $Q_n = R_{n,1} - R_{n,2}$, we get

$$E\left\langle Q_{n+1} x_{n+1,1}, x_{n+1,2}\right\rangle = E\left\langle Q_n x_{n,1}, x_{n,2}\right\rangle, \text{ for all } n \geq k.$$

It is easy to see that $E\left\langle Q_{n+1} x_{n+1,1}, x_{n+1,2}\right\rangle = \langle Q_k x, x\rangle$ for all $n \geq k$, $x \in H$.

Since $R_{n,i}, i = 1, 2$ are bounded on $\mathbb{N}$ we deduce that there exists $M > 0$ such that $\|Q_n\| \leq M$ for all $n \in \mathbb{N}$. Thus,

$$0 \leq |\langle Q_k x, x\rangle| \leq M\sqrt{E\left\|x_{n+1,1}\right\|^2 E\left\|x_{n+1,2}\right\|^2}.$$

From the hypothesis and from the Definition 17 it follows that the systems (16) are uniformly exponentially stable and $E\left\|x_{n+1,i}\right\|^2 \underset{n\to\infty}{\rightarrow} 0$, $i = 1, 2$, uniformly with respect to x.

As $n \to \infty$ in the last inequality, we deduce that $Q_k = 0$ and $R_{k,1} = R_{k,2}$ for all $k \in \mathbb{N}$. The proof is complete. ∎

Let x_n be the solution of system $\{A : D, B\}$. By $U_{k,M}, M \in \mathbb{N}^*$ we denote the set of all finite sequences $u_k^M = \{u_k, u_{k+1}, ...u_{M-1}\}$ of U-valued and $\mathcal{F}_i$ measurable random variables u_i, $i = k, ..., M-1$ with the property $E\left\|u_i\right\|^2 < \infty$. Now, we introduce the performance

$$V(M, k, x, u) = E\sum_{n=k}^{M-1}[\|C_n x_n\|^2 + < K_n u_n, u_n >].$$

Let us consider the sequence $R(M, M) = 0 \in \mathcal{K}$,

$$R(M, n) = \mathcal{U}_n(R(M, n+1)) + C_n^* C_n - \mathcal{G}_n(R(M, n+1))$$

for all $n \leq M-1$.

The following lemma prove that the sequence $R(M, n)$ is well defined for all $0 \leq n \leq M$. It is called the *solution of the Riccati equation (14) with the final condition* $R(M, M) = 0$.

Lemma 19.

a) *$R(M,n)\in\mathcal{K}$ for all $0\leq n\leq M$;*

b) *$0\leq R(M-1,n)\leq R(M,n)$ for all $0\leq n\leq M-1$.*

Moreover, if H_1 is satisfied then

$$R(M+\tau,n+\tau)=R(M,n),0\leq n\leq M. \tag{17}$$

Proof: We will prove the first assertion by induction. For $n=M$, $R(M,n)=0\in\mathcal{K}$. Let us assume $R(M,n)\in\mathcal{K}$ for all $n\in\mathbb{N},k<n\leq M$.

We will prove $R(M,k)\in\mathcal{K}$. Let x_n be the solution of system $\{A:D,B\}$ with the initial condition $x_k=x$ and let us denote $F_n=-[K_n+D_n^*R(M,n+1)D_n]^{-1}D_n^*R(M,n+1)A_n$ and $z_n=u_n-F_nx_n$. We have

$$\begin{aligned}
&E\left\langle R(M,n+1)x_{n+1},x_{n+1}\right\rangle=\\
&E\left\langle R(M,n)x_n,x_n\right\rangle-E\left\langle C_n^*C_nx_n,x_n\right\rangle-E\left\langle K_nu_n,u_n\right\rangle+\\
&E\left\langle (K_n+D_n^*R(M,n+1)D_n)z_n,z_n\right\rangle.
\end{aligned}$$

Now, we consider the last equality for $n=k,k+1,...,M-1$ and summing, we obtain

$$\begin{aligned}
V(M,k,x,u)\;=\;&\left\langle R(M,k)x,x\right\rangle+\\
&E\sum_{n=k}^{M-1}\left\langle (K_n+D_n^*R(M,n+1)D_n)z_n,z_n\right\rangle
\end{aligned} \tag{18}$$

Let $\widetilde{x}_n$ be the solution of system

$$\begin{cases} x_{n+1}=(A_n+D_nF_n)x_n+\xi_nB_nx_n \\ x_k=x\in H \end{cases}, \tag{19}$$

where F_n was introduced above.

