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SHOCK WAVES AND GRAVITATIONAL WAVES IN MATTER SPACETIMES WITH GOWDY SYMMETRY

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Abstract: In the context of classical general relativity, we consider the matter Einstein equations for perfect fluids on Gowdy spacetimes with plane-symmetry. Such spacetimes admit two commuting Killing vector fields and contain gravitational waves; the fluid variable may exhibit shock waves. We establish the existence of a bounded variation solution to the Cauchy problem, which is defined globally until either a true singularity occurs in the geometry (e.g. the vanishing of the area of the 2-dimensional space-like orbits of the symmetry group) or a blow-up of the energy density takes place.

1 – Introduction

Vacuum Gowdy spacetimes are inhomogeneous spacetimes admitting two commuting spatial Killing vector fields [7]. The existence of vacuum spacetimes with Gowdy symmetry was established by Moncrief [12]. Much attention has been focused on these solutions of the Einstein equations, which play an important role in cosmology for instance. Numerical work was performed recently to understand the formation and properties of the singularities which arise even in the vacuum. The dynamics of solutions and, in particular, the long-time asymptotics of solutions have been found to be particularly complex [1]. In comparison, much less emphasis has been put on matter models. Recently, in [3] the authors initi-

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ated a rigorous mathematical treatment of the coupled Einstein–Euler system on Gowdy spacetimes. A Cauchy problem was considered in which the unknowns are the density and velocity of the fluid together with the components of the metric tensor. The local existence of a solution to the Cauchy problem in the class of solutions with (arbitrary large) bounded total variation was proved. The formation of shock waves in the fluid is not an obstacle to the existence of solutions, understood in a weak sense.

In the present paper we continue the analysis of [3] and establish a global existence result: the solution of the Euler–Einstein equation is globally defined until the geometry becomes truly singular or the sup norm of the density blows up. Our result can be interpreted as a global stability result of the Gowdy spacetimes in presence of matter allowing shock waves. For previous work in this direction with a different matter model see [2].

2 – The Euler–Einstein system

In this section we present the model to be studied in this paper and explain several basic properties. We are interested in the evolution of a compressible perfect fluid in a plane-symmetric spacetime, under the assumption that the metric has the **polarized Gowdy symmetry**, characterized by three scalar coefficients a, b, c

$$ds^2 \,=\, e^{2a}\,(-dt^2 + dx^2) \,+\, e^{2b}\,(e^{2c}\,dy^2 + e^{-2c}\,dz^2). \tag{2.1}$$

That is, the only non-zero covariant components of the metric $(g_{\alpha\beta})_{\alpha,\beta=0,\ldots,3}$ are

$$g_{00} = -e^{2a}\,, \qquad g_{11} = e^{2a}\,, \qquad g_{22} = e^{2b+2c}\,, \qquad g_{33} = e^{2b-2c}.$$

All variables are assumed to depend on the time variable t and the space variable x, only. The coefficient e^{2b} is essentially the area of the 2-surfaces of the group of symmetry.

We consider perfect fluids with energy density $\mu > 0$ and pressure p. These thermodynamical variables are related via the equation of state of the fluid

$$p = p(\mu).$$

Although the remaining results of this section do not depend on a specific choice of the equation of state, we shall, in subsequent sections make use of the "ultra-relativistic" equation of state where $p = \mu\, c_s^2$ where the sound speed c_s is a constant with $0 < c_s < 1$. The 4-velocity vector $(u^\alpha)_{\alpha=0,\ldots,3}$ of the fluid is normalized to be of unit length

$$u^\alpha\, u_\alpha = -1,$$

where the Einstein convention on repeated indices is used and, as usual, indices are raised and lowered with the metric, for instance

$$u^\alpha = \, g^{\alpha\beta}\, u_\beta.$$

We then define the scalar velocity v and the relativistic factor $\xi = \xi(v)$ by

$$(u^\alpha)_{\alpha=0,\dots,3} = e^{-a}\, \xi(1, v, 0, 0)\,, \qquad \xi = (1-v^2)^{-1/2},$$

where it should be observed that $|v|<1$.

The matter is described by the **energy-momentum tensor**

$$T^{\alpha\beta} \,=\, (\mu + p)\, u^\alpha u^\beta + \, p\, g^{\alpha\beta}, \tag{2.2}$$

from which we extract fields τ, S and Σ defined from the "first three" components

$$\begin{aligned} T^{00} \;&=\; e^{-2a}\Big((\mu+p)\,\xi^2 - p\Big) \;\;=:\; e^{-2a}\tau, \\ T^{01} \;&=\; e^{-2a}(\mu+p)\,\xi^2 v \;\;=:\; e^{-2a} S, \\ T^{11} \;&=\; e^{-2a}\Big((\mu+p)\,\xi^2 v^2 + p\Big) \;\;=:\; e^{-2a}\,\Sigma. \end{aligned} \tag{2.3}$$

We shall also need $T^{22} = pe^{-2(b+c)}$ to compute the evolution equations. Note that given μ and v (which we consider as our primary unknowns) it is easy to determine the conservative variables τ, S and Σ. These calculations take place in Minkowski spacetime, i.e., the expressions are independent of the geometry variables a, b and c.

The **Einstein field equations** read

$$G^{\alpha\beta} = \kappa\, T^{\alpha\beta}, \tag{2.4}$$

where $G^{\alpha\beta}$ is the Einstein tensor and κ is a normalization constant (of the order of $1/c_l^4$). Recall that the Einstein tensor is determined from the Ricci tensor which itself depends upon second order derivatives of the metric coefficients. By making explicit the equations (2.1)–(2.4) and after very tedious calculations we arrive at the following **constraint equations**

$$\begin{aligned} 2\, a_t\, b_t + 2\, a_x\, b_x + b_t^2 - 2\, b_{xx} - 3\, b_x^2 - c_t^2 - c_x^2 \;&=\; \kappa\, e^{2a}\, \tau, \\ -2\, a_t\, b_x - 2\, a_x\, b_t + 2\, b_{tx} + 2\, b_t\, b_x + 2\, c_t\, c_x \;&=\; \kappa\, e^{2a}\, S, \end{aligned} \tag{2.5}$$

and **evolution equations** for the geometry

$$
\begin{aligned}
a_{tt} - a_{xx} &= b_t^2 - b_x^2 - c_t^2 + c_x^2 + \frac{\kappa}{2}\, e^{2a}\,(-\tau + \Sigma - 2\,p), \\
b_{tt} - b_{xx} &= -2b_t^2 + 2b_x^2 + \frac{\kappa}{2}\, e^{2a}\,(\tau - \Sigma), \\
c_{tt} - c_{xx} &= -2\,b_t\,c_t + 2\,b_x\,c_x.
\end{aligned}
\tag{2.6}
$$

Note that the evolution equations contain second-order time derivatives of a, b and c, whereas the constraint equations contain only zero- or first-order time derivatives.

