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ON THE TOPOLOGY OF LAGRANGIAN SUBMANIFOLDS EXAMPLES AND COUNTER-EXAMPLES

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Roughly speaking, the two natural questions that can be asked on the topology of Lagrangian submanifolds are the following:

- Given a symplectic manifold W, which closed manifolds L is it possible to embed in W as Lagrangian submanifolds?
- Given a closed manifold L, in which symplectic manifolds W can it be embedded as a Lagrangian submanifold?

The main tools we can use to approach these questions are the pseudo-holomorphic-curves-Floer-homology on the one hand and the Stein-manifolds-subcritical-polarizations on the other one. Inside these technologies or parallel to them, it seems that there is also some place left for a little amount of rather elementary topology, as is the beautiful addition of a grading to Floer homology by Seidel [22].

I will present here a few examples of Lagrangian submanifolds of compact symplectic manifolds and some of the results I evoked. A large part of the results in this paper are due to Biran, Cieliebak and Seidel and are already known. To make this repetition acceptable, I have tried to give a simple presentation together with many examples (in §§ 1 and 2.2) that illustrate the results presented in the paper, and a few side results that might be original (in §§ 2.3 and 2.4).

The paper is organized as follows. I start, in § 1, by the presentation of the basic examples of Lagrangian submanifolds in compact symplectic manifolds and some hints on how to construct them. In § 2, I consider the specific situation of the Lagrangian skeleton of a symplectic manifold, a notion introduced by Biran

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and that has already given a lot of beautiful results (see the survey [10]). Again, I give the examples I know and try to give some evidence that these might be the only possible examples — the results I have in this direction are very partial and based mainly on (soft) topological considerations. In the last section, § 3, I explain Seidel's idea of "grading" Lagrangian submanifolds and a few applications to non embedding theorems.

1 – Lagrangian submanifolds and how to produce them

I will use the standard notation (standard structures on the usual spaces $\mathbf{C}^n$, $\mathbf{P}^n(\mathbf{C})$, $T^\star L$, ...) and the basic results on symplectic geometry (Darboux and Darboux–Weinstein theorems, ...). See *e.g.* [2, 17, 24].

A Lagrangian submanifold of a symplectic manifold is a submanifold the tangent space of which is, at any point, a maximal totally isotropic subspace. If the symplectic form on the manifold W is denoted ω and the inclusion of the submanifold is

$$j : L \hookrightarrow W,$$

to say that L is Lagrangian is to say that $j^\star\omega = 0$ and $\dim L = \frac{1}{2}\dim W$.

1.1. The basic examples

Any manifold L is a Lagrangian submanifold of its cotangent bundle, by the zero section

$$L \hookrightarrow T^\star L.$$

Moreover, a neighborhood of a Lagrangian submanifold in any symplectic manifold is isomorphic to a neighborhood of the zero section in $T^\star L$ [24, 2]. We look for more global examples.

1.1.a. Tori

Starting from the fact that any curve in a symplectic surface is Lagrangian, we get that any product of closed embedded curves in $\mathbf{C}$ is a Lagrangian submanifold of $\mathbf{C}^n$ (n is the number of factors). Hence, any n-dimensional torus can be embedded as a Lagrangian submanifold in $\mathbf{C}^n$. So that we have Lagrangian tori in all symplectic manifolds (using Darboux theorem, anything we are able to embed in $\mathbf{C}^n$ can be embedded in any symplectic manifold).

More generally, if the Lagrangian submanifold we are considering is a leaf in a Lagrangian foliation, then this is a torus (this is part of the celebrated Arnold-Liouville theorem, see [1, 2]), so that the theory of integrable systems is full of Lagrangian tori.

1.1.b. Spheres

In contrast to the case of tori, in $\mathbf{C}^n$, there are no Lagrangian spheres (except for the circle S^1 in $\mathbf{C}$). This statement was one of the first applications of the holomorphic techniques introduced by Gromov in 1985 [15]. More generally, no Lagrangian embedding into $\mathbf{C}^n$ can be *exact*, in the sense that the Liouville form pulls back to an exact form, so that no manifold L with $H^1(L;\mathbf{R}) = 0$ and in particular no simply connected manifold can be embedded as a Lagrangian submanifold into $\mathbf{C}^n$.

Hence, unlike what happens in the case of tori, the question of embedding a sphere as a Lagrangian submanifold of a compact symplectic manifold is a global question (with respect to the symplectic manifold in question). We will see below that, for instance, the n-sphere is not a Lagrangian submanifold of the complex projective space (this is one of the results of [22], here Corollary 3.1.2), but that there are Lagrangian spheres in other compact symplectic manifolds (there are examples in § 2.2.b). An example of Lagrangian sphere comes from the mapping

$$\begin{aligned} S^{2n+1} &\longrightarrow \mathbf{P}^n(\mathbf{C}) \times \mathbf{C}^{n+1} \\ z &\longmapsto ([z], \overline{z}), \end{aligned}$$

which is a Lagrangian embedding (this was noticed by Polterovich [8]). Notice that, by compactness of the sphere, this is indeed an embedding in $\mathbf{P}^n(\mathbf{C})\times$ some bounded subset of $\mathbf{C}^n$ and that this can be considered as a Lagrangian embedding into $\mathbf{P}^n(\mathbf{C}) \times T^{2n+2}$, a compact symplectic manifold.

Notice also that the same mapping defines a Lagrangian embedding of $\mathbf{P}^n(\mathbf{C})$ into $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$, which is simply $[z] \mapsto ([z], [\overline{z}])$.

1.1.c. Surfaces

A surface which is not a torus cannot be embedded as a Lagrangian submanifold in $\mathbf{C}^2$. There are simple topological obstructions (see *e.g.* [4])... except in the case of the Klein bottle, that can be embedded in $\mathbf{C}^2$ as a totally real submanifold [21] but probably not as a Lagrangian, as announced by Mohnke [18].

However, it is not very difficult to embed all the surfaces as Lagrangian submanifolds of compact symplectic manifolds, as I show it now. Let L be an orientable surface, ω a volume form on L and τ a fixed point free involution reversing the orientation (such an involution exists because any orientable surface is a twofold covering of a non orientable one, according to the classification of surfaces). We can assume that the volume form has been chosen so that $\tau^\star\omega = -\omega$.

If L is an orientable surface, let $W = L \times L$, endowed with the symplectic form $\omega \oplus \omega$. It contains

$$\begin{aligned} \Delta : L &\hookrightarrow L \times L \\ x &\longmapsto (x, x) \end{aligned}$$

as a symplectic submanifold and

$$\begin{aligned} L &\hookrightarrow L \times L \\ x &\longmapsto (x, \tau(x)) \end{aligned}$$

as a Lagrangian submanifold. See Figure 1.

