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# ORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE NEHARI PROBLEM

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**Abstract:** The aim of this note is to make explicit the connection between the Nehari problem and certain class of orthogonal polynomials on the unit circle obtained in a recursive way. A sequence of Schur-type parameters for this problem is also obtained.

## Introduction

The well-known Nehari interpolation problem can be stated as follows:

Given a sequence $\{1 = s_0, s_1, \ldots, s_n, \ldots\}$ of complex numbers, find a necessary and sufficient condition for the existence of a measurable bounded function $f$ in the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ such that

$$\widehat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t)\, dt = s_n, \quad \text{for } n \geq 0.$$

This problem was first solved by Nehari [10] but a theorem due to Adamjan, Arov and Krein [1] parametrizes the set of all the solutions.

The approach given by Cotlar and Sadosky [7] allows to generalize the Nehari problem by using the fruitful notion of generalized Toeplitz kernels. In this setting, the Nehari problem has a solution if and only if the kernel $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$

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defined by

$$K(m,n) = \begin{cases} s_{m-n} & \text{if } m \geq 0, n < 0 \\ \overline{s_{n-m}} & \text{if } m < 0, n \geq 0 \\ \delta_{m,n} & \text{elsewhere} \end{cases} \tag{1}$$

is positive definite.

The Nehari problem can be seen as a generalization of the Carathéodory-Fejér problem and also a modification of the Schur algorithm provides in a recursive way the set of all the solutions of the problem [3]. These problems are also related to the theory of orthogonal polynomials on $\mathbb{T}$ (see [6] and the references therein).

In several papers (see for example [2], [5], [9]), basic results on interpolation theory have been exposed from the point of view of the Schur analysis and extended to the domain of the orthogonal polynomials. Landau's approach suggests to define a scalar product in the space of trigonometric polynomials and apply an orthogonal decomposition. In this way, many aspects about orthogonality, coefficients of reflection, prediction formulas, and so on, are highlighted.

Here we show that certain results concerning the Nehari problem can also be deduced from an orthogonal decomposition in a finite-dimensional space. More precisely, we show a recursive method in order to obtain the solutions of the reduced Nehari problem, by using a family of Schur-type parameters. Therefore, we will study a method to find a recurrence formula for the sequence $\{s_n\}$, given the $n$ first terms and the positive definite kernel $K$ defined in (1).

In section 1 we adapt the well-known results about orthogonal polynomials in the unit circle to the case of the generalized Toeplitz kernel in a slightly different way from that one stated by Gohberg and Landau [8].

## 1 – Orthogonal polynomials

In the sequel, given the sequence $\{1 = s_0, s_1, \ldots, s_n\}$ and the generalized Toeplitz kernel K defined in (1), we will consider the matrix

$$C_n = [K(i,j)]_{i,j=-1,0,\ldots,n-1} = \begin{pmatrix} 1 & \overline{s_1} & \overline{s_2} & \ldots & \overline{s_n} \\ s_1 & 1 & 0 & \ldots & 0 \\ s_2 & 0 & 1 & & \\ \vdots & & & \ddots & \\ s_n & 0 & & \ldots & 1 \end{pmatrix}$$

and denote $\Delta_n = \det C_n$. By the way, the matrix $C_n$ is Hermitian but it is not a Toeplitz matrix unlike the case of the Carathéodory-Fejér problem.

Since $K$ is positive definite, $\Delta_n > 0$ i.e. $1 - \sum_{i=1}^{n} |s_i|^2 > 0$ and, in particular, $|s_i| < 1$ for all $i = 1, \dots, n$.

Let $\mathcal{P}_n$ be the space of the analytic trigonometric polynomials of degree less than or equal to $n$, $\mathcal{P}_n = \{p_n(z) = \sum_{k=0}^{n} a_k z^k : a_k \in \mathbb{C}, |z| = 1\}$. We define in $\mathcal{P}_n$ the inner product

$$\langle p_n, q_n \rangle = \mathbf{a} C_n \mathbf{b}^* = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i K(i-1, j-1) \overline{b_j}, \tag{2}$$

for all $p_n(z) = \sum_{i=0}^{n} a_i z^i$, $q_n(z) = \sum_{j=0}^{n} b_j z^j$, where $\mathbf{a} = (a_0, a_1, \dots, a_n)$ and $\mathbf{b} = (b_0, b_1, \dots, b_n)$ are their respective coefficient vectors.

The following properties are straightforward.