The evolution equations for the fluid are a consequence of the Einstein field equations and are obtained by expressing the Bianchi identities

$$
G^{\alpha\beta}{}_{;\beta} = 0
$$

(satisfied by any metric) in terms of the energy-momentum tensor

$$
T^{\alpha\beta}{}_{;\beta} = 0,
$$

where $_{;\beta}$ denotes covariant differentiation. This leads us to the **Euler equations** for the fluid

$$
\begin{aligned}
\tau_t + S_x &= T_1, \\
S_t + \Sigma_x &= T_2,
\end{aligned}
\tag{2.7}
$$

in which the source terms are

$$
\begin{aligned}
T_1 &= -\tau\,(a_t + 2b_t) - S\,(2a_x + 2b_x) - \Sigma\,a_t - 2\,p\,b_t, \\
T_2 &= -\tau a_x - S\,(2\,a_t + 2\,b_t) - \Sigma\,(a_x + 2b_x) + 2\,p\,b_x.
\end{aligned}
\tag{2.8}
$$

Note that the principal part of (2.7) (obtained by replacing T_1 and T_2 by 0) is nothing but the *special relativistic Euler equations*, which model the dynamics of the fluid in flat Minkowski spacetime. Note also $T_1 = T_2 = 0$ precisely when a, b and c are constants, in which case (2.1) becomes the flat metric.

This completes the description of the matter Einstein equations under study in the present paper. Let us observe a key property of the constraints. By defining

$$
\begin{aligned}
H &:= e^{2b}\left(2\,a_t\,b_t + 2\,a_x\,b_x + b_t^2 - 2\,b_{xx} - 3\,b_x^2 - c_t^2 - c_x^2 - \kappa\,e^{2a}\,\tau\right), \\
K &:= e^{2b}\left(-2\,a_t\,b_x - 2\,a_x\,b_t + 2\,b_{tx} + 2\,b_t\,b_x + 2\,c_t\,c_x - \kappa\,e^{2a}\,S\right),
\end{aligned}
\tag{2.9}
$$

the equations (2.5) are equivalent to

$$
H = K = 0. \tag{2.10}
$$

It is straightforward to check that if equations (2.6) and (2.7) hold everywhere then H and K satisfy the linear hyperbolic system

$$\begin{aligned} H_t + K_x &= 0, \\ K_t + H_x &= 0. \end{aligned} \tag{2.11}$$

Thus if the constraint equations (2.5) (that is, (2.10)) are satisfied at $t = 0$ and if we then solve the evolution equations (2.6) and (2.7), then the constraint equations are satisfied for all times $t \geq 0$.

3 – Existence result

We propose here to reformulate the Einstein–Euler equations in the form of a nonlinear hyperbolic system of balance laws with integral source-term, in the variables (μ, v) and

$$w := (a_t, a_x, \beta_t, \beta_x, c_t, c_x),$$

where $\beta = e^{2b}$. It is convenient to also set $\alpha = e^{2a}$. The system of equations for the fluid variables has the form

$$\begin{aligned} \tau(\mu, v)_t + S(\mu, v)_x &= T_1(\mu, v, w, \alpha, b), \\ S(\mu, v)_t + \Sigma(\mu, v)_x &= T_2(\mu, v, w, \alpha, b), \end{aligned} \tag{3.1}$$

in which the source terms are

$$\begin{aligned} T_1(\mu, v, w, \alpha, b) &= -\tau(\mu, v)\,(w_1 + e^{-2b}\,w_3) - S(\mu, v)\,(2w_2 + e^{-2b}\,w_4) \\ &\quad - \Sigma(\mu, v)\,w_1 - p(\mu)\,e^{-2b}\,w_3, \\ T_2(\mu, v, w, \alpha, b) &= -\tau(\mu, v) w_2 - S(\mu, v)\,(2w_1 + e^{-2b}\,w_3) \\ &\quad - \Sigma(\mu, v)\,(w_2 + e^{-2b}\,w_4) + p(\mu)\,e^{-2b}\,w_4. \end{aligned} \tag{3.2}$$

Choosing the equation of state to be $p = \mu c_s^2$, the evolution equations for the geometry read

$$\begin{aligned} w_{1t} - w_{2x} &= \frac{e^{-4b}}{4}\,(w_3^2 - w_4^2) - w_5^2 + w_6^2 - \frac{\kappa}{2}\,(1 + c_s^2)\,\alpha\,\mu, \\ w_{2t} - w_{1x} &= 0, \\ w_{3t} - w_{4x} &= \kappa\,(1 - c_s^2)\,\alpha e^{2b}\,\mu, \\ w_{4t} - w_{3x} &= 0, \\ w_{5t} - w_{6x} &= e^{-2b}\,(-\,w_3\,w_5 + w_4\,w_6), \\ w_{6t} - w_{5x} &= 0, \end{aligned} \tag{3.3}$$

while the constraints are

$$
(3.4)\qquad
\begin{aligned}
w_{3x} &= D(\mu, v, w, \alpha, b) \\
&= w_1\, w_4 + w_2\, w_3 + \frac{e^{-2b}}{2}\, w_3\, w_4 - 2e^{2b}\, w_5\, w_6 + \kappa\, \alpha\, e^{2b}\, S(\mu, v), \\
w_{4x} &= E(\mu, v, w, \alpha, b) \\
&= w_1\, w_3 + w_2\, w_4 + \frac{e^{-2b}}{4}(w_3^2 + w_4^2) - e^{2b}\,(w_5^2 + w_6^2 + \kappa\, \alpha\, \tau(\mu, v)).
\end{aligned}
$$

Moreover, the functions α, b (and a, β) are determined by imposing that

$$\lim_{x \to -\infty} (a, b, c) = (0, 0, 0)$$

thus

$$
(3.5)\qquad
\begin{aligned}
&\alpha(t, x) = e^{2a(t,x)}\,, \qquad a(t, x) = \int_{-\infty}^{x} w_2(t, y)\, dy, \\
&b(t, x) = \frac{1}{2}\, \ln \beta(t, x)\,, \qquad \beta(t, x) = 1 + \int_{-\infty}^{x} w_4(t, y)\, dy.
\end{aligned}
$$

Obviously, we are interested in solutions such that β remains positive.