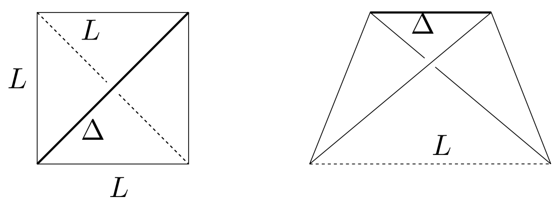

Figure 1

If L is a non orientable surface, let V be its orientation covering. The surface V can be considered as a complex curve, and it is well known (and easy to check) that the symmetric product of a complex curve is a smooth (complex) manifold, so that

$$W = V \times V/\mathfrak{S}_2$$

is a smooth complex surface. For instance, if $V = S^2 = \mathbf{P}^1(\mathbf{C})$, this is the projective plane $\mathbf{P}^2(\mathbf{C})$ (see footnote 2). For the involution σ defined by $\sigma(x, y) = (y, x)$, we have

$$\sigma^\star(\omega \oplus \omega) = \omega \oplus \omega.$$

Looking at the diagonal, it is easy to prove:

Lemma 1.1.1. *Given any neighbourhood of the diagonal in $V \times V$, there exists a symplectic form on W, which lifts to $\omega \oplus \omega$ outside this neighbourhood and such that the diagonal*

$$\Delta : V \to V \times V \to W$$

embeds V as a symplectic submanifold of W. The anti-diagonal map defines a Lagrangian embedding of V/σ in W. ∎

In the case where $V = S^2 = \mathbf{P}^1(\mathbf{C})$ with the antipodal involution (namely the map $x \mapsto -x$ on S^2 or $z \mapsto -1/\overline{z}$ on $\mathbf{P}^1$), the Lagrangian submanifold is $\mathbf{P}^2(\mathbf{R}) \subset \mathbf{P}^2(\mathbf{C})$ while the symplectic manifold is a sphere (this is actually a conic in $\mathbf{P}^2(\mathbf{C})$), as we shall see it more generally in § 2.2.a.

Proposition 1.1.2. *Any compact surface L is a Lagrangian submanifold of a symplectic 4-dimensional manifold W, which can be taken as the product $L \times L$ if L is orientable or to a suitable quotient of $\widetilde{L} \times \widetilde{L}$, $\widetilde{L}$ being its orientation covering, if it is not.* ∎

1.2. Real parts of real manifolds

Most of the Lagrangian submanifolds we know are real parts of complex (Kähler) manifolds endowed with real structures.

1.2.a. The real structure

Let W be a complex Kähler manifold of complex dimension n, the Kähler form of which is called ω. We assume moreover that W is endowed with a real structure, namely an anti-holomorphic involution S.

Examples 1.2.1.

(**1**) $W = \mathbf{C}$, ω as usual, $S(z) = -1/\overline{z}$,

(**2**) $W = \mathbf{C}^n$ or $W = \mathbf{P}^n(\mathbf{C})$, ω standard, $S(z) = \overline{z}$,

(**3**) $W = (\mathbf{C}^\star)^n$, ω standard, $S(z_1, \ldots, z_n) = (1/\overline{z}_1, \ldots, 1/\overline{z}_n)$. □

1.2.b. The real part

Let us call $L = W_{\mathbf{R}}$ the "real part" — the fixed points of S. This can be empty (as in the first example above). Otherwise, this is a real analytic submanifold of W, all the components of which have dimension n (this is elementary and very classical; there is a proof, for instance in [3]).

1.2.c. Lagrangians

Moreover, if $S^\star\omega = -\omega$, the real part is Lagrangian. This is what happens in the second example above, proving that $\mathbf{R}^n$ is a Lagrangian submanifold of $\mathbf{C}^n$... and $\mathbf{P}^n(\mathbf{R})$ a Lagrangian submanifold of $\mathbf{P}^n(\mathbf{C})$ (I will come back to this simple example below).

Another example is the case of $W \times W$, endowed with the real structure $\widetilde{S}$ defined by

$$\widetilde{S}(x, y) = (S(y), S(x))$$

and the product Kähler structure. One has $\widetilde{S}^\star(\omega \oplus \omega) = -(\omega \oplus \omega)$ if S satisfies $S^\star\omega = -\omega$. This shows W as a Lagrangian submanifold of $(W \times W, \omega \oplus \omega)$ by

$$\begin{aligned} W &\longrightarrow W \times W \\ x &\longmapsto (x, S(x)), \end{aligned}$$

a kind of "anti-diagonal". We have met an example of this situation, the case where $W = \mathbf{P}^n(\mathbf{C})$, at the end of § 1.1.b.

But of course, we do not really need to have $S^\star\omega = -\omega$ at all points of W, the points of L are enough. This is also classical, but I nevertheless include a proof.

Proposition 1.2.2. *Assume $(S^\star\omega)_x = -\omega_x$ for all points x in L, then L is a Lagrangian submanifold of W.*

Proof: Let $j : L \to W$ denote the inclusion. We have $S \circ j = j$, so that,

$$j^\star\omega = (S \circ j)^\star\omega = j^\star(S^\star\omega) = j^\star(-\omega) = -j^\star\omega. \blacksquare$$

This is what happens in our third example, showing again T^n as a Lagrangian submanifold of $(\mathbf{C}^\star)^n$.

1.2.d. Complex hypersurfaces

Let $P \in \mathbf{R}[x_0, \ldots, x_{n+1}]$ be a homogeneous polynomial such that the complex hypersurface W of $\mathbf{P}^{n+1}(\mathbf{C})$ defined by the annulation of P is nonsingular. The coefficients of P being real, W is stable under complex conjugation and its real part must thus be a Lagrangian submanifold.

Consider for instance the case of the quadric defined by

$$P(x_0, \ldots, x_{n+1}) = -x_0^2 + \sum_{i=1}^{n+1} x_i^2.$$

Its real part is a Lagrangian sphere. I will come back to projective quadrics in § 2.2.

This is a good way to construct examples having additional properties. For instance, using hypersurfaces of degree 4, Bryant has constructed in [12] examples of Calabi-Yau projective hypersurfaces, which contain special Lagrangian tori.

1.2.e. Real structures on Lie groups

Let us try now to use this in (a little) more elaborate examples. We apply this to $W = \mathfrak{gl}(n; \mathbf{C})$, the Lie algebra of all complex $n \times n$ matrices, endowed with its natural complex structure, the Kähler form

$$\omega(X, Y) = \operatorname{Im} \operatorname{tr}({}^t X \overline{Y})$$

(I use the clearer French notation ${}^t A$ for the transpose of the matrix A; the form ω is of course the standard symplectic form on the complex vector space $\mathfrak{gl}(n; \mathbf{C})$).

Consider the real structures

(**1**) $S(A) = \overline{A}$, for which $V = \mathfrak{gl}(n; \mathbf{R}) \subset \mathfrak{gl}(n; \mathbf{C})$, as a special case of our second example above,

(**2**) $S(A) = ({}^t\overline{A})^{-1}$ (this is similar to the third example above, here we use the open subset $\mathrm{GL}(n; \mathbf{C})$ of invertible matrices), in this case

$$V = \mathrm{U}(n) \subset \mathrm{GL}(n; \mathbf{C}) \subset \mathfrak{gl}(n; \mathbf{C}).$$

In the second case, the tangent map to S at a point A is given by

$$T_A S(X) = -({}^t\overline{A})^{-1}({}^t\overline{X})({}^t\overline{A})^{-1}$$

so that

$$\begin{aligned}(S^\star\omega)_A(X, Y) &= \omega_{S(A)}(T_A S(X), T_A S(Y)) \\ &= \operatorname{Im} \operatorname{tr}((\overline{A})^{-1}(\overline{X})(\overline{A})^{-1}({}^t A)^{-1}({}^t Y)({}^t A^{-1})),\end{aligned}$$

which, if $A \in \mathrm{U}(n)$, gives

$$(S^\star\omega)_A(X, Y) = \operatorname{Im} \operatorname{tr}(\overline{X}{}^t Y) = \operatorname{Im}\left(\overline{\operatorname{tr}({}^t X\overline{Y})}\right) = -\operatorname{Im} \operatorname{tr}({}^t X\overline{Y}) = -\omega_A(X, Y).$$