**Proposition 1.1.** *Let $p_n, q_n \in \mathcal{P}_n$ and let $z : \mathcal{P}_n \to \mathcal{P}_{n+1}$ denotes the multiplication by the independent variable. If $C_n$ is positive definite, then $\langle \cdot, \cdot \rangle$ is a hermitian and non-degenerate sesquilinear form in $\mathcal{P}_n$. Moreover,*

$$\langle zp_n, zq_n \rangle (= \sum_{i=0}^{n} a_i \overline{b_i}) = \langle p_n, q_n \rangle - a_0 \sum_{i=1}^{n} \overline{b_i}\, \overline{s_i} - \overline{b_0} \sum_{i=1}^{n} a_i s_i.$$

**Remark.** This modification of the Toeplitz condition gives rise to a Liapunov equation

$$\langle zp_n, zq_n \rangle = \langle p_n, q_n \rangle - c(p_n)\, \overline{d(q_n)} - d(p_n)\, \overline{c(q_n)},$$

where $c(p_n) = a_0$, $d(p_n) = \sum_{i=1}^{n} a_i s_i$, providing a connection between the Generalized Toeplitz Forms and the Potapov colligations in the sense shown in [4]. □

In our approach, we will use the following formulation:

$$\langle zp_n, zq_n \rangle = \langle p_n, q_n \rangle - \langle a_0, q_n \rangle - \langle p_n, b_0 \rangle + 2\langle a_0, b_0 \rangle. \tag{3}$$

We observe that, if the Nehari problem has a solution $f$, then:

$$\begin{aligned} \langle p_n, q_n \rangle = \sum_{j=0}^{n} a_j \overline{b_j} \; &+ \; \frac{1}{2\pi} \int_0^{2\pi} a_0\, \overline{q_n(e^{-it}) f(t)}\, dt \\ &+ \; \frac{1}{2\pi} \int_0^{2\pi} \overline{b_0} p_n(e^{-it}) f(t)\, dt - 2 a_0\, \overline{b_0}. \end{aligned}$$

Conversely, if $\langle p_n, q_n \rangle$ satisfies the last equation for all $p_n, q_n \in \mathcal{P}_n$ and some $f$, then the problem has a solution because

$$s_n = \langle z^n, 1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t)\, dt = \widehat{f}(n).$$

Therefore, the problem can be stated looking for the relation between the inner product (2) and its values.

With the product (2), $\mathcal{P}_n$ is an $(n+1)$-dimensional Euclidean space. Moreover for each $k$ $(0 \le k \le n)$, the matrix $C_k$ defines an inner product over $\mathcal{P}_k$ which is the restriction of the inner product defined in all $\mathcal{P}_n$. To find an orthogonal basis, we can span a polynomial sequence by applying the Gram-Schmidt process to the sequence $\{1, z, \ldots, z^n\}$.

For each $k \in \{1, \ldots, n\}$, we consider the polynomial $T_k = \sum_{i=0}^{k} t_{ki} z^i \in \mathcal{P}_k$ such that

$$\tau_{\mathbf{k}} C_k = (0, \ldots, 0, 1), \tag{4}$$

where $\tau_{\mathbf{k}} = (t_{k0}, t_{k1}, \ldots, t_{kk})$. By solving the system (4), we have explicitly:

$$\begin{aligned} T_0(z) &= 1, \\ T_k(z) &= \frac{-s_k + s_k\, \overline{s_1} z + \cdots + s_k\, \overline{s_{k-1}} z^{k-1} + \Delta_{k-1} z^k}{\Delta_k}, \quad k \ge 1. \end{aligned}$$

By its own definition, it is clear that $\langle T_k, p_{k-1} \rangle = 0$, $\forall p_{k-1} \in \mathcal{P}_{k-1}$; therefore, the set $\{T_0, T_1, \ldots, T_n\}$ is an orthogonal basis of $\mathcal{P}_n$. Taking into account that $\|T_k\|^2 = \tau_{\mathbf{k}} C_k \tau_{\mathbf{k}}{}^* = \overline{t_{kk}} = t_{kk}$, the set

$$\{T_k / \sqrt{t_{kk}},\ k = 0, \ldots, n\}$$

is an orthonormal basis of $\mathcal{P}_n$ with respect to the product defined in (2).

## 2 – Reproducing kernels

An important fact of the inner product defined above is that we can construct on $\mathcal{P}_n$ a reproducing kernel. Next, we are going to see the properties that this kernel takes for this problem.