It is clear that $\widetilde{x}_n$ is also the solution of $\{A:D,B\}$ with $\widetilde{u}_n=F_n\widetilde{x}_n$, $k\leq n\leq M-1$ and $\{\widetilde{u}_n,k\leq n\leq M-1\}\in U_{k,M}$.

Thus we obtain, for all $0\leq k<M$

$$\min_{u\in U_{k,M}}V(M,k,x,u)=V(M,k,x,\widetilde{u})=\left\langle R(M,k)x,x\right\rangle. \tag{20}$$

We deduce that $R(M,k)\geq 0$ and the induction is complete.

b) Let $u_k^{M-1}=\{\widetilde{u}_k,\widetilde{u}_{k+1},...,\widetilde{u}_{M-2}\}$. It is clear that $u_k^{M-1}\in U_{k,M-1}$ and from the definition of $V(M,k,x,u)$ we get $V(M-1,k,x,u)\leq V(M,k,x,\widetilde{u})$.

If we consider (20) for $M-1$, we have, for all $0 \le k < M$

$$\langle R(M-1,k)x,x\rangle = \min_{\widehat{u}\in U_{k-1,M}} V(M-1,k,x,\widehat{u}) \le V(M-1,k,x,u).$$

From (20) and the last inequalities it follows the conclusion. The proof of last statement is trivial. ∎

Proposition 20. *Assume that $\{A : D, B\}$ is stabilizable. Then the Riccati equation (14) admits a bounded on $\mathbb{N}$ solution. If H_1 is satisfied then the solution of the Riccati equation is τ-periodic.*

Proof: Since $\{A : D, B\}$ is stabilizable it follows that there exists a bounded on $\mathbb{N}$ sequence $F = \{F_n\}_{n\in\mathbb{N}}$, $F_n \in L(H,U)$ such that $\{A + DF, B\}$ is uniformly exponentially stable.

Let us consider $\overline{u}_n = F_n x_n$, where x_n is the solution of $\{A + DF, B\}$ with the initial condition $x_k = x$. Since F_n is bounded on $\mathbb{N}$, it is not difficult to see that $\overline{u}_n \in \widetilde{U}_k$. We have

$$V(M,k,x,\overline{u}) \le \eta \sum_{n=k}^{\infty} E\|x_n\|^2$$

for all $M > k$, where $\eta = \widetilde{C}^2 + \widetilde{K}\widetilde{F}^2$. Since $\{A+DF, B\}$ is uniformly exponentially stable, it is not difficult to see that there exists $\lambda_1 > 0$ such that $V(M,k,x,\overline{u}) \le \eta\lambda_1 \|x\|^2 = \lambda \|x\|^2, x \in H$.

Let $R(M,n)$ be the solution of the Riccati equation (14) with $R(M,M) = 0$. Using (20) and the above inequality, we deduce that

$$\langle R(M,k)x,x\rangle \le \lambda \|x\|^2 .$$

Using Lemma 19 it follows that there exists $R(k) \in L(H)$ such that $0 \le R(M,k) \le R(k) \le \lambda I$ for $M \in \mathbb{N}$, $M \ge k$ and the sequence $\{R(M,k)\}_{M\in\mathbb{N},M\ge k}$ converges to $R(k)$ in the strong operator topology.

We denote $L = \lim\limits_{M\to\infty} (< \mathcal{G}_n(R(M,n+1))x, x > - < \mathcal{G}_n(R(n+1))x, x >)$ and $P_{M,n} = K_n + D_n^* R(M,n+1)D_n$, $P_n = K_n + D_n^* R(n+1)D_n$. If

$$\begin{aligned} L_1 &= \lim_{M\to\infty} \left\|P_{M,n}^{-1}\right\| \|D_n^* R(M,n+1)A_n x - D_n^* R(n+1)A_n x\| \cdot \\ &\quad \|D_n^* R(M,n+1)A_n x + D_n^* R(n+1)A_n x\| \text{ and} \\ L_2 &= \lim_{M\to\infty} \left\langle \left(P_{M,n}^{-1} - P_n^{-1}\right) D_n^* R(n+1)A_n x, D_n^* R(n+1)A_n x \right\rangle, \end{aligned}$$