Hence $\mathrm{U}(n)$ is a Lagrangian submanifold of $\mathfrak{gl}(n; \mathbf{C})$ (using Proposition 1.2.2). As it is contained in $\mathfrak{gl}(n; \mathbf{C}) - \{0\}$, the inclusion gives a map to $\mathbf{P}^{n^2-1}(\mathbf{C})$

$$\begin{tikzcd}
\mathrm{SU}(n) \arrow[r, hook] \arrow[d, "\mathbf{Z}/n"'] & \mathrm{U}(n) \arrow[r] \arrow[d, "S^1"'] \arrow[rd, "f"] & \mathfrak{gl}(n; \mathbf{C}) - \{0\} \arrow[d, "\pi"] \\
\mathbf{P}\,\mathrm{SU}(n) \arrow[r, equal] & \mathbf{P}\,\mathrm{U}(n) \arrow[r, "g"'] & \mathbf{P}^{n^2-1}(\mathbf{C})
\end{tikzcd}$$

which in turn gives the injective map g in the diagram, which is a Lagrangian embedding.

In the case where $n = 2$, $\mathbf{P}\,\mathrm{SU}(2)$ is a real projective space $\mathbf{P}^3(\mathbf{R})$ and we might expect to get an "exotic" (whatever it means) Lagrangian embedding of this manifold into $\mathbf{P}^3(\mathbf{C})$. This is unfortunately not the case. We are looking at

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\} \subset \mathfrak{gl}(2; \mathbf{C}).$$

This is contained in the 4-dimensional (real) subspace $\mathbf{H}$ generated by the "Pauli matrices"

$$\mathbf{H} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle.$$

We can thus include $\mathbf{H}$ in the diagram above, getting

$$\begin{tikzcd}
\mathrm{SU}(2) \arrow[r, hook] \arrow[d, "\{\pm 1\}"'] & \mathbf{H} - \{0\} \arrow[r, hook] \arrow[d, "\mathbf{R}^\star"] & \mathfrak{gl}(2; \mathbf{C}) - \{0\} \arrow[d, "\mathbf{C}^\star"] \\
\mathbf{P}\,\mathrm{SU}(2) \arrow[r, equal] & \mathbf{P}^3(\mathbf{R}) \arrow[r, hook] & \mathbf{P}^3(\mathbf{C})
\end{tikzcd}$$

so that the Lagrangian $\mathbf{P}^3(\mathbf{R})$ got this way is a standard one, in the sense that it comes from a linear real subspace.

1.3. Group actions

The example of the Lagrangian embedding of S^{2n+1} into $\mathbf{P}^n(\mathbf{C}) \times \mathbf{C}^{n+1}$ in § 1.1.b can also be understood as a special case of the Lagrangian inclusion

$$\mu^{-1}(0) \longrightarrow (\mu^{-1}(0)/G) \times W$$

where μ is the momentum mapping(1) $W \to \mathfrak{g}^\star$ of a Hamiltonian G-action on W and the manifold on the right is endowed with the symplectic form $\omega_{\text{reduced}} \oplus -\omega$. In the sphere example, $G = S^1$ (and we have replaced complex conjugation with a $-$ sign in the symplectic form).

There is another way to use momentum mappings to produce Lagrangian submanifolds. If $\mu : W \to \mathfrak{g}^\star$ is the momentum mapping for the Hamiltonian action of the compact Lie group G on W, for any $x \in \mu^{-1}(0)$, the orbit $G \cdot x$ of x in W is an isotropic submanifold. The most classical application of this remark

(1) For the basic properties of momentum mappings used here, see *e.g.* [5, 16].

is the fact that the regular levels of the momentum mapping of a torus action are Lagrangian tori.

Here is a more original example. The symplectic manifold is $W = \mathbf{P}^3(\mathbf{C})$. The Lagrangian will be the quotient of a 3-sphere by a subgroup of order 12 of SU(2). This example, due to River Chiang [13], will rather be used as a counter-example in this paper.

Proposition 1.3.1 (Chiang [13]). *There exists a Lagrangian submanifold of* $\mathbf{P}^3(\mathbf{C})$ *which is a quotient of* SO(3) *(or* $\mathbf{P}^3(\mathbf{R})$*) by the symmetric group* $\mathfrak{S}_3$.

Proof: Recall first that, if $\mu : W \to \mathfrak{g}^\star$ is the momentum mapping of the Hamiltonian action of the compact group G on a symplectic manifold W, then for any $x \in \mu^{-1}(0)$, the orbit $G \cdot x$ of x in W is an isotropic submanifold (the proof is by straightforward verification, see for instance [5]).

In the present example, we make SO(3) act on the 2-sphere (by rotations!), then on $S^2 \times \cdots \times S^2$ (n factors) by the diagonal action. This is a Hamiltonian action, the momentum mapping of which is

$$\begin{aligned} \widetilde{\mu} : S^2 \times \cdots \times S^2 &\longrightarrow \mathbf{R}^3 \\ (x_1, \ldots, x_n) &\longmapsto x_1 + \cdots + x_n. \end{aligned}$$

If $n = 3$, $\widetilde{\mu}^{-1}(0)$ is an orbit of SO(3), since the two conditions

$$x_1, x_2 \text{ and } x_3 \in S^2 \text{ and } x_1 + x_2 + x_3 = 0$$

imply

$$x_1 \cdot x_2 = -\frac{1}{2}$$

so that x_1, x_2 and x_3 are the vertices of an equilateral triangle centered at 0. Moreover, any element of the stabilizer of a point (x_1, x_2, x_3) in this orbit must fix the plane generated by the three vectors and hence be the identity. Hence $\widetilde{\mu}^{-1}(0) \cong \mathrm{SO}(3)$. According to the previous remark, this is an isotropic submanifold and it is Lagrangian for dimensional reasons.

Notice that the momentum mapping $\widetilde{\mu}$ is invariant by the action of the symmetric group $\mathfrak{S}_n$ on $S^2 \times \cdots \times S^2$, so that it defines a map

$$\mu : S^2 \times \cdots \times S^2/\mathfrak{S}_n \longrightarrow \mathbf{R}^3$$

that is the momentum mapping of an SO(3)-action. Recall (using $S^2 = \mathbf{P}^1(\mathbf{C})$) that the quotient in the left hand side([2]) is $\mathbf{P}^n(\mathbf{C})$. For $n = 3$, we get that

$$\mu^{-1}(0) = \widetilde{\mu}^{-1}(0)/\mathfrak{S}_3 \cong \mathrm{SO}(3)/\mathfrak{S}_3$$

is a Lagrangian submanifold of $\mathbf{P}^3(\mathbf{C})$ (notice that the symmetric group $\mathfrak{S}_3$ is embedded as a subgroup in SO(3), the transposition $(x, y, z) \mapsto (y, x, z)$ corresponding to the half-turn about z). ∎

Remark 1.3.2. The principal stabilizer of the SO(3)-action in this example is $\mathbf{Z}/3$, this group, realized as the group of order-3 rotations about the axis $x + y + z$, being the actual stabilizer of the image of a (generic) point (x, y, z). □

Remark 1.3.3. The fundamental group Γ of this manifold is the inverse image of $\mathfrak{S}_3$ by the covering map φ:

$$\begin{tikzcd} \mathrm{SU}(2) \arrow[r, "\varphi"] & \mathrm{SO}(3) \\ \Gamma \arrow[u] \arrow[r] & \mathfrak{S}_3 \arrow[u] \end{tikzcd}$$

Recall that φ maps an element with eigenvalues $(e^{i\theta}, e^{-i\theta})$ to a rotation of angle $\pm 2\theta$ (and in particular the elements of square -1 of SU(2) to half-turns), so that Γ is generated by an element α of order 6 (mapped to an element of order 3 in $\mathfrak{S}_3$) and an element β of order 4 (mapped to an element of order 2 in $\mathfrak{S}_3$) such that

$$\alpha^3 = \beta^2 (= -\mathrm{Id}) \text{ and } \beta\alpha\beta^{-1} = \alpha^{-1}. \ \square$$

I will come back to this example in § 2.5 and in § 3.3.b.