The equations (3.3) consist of three sets of two equations associated with the propagation speeds ± 1, the speed of light (after normalization). The left-hand sides of (3.1) are the standard relativistic fluid equations in a Minkowski background, with wave speeds

$$\lambda_{\pm} = \frac{v \pm c_s}{1 \pm v\, c_s}.$$

Note that $\lambda_- < \lambda_+$ for all v with $|v| < 1$. To formulate the initial-value problem it is natural to prescribe the values of μ, v, w on the initial hypersurface at $t = 0$, denoted by (μ^0, v^0, w^0), and to set

$$
(3.6)\qquad
\begin{aligned}
&\alpha^0(x) = e^{2a^0(x)}\,, \qquad a^0(x) = \int_{-\infty}^{x} w_2^0(y)\, dy, \\
&b^0(x) = \frac{1}{2}\, \ln \beta^0(x)\,, \qquad \beta^0(x) = 1 + \int_{-\infty}^{x} w_4^0(y)\, dy.
\end{aligned}
$$

Our main result is the following:

Theorem 3.1. *Consider the (μ, v, w)-formulation of the Einstein–Euler equations on a polarized Gowdy spacetime with plane-symmetry and restrict attention to perfect fluids governed by the linear pressure law*

$$(3.7)\qquad p(\mu) = c_s^2\, \mu\,, \qquad c_s \in (0, 1),$$

where $c_s > 0$ denotes the sound speed.

Let the initial data (μ^0, v^0, w^0) be functions with bounded total variation,

$$TV(\mu^0, v^0, w^0) < \infty,$$

satisfying the constraints (3.4). Suppose also that the corresponding functions α^0, b^0 given by (3.6) are measurable and bounded,

$$\sup |(\alpha^0, b^0)| < \infty.$$

Then the Cauchy problem associated with (3.1)–(3.5) admits a solution μ, v, w (in the sense of distributions) which are measurable and bounded functions such that for some increasing function $C(t)$

$$TV(\mu, v, w)(t) + \sup |(\alpha, b)(t, \cdot)| \leq C(t)\,, \qquad t \geq 0,$$

and are defined up to a maximal time $T \leq \infty$. If $T < \infty$ then either the geometry variables α, b given by (3.5) blow up, that is:

$$\lim_{t \to T} \Big(\sup_{\mathbb{R}} |\alpha(t, \cdot)| + |b(t, \cdot)| \Big) = \infty,$$

or the energy density blows up:

$$\lim_{t \to T} \sup_{\mathbb{R}} |\mu(t, \cdot)| = \infty.$$

Hence, the solution exists until either a singularity occurs in the geometry (e.g. the area β of the 2-dimensional space-like orbits of the symmetry group vanishes) or the matter collapses to a point. To our knowledge this is the first *global* existence result for the Euler–Einstein equations on spacetimes with Gowdy symmetry. If a shock wave forms in the fluid, the functions μ, v will be discontinuous and, as the consequence of (3.4), $w_{3x} = \beta_{tx}$ and $w_{4x} = \beta_{xx}$ might also be discontinuous. In fact, Theorem 3.1 allows not only such discontinuities in second-order derivatives of the geometry components (i.e. at the level of the curvature of the metric), but also discontinuities in the first-order derivatives which propagate at the speed of light. The latter correspond to *Dirac distributions* in the curvature of the metric.

Remark 3.2. 1. The first results on shock waves and the Glimm scheme in special and general relativity are due to Smoller and Temple [13] (flat Minkowski spacetime) and Groah and Temple [8, 9] (spherically symmetric spacetimes). The novelty in [3] and in the present paper is the generalization to a model allowing gravitational waves in addition to the shock waves.

2. It would be interesting to extend Theorem 3.1 in the following direction. It was checked in [12] that, when the initial (Riemannian) metric is close to the flat metric, the equations for a vacuum, polarized Gowdy spacetime have actually globally defined solution up to time $t = +\infty$ (where a physical singularity is expected). It is natural to ask the question whether such Gowdy spacetimes are globally stable when matter is included. It is conceivable that if the geometry is almost flat initially and the density is small and supported in a small compact interval (or decay rapidly at spatial infinity), the weak solution of the Euler–Einstein system is actually globally defined in time. Such a result would be consistent with theoretical results [5] and numerical experiments obtained with spherically symmetric spacetimes [10]. □

4 – Glimm scheme and uniform estimates

We follow the notation and the general strategy in [3]. The main difference with [3] is that we are not introducing an augmented system for the second order derivatives, and we are not writing a separate equation for the component a of the metric. The main new difficulty is to establish uniform bounds for the geometry variables.

The Glimm scheme for a general hyperbolic system of the form ($u = h(\mu, v, w)$ being the conservative variables)

$$\partial_t u + \partial_x f(u) = g(u, \alpha, b), \tag{4.1}$$

is decomposed into a step based on solving the Riemann problem for the homogeneous system

$$\partial_t u + \partial_x f(u) = 0$$

and a step based on solving the ordinary differential equation

$$\partial_t u = g(u, \alpha, b).$$

Given a vector u_* and constants α, b we denote by $u(t) = \mathbf{S}_t(u_*, \alpha, b)$ the solution of

$$\begin{aligned} u'(t) &= g(u(t), \alpha, b)\,, \qquad t \geq 0, \\ u(0) &= u_*. \end{aligned} \tag{4.2}$$

Consider first the Riemann problem. In our formulation (3.1)–(3.5) of the Einstein–Euler system, the source-term depends on the integral quantities α, b, which should be updated at each discrete time. We take them to be constants in each cell of the mesh, as will be defined shortly.