then

$$|L| \leq L_1 + L_2$$

Since $P_{M,n} \geq K_n \geq \delta I$, $\delta > 0$ we deduce that $\left\| P_{M,n}^{-1} \right\| \leq \frac{1}{\delta}$ for all $M \geq n+1 \geq k$ and from the strong convergence of $\{R(M,n)\}_{M\in\mathbb{N},\, M\geq n}$ it follows $L_1 = 0$.

We see that $\left\| P_{M,n}^{-1}x - P_n^{-1}x \right\| \leq \left\| P_{M,n}^{-1} \right\| \|P_{M,n}y - P_n y\|$, where $y = P_n^{-1}x$. Since $\lim_{M\to\infty} \|P_{M,n}y - P_n y\| = 0$ we get $\lim_{M\to\infty} \left\| P_{M,n}^{-1}x - P_n^{-1}x \right\| = 0$.

Now it is clear that $L_2 = 0$. Hence $L = 0$ and

$$\lim_{M\to\infty} \langle \mathcal{G}_n(R(M, n+1))x, x\rangle = \langle \mathcal{G}_n(R(n+1))x, x\rangle .$$

From the definition of $R(M,n)$ and the above result we deduce that $R(n)$ is a solution of (14). If H_1 holds then we take $M \to \infty$ in (17) and it follows that $R(n)$ is τ-periodic. ∎

Theorem 21. *Let us assume that the system $\{A, B; C\}$ is uniformly observable. If R_n is a nonnegative bounded on $\mathbb{N}$ solution of (14) then:*

a) *there exist $m > 0$ such that $R_n \geq mI$, for all $n \in \mathbb{N}$ (R_n is uniformly positive on $\mathbb{N}$).*

b) *R_n is stabilizing for (1).*

Proof: The main idea is the one of [6] .

Let R_n be a nonnegative, τ-periodic solution of (14) and let $\widetilde{X}(n,k)$ be the random evolution operator associated to system $\{A + DF, B\}$ with

$$F_n = -(K_n + D_n^* R_{n+1} D_n)^{-1} D_n^* R_{n+1} A_n. \tag{21}$$

Let n_0 and ρ be the number introduced by the Definition 9. We have (see the proof of Lemma 19)

$$\begin{aligned} \langle T_n x, x\rangle &= \langle R_n x, x\rangle - \\ &\quad E\left\langle R_{n_0+n+1}\widetilde{X}(n_0+n+1,n)x, \widetilde{X}(n_0+n+1,n)x \right\rangle, \end{aligned} \tag{22}$$

where the operator $T_n \in \mathcal{H}$ is

$$\langle T_n x, x\rangle = \sum_{j=n}^{n+n_0} (E\left\| C_j \widetilde{X}(j,n)x \right\|^2 + E\left\langle K_j F_j \widetilde{X}(j,n)x, F_j \widetilde{X}(j,n)x \right\rangle). \tag{23}$$

From (6), we deduce that for all $j \geq n+1$ we have

$$\widetilde{X}(j,n)x = X(j,n)x + \sum_{i=n}^{j-1} X(j,i+1)D_i\widetilde{u}_i,$$

where $\widetilde{u}_i = F_i\widetilde{X}(i,n)x$ and $X(n,k)$ is the random evolution associated to $\{A,B\}$. Thus

$$\begin{aligned} \langle T_n x, x\rangle &> \sum_{j=n+1}^{n+n_0} E\left\| C_jX(j,n)x + C_j\sum_{i=n}^{j-1} X(j,i+1)D_i\widetilde{u}_i \right\|^2 + \frac{1}{2}\|C_n x\|^2 \\ &\geq \frac{1}{2}(\sum_{j=n}^{n+n_0} E\|C_jX(j,n)x\|^2) - \widetilde{C}^2\sum_{j=n+1}^{n+n_0} E\left\|\sum_{i=n}^{j-1} X(j,i+1)D_i\widetilde{u}_i\right\|^2. \end{aligned}$$

Using H_0, Lemma 3 and (23) it follows

$$E\left\|\sum_{i=n}^{j-1} X(j,i+1)D_i\widetilde{u}_i\right\|^2 \leq \widetilde{D}^2\mu_{n_0}\sum_{i=n}^{n+n_0} E\left\|F_i\widetilde{X}(i,n)x\right\|^2 \leq c\langle T_n x, x\rangle,$$

where $\mu_{n_0} = \left(n_0\max\{1,(\widetilde{A}^2+\widetilde{b}\widetilde{B}^2)^{n_0}\}\right)$ and $c = \frac{\widetilde{D}^2\mu_{n_0}}{\delta}$.