2 – Lagrangian skeletons

2.1. Biran's barriers

In [9], Biran considers the situation of a Kähler manifold W (the cohomology class of the Kähler form of which is integral) endowed with a complex hypersurface V, the fundamental class of which is dual to some integral multiple of

([2]) Map $\mathbf{C}^n/\mathfrak{S}_n$ to $\mathbf{C}^n$, sending an n-tuple of points $(z_1, \ldots, z_n)$ to the (coefficients of the) polynomial $(z - z_1) \cdots (z - z_n)$ and compactify $\mathbf{C}^n/\mathfrak{S}_n$ as $(\mathbf{P}^1(\mathbf{C}))^n/\mathfrak{S}_n$, $\mathbf{C}^n$ as $\mathbf{P}^n(\mathbf{C})$.

the Kähler class. Then he proves that there is an isotropic CW-complex in W, the complement of which is a disc bundle of the symplectic normal bundle of V in W. Biran derives quite a few consequences of this theorem, mainly on symplectic embeddings in W (see [9]). An important tool in the proof is a function, the minimum of which is the Kähler submanifold V, and which is used to construct the isotropic skeleton.

We will consider here something similar, but from the opposite point of view, in the sense that we start from the Lagrangian skeleton, assumed to be, not only Lagrangian, but also smooth. The precise question I will consider in this section is the following one: given a closed manifold L of dimension n, is it possible to find a compact symplectic manifold W of dimension $2n$ and a Morse-Bott function $f : W \to \mathbf{R}$ with exactly two critical values

- a minimum, reached on a Lagrangian submanifold diffeomorphic with L,
- a maximum, reached on a symplectic submanifold V of dimension $2n-2$?

Remarks 2.1.1.

(**1**) If L is a Lagrangian submanifold of a symplectic manifold W, it has a tubular neighborhood which is symplectomorphic with a neighborhood of the zero section in the cotangent bundle $T^\star L$. The question is thus a weak version of the following: is it possible to compactify $T^\star L$ into a symplectic manifold by adding to it a symplectic "hypersurface"?

(**2**) A cotangent bundle cannot be compactified into a manifold by adding a point to it (there is an easy topological argument). In the symplectic framework, one could argue that $T^\star L$ has a vector field (the Liouville vector field) which expands the symplectic form ($\mathcal{L}_X\omega = \omega$), a property which is incompatible with Darboux' theorem in a neighborhood of the point we try to add at infinity. This argument can be adapted to prove that, in a symplectic compactification of $T^\star L$ by a symplectic manifold, the latter must have codimension 2 (this generalization was explained to me by Kai Cieliebak). □

According to these remarks, we are looking for a compactification of $T^\star L$ by a symplectic manifold V of dimension $2\dim L - 2$, such that L and V are the (only) critical submanifolds of some Morse-Bott function on the resulting compact symplectic manifold.

2.2. Examples

2.2.a. The most classical one

This is the first example of a Lagrangian barrier given in [9]. The symplectic manifold is $W = \mathbf{P}^n(\mathbf{C})$, the complex projective space, the Lagrangian L is its real part $\mathbf{P}^n(\mathbf{R})$ (as in Example 1.2.1 above) and V the complex hypersurface (quadric) of equation

$$Q_{n-1} : z_0^2 + \cdots + z_n^2 = 0.$$

The assertion (well known to real algebraic geometers) is that the complement of $\mathbf{P}^n(\mathbf{R})$ in $\mathbf{P}^n(\mathbf{C})$ retracts to a quadric hypersurface of $\mathbf{P}^n(\mathbf{C})$ as, viewed from the opposite side, the complement of a quadric retracts to $\mathbf{P}^n(\mathbf{R})$. See Figure 2 where the case $n = 1$ is depicted.

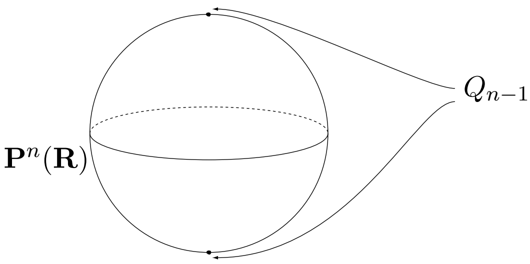

Figure 2 – A barrier in $\mathbf{P}^n(\mathbf{C})$

For the sake of completeness, let me give a symplectic (and non computational) proof of these two facts here. We consider the following diagram

$$\begin{tikzcd}
\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \arrow[r, "(x,y)\mapsto x+iy"] & \mathbf{C}^{n+1} \arrow[r, "x+iy\mapsto x\wedge y"] & \bigwedge^2 \mathbf{R}^{n+1} \ni & \frac{x\wedge y}{\|x\|^2+\|y\|^2} \\
TS^n \arrow[u, hook] \arrow[r] \arrow[d] & \mathbf{C}^{n+1} - \{0\} \arrow[u, hook] \arrow[d] & & \\
T\mathbf{P}^n(\mathbf{R}) \arrow[r, "j"'] & \mathbf{P}^n(\mathbf{C}) \arrow[uur] & \ni [x+iy] \arrow[uur, mapsto] &
\end{tikzcd}$$

in which the tangent bundle TS^n is considered as

$$TS^n = \left\{(x, y) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \mid \|x\|^2 = 1 \text{ and } x \cdot y = 0\right\}$$

and $T\mathbf{P}^n(\mathbf{R})$ is its quotient by the involution $(x, y) \mapsto (-x, -y)$.

To begin with, we only consider the left part of the diagram. One easily checks that any point of $\mathbf{P}^n(\mathbf{C})$ can be written $[x+iy]$ for real orthogonal vectors x and y with $\|x\|^2 = 1$ and $\|y\|^2 \leq 1$. Moreover, such an x and a y are unique, except that $[x+iy] = [-x-iy]$. This shows

- that the map j in the diagram is an embedding, when restricted to the unit open disc bundle of $T\mathbf{P}^n(\mathbf{R})$
- and that the complement of its image consists of the points of the form $[x+iy]$ with x and y orthogonal and both unit vectors, namely, the complement is the quadric hypersurface of equation

$$\sum(x_j^2 - y_j^2) + 2i \sum x_j y_j = \sum z_j^2 = 0.$$

Let us look now at the right part of the diagram. Notice that, identifying $\bigwedge^2 \mathbf{R}^{n+1}$ with the vector space $\mathfrak{so}(n+1)$ of skew-symmetric matrices, the horizontal map $(x,y) \mapsto x \wedge y$ is the momentum mapping μ for the diagonal action of $\mathrm{SO}(n+1)$ on $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, or on $\mathbf{C}^{n+1}$, or even on $\mathbf{P}^n(\mathbf{C})$, provided we homogenize the formula. Let us consider now the function $f = \|\mu\|^2$, namely the function

$$\begin{aligned} f : \mathbf{P}^n(\mathbf{C}) &\longrightarrow \mathbf{R} \\ [x+iy] &\longmapsto \left\| \frac{x \wedge y}{\|x\|^2 + \|y\|^2} \right\|^2. \end{aligned}$$

Taking, as above, x and y orthogonal and x a unit vector, this is

$$f([x+iy]) = \frac{\|y\|^2}{(1+\|y\|^2)^2}$$

a Morse-Bott function which achieves its minimum on $\mathbf{P}^n(\mathbf{R})$, its maximum on the quadric hypersurface, and has no other critical point.