Given two constant vectors u_l, u_r, a point (t_*, x_*), and some constants α, β, the *generalized Riemann problem* is the Cauchy problem for the system (4.1) with initial data

$$u(t_*, x) = \begin{cases} u^-, & x < x_*, \\ u^+, & x > x_*. \end{cases} \tag{4.3}$$

The *classical Riemann problem* is obtained by neglecting the source term $g(u, \alpha, b)$; let us denote its solution by $u^C(t, x; u^-, u^+; t_*, x_*)$. Let $u^G(t, x; u^-, u^+, \alpha, b; t_*, x_*)$ be the *approximate solver* of the generalized Riemann problem defined for all $t > t_*$ and $x \in \mathbb{R}$ by

$$\begin{aligned} u^G(t, x; u^-, u^+, \alpha, b; t_*, x_*) \;=\; & u^C(t, x; u^-, u^+; t_*, x_*) \\ & + \int_0^{t-t_*} g\Big(\mathbf{S}_\tau u^C(t, x; u^-, u^+; t_*, x_*), \alpha, b\Big)\, d\tau. \end{aligned} \tag{4.4}$$

Observe that u^G at a given time t only depends upon u^C at the same time t.

Our generalization of the Glimm method is based on the approximate Riemann solver just introduced. Let s and r denote time and space mesh lengths satisfying $s/r < 1$, the ratio s/r being kept constant while $r, s \to 0$. Let $a = (a_k)_{k=0,1,\ldots}$ be an equidistributed sequence in $(-1, 1)$. We define an approximate solution $u_r = u_r(t, x)$ of the general Cauchy problem consisting of the system (4.1) and prescribed initial data u^0:

$$u(0, x) = u^0(x)\,, \qquad x \in \mathbb{R}. \tag{4.5}$$

To $u^0 = h(\mu^0, v^0, w^0)$ we also associate the function α^0 and b^0 determined by (3.6).

First, $u_r(0, x)$ is defined to be a piecewise constant approximation of u^0, say

$$u_r(0, x) = u^0\Big((h + 1)r\Big)\,, \qquad x \in [hr, (h{+}2)r), \quad h \text{ even}. \tag{4.6}$$

The piecewise constant functions α_r, b_r are defined in the first time slab by

$$\begin{aligned} \alpha_r(t, x) = \;& e^{2\int_{-\infty}^{(h+1)r} w_{2,r}(0,y)\,dy}, \\ b_r(t, x) = \;& \frac{1}{2}\ln\Big(1 + \int_{-\infty}^{(h+1)r} w_{4,r}(0, y)\, dy\Big)\,, \qquad x \in [hr, (h{+}2)r), \quad t \in [0, s). \end{aligned}$$

If $u_r(t, x)$ is known for $t < ks$ for some integer $k \geq 0$ and if α_r, β_r are known for all $t < (k{+}1)s$ we set

$$u_r(ks+, x) = u_r\Big(ks-, (h{+}1{+}a_k)r\Big)\,, \qquad x \in [hr, (h{+}2)r), \quad k{+}h \text{ even}. \tag{4.7}$$

Then, in each region $ks \le t < (k+1)s$, $(h-1)r \le x < (h+1)r$ ($k+h$ even), the function u_r is defined as the approximate solution to the generalized Riemann problem with data $u_r(ks,(h-1)r)$, $u_r(ks,(h+1)r)$ and $\alpha(ks,hr)$ centered at the point (ks,hr), that is

$$(4.8)\qquad \begin{aligned} u_r(t,x) \;&=\; u^G\Big(t,x;\, u_r(ks,(h-1)r),\; u_r(ks,(h+1)r);\, ks,hr\Big),\\ &\qquad t\in[ks,(k+1)s),\quad x\in[(h-1)r,(h+1)r),\quad k+h \text{ even}.\end{aligned}$$

The functions α_r, β_r are then defined by

$$\begin{aligned} \alpha_r(t,x) &= e^{2\int_{-\infty}^{(h+1)r} w_{2,r}((k+1)s,y)\,dy},\\ b_r(t,x) &= \frac{1}{2}\ln\Big(1+\int_{-\infty}^{(h+1)r} w_{4,r}((k+1)s,y)\,dy\Big),\\ &\qquad x\in[hr,(h+2)r),\quad t\in[(k+1)s,(k+2)s).\end{aligned}$$

This completes the description of our generalization to the Glimm scheme.

We are now in position to state our convergence result:

Theorem 4.1. *Let the initial data u^0 in (4.5) be a function with bounded variation and α^0, b^0 be bounded functions. Consider the approximate solutions $u_r = (\mu_r, v_r, w_r)$ constructed by the generalized version of the Glimm scheme, as defined above. Then the solutions are well-defined (for all r) on any interval $[0,T]$ in which the variables μ_r, α_r, b_r satisfy the uniform bound*

$$(4.9)\qquad \sup_{t\in[0,T],\, x\in\mathbb{R}} \mu_r + |\alpha_r| + |b_r| \;\le\; C_1$$

for some constant C_1 independent of r. Moreover, there exists constants $c_2, C_2 > 0$ (depending on C_1 and T), such that the approximate solutions $u_r = u_r(t,x)$ satisfies for all $t\in[0,T]$ and $x\in\mathbb{R}$

$$(4.10)\qquad c_2 \le \mu_r(t,x) \le C_2\,, \qquad |v_r(t,x)| \le 1-c_2,$$

$$(4.11)\qquad |w_r(t,x)| \le C_2,$$

$$(4.12)\qquad TV\Big((\mu_r, v_r, w_r)(t)\Big) \le C_2.$$

This theorem can be proven along similar lines as the ones in [3] and therefore we will only indicate the key steps and stress the differences with [3]. There are three main issues:

- First of all we must check that, as long as the condition (4.9) hold, no further singularity can occur in the approximate solution. That is, we must check that the density μ remains bounded away from the vacuum, corresponding to $\mu = 0$), and that the velocity v is bounded away from the speed of light ± 1. Both values are singularities to be avoided in the fluid equations.
- Second, we must derive a uniform bound on the amplitude of the solution, i.e. in addition to the bounds above, we must get an upper bound for the density together with the estimate (4.11).
- Most importantly, we must control the total variation of the solution u_r. See the discussion at the end of this section.