For each $\zeta \in \mathbb{C}$ we define the linear functional $\phi_\zeta : \mathcal{P}_n \to \mathbb{C}$ by

$$\phi_\zeta(p_n) = p_n(\zeta), \ \forall p_n \in \mathcal{P}_n.$$

It is plain that $\phi_\zeta$ is bounded; by the Riesz representation theorem, there exists a unique $E_n^\zeta \in \mathcal{P}_n$ such that $p_n(\zeta) = \langle p_n, E_n^\zeta \rangle$, $\forall p_n \in \mathcal{P}_n$. This polynomial, so called the *evaluating polynomial* or *reproducing kernel* in $\zeta$, has the following properties:

**Proposition 2.1.** *Let $E_n^\zeta$ be defined as above. Then:*

**i)** $E_n^\zeta(\eta) = \overline{E_n^\eta(\zeta)}$.

**ii)** $E_n^\zeta(\zeta) = \|E_n^\zeta\|^2$.

**iii)** $E_n^\zeta(z) = \sum_{k=0}^n \frac{T_k(z)\overline{T_k(\zeta)}}{t_{kk}}$.

**iv)** $\|E_n^\zeta\|^{-2} = \inf_{S_n(\zeta)=1} \|S_n\|^2$.

**v)** *If $\epsilon_{\mathbf{n}} = (e_{n0}, \ldots, e_{nn})$ is the coefficient vector of $E_n^0(z)$, then*

$$\epsilon_{\mathbf{n}} C_n = (1, 0, \ldots, 0). \tag{5}$$

**vi)** $\|E_n^0\|^2 = (\Delta_n)^{-1} = (\Delta_{n-1})^{-1} \cdot \|T_n\|^2$.

**Proof:** Items i), ii), iii) and iv) are straightforward.

In order to prove v), we consider the vector $\eta_{\mathbf{k}} = (\delta_{kj})_{j=0,\ldots,n}$ of coefficients of $z^k$, for $k = 0, 1, \ldots, n$, and observe that, by definition,

$$\langle E_n^0, z^k\rangle = \epsilon_{\mathbf{n}} C_n {\eta_{\mathbf{k}}}^* = (\epsilon_{\mathbf{n}} C_n)_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

From this fact, we obtain explicitly the coefficients of $E_n^0$, namely:

$$e_{n0} = 1/\Delta_n,\ e_{nk} = -\,\overline{s_k}/\Delta_n,\ k = 1, \ldots n. \tag{6}$$

Finally, item vi) is a direct consequence of ii) and (6). ∎

**Remark.** Taking into account the relations (4) and (5), we have, for each $n \geq 1$:

$$\begin{aligned} t_{nk} &= -s_n e_{nk},\ k = 0, \ldots, n-1, \\ t_{nn} &= 1 - s_n e_{nn}, \end{aligned}$$

and by (6),

$$\begin{aligned} t_{nk} &= \frac{e_{nk} \cdot \overline{e_{nn}}}{e_{n0}},\ k = 0, 1, \ldots, n-1, \\ t_{nn} &= 1 + \frac{|e_{nn}|^2}{e_{n0}}. \end{aligned}$$

In particular, we have $\sum_{i=0}^n s_i t_{ni} = 0$ and $\sum_{i=0}^n s_i e_{ni} = 1$. ◻

From these formulas we deduce that each $T_n$ determines $E_n^0$ and conversely. Moreover, the following result holds.

**Proposition 2.2.** *The map $\{1 = s_0, s_1, \dots, s_n\} \mapsto E_n^0$ is one-to-one.*

**Proof:** From the previous remark and the formula

$$E_{n-1}^0(z) = E_n^0(z) - \frac{T_n(z)\,\overline{T_n(0)}}{t_{nn}},$$

we deduce that, for each $n \in \mathbb{N}$, $E_n^0(z)$ defines $E_{n-1}^0(z)$; this one defines $T_{n-1}(z)$ and so on. In this way, the polynomial $E_n^0$ generates the sequence $\{T_k(z), k = 0, \dots, n\}$.

Now, from the expansion $z^j = \sum_{k=0}^j a_{jk} T_k(z)$ and taking into account that $s_j = \langle z^j, 1\rangle$, we get $s_j = a_{j0}$ for $j = 1, \dots, n$. ∎

Now, we will see that a system $\{E_n^{\zeta_k} : k = 0, 1, \dots, n\}$ of evaluating polynomials determines the $n$-th orthogonal polynomial $T_n$.