2.2.b. The simplest one

Let now W be Q_n, that is, again, the quadric hypersurface above, now in complex dimension n. This manifold is also the Grassmannian of oriented 2-planes in $\mathbf{R}^{n+2}$, that is,

$$\widetilde{G}_2(\mathbf{R}^{n+2}) = \mathrm{SO}(n+2)/\,\mathrm{SO}(n) \times \mathrm{SO}(2)$$

(recall the action of the orthogonal group we have used above). The Lagrangian L is the n-sphere of points $[i, x_1, \ldots, x_{n+1}]$ (the x_j's are real) and the symplectic manifold V is the quadric Q_{n-1} obtained as the intersection of Q_n with the hyperplane $z_0 = 0$. A possible Morse-Bott function g is simply

$$g([z_0, \dots, z_{n+1}]) = 1 - \frac{|z_0|^2}{\sum_{i=0}^{n+1} |z_i|^2}.$$

Remarks 2.2.1. Notice that:

(**1**) The quadric Q_n is also a small coadjoint orbit of $\mathrm{SO}(n+2)$, another way to give it a symplectic structure. The latter coincides with some multiple of the former.

(**2**) It is endowed with a Hamiltonian $\mathrm{SO}(n+1)$-action (this is the subgroup corresponding to the hyperplane $x_0 = 0$ in $\mathbf{R}^{n+2}$), the momentum mapping of which is

$$z_0 e_0 + x + iy \longrightarrow \frac{x \wedge y}{|z_0|^2 + \|x\|^2 + \|y\|^2}.$$

The function g is just the square of the norm of this momentum mapping.

(**3**) Writing $n+2 = 2k$ or $2k+1$, the symplectic manifold $\widetilde{G}_2(\mathbf{R}^{n+2})$ is endowed with an action of $\mathrm{SU}(k)$, using the inclusions

$$\mathrm{SU}(k) \subset \mathrm{SO}(2k) \subset \mathrm{SO}(n+2).$$

I will come back to the case of $\mathrm{SU}(2)$ in § 2.4. □

2.2.c. Relations

I believe that this example (the sphere in the quadric) is more elementary than the previous one (the real projective space in the complex projective space), because the normal bundle to the symplectic submanifold Q_{n-1} in Q_n has Chern class 1 here, while that of Q_{n-1} in $\mathbf{P}^n(\mathbf{C})$ above has Chern class 2 or, which amounts to the same thing, because the Lagrangian submanifold S^n here is simply connected, while the $\mathbf{P}^n(\mathbf{R})$ above is not. Also, the "projective space example" is the quotient of the "sphere example" by an involution.

Start with the projection

$$\mathbf{P}^{n+1}(\mathbf{C}) - \{[1, 0, \dots, 0]\} \longrightarrow \mathbf{P}^n(\mathbf{C})$$

and restrict it to the quadric Q_n, so that it becomes a two-fold covering map

$$Q_n \longrightarrow \mathbf{P}^n(\mathbf{C})$$

branched along the $(n-1)$-dimensional quadric Q_{n-1}. We have a diagram

$$\begin{tikzcd}
 & Q_{n-1} \arrow[dr] \arrow[dd] & & \\
S^n \arrow[rr] \arrow[dd] & & \widetilde{G}_2(\mathbf{R}^{n+2}) \arrow[dd] \arrow[ddr, dotted] & \\
 & Q_{n-1} \arrow[dr] & & \\
\mathbf{P}^n(\mathbf{R}) \arrow[rr] & & \mathbf{P}^n(\mathbf{C}) \arrow[r, "f"'] & \mathbf{R}
\end{tikzcd}$$

from which we deduce that our two submanifolds are indeed the minimum and the maximum of a Morse function (notice however that the symplectic form of $\mathbf{P}^n(\mathbf{C})$ pulls back to a form that is singular along the quadric Q_{n-1}). This is shown on Figure 3. Notice that the quotient mapping $S^2 \times S^2 \to \mathbf{P}^2(\mathbf{C})$ considered in § 1.1.c is the case $n = 2$ of this construction.

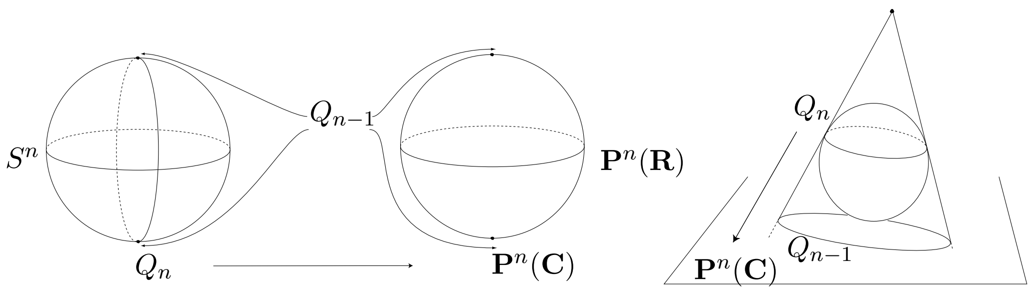

Figure 3 – Two viewpoints on the double covering $Q_n \to \mathbf{P}^n(\mathbf{C})$

2.3. General topological remarks

In this part, we give a few constraints on the topology of a manifold L which is a Lagrangian skeleton. Let us fix a few notations. We consider a symplectic manifold W endowed with a Morse-Bott function $f : W \to \mathbf{R}$. We assume that f has only

- a minimum, reached on a Lagrangian submanifold L
- a maximum, reached along a symplectic submanifold V.

We write $f(W) = [a, b] \subset \mathbf{R}$ and call

$$\begin{cases} \mathcal{U} = f^{-1}([a, b[) & \text{a neighborhood of } L \\ \mathcal{V} = f^{-1}(]a, b]) & \text{a neighborhood of } V, \end{cases}$$

so that $\mathcal{U} \cap \mathcal{V}$ retracts on $\mathcal{E}$, a regular level of f, which is a sphere bundle both on V and L. We can choose an almost complex structure on W with is calibrated by the symplectic form and such that V is a complex submanifold of W. This gives us a Riemannian metric on W, with respect to which the gradient of the Morse function can be considered.

2.3.a. In dimension 4

Let us prove now that the only surfaces that can occur as Lagrangian skeletons of 4-dimensional symplectic manifolds are the sphere and the projective plane.

Proposition 2.3.1. *Let W be a compact connected symplectic manifold of dimension 4 endowed with a Morse-Bott function with two critical values, the minimum being reached along a Lagrangian surface L and the maximum along a symplectic surface V. Then V is a 2-sphere and*

- *either L is a 2-sphere and W is homeomorphic with $S^2 \times S^2$,*
- *or L is a real projective plane and W is homeomorphic with $\mathbf{P}^2(\mathbf{C})$.*

From what we deduce in particular that the the only Lagrangian surfaces constructed in § 1.1.c that are Lagrangian skeletons are the sphere and the projective plane.