First, the derivation of (4.10)–(4.11) follows from the following key observation concerning the source-term.

Lemma 4.2. *For α, b fixed, the trajectories of the ordinary differential equation (4.2) are globally defined in time unless the energy density μ blows up. In particular, the fluid variables remain bounded away from the singularities $\mu = 0$ and $v = \pm 1$.*

Proof: Given constants α, b, we consider the ordinary differential equations

$$\tau(\mu, v)_t = T_1(\mu, v, w, \alpha, b),$$
$$S(\mu, v)_t = T_2(\mu, v, w, \alpha, b),$$

coupled with

$$\begin{aligned}
w_{1t} &= \frac{e^{-4b}}{4}\,(w_3^2 - w_4^2) - w_5^2 + w_6^2 - \frac{\kappa}{2}\,(1 + c_s^2)\,\alpha\,\mu, \\
w_{2t} &= 0\varphi \\
w_{3t} &= \kappa\,(1 - c_s^2)\,\alpha\,e^{2b}\,\mu, \\
w_{4t} &= 0, \\
w_{5t} &= e^{-2b}\,(-\,w_3\,w_5 + w_4\,w_6), \\
w_{6t} &= 0,
\end{aligned}$$

We can derive a system for μ, v:

$$\mu_t = f_1(\mu, v),$$
$$v_t = f_2(v),$$

where

$$f_1(\mu,v) := -\mu \frac{1+c_s^2}{1-c_s^2 v^2} \Big((1-v^2)\, w_1 + e^{-2b}\,(w_3 + v\, w_4)\Big),$$
$$f_2(v) := \frac{1-v^2}{1-c_s^2 v^2} \Big(-v\,(1-c_s^2)\, w_1 - (1-c_s^2 v^2)\, w_2 + v\, c_s^2\, e^{-2b}\,(w_3 + v\, w_4)\Big).$$

The function $f_2(v)$ depends smoothly upon $v \in (-1,1)$ and vanishes at $v=\pm 1$. Trajectories $t \mapsto v(t)$ cannot exit the interval $(-1,1)$. On the other hand, the function $f_1(\mu,v)$ is *linear* in μ,

$$f_1(\mu,v) = \widetilde{f}_1(v)\,\mu,$$

where $\widetilde{f}_1(v)$ is smooth for $|v|<1$, and in particular f_1 vanishes at $\mu=0$. It follows that μ stays positive. Hence we have

$$\mu > 0\,, \qquad -1 < v < 1.$$

To see that (μ,v,w) do not blow up in finite time we consider the variables

$$z_1 = \log\Big(\frac{1+v}{1-v}\Big),$$
$$z_2 = \log\mu.$$

which satisfy

$$z_{1t} = \frac{2}{1-c_s^2 v^2}\Big(-v(1-c_s^2)\, w_1 - (1-c_s^2 v^2)\, w_2 + v\, c_s^2 e^{-2b}\,(w_3 + v\, w_4)\Big),$$
$$z_{2t} = -\frac{1+c_s^2}{1-c_s^2 v^2}\Big((1-v^2)\, w_1 + e^{-2b}\,(w_3 + v\, w_4)\Big),$$

where $v = v(z_1) \in (-1,1)$. Note that the coefficients in front of w_1, w_2, w_3, w_4 are uniformly bounded a priori.

The functions w_2, w_4, w_6 are obviously constant, so, for some constants C_1, C_2, etc, and some uniformly bounded functions $B_1(t), B_2(t)$ which need not be positive, we end up with the system

$$\begin{aligned}
w_{1t} &= C_1\, w_3^2 - w_5^2 + C_2\, e^{z_2} + C_3,\\
w_{3t} &= C_4\, e^{z_2},\\
w_{5t} &= C_5\, w_3\, w_5 + C_6,\\
z_{1t} &= B_1(t)\, w_1 + B_2(t)\, w_3 + B_3(t),\\
z_{2t} &= B_4(t)\, w_1 + B_5(t)\, w_3 + B_6(t).
\end{aligned}$$

Suppose that the density variable z_2 remains bounded, and let us check that the other components of the solution cannot blow up. First of all, it follows from the w_3-equation that w_3 is bounded or all times, and the above system takes the form

$$\begin{aligned} w_{1t} &= -w_5^2 + B_7(t), \\ w_{5t} &= B_8(t)\, w_5 + C_6, \\ z_{1t} &= B_1(t)\, w_1 + B_9(t), \\ z_{2t} &= B_4(t)\, w_1 + B_{10}(t). \end{aligned}$$

The equation for w_5 is affine in w_5 and thus w_5 cannot blow up in finite time. In turn the w_1-equation yields also a uniform bound for w_1. Finally, the right-hand sides of the equations for z_1, z_2 are bounded in t, and therefore z_1, z_2 cannot blow up. This completes the proof of Lemma 4.2. ∎

Second, let us emphasize that the system (4.1) under consideration has the form

$$\partial_t h^1(\mu, v) + \partial_x f^1(\mu, v) \;=\; g^1(u, \alpha, b), \tag{4.13}$$

$$\partial_t h^2(w) + \partial_x f^2(w) \;=\; g^2(u, \alpha, b), \tag{4.14}$$

in which the map f^2 is linear. The derivation of the uniform total variation bound is based on the following two key observations.