**Proposition 2.3.** *There exists a set $\{\zeta_0, \zeta_1, \dots, \zeta_n\}$ of complex numbers such that*

$$T_n(z) = \sum_{k=0}^n \frac{E_n^{\zeta_k}(z)}{\overline{R'(\zeta_k)}},$$

*where we define $R(z) = \prod_{k=0}^n (z - \zeta_k)$ and $R'(z)$ denotes the derivative of $R(z)$.*

**Proof:** We choose $\{\zeta_0, \zeta_1, \dots, \zeta_n\}$ such that

$$T_j(\zeta_k) \neq 0,\ 1 \leq j \leq n-1,\ 0 \leq k \leq n.$$

Then, for each $j \in \{1, \dots, n-1\}$ and an appropriate $r > 0$, by the residue theorem,

$$\int_{|z|<r} \frac{T_j(z)}{R(z)}\, dz = 2\pi i \sum_{k=0}^n \frac{T_j(\zeta_k)}{R'(\zeta_k)}.$$

Since $\lim_{r\to\infty} \int_{|z|<r} \frac{T_j(z)}{R(z)}\, dz = 0$, then

$$0 = \sum_{k=0}^n \frac{T_j(\zeta_k)}{R'(\zeta_k)} = \sum_{k=0}^n \frac{\langle T_j, E_{n-1}^{\zeta_k}\rangle}{R'(\zeta_k)} = \Big\langle T_j, \sum_{k=0}^n \frac{E_{n-1}^{\zeta_k}}{\overline{R'(\zeta_k)}}\Big\rangle.$$

Now, by using the equality $\langle T_j, E_n^{\zeta_k}\rangle = \langle T_j, E_{n-1}^{\zeta_k}\rangle$ $(j = 0, 1, \ldots, n-1)$, it results that there exists a constant $\alpha$ such that

$$\alpha T_n(z) = \sum_{k=0}^{n} \frac{E_n^{\zeta_k}(z)}{\overline{R'(\zeta_k)}}.$$

But, since

$$\Big\langle R', \sum_{k=0}^{n} \frac{E_n^{\zeta_k}}{\overline{R'(\zeta_k)}}\Big\rangle = \sum_{k=0}^{n} \frac{1}{R'(\zeta_k)} \langle R', E_n^{\zeta_k}\rangle = \sum_{k=0}^{n} \frac{R'(\zeta_k)}{R'(\zeta_k)} = n+1,$$

and

$$\langle R', T_n\rangle = \sum_{k=0}^{n} \langle \prod_{j\neq k} (z-\zeta_j), T_n\rangle = n+1,$$

it must be $\alpha = 1$, so giving the theorem. ∎

## 3 – Levinson algorithm and Schur-type parameters

The following formula gives a representation of the evaluating polynomials from the orthogonal basis $\{T_k\}_{k=0}^{n}$.

**Theorem 3.1.** *(generalized Christoffel-Darboux formula). Let $z, \zeta \in \mathbb{C}$ be such that $\overline{\zeta} z \neq 1$. For every $n \in \mathbb{N}$, there exists a polynomial $\widetilde{W} \in \mathcal{P}_{n-1}$ such that*

$$E_n^{\zeta}(z) = \frac{\Delta_n \cdot \overline{E_n^0(\zeta)} E_n^0(z) - \overline{\zeta} z^{n+1} \cdot \overline{T_n(\zeta)} + \overline{\zeta} z \cdot \widetilde{W}(z)}{1 - \overline{\zeta} z}. \tag{7}$$

**Proof:** If we call

$$W(z) = t_{nn} z^n - T_n(z), \tag{8}$$

then

$$z T_n + z W \perp z(\mathcal{P}_{n-1}).$$

On the other hand, taking into account the equality

$$0 = \langle zQ - \zeta Q, E_n^{\zeta}\rangle,\ \forall \zeta, \forall Q \in \mathcal{P}_{n-1}$$

we have

$$\langle zQ, E_n^{\zeta}\rangle = \zeta \langle Q, E_n^{\zeta}\rangle.$$

Again, formula (3) gives

$$\langle Q, E_n^\zeta\rangle = \langle zQ, zE_n^\zeta\rangle + \langle \beta_0, E_n^\zeta\rangle + \langle Q, e_{n0}^\zeta\rangle - 2\langle \beta_0, e_{n0}^\zeta\rangle,$$

where $(e_{n0}^\zeta, \dots, e_{nn}^\zeta)$ and $(\beta_0, \dots, \beta_{n-1})$ denote the coefficient vectors of $E_n^\zeta(z)$ and $Q(z)$, respectively.