Since the system $\{A,B;C\}$ is uniformly observable then we have

$$\langle T_n x, x\rangle > \frac{1}{2}\rho\|x\|^2 - \widetilde{C}^2 n_0 c\langle T_n x, x\rangle).$$

From the last equality and from the hypothesis we deduce that there exist $M > m$ such that

$$m\|x\|^2 \leq \langle T_n x, x\rangle \leq \langle R_n x, x\rangle \leq M\|x\|^2. \tag{24}$$

We obtain from (22) and (24)

$$\begin{aligned} &-m/M\langle R_n x, x\rangle \geq -\langle R_n x, x\rangle + \\ &E\left\langle R_{n_0+n+1}\widetilde{X}(n_0+n+1,n)x, \widetilde{X}(n_0+n+1,n)x\right\rangle. \end{aligned}$$

Thus $E\left\langle R_{n_0+n+1}\widetilde{X}(n_0+n+1,n)x, \widetilde{X}(n_0+n+1,n)x\right\rangle \leq q\langle R_n x, x\rangle$ for all $n \in \mathbb{N}$ and $x \in H$, where $q = 1 - m/M, q \in (0,1)$.

Let $\widetilde{T}(n,k)$ be the operator introduced by Theorem 4 for the system $\{A + DF, B\}$, where the sequence F_n is given by (21). Then the previous inequality can be written

$$\widetilde{T}(n_0+n+1,n)(R_{n_0+n+1}) \leq qR_n.$$

Since $\widetilde{T}(n,k)$ is monotone ($\widetilde{T}(n,k)(P) \leq \widetilde{T}(n,k)(R)$ for $P \leq R$, $n \geq k$) we deduce from (24) that $\widetilde{T}(n,k)\left(\widetilde{T}(n_0+n+1,n)\left(R_{n_0+n+1}\right)\right) \leq q\widetilde{T}(n,k)(R_n)$ and $\widetilde{T}(n_0+n+1,k)\left(R_{n_0+n+1}\right) \leq q\widetilde{T}(n,k)(R_n)$ for all $n \geq k$.

Let $n \geq k$ arbitrary. Then there exists $c, r \in \mathbb{N}$ such that $n-k = (n_0+1)c+r$ and $0 \leq r \leq n_0$. We obtain by induction:

$$\widetilde{T}(n,k)(R_n) \leq q^c\widetilde{T}(r+k,k)(R_r).$$

From (24) and Theorem 4 we get $m\widetilde{T}(n,k)(I) \leq Mq^c\left\|\widetilde{X}(r+k,k)\right\|^2 I$.

Using Lemma 3 we put $G = M \max\limits_{0\leq r\leq n_0}\{(\widetilde{A}^2+\widetilde{b}\widetilde{B}^2)^r\}$ and we get $m\widetilde{T}(n,k)(I) \leq q^cGI$. Now we take $a = q^{1/(n_0+1)}$, $b = q^{-n_0/(n_0+1)}(G/m) \geq 1$ and it follows $\widetilde{T}(n,k)(I) \leq ba^{n-k}I$.

From Theorem 4 we deduce $E\left\|\widetilde{X}(n,k)x\right\|^2 \leq ba^{n-k}\left\|x\right\|^2$ for all $x \in H$ and $0 \leq k \leq n$, $k, n \in \mathbb{N}$. Therefore R_n is stabilizing for (1). The proof is complete. ∎

Now, we can state the main result of this section.

Theorem 22. *Assume that*

1) *the system $\{A : D, B\}$ is stabilizable and*

2) *the system $\{A, B; C\}$ is uniformly observable.*

Then the Riccati equation (14) admits a unique uniformly positive, bounded on $\mathbb{N}$ and stabilizing solution. Moreover, if H_1 holds then the solution of the Riccati equation is τ-periodic.