On one hand, the homogeneous system associated with the fluid variables (μ, v),

$$\partial_t h^1(\mu, v) + \partial_x f^1(\mu, v) \;=\; 0,$$

is the Euler system in Minkowski spacetime, which enjoys the following total variation diminishing property: if $(t, x) \mapsto (\mu_r, v_r)$ is an (approximate) solution (generated by a Glimm scheme) then $TV(\mu_r(t))$ is a non-increasing function t (cf. [13]). Moreover, the total variation $TV(v_r(t))$ is controled also by $TV(\mu_r(t))$. In turn, for the full equations with source-terms (4.13), we can write (for some constant $C > 0$)

$$TV((\mu_r, v_r)(t)) \;\le\; C\, TV((\mu_r, v_r)(0)) + C \int_0^t TV(g^1(u_r, \alpha_r, b_r)(t'))\, dt'.$$

Hence, since the solutions are already known to be uniformly bounded in amplitude, we obtain

$$\begin{aligned} TV((\mu_r, v_r)(t)) \;\le\; & C\, TV((\mu_r, v_r)(0)) \\ & + C \int_0^t \Big(TV(\mu_r, v_r)(t') + TV(w_r)(t') \Big)\, dt'. \end{aligned} \tag{4.15}$$

On the other hand, the homogeneous system associated with the geometry variables,

$$\partial_t h^2(w) + \partial_x f^2(w) \,=\, 0,$$

consists of linear hyperbolic equations, and it is immediate that the total variation of the characteristic variables $w_1 \pm w_2$, etc, is conserved in time. In turn, for the full equations with source-terms we obtain

$$TV(w_r(t)) \,\leq\, C\,TV(w_r(0)) + C \int_0^t \Big(TV(\mu_r, v_r)(t') + TV(w_r)(t') \Big)\, dt'. \tag{4.16}$$

Applying Gronwall's lemma to (4.15)–(4.16) we conclude that there exists a constant $C > 0$ so that

$$TV((\mu_r, v_r)(t)) + TV(w_r(t)) \leq C\, e^{C\,t} \Big(TV((\mu_r, v_r)(0)) + TV(w_r(0)) \Big), \quad t \in [0, T],$$

which completes the derivation of the uniform total variation bounds.

5 – Uniform estimates

5.1. Vacuum Einstein equations

In this section we focus our attention on the Einstein equations in vacuum. Taking the coupling constant $\kappa = 0$ in (2.6) we find the evolution equations for the Gowdy metric in the vacuum

$$\begin{aligned} a_{tt} - a_{xx} \,&=\, b_t^2 - b_x^2 - c_t^2 + c_x^2, \\ b_{tt} - b_{xx} \,&=\, -2 b_t^2 + 2 b_x^2, \\ c_{tt} - c_{xx} \,&=\, -2\, b_t\, c_t + 2\, b_x\, c_x. \end{aligned} \tag{5.1}$$

The b-equation decouples from the a- and c-equation. By defining $\beta := e^{2b}$, so that $\beta_t = 2\, b_t\, e^{2b}$ and $\beta_{tt} = (2\, b_{tt} + 4\, b_t^2)\, e^{2b}$, the second equation in (4.1) takes the form

$$\beta_{tt} - \beta_{xx} = 0, \tag{5.2}$$

a linear wave equation. This motivates the choice of β (and its first order derivatives) as one of the main unknowns in the system (3.1)–(3.5).

It is easily seen that given some initial data β_t^0 and β_x^0 at time $t = 0$, the solution β of the initial-value problem associated with (4.5) may vanish in finite time *unless* the initial data are sufficiently close to the flat metric, that is β_t^0 and β_x^0 are sufficiently small. This shows that, in Theorem 3.1, we could not exclude that the function b could blow up (i.e. tends to $-\infty$) in finite time. As long as the solution β of (5.2) is positive, we can plug the expression of b in the right-hand side of the c-equation in (5.1) and prescribe arbitrary initial data c_x^0, c_t^0; this allows us to determine the solution c uniquely. Finally, after b and c are computed, the first equation in (5.1) determines uniquely the function a from any given initial data.

In the above discussion, only the evolution equations were considered and the contraint equations did not play a role. By taking the contraint equations and suitable boundary conditions at $x = \pm\infty$ into account, it might be possible to exclude the blow-up of b and to obtain globally defined even if b is "large" initially. To this end the following reformulation of the equation, based on the characteristic coordinates $x \pm t$, is useful to derive uniform estimates on the solutions.

Define $u := x + t$ and $v := x - t$, so that

$$\begin{aligned} a_{uv} &= b_u\, b_v - c_u\, c_v, \\ b_{uv} &= -2b_u\, b_v, \\ c_{uv} &= -b_u\, c_v - b_v\, c_u, \end{aligned}$$

while the constraints can be reduced to

$$\begin{aligned} b_{uu} &= 2\, a_u\, b_u - b_u^2 - c_u^2, \\ b_{vv} &= 2\, a_v\, b_v - b_v^2 - c_v^2. \end{aligned}$$

Defining $d = b + 2\, a$ we have

$$d_{uv} = -2\, c_u\, c_v,$$

while the constraints become

$$\begin{aligned} b_{uu} &= -2\, b_u^2 + d_u\, b_u - c_u^2, \\ b_{vv} &= -2\, b_v^2 + d_v\, b_v - c_v^2. \end{aligned}$$

Defining $\beta := e^{2b}$ we obtain $\beta_{uv} = 0$ and also

$$\beta(u,v) = f(u) + g(v),$$

and therefore b, c, d satisfy

$$\begin{aligned}
f_{uu} &= \beta_{uu} = d_u\, f_u - 2(f+g)\, c_u^2, \\
g_{vv} &= \beta_{vv} = d_v\, g_v - 2(f+g)\, c_v^2, \\
c_{uv} &= -b_u\, c_v - b_v\, c_u = -\frac{(f_u\, c_v + g_v\, c_u)}{2\,(f+g)}, \\
d_{uv} &= -2c_u\, c_v.
\end{aligned} \tag{5.3}$$

To illustrate how uniform bounds can be derived from (5.3) let us observe for instance that

$$\begin{aligned}
\left(e^{-d}\, f_u\right)_u &= -2\, e^{2b-d}\, c_u^2 < 0, \\
\left(e^{-d}\, g_v\right)_v &= -2\, e^{2b-d}\, c_v^2 < 0.
\end{aligned}$$

Therefore, by integrating from the initial data line $t = 0$, that is $u = v$, we arrive at the upper bounds

$$(e^{-d}\, f_u)(u,v) \le C\,, \qquad (e^{-d}\, g_v)(u,v) \le C, \tag{5.4}$$

where the constant $C > 0$ depends upon initial data at $t = 0$ only.