Proof: The symplectic submanifold V is an oriented surface. Let us assume that this is not a sphere. This is then a surface of genus $g \geq 1$. A regular level $\mathcal{E}$ of f is the total space of a principal S^1-bundle over V. Using van Kampen's theorem, we get that

$$\pi_1(\mathcal{E}) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, c; c^m = [a_1, b_1] \cdots [a_g, b_g] \rangle$$

where c is the image of the fiber and m is (up to sign) the Euler class of the bundle $p : \mathcal{E} \to V$.

Notice that, as V is symplectic, m cannot be zero. If m were zero, the projection p would induce an injective map

$$p^\star : H^2(V; \mathbf{Z}) \longrightarrow H^2(\mathcal{E}; \mathbf{Z})$$

(this is the Gysin exact sequence of the fibration p), so that the map $p^\star$ would also be injective on $H^2(V;\mathbf{R})$. But, in the Mayer-Vietoris exact sequence associated with the decomposition $W = \mathcal{U}\cup\mathcal{V}$, since $\mathcal{U}$ retracts onto L, $\mathcal{V}$ onto V and $\mathcal{U}\cap\mathcal{V}$ onto $\mathcal{E}$, we see a piece

$$\begin{array}{ccccc} H^2(W;\mathbf{R}) & \longrightarrow & H^2(L;\mathbf{R})\oplus H^2(V;\mathbf{R}) & \longrightarrow & H^2(\mathcal{E};\mathbf{R}) \\ [\omega] & \longmapsto & (0,[\omega]) & \longmapsto & p^\star[\omega] \end{array}$$

(where $[\omega]$ is the class of the symplectic form in W and in V), so that $[\omega]$ is a non zero element of $\operatorname{Ker} p^\star$, which shows that $p^\star$ cannot be injective.

Hence $H_1(\mathcal{E};\mathbf{Z}) \cong \mathbf{Z}^{2g}\oplus\mathbf{Z}/m$ for a non zero integer m which is (plus or minus) the Euler class of $\mathcal{E}\to V$.

Let us prove now that the Lagrangian surface L must be orientable. Assume that it is not. Then

$$H_1(L;\mathbf{Z}) \cong \mathbf{Z}^{g'}\oplus\mathbf{Z}/2$$

for some integer g'. We look firstly at the Gysin exact sequence of the fibration $\mathcal{E}\to V$:

$$0 \longrightarrow H^1(V;\mathbf{Z}) \xrightarrow{p^\star} H^1(\mathcal{E};\mathbf{Z}) \xrightarrow{p_\star} H^0(V;\mathbf{Z}) \xrightarrow{\times m} H^2(V;\mathbf{Z}) \longrightarrow$$

so that this $p^\star$ is an isomorphism. We look then at the Mayer-Vietoris exact sequence for $W = \mathcal{U}\cup\mathcal{V}$ again. It splits in two parts:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(W;\mathbf{R}) & \longrightarrow & H^1(L;\mathbf{R})\oplus H^1(V;\mathbf{R}) & \longrightarrow & H^1(\mathcal{E};\mathbf{R}) & \longrightarrow & 0 \\ & & & & \mathbf{R}^{g'}\oplus\mathbf{R}^{2g} & & \mathbf{R}^{2g} & & \end{array}$$

which gives $b_1(W) = g'$, and

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(W;\mathbf{R}) & \longrightarrow & H^2(L;\mathbf{R})\oplus H^2(V;\mathbf{R}) & \longrightarrow & H^2(\mathcal{E};\mathbf{R}) \longrightarrow \\ & & & & 0\oplus\mathbf{R} & & \mathbf{R}^{2g} \\ & \longrightarrow & H^3(W;\mathbf{R}) & \longrightarrow & 0 & & \end{array}$$

which implies that $b_2(W) = 1$ (it must be at least 1, since W is symplectic) and $H^3(W;\mathbf{R}) \cong \mathbf{R}^{2g}$, hence $b_3(W) = 2g$ and, according to Poincaré duality, $g' = 2g$. The fundamental group of our non orientable surface L has a presentation

$$\pi_1(L) = \langle \alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g,\varepsilon;[\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]\varepsilon^2\rangle,$$

so that, using van Kampen again,

$$\pi_1(\mathcal{E}) = \langle \alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g,\varepsilon,d;d^k = [\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]\varepsilon^2\rangle$$

for some integer k. But then,

$$H_1(\mathcal{E};\mathbf{Z}) \cong \mathbf{Z}^{2g} \times (\mathbf{Z} \times \mathbf{Z})/(2,k)$$

has rank $2g+1$, in contradiction with the fact that

$$H_1(L;\mathbf{Z}) \cong \mathbf{Z}^{2g} \oplus \mathbf{Z}/m.$$

Hence, L is an orientable surface. We can look at the S^1-bundle $\mathcal{E} \to L$ and at its Gysin exact sequence, which gives exactly the same thing as it gave on the side of V. In particular, V and L have the same genus and the two bundles have (up to sign) the same Euler class. The difference is that we know what the bundle on L is: this is the sphere bundle of the cotangent bundle $T^\star L$. So that its Euler class is $2g-2$. Hence $m = 2g-2$ (and $g \neq 1$!).

We are thus left to investigate the case of a manifold W which is obtained by taking the disc bundles of $T^\star V$ and TV and gluing them along the boundary. Our manifold W is thus homeomorphic with the manifold $\mathbf{P}(T^\star V \oplus \mathbf{C})$, where, for the simplicity of the notation, I have considered TV as a complex line bundle over V. This is a manifold which is fibered over V, with fiber S^2; the bundle has two sections, one of which is our Lagrangian L and the other one is the symplectic surface V. Notice that the "complex" description is not completely irrelevant, at least outside L, since the normal bundle to the symplectic surface V is a symplectic (or complex) line bundle. The fibers of $W \to V$ are thus (topologically) 2-spheres that have a symplectic part (almost all, if we wish) and a Lagrangian one (as small as we wish). See Figure 4.

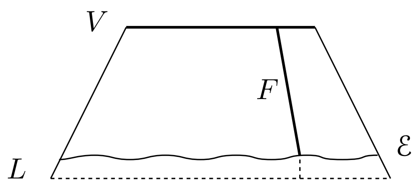

Figure 4

In particular, if F is the homology class of such a fiber, the symplectic form integrates on F to a positive number. This is what will give us the expected contradiction: $H_2(W;\mathbf{Z})$ is a $\mathbf{Z}^2$ generated by the class $[V]$ of the symplectic surface V and the class F of the fiber, satisfying

$$\begin{cases} F \cdot F = 0 \\ [V]\cdot[V] = 2-2g \qquad \text{(Euler class of } TV\text{)} \\ [V] \cdot F = 1. \end{cases}$$

The homology class $[L]$ of L is then

$$[L] = (2g-2)F + [V].$$

Because of the remark above, we must then have

$$\int_L \omega = (2g-2)\int_F \omega + \int_V \omega > 0$$

in contradiction with the fact that L is Lagrangian.