Proof: From the Proposition 20 and the assumption 1) we deduce that (14) admits a nonnegative, bounded on $\mathbb{N}$ (or τ-periodic, if H_1 holds) solution. Now, using the above theorem and 2), we deduce that this solution is stabilizing. A stabilizing and bounded on $\mathbb{N}$ solution of the Riccati equation is unique by Proposition 18. The proof is complete. ∎

The above theorem is proved in [6] for the discrete time stochastic systems in finite dimensional spaces. The continuous case, for stochastic systems on infinite dimensional spaces, is treated in [9].

Definition 23. *The system $\{A, B; C\}$ is detectable if there exists a bounded on $\mathbb{N}$ sequence $P = \{P_n\}_{n\in\mathbb{N}}, P_n \in L(U, H)$ such that $\{A + PC, B\}$ is uniformly exponentially stable.* □

The next result is the infinite dimensional version of Proposition 9 from [7], where we replace the Markov perturbations with independent random perturbations. So, it can be proved similarly the following proposition.

Proposition 24. *If $\{A, B; C\}$ is detectable then every nonnegative bounded solution of (14) is stabilizing.*

Now, it is clear that if we replace the observability condition in Theorem 22 with the detectability property we deduce that the Riccati equation (14) has a unique nonnegative, bounded on $\mathbb{N}$ (τ-periodic, if H_1 holds) and stabilizing solution. The obtained result is already known for the time invariant case (see [10]) and for the continuous, time-varying case (see [1]).

We only will prove that observability does not imply detectability and it follows that our result is different to those mentioned above. Before to give the counter-example, which will solve this problem we need the following remarks.

Remark 25. Let us consider the time invariant case $A_n = A$, $B_n = B$, $b_n = b$, $C_n = C$ and $K_n = I$. It is not difficult to see that, in the finite dimensional case, the system $\{A, B; C\}$ is detectable if and only if the controlled system $\{A^* : C^*, B^*\}$ is stabilizable. Thus, it follows from Proposition 20 that if $\{A, B; C\}$ is detectable then the Riccati equation (14), where we replace the operators A with A^*, B with B^*, C with I, and D with C^* has a nonnegative bounded on $\mathbb{N}$ solution. Using Lemma 3.1 from [11] we deduce that the Riccati equation associated to the above detectable system becomes

$$R_n = \mathcal{A}(R_{n+1}) \tag{25}$$

where $\mathcal{A}\colon \mathcal{K} \to \mathcal{K}$ $\mathcal{A}(S) = bBSB^* + I + AS(I + C^*CS)^{-1}A^*$. By Proposition 20 it follows that if the system $\{A, B; C\}$ is detectable then the algebraic Riccati equation

$$R = \mathcal{A}(R) \tag{26}$$

has a nonnegative solution. □

The following counter-example prove that the stochastic observability doesn't imply detectability.

Counter-example

Let us consider the stochastic system $\{A, B; C\}$ where $H = \mathbb{R}^2, V = \mathbb{R}$ ($\mathbb{R}^2$ is the real 2-dimensional space), $A_n = A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, C_n = C = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $b_n = 1$ and $B_n = B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for all $n \in \mathbb{N}$.

Since $rank(C^*, A^*C^*) = 2$ then the deterministic system $\{A; C\}$ [3] is observable. Therefore (see Conclusion 13) the stochastic system $\{A, B; C\}$ is uniformly observable. It is easy to see that if we look for a solution $K = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$ of (26), which satisfies the conditions $x_1 x_3 \geq x_2^2, x_1 \geq 0$, we obtain $x_2^2 = 3x_3 x_1 + 3x_3 + x_1 + 1 \geq 3x_2^2 + 1$, that is impossible. Thus the equation (26) has not a nonnegative solution. Then, from Remark 25, we deduce that $\{A, B; C\}$ cannot be detectable.

4.2. Optimal quadratic control and the uniform observability

The following theorem gives the optimal control, which minimize the cost function (3). Let H_0 and H_1 hold.