5.2. Sup norm and total variation bounds

One open problem is to show that no blow-up can occur in μ — unless the variables a, b themselves blow up. We will derive here some estimates that should be useful to tackle this issue. Let us begin by discussing the b-equation in (2.6). We introduce the change of unknown $\beta := e^{2b}$ and set $w = \kappa\, e^{2a}\,(\tau - \Sigma)$, so that

$$\beta_{tt} - \beta_{xx} = w\, \beta. \tag{5.5}$$

Note that w is proportional to the density μ. We introduce

$$\beta' = \beta_t + \beta_x\,, \qquad \beta'' = \beta_t - \beta_x,$$

and rewrite (5.5) as a system of two equations

$$\begin{aligned}
\beta'_t - \beta'_x &= w\, \beta > 0, \\
\beta''_t + \beta''_x &= w\, \beta > 0,
\end{aligned} \tag{5.6}$$

in which β can be recovered from β', β'' by

$$\beta(t,x) \;=\; 1 + \frac{1}{2}\int_{-\infty}^{x}\Big(\beta'(t,y) - \beta''(t,y)\Big)\,dy, \tag{5.7}$$

provided $\lim_{x\to-\infty}\beta(t,x) = 1$.

Given Cauchy data β'_0 and β''_0 (or equivalently β^0, β^0_t) we obtain

$$\begin{aligned} \beta'(t,x) \;&=\; \beta'_0(x+t) \;+ \int_0^t (w\,\beta)(s, x+t-s)\,ds, \\ \beta''(t,x) \;&=\; \beta''_0(x-t) \;+ \int_0^t (w\,\beta)(s, x-t+s)\,ds, \end{aligned} \tag{5.8}$$

thus

$$\begin{aligned} 2\,\beta_t(t,x) \;=\;& \beta'_0(x+t) + \beta''_0(x-t) \\ &+ \int_0^t \Big((w\,\beta)(s, x-t+s) + (w\,\beta)(s, x+t-s)\Big)\,ds. \end{aligned} \tag{5.9}$$

On the other hand, we can write

$$\beta(t,x) \;=\; \beta_0(0,x) + \int_0^t \beta_t(s,x)\,ds,$$

and, therefore,

$$\begin{aligned} |\beta(t,x)| + 2\,|\beta_t(t,x)| \;\le\;& |\beta_0(0,x)| + |\beta'_0(x+t)| + |\beta''_0(x-t)| + \int_0^t |\beta_t(s,x)|\,ds \\ &+\; \sup w \int_0^t \Big(|\beta|(s, x-t+s) + |\beta|(s, x+t-s)\Big)\,ds \end{aligned}$$

thus

$$\begin{aligned} \sup|\beta(t,\cdot)| + 2\,\sup|\beta_t(t,\cdot)| \;\le\;& 3\,\sup|\beta^0, \beta^0_t, \beta^0_x| + \int_0^t \sup|\beta_t(s,\cdot)|\,ds \\ &+\; 2\,(\sup w)\int_0^t \sup|\beta|\,ds. \end{aligned}$$

By applying Gronwall's inequality we deduce the sup norm estimate

$$\sup|\beta(t,\cdot)| + 2\,\sup|\beta_t(t,\cdot)| \;\le\; 3\,e^{2(1+\sup w)\,t}\,\sup|\beta^0, \beta^0_t, \beta^0_x|. \tag{5.10}$$

To control the sup norm of β_x we return to (5.6) and observe that

$$|\beta_x(t,x)| \;\le\; |\beta_t(t,x)| + |\beta_0''(x-t)| + \int_0^t w\,|\beta|(s,x-t+s)\,ds,$$

thus using (5.10)

$$\begin{aligned}\sup|\beta_x(t,\cdot)| \;\le\; & 3\,e^{2(1+\sup w)\,t}\sup|\beta^0,\beta^0_t,\beta^0_x|\\ & + \sup|\beta^0_t| + \sup|\beta^0_x| + (\sup w)\int_0^t 3\,e^{2(1+\sup w)s}\sup|\beta^0,\beta^0_t,\beta^0_x|\;ds,\end{aligned}$$

which yields the sup norm estimate

$$\sup|\beta_x(t,\cdot)| \;\le\; 6\,e^{2(1+\sup w)\,t}\,\sup|\beta^0,\beta^0_t,\beta^0_x|. \tag{5.11}$$

We can also control the total variation of β_t and β_x, as follows. By differentiating in x the identity (5.5) derived earlier for β_t we get

$$TV(\beta_t(t)) \le TV(\beta^0_t)+TV(\beta^0_x)+\int_0^t\Big((\sup w)\,TV(\beta(s))+(\sup|\beta(s)|)\,TV(w(s))\Big)\,ds.$$

The function β_x satisfies the same estimate. On the other hand we can write

$$TV(\beta(t)) \;\le\; TV(\beta^0) + \int_0^t TV(\beta_t(s))\,ds,$$

therefore

$$\begin{aligned}TV(\beta,\beta_t,\beta_x)(t) \;\le\; & TV(\beta^0) + 2\,TV(\beta^0_t) + 2\,TV(\beta^0_x)\\ & + (1+\sup w)\int_0^t\Big(TV(\beta,\beta_t,\beta_x)(s) + (\sup|\beta(s)|)\,TV(w(s))\Big)\,ds.\end{aligned}$$

Using the Gronwall inequality and the sup norm estimate (5.9) for β we arrive at the total variation estimate

$$\begin{aligned}TV(\beta,\beta_t,\beta_x)(t) \;\le\; & 2\,e^{(1+\sup w)}\,TV(\beta^0,\beta^0_t,\beta^0_x)\\ & + \sup_s TV(w(s))\,\sup|\beta^0,\beta^0_t,\beta^0_x|\,\frac{e^{2(1+\sup w)t}-e^{-2(1+\sup w)t}}{2(1+\sup w)}.\end{aligned} \tag{5.12}$$

We observe that the upper-bounds in (5.10)–(5.12) depend upon the sup norm and the total variation of w. Since $w = \kappa\, e^{2a}\,(\tau-\Sigma)$ and since $\tau-\Sigma$ is proportional to the density μ, it follows that the sup norm of w is controled by the sup norms of e^a and μ, and that the total variation of w is controlled if in addition we control the total variation of a and μ.