We have thus proved that V is a 2-sphere. The same "symplectic-Mayer-Vietoris" argument as above gives that the Euler class m of its normal bundle is non zero. The exact homotopy sequence gives that $\pi_1(\mathcal{E}) = \mathbf{Z}/m$. On the other side, this gives that $m = 2$ and, either $L = S^2$ and W is homeomorphic with $\mathbf{P}(T^\star S^2 \oplus \mathbf{C}) \cong S^2 \times S^2$ (this is why Figures 1 and 4 look so similar), or $L = \mathbf{P}^2(\mathbf{R})$ and W is homeomorphic with $\mathbf{P}^2(\mathbf{C})$ using the gradient flow. ∎

Remark 2.3.2. As I understand from what Kai Cieliebak explains me, the fact that the fundamental group of a Lagrangian skeleton must be small (and in particular, due to the classification of surfaces, Proposition 2.3.1), can also be derived from the techniques in [11]. However, I have chosen to present here these topological proofs. □

Remark 2.3.3. I believe, but have not proven here, that the symplectic 4-folds admitting a surface as a Lagrangian skeleton are actually *symplectomorphic* to the complex projective plane or to $S^2 \times S^2$ with the product symplectic form. □

2.3.b. In higher dimensions

The first remark is that, in the situation we are considering, the symplectic submanifold must have codimension 2.

Proposition 2.3.4. *Let W be a compact connected symplectic manifold of dimension $2n$. Assume that*

$$f : W \longrightarrow \mathbf{R}$$

is a function with only two critical values, the minimum, reached along a Lagrangian submanifold $L \subset W$, and the maximum, reached along a symplectic submanifold V. Then the latter has codimension 2.

Proof: This is based on a similar Gysin-Mayer-Vietoris argument. If

$$\operatorname{codim} V = 2k \geq 4,$$

the bundle $\mathcal{E} \to V$ is an oriented S^{2k-1}-bundle so that the Gysin exact sequence gives the injectivity of the map

$$p^\star : H^2(V; \mathbf{Z}) \longrightarrow H^2(\mathcal{E}; \mathbf{Z}).$$

And this is in contradiction with the Mayer-Vietoris exact sequence

$$\begin{array}{ccccc} H^2(W;\mathbf{R}) & \longrightarrow & H^2(L;\mathbf{R}) \oplus H^2(V;\mathbf{R}) & \longrightarrow & H^2(\mathcal{E};\mathbf{R}) \\ [\omega] & & (0, j^\star[\omega]) & & \end{array}$$

which gives a non zero element $j^\star[\omega]$ in the kernel of $p^\star$. ∎

The next result gives restrictions on the Lagrangian skeleton L: it must have a very small fundamental group.

Proposition 2.3.5. *Let W be a compact connected symplectic manifold of dimension $2n$. Assume that*

$$f : W \longrightarrow \mathbf{R}$$

is a function with only two critical values

- *the minimum, reached along a Lagrangian submanifold $L \subset W$*
- *and the maximum, reached along a symplectic simply connected submanifold V.*

Then W is simply connected, the symplectic submanifold V is dual to multiple of the symplectic form of W and the fundamental group of L is (at most) a (finite) cyclic group.

Proof: The case $n = 1$ is trivial and the case $n = 2$ is included in Proposition 2.3.1 above. We can thus assume that $n \geq 3$. We consider a regular value $\mathcal{E}$ of the function f. This $(2n-1)$-dimensional submanifold of W is a sphere bundle of the two normal bundles of L and V, so that we have two fibrations

$$S^{n-1} \subset \mathcal{E} \longrightarrow L \text{ and } S^1 \subset \mathcal{E} \longrightarrow V$$

(according to Proposition 2.3.4, $\operatorname{codim} V = 2$). The exact homotopy sequence for the fibration $\mathcal{E} \to L$ is

$$\pi_1(S^{n-1}) \longrightarrow \pi_1(\mathcal{E}) \longrightarrow \pi_1(L) \longrightarrow 0.$$

As $n \geq 3$, this implies that $\pi_1(\mathcal{E})$ is isomorphic to $\pi_1(L)$. For the fibration $\mathcal{E} \to V$, we have

$$\pi_2(V) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(\mathcal{E}) \longrightarrow \pi_1(V) \longrightarrow 0.$$

This gives, as V is simply connected, that $\pi_1(\mathcal{E})$ is $\mathbf{Z}$ or a quotient of $\mathbf{Z}$. This also proves that W is simply connected, using van Kampen.

We evaluate $H^2(\mathcal{E};\mathbf{Z})$ using pieces of the Gysin exact sequence for the fibration $\mathcal{E} \to V$

$$H^1(\mathcal{E};\mathbf{Z}) \longrightarrow H^0(V;\mathbf{Z}) \xrightarrow{\smile c_1} H^2(V;\mathbf{Z}) \xrightarrow{p^\star} H^2(\mathcal{E};\mathbf{Z}) \longrightarrow 0$$

from which we deduce that $H^2(V;\mathbf{Z})$ maps onto $H^2(\mathcal{E};\mathbf{Z})$, the kernel of the onto map $p^\star$ being spanned by the multiples of $c_1(N_V)$.

Notice that, if $\pi_1(L) \cong \mathbf{Z}$, the exact sequence

$$\pi_2(V) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(\mathcal{E}) \longrightarrow 0$$

gives that $\pi_2(V) \to \pi_1(S^1)$ is trivial, so that $c_1(N_V) = 0$.

We look now at the Mayer-Vietoris exact sequence for $W = \mathcal{U} \cup \mathcal{V}$ (with real coefficients)

$$0 \longrightarrow H^2(W) \longrightarrow H^2(L) \oplus H^2(V) \longrightarrow H^2(\mathcal{E}) \longrightarrow 0.$$

Since L is Lagrangian, the class $[\omega]$ is mapped to an element $(0,a)$ for some element a in the kernel of $H^2(V) \to H^2(\mathcal{E})$, which is thus a multiple of $c_1(N_V)$, say $\lambda c_1(N_V)$.

As V is a symplectic submanifold, $[\omega]$ cannot be mapped to $0 \in H^2(V)$, thus $c_1(N_V)$ cannot be 0 and thus $\pi_1(L)$ cannot be isomorphic to $\mathbf{Z}$, this can only be a finite (maybe trivial) cyclic group.

The only thing which is left to prove is that V is dual to a multiple of the symplectic form. To achieve this, we extend the complex line bundle $N_V \to V$ to a complex line bundle $E \to W$, which coincides with N_V along V and is trivial on the complement. This is fairly standard: over $\mathcal{V} - V$, the bundle $p^\star N_V$ has a natural trivialization

$$\begin{array}{ccc} p^\star N_V & \longrightarrow & N_V \times \mathbf{C} \\ (x,v,w) & \longmapsto & \left(x, v, \dfrac{w}{v}\right) \end{array}$$

so that $p^\star N_V$ can be glued to the trivial bundle over $\mathcal{U}$. By construction, the homology class of V is dual to the first Chern class of E. In the Mayer-Vietoris exact sequence above, this class $c_1(E) \in H^2(W;\mathbf{Z})$, which is mapped to $(0, c_1(N_V))$ as is $\frac{1}{\lambda}[\omega]$, must be equal to $\frac{1}{\lambda}[\omega]$. And this ends the proof. ∎

2.4. In dimension 6, with an action of SU(2)

In this dimension, we know that, in order to compactify a cotangent bundle $T^\star L$ by the addition of a simply connected symplectic submanifold, we must

- start with a 3-manifold L with cyclic fundamental group (according to Proposition 2.3.5)
- try to add to its cotangent bundle a symplectic manifold of dimension 4 (according to Remark 2.1.1).