Taking into account that

$$\begin{aligned}
\langle \beta_0, E_n^\zeta\rangle &= \langle zQ, z\widetilde{P}\rangle, \text{ if } \widetilde{P}(z) = \sum_{i=0}^{n} s_i \cdot e_{ni}^\zeta \ , \\
\langle Q, e_{n0}^\zeta\rangle &= \langle zQ, z\widetilde{R}\rangle, \text{ if } \widetilde{R}(z) = e_{n0}^\zeta \cdot \sum_{i=0}^{n-1} \overline{s_i} z^i \ , \\
\langle \beta_0, e_{n0}^\zeta\rangle &= \langle zQ, z\widetilde{V}\rangle, \text{ if } \widetilde{V}(z) = e_{n0}^\zeta,
\end{aligned}$$

we obtain

$$\langle zQ, E_n^\zeta\rangle = \langle zQ, \overline{\zeta} z E_n^\zeta\rangle + \langle zQ, \overline{\zeta} z\widetilde{P}\rangle + \langle zQ, \overline{\zeta} z\widetilde{R}\rangle - 2\langle zQ, \overline{\zeta} z\widetilde{V}\rangle,$$

i.e.

$$(1 - \overline{\zeta} z)E_n^\zeta - \overline{\zeta} z\widetilde{W} \perp z(\mathcal{P}_{n-1}),$$

where we call $\widetilde{W}(z) = \widetilde{P}(z) + \widetilde{R}(z) - 2\widetilde{V}(z) = \sum_{i=1}^n s_i e_{ni}^\zeta + e_{n0}^\zeta \cdot \sum_{i=1}^{n-1} s_i z^i$.

Since $\dim(\mathcal{P}_{n+1} \ominus z(\mathcal{P}_{n-1})) = 2$, they must exist $\alpha, \beta \in \mathbb{C}$ such that

$$(1 - \overline{\zeta} z)E_n^\zeta(z) - \overline{\zeta} z\widetilde{W}(z) = \alpha E_n^0(z) + \beta \cdot t_{nn} z^{n+1}. \tag{9}$$

By comparing corresponding terms in both sides of the equation, we obtain

$$\alpha = \frac{-s_n \cdot \overline{E_n^0(\zeta)}}{t_{n0}} \ , \quad \beta = \frac{-\overline{\zeta} \cdot \overline{T_n(\zeta)}}{t_{nn}},$$

thus giving formula (7). ∎

The following result gives an algorithm to compute the sequence of polynomials $\{T_k\}_{k=0}^n$ in a recursive way.

**Theorem 3.2.** *(Levinson-type algorithm). For each $n \in \mathbb{N}$, there exist a constant $\alpha_n \in \mathbb{C}$ such that*

$$\frac{T_{n+1}(z)}{t_{n+1,n+1}} = z^{n+1} + \alpha_n \cdot \frac{E_n^0(z)}{t_{nn}} \tag{10}$$

*holds.*

**Proof:** Since the polynomials

$$S(z) = \frac{T_{n+1}(z)}{t_{n+1,n+1}} - z^{n+1}$$

and $E_n^0$ are both orthogonal to $z\mathcal{P}_{n-1}$, then there exists $\alpha_n$ such that $S(z) = \alpha_n \cdot E_n^0(z)/t_{nn}$. ∎

In order to compute the coefficient $\alpha_n$, it is enough to evaluate (10) at $z = 0$. So we have:

$$\frac{T_{n+1}(0)}{t_{n+1,n+1}} = \alpha_n \cdot \frac{E_n^0(0)}{t_{nn}},$$

whence

$$\alpha_n = \frac{t_{nn}}{t_{n+1,n+1}} \cdot \frac{t_{n+1,0}}{\|E_n^0\|^2} = \frac{-s_{n+1} \cdot \Delta_{n-1}}{\Delta_n}.$$

Formula (10) provides a method for generating the sequence of orthogonal polynomials $\{T_k\}$ by using the parameters $\{\alpha_k\}$. In fact, rewriting (10) as follows:

$$\frac{T_{n+1}(z)}{t_{n+1,n+1}} - \alpha_n \cdot \frac{E_n^0(z)}{t_{nn}} = z^{n+1}$$

and taking norms in both sides, we obtain:

$$\begin{aligned} \left\| \frac{T_{n+1}}{t_{n+1,n+1}} - \alpha_n \cdot \frac{E_n^0}{t_{nn}} \right\|^2 &= \frac{1}{|t_{n+1,n+1}|^2} \cdot \|T_{n+1}\|^2 + \frac{|\alpha_n|^2}{|t_{nn}|^2} \cdot \|E_n^0\|^2 \\ &= \frac{1}{t_{n+1,n+1}} + \frac{|\alpha_n|^2}{t_{nn} \cdot \Delta_{n-1}}, \\ \left\| z^{n+1} \right\| &= 1. \end{aligned}$$