Let us now turn attention to the a-equation in (2.6). Setting

$$z \,:=\, \frac{\kappa}{2}\, e^{2a}\, (\tau - \Sigma + 2\,p) \,\geq\, 0,$$

which is proportional to the density μ, we find

$$a_{tt} - a_{xx} \,=\, b_t^2 - b_x^2 - c_t^2 + c_x^2 - z. \tag{5.13}$$

Setting $\alpha' := a_t + a_x$ and $\alpha'' := a_t - a_x$ and Cauchy data α'_0 and α''_0 being given we obtain

$$\begin{aligned} \alpha'(t,x) \,&=\, \alpha'_0(x+t) \,+ \int_0^t (b_t^2 - b_x^2 - c_t^2 + c_x^2 - z)\,(s, x+t-s)\; ds, \\ \alpha''(t,x) \,&=\, \alpha''_0(x-t) \,+ \int_0^t (b_t^2 - b_x^2 - c_t^2 + c_x^2 - z)\,(s, x-t+s)\; ds, \end{aligned} \tag{5.14}$$

which allows us to determine the time derivative

$$\begin{aligned} 2\,a_t(t,x) \,=\, &\alpha''_0(x-t) + \alpha'_0(x+t) \,+ \int_0^t (b_t^2 - b_x^2 - c_t^2 + c_x^2 - z)\,(s, x+t-s)\; ds \\ &+ \int_0^t (b_t^2 - b_x^2 - c_t^2 + c_x^2 - z)\,(s, x-t+s)\; ds, \end{aligned} \tag{5.15}$$

and therefore

$$\begin{aligned} 2\,a(t,x) \,=\, &2\,a_0(x) \,+ \int_0^t \Big(\alpha''_0(x-s) + \alpha'_0(x+s)\Big)\, ds \\ &+ \int_0^t\!\!\int_0^t (b_t^2 - b_x^2 - c_t^2 + c_x^2 - z)\,(s, x+t'-s)\; ds\, dt' \\ &+ \int_0^t\!\!\int_0^t (b_t^2 - b_x^2 - c_t^2 + c_x^2 - z)\,(s, x-t'+s)\; ds\, dt'. \end{aligned} \tag{5.16}$$

Taking advantage of the fact that z is non-negative we obtain the upper-bound

$$\begin{aligned} \sup_x a(t,x) \,\leq\, &\|a_0\|_{L^\infty(\mathbb{R})} + (1/2)\,\|\alpha'_0\|_{L^\infty(\mathbb{R})} \\ &+ (1/2)\,\|\alpha''_0\|_{L^\infty(\mathbb{R})} + (1/2)\,\mathcal{I}(t), \end{aligned} \tag{5.17}$$

where

$$\mathcal{I}(t) \,=\, \sup_x \sum_{\pm} \int_0^t\!\!\int_0^t \Big(|b_t^2 - b_x^2| + |c_t^2 - c_x^2|\Big)\,(s, x \pm (t'-s))\; ds\, dt'.$$

are integrals over a characteristic square of length t.

This inequality shows that the function e^a (arising in the right-hand side of the evolution equations) remains globally bounded provided we can control $\mathcal{I}(t)$.

Using (5.16)–(5.17) we can also derive sup norm estimates for a, a_t, a_x

$$\sup|a(t,\cdot)| \le \sup|a^0| + t\sup|a_t^0, a_x^0| + t^2\sup|b_t, b_x, c_t, c_x|^2 + t^2\sup|z| \tag{5.18}$$

and

$$\sup|(a_t, a_x)(t,\cdot)| \le \sup|a_t^0, a_x^0| + t\sup|b_t, b_x, c_t, c_x|^2 + t\sup|z|. \tag{5.19}$$

We can also write for instance

$$\begin{aligned} TV(a_t(t)) \le\ & TV(a_t^0) + TV(a_x^0) \\ & + \int_0^t \Big((\sup|b_t|)\, TV(b_t) + (\sup|b_x|)\, TV(b_x) \\ & \qquad + (\sup|c_t|)\, TV(c_t) + (\sup|c_x|)\, TV(c_x) + TV(z) \Big)\, ds, \end{aligned}$$

so that using again Gronwall's inequality, the total variation of a, a_t, a_x can be estimated in the form

$$TV(a, a_t, a_x)(t) \le C\Big(t, \sup_s TV(w(s)), \sup w\Big). \tag{5.20}$$

The bounds for the function c are analogous.

In summary, we are able to bound the sup norm and total variation of a, b, c and their first order derivatives, provided similar bounds are available on the density and the integral term $\mathcal{I}(t)$ can be controlled.

In view of formula (5.8), the integral term (arising in $\mathcal{I}(t)$)

$$\begin{aligned} \mathcal{I}^\beta :=\ & \sum_\pm \int_0^t\int_0^t |\beta_t^2 - \beta_x^2|\,(s, x\pm(t'-s))\; ds\, dt' \\ =\ & \sum_\pm \int_0^t\int_0^t |\beta'\beta''|\,(s, x\pm(t'-s))\; ds\, dt'. \end{aligned}$$

can be estimated as follows:

$$\begin{aligned} & \int_0^t\int_0^t |\beta'\beta''|\,(s, x\pm(t'-s))\; ds\, dt' \\ & \le \sum_\pm \int_0^t\int_0^t \Big|\beta_0'(x\pm(t'-s)-s) + \int_0^s (w\,\beta)\,(s', x\pm(t'-s)-s+s')\, ds'\Big| \\ & \qquad\qquad \Big|\beta_0''(x\pm(t'-s)+s) + \int_0^s (w\,\beta)\,(s', x\pm(t'-s)+s-s')\, ds'\Big|\; ds\, dt' \\ & \le C_1(t) + C_2(t)\int_0^t \sup|w\,\beta|(s)\, ds, \end{aligned}$$

where the expressions $C_1(t), C_2(t)$ depend only on the initial data.

Finally we note that it is an open problem to derive a uniform control for $\sup_x a(t)$, that is a uniform control of e^a. The main difficulty comes from the terms $w\,\beta$ above, since w contains e^a, which may lead to blow-up of a in finite time.

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