Theorem 26. *Assume that the hypothesis 1) of the Theorem 22 holds and the system $\{A, B; C\}$ is either uniformly observable or detectable. Let R_n be the unique solution of Riccati equation (14). If g_n is the unique τ-periodic solution of the Lyapunov equation*

$$g_n = (A_n + D_n F_n)^* g_{n+1} + R_n f_{n-1} \tag{27}$$

where F_n is given by (21), then

$$\begin{aligned} \min_{u \in U_{k,x}} I_k(x, u) &= I(x, \widetilde{u}) \\ &= \frac{1}{\tau} \sum_{i=0}^{\tau-1} [2 \langle g_{i+1}, f_i \rangle - \left\| V_i^{-1/2} D_i^* g_{i+1} \right\|^2 - \langle R_{i+1} f_i, f_i \rangle], \end{aligned} \tag{28}$$

where the optimal control is

$$\widetilde{u}_n = -V_n{}^{-1} D_n^* \left(R_{n+1} A_n \widetilde{x}_n + g_{n+1} \right), \tag{29}$$

$n \geq k \geq 0$, $\widetilde{x}_n$ is the solution of the system (1) and $V_n = K_n + D_n^* R_{n+1} D_n$.

Proof: First we note that if the above hypotheses hold, then the Riccati equation (14) has a unique nonnegative, bounded on $\mathbb{N}$ and stabilizing solution R_n, according Theorem 22 and Proposition 24. Since the Riccati equation is stabilizing we can apply Proposition 12 to deduce that $\{A+DF\}$ (see Definition 7) is uniformly exponentially stable, where $F_{n,\ n\in\mathbb{N}}$ is given by (21). Using the Proposition 8 it follows that (27) has a unique τ-periodic solution. Let x_n be the solution of the system (1) and let us consider the function

$$v_n : H \to \mathbb{R}, v_n(x) = \langle R_n x, x\rangle + 2\langle g_{n+1}, (A_n + D_n F_n)x\rangle .$$

Arguing as in the proof of Lemma 19 we have

$$\begin{aligned} Ev_{n+1}(x_{n+1}) \;=\; & Ev_n(x_n) - E[\|C_n x_n\|^2 + \langle K_n u_n, u_n\rangle]+ \\ & E\langle V_n(u_n - F_n x_n), u_n - F_n x_n\rangle + \\ & 2E\langle D_n^* g_{n+1}, u_n - F_n x_n\rangle + 2\langle g_{n+1}, f_n\rangle - \langle R_{n+1} f_n, f_n\rangle . \end{aligned}$$

If we put $a_n = V_n^{-1} D_n^* g_{n+1} + u_n - F_n x_n$ we have

$$\begin{aligned} E\langle V_n a_n, a_n\rangle - E\left\|V_n^{-1/2} D_n^* g_{n+1}\right\|^2 \;=\; & E\langle V_n(u_n - F_n x_n), (u_n - F_n x_n)\rangle + \\ & 2E\langle D_n^* g_{n+1}, u_n - F_n x_n\rangle . \end{aligned}$$

Hence we deduce that

$$\begin{aligned} Ev_{n+1}(x_{n+1}) \;=\; & Ev_n(x_n) - E[\|C_n x_n\|^2 + \langle K_n u_n, u_n\rangle] + E\langle V_n a_n, a_n\rangle - \\ & \left\|V_n^{-1/2} D_n^* g_{n+1}\right\|^2 + 2\langle g_{n+1}, f_n\rangle - \langle R_{n+1} f_n, f_n\rangle . \end{aligned} \tag{30}$$

Let $\widetilde{x}_n$ be the solution of system (1), where $\widetilde{u}_n = F_n\widetilde{x}_n - V_n^{-1} D_n^* g_{n+1}$. It is not difficult to see that $\widetilde{x}_n$ and $\widetilde{u}_n$ are bounded on $\mathbb{N}$. Thus $\widetilde{u} \in U_{k,x}$.