By comparing these results, we get the following representation of the leading coefficient $t_{n+1,n+1}$ in terms of $t_{nn}$ and the parameter $\alpha_n$:

$$\frac{\Delta_{n-1}}{t_{n+1,n+1}} = \Delta_{n-1} - \frac{|\alpha_n|^2}{t_{nn}}. \tag{11}$$

Now, for each $\alpha_n$ such that $|\alpha_n| < \Delta_{n-1}/\Delta_n^{1/2}$, from (11) we obtain $t_{n+1,n+1}$ and, from (10), the polynomial $T_{n+1}$.

A similar procedure allows to generate the evaluating polynomials $\{E_n^0\}$ by using analogous formulas to (10) and (11).

Summing up, if the positive definite matrix $C_n$ is given, we can establish a one-to-one correspondence between the set $\{\alpha_1, \dots, \alpha_n\}$, with $|\alpha_k| < \Delta_{k-1}/\Delta_k^{1/2}$, and the positive definite matrix $C_{n+1}$.

In fact, if $|\alpha_n| < \Delta_{n-1}/\Delta_n^{1/2}$, we obtain $t_{n+1,n+1}$ from (11) and $T_{n+1}$ from (10). Such a definition makes $T_{n+1}$ to be orthogonal to $z\mathcal{P}_{n-1}$. In order to be orthogonal to $\mathcal{P}_n$, it is enough that $\langle T_{n+1}, 1\rangle = 0$, condition that determines $s_{n+1}$. With this value, the matrix $C_{n+1}$ will be positive definite and $\{T_k\}_{k=0}^{n+1}$ will be an orthogonal family of polynomials.

**Remark.** The coefficients $\alpha_n$, called the Schur parameters associated with the Nehari problem, are also called either *partial correlation coefficients*, due to their interpretation in time series or control theory, or *reflection coefficients*, in view of their physical interpretation. For example, rewriting $\alpha_n$ in the following way,

$$\begin{aligned}\alpha_n &= \Delta_{n+1} \cdot T_{n+1}(0) \cdot \frac{\Delta_{n-1}}{\Delta_n} = \frac{T_{n+1}(0)}{\|E_{n+1}^0\|^2} \cdot \frac{\Delta_{n-1}}{\Delta_n} \\ &= \frac{\Delta_{n-1}}{\Delta_n} \cdot \frac{\|T_{n+1}\|}{\|E_{n+1}^0\|} \Big\langle \frac{T_{n+1}}{\|T_{n+1}\|}, \frac{E_{n+1}^0}{\|E_{n+1}^0\|} \Big\rangle \\ &= \frac{\Delta_{n-1}}{\sqrt{\Delta_n}} \Big\langle \frac{T_{n+1}}{\|T_{n+1}\|}, \frac{E_{n+1}^0}{\|E_{n+1}^0\|} \Big\rangle,\end{aligned}$$

we deduce that $(\Delta_n^{1/2}/\Delta_{n-1})\alpha_n$ can be interpreted as a correlation coefficient.

Likely, from (10) we also obtain that

$$\frac{1}{t_{n+1,n+1}} \langle T_{n+1}, E_n^0\rangle = \langle z^{n+1}, E_n^0\rangle + \alpha_n \cdot \frac{\|E_n^0\|^2}{t_{nn}},$$

and taking into account that $\langle T_{n+1}, E_n^0\rangle = \langle zW, E_n^0\rangle = 0$, we get:

$$\alpha_n = -\frac{\langle zT_n, E_n^0\rangle}{\|E_n^0\|^2} = -\frac{\|zT_n\|}{\|E_n^0\|} \Big\langle \frac{zT_n}{\|zT_n\|}, \frac{E_n^0}{\|E_n^0\|} \Big\rangle.$$

Then $-(\|E_n^0\|/\|zT_n\|)\alpha_n$ represents the correlation between the forward prediction error of length $n$ advanced by one step, and the backward prediction error of length $n$. $\square$

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