If we assume moreover that the 3-dimensional Lagrangian L is orientable, it must be parallelizable, so that the regular level hypersurface $\mathcal{E}$ is diffeomorphic to $S^2 \times L$. The Gysin exact sequence for the S^1-bundle $\mathcal{E} \to V$ gives that $H^2(V;\mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$ and the exact homotopy sequence for the same bundle gives that $\pi_2(L) = 0$, so that L has the same homotopy type as a lens space.

I believe this can only happen in the cases of the examples above, that is, I think that L must be a sphere or a real projective space. I will only give here a proof under some additional assumptions: I need the situation to be more rigid, in order to understand not only the topology but also the symplectic properties of the regular level hypersurface.

Notice that the two 6-dimensional examples of §§ 2.2.a and 2.2.b are endowed with an action of SU(2). We will thus assume that the 6-dimensional symplectic manifold W is endowed with an action of SU(2) (this is the case, in particular, if W is endowed with an action of the quotient SO(3)) and that the function we consider is the square of the norm of the momentum mapping(3)

$$\mu : W \longrightarrow \mathfrak{so}(3)^\star, \quad f = \|\mu\|^2,$$

then L must be an orbit of SU(2), thus a quotient of S^3, and hence, since we know that it has a cyclic fundamental group, this is indeed a lens space. We are going to prove:

Proposition 2.4.1. *Let W be a compact connected symplectic manifold of dimension 6 endowed with a Hamiltonian action of* SU(2) *or* SO(3) *with momentum mapping μ. Assume the function $f = \|\mu\|^2$ has only two critical values, one of which being reached along a Lagrangian submanifold L. Then*

- *The Lagrangian submanifold L is the minimum of f and is mapped to* 0.

(3) For all the symplectic geometry used here and especially the Morse theoretical properties of $\|\mu\|^2$, see [16].

– *The Lagrangian submanifold L is an orbit of the* $\mathrm{SU}(2)$*-action and is either a 3-sphere or a 3-dimensional real projective space.*
– *The maximum is reached along a symplectic $S^2 \times S^2$.*

The manifold W is isomorphic to $\widetilde{G}_2(\mathbf{R}^5)$ or to $\mathbf{P}^3(\mathbf{C})$.

Before we proceed to prove this proposition, let us check that the two manifolds in question actually satisfy the assumptions of the theorem. The two functions f and g given in §§ 2.2.a and 2.2.b, squares of the norm of momentum mappings to $\mathfrak{so}(4)^\star$ are also, up to a multiplicative constant, the squares of the norm of the momentum mappings to $\mathfrak{su}(2)^\star$. This is a direct computation, using the fact that the natural projection $\mathfrak{so}(4)^\star \to \mathfrak{su}(2)^\star$ is

$$\begin{bmatrix} 0 & -a & c & d \\ a & 0 & e & f \\ -c & -e & 0 & -b \\ -d & -f & b & 0 \end{bmatrix} \mapsto \begin{bmatrix} i(f-c) & -(a+b)-i(d+e) \\ a+b-i(d+e) & -i(f-c) \end{bmatrix}.$$

Notice that the $\mathrm{SU}(2)$-action on $\mathbf{P}^3(\mathbf{C})$ is not effective, as $-\mathrm{Id}$ acts trivially. It can thus be considered as an $\mathrm{SO}(3)$-action.

Proof of Proposition 2.4.1: The idea is to look at the properties of a regular level hypersurface with respect to the symplectic form. We will need to describe this hypersurface and, firstly, to describe the two critical levels. Denote $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$.

– We first describe the two critical levels, applying the slice theorem and the Darboux–Weinstein theorem.
– We then look at the orbits of the Hamiltonian vector field X_f in a regular level hypersurface $\mathcal{E}$.

Notice first that the two critical levels are connected.

The Lagrangian is the minimum

Let $a^2 \in \mathbf{R}$ be the critical value corresponding to the Lagrangian L. As this is either a maximum or a minimum, $f^{-1}(a^2) = L$, so that

$$L = \left\{ x \in W \mid \|\mu(x)\|^2 = a^2 \right\} = \mu^{-1}(S_a^2)$$

is the inverse image of the sphere of radius a in $\mathbf{R}^3 = \mathfrak{g}^\star$. Now μ is a Poisson map $W \to \mathfrak{g}^\star$ and it cannot map a Lagrangian onto a symplectic 2-sphere, so that $a = 0$ (and L is the minimum of f). This proves the first assertion([4]).

The maximum is a symplectic submanifold

Let us look now at the other side. Following Kirwan [16], we consider a maximal torus $T = S^1 \subset G$ and the associated periodic Hamiltonian H. Let $\xi \in \mathfrak{t} \subset \mathbf{R}^3$ be the maximal value of H. The sphere through ξ has radius $\|\xi\|$ and this is the maximal sphere for f. Being the maximum of a periodic Hamiltonian, $H^{-1}(\xi)$ is a symplectic submanifold. Now, the manifold V is obtained form $H^{-1}(\xi)$ by letting the group G act

$$V = G \cdot (H^{-1}(\xi)).$$

We deduce that $H^{-1}(\xi)$ can be identified with the orbit space V/G, which is thus symplectic and hence an orientable surface or a point. Now the restriction of the momentum mapping

$$\mu : V \longrightarrow S^2_{\|\xi\|}$$

is a Poisson mapping with basis a symplectic manifold and with symplectic fibers, so that V is indeed a symplectic submanifold. Hence, according to Proposition 2.1.1, its dimension is 4. Thus V/G is an orientable surface. Moreover, the mapping

$$\begin{aligned} V &\longrightarrow S^2_{\|\xi\|} \times V/G \\ x &\longmapsto (\mu(x), G \cdot x) \end{aligned}$$

is a diffeomorphism, so that V is diffeomorphic with $S^2 \times \Sigma$ for some oriented surface Σ.

The Lagrangian is an orbit

This is a consequence of the following (certainly classical) lemma.

Lemma 2.4.2. *Let μ be the momentum mapping of the Hamiltonian action of a connected Lie group G on the symplectic manifold W. Then $\mu^{-1}(0)$ is a Lagrangian submanifold of W if and only if it is a G-orbit.*

([4]) Notice that we know, more generally, following Kirwan [16, Cor. 3.16], that μ maps each connected component of the critical locus of f to a single coadjoint orbit in $\mathfrak{g}^\star$ (for any compact Lie group G), from what it is easily deduced that a Lagrangian critical manifold must always be mapped to 0.

Proof: According to the equivariant version of Darboux–Weinstein theorem, the G-action is given, in a neighborhood of L, by the standard formula

$$g \cdot (z, \psi) = (g \cdot z, {}^t(T_z g)^{-1}(\psi)).$$

The momentum mapping $\mu : T^\star L \to \mathfrak{g}^\star$ is

$$\langle \mu(z, \psi), X \rangle = \alpha_{(z,\psi)}(\underline{X}^{T^\star L}_{(z,\pi)}) = \psi(\underline{X}^L_z)$$

(α is the Liouville form and $\underline{X}^M_x$ denotes the value at $x \in M$ of the fundamental vector field $\underline{X}^M$ associated to $X \in \mathfrak{g}$ by the G-action on the manifold M).

Hence $\mu^{-1}(0) = L$ if and only if $\psi(\underline{X}_z) = 0$ for all $X \in \mathfrak{g}$ implies $\psi = 0$, that is, if and only if the values of the fundamental vector fields $\underline{X}$ at z span