Using (30) we get

$$\begin{aligned} \frac{1}{n-k}[v_k(x) - Ev_{n+1}(\widetilde{x}_{n+1})] \;=\; & \frac{1}{n-k}\sum_{i=k}^{n-1} E[\|C_i\widetilde{x}_i\|^2 + \langle K_i\widetilde{u}_i, \widetilde{u}_i\rangle] - \\ & 2\langle g_{i+1}, f_i\rangle + \left\|V_i^{-1/2} D_i^* g_{i+1}\right\|^2 + \langle R_{i+1} f_i, f_i\rangle . \end{aligned} \tag{31}$$

Since g_n, R_n are τ periodic and R_n is stabilizing we deduce that there exists $P > 0$ such that $Ev_{n+1}(\widetilde{x}_{n+1}) \leq P$ for all $n \in \mathbb{N}$.

As $n \to \infty$ in (31), it follows

$$I_k(x, \widetilde{u}) = \overline{\lim_{n\to\infty}}\, \frac{1}{n-k}\sum_{i=k}^{n-1} [2\langle g_{i+1}, f_i\rangle - \left\|V_i^{-1/2} D_i^* g_{i+1}\right\|^2 - \langle R_{i+1} f_i, f_i\rangle]$$

Thus

$$\min_{u\in U_{i,x}} I_k(x,u) \leq I_k(x,\widetilde{u}) = \overline{\lim_{n\to\infty}} \frac{1}{n-k}\sum_{i=k}^{n-1}[2\langle g_{i+1}, f_i\rangle -$$
$$\left\|V_i^{-1/2}D_i^* g_{i+1}\right\|^2 - \langle R_{i+1}f_i, f_i\rangle].$$

If $u \in U_{k,x}$ it is not difficult to deduce from (30) that $I_k(x,u) \geq I_k(x,\widetilde{u})$.

Thus $\min\limits_{u\in U_{i,x}} I_k(x,u) = I_k(x,\widetilde{u})$. Using H_1 we see that for $n = p\tau + k$ then $I_k(x,\widetilde{u}) = \frac{1}{\tau}\sum\limits_{i=0}^{\tau-1}[2\langle g_{i+1}, f_i\rangle - \left\|V_i^{-1/2}D_i^* g_{i+1}\right\|^2 - \langle R_{i+1}f_i, f_i\rangle]$ and the conclusion follows. ∎

From the above theorem it follows that the optimal cost does not depend on the initial condition. It is not difficult to see that the conclusions of the above theorem stay true if we consider the initial condition $x_k = \xi \in L_k^2(H)$. Thus, using Proposition 14 we have the following result:

Proposition 27. *If the hypothesis of the Theorem 26 holds and the sequence* $\{\xi_n\}, n \in \mathbb{Z}$ *is τ-periodic, then the optimal cost is given by (28) and the optimal control is (29), where* $\widetilde{x}_n = \sum\limits_{p=-\infty}^{n-1}\widetilde{X}(n,p+1)\left(f_p - D_pV_p^{-1}D_p^* g_{p+1}\right), g_p$ *is the τ-periodic solution of (27) and* $\widetilde{X}(n,k), n \geq k$ *is the random evolution operator associated with the system* $\{A + DF, B\}$ *considered on* $\mathbb{Z}$.

The time invariant case

In this subsection we work under the hypotheses H_0 and

H_2: $\mathcal{Z}_n = \mathcal{Z}$ for all $n \in \mathbb{N}$ (or $n \in \mathbb{Z}$) and $\mathcal{Z} = A, B, D, C, F, K, b, f$.

We consider the algebraic Riccati equation

$$R = \mathcal{U}(R) + C^*C - \mathcal{G}(R), \tag{32}$$

where $\mathcal{U}(R) = A^*RA + bB^*RB$ and $\mathcal{G}(R) = A^*RD(K + D^*RD)^{-1}D^*RA$.

Remark 28. It is easy to see that if the hypotheses 1) and 2) of the Theorem 22 hold then

a) the algebraic equation (32) has a unique positive solution;

b) the system (10) has a unique time-invariant solution given by

$$g = \sum_{p=0}^{\infty} (A^* + F^* D^*)^p f. \quad \square \tag{33}$$

Corollary 29. *If the hypotheses of the Theorem 26 are verified then the Riccati equation (32) has a unique nonnegative solution R and the optimal cost is*

$$I(\widetilde{u}) = 2\langle g, f\rangle - \left\| V^{-1/2} D^* g \right\|^2 - \langle Rf, f\rangle, \tag{34}$$

where g and the optimal control $\widetilde{u}$ are given by (33) respectively (29) and $V = K + D^*RD$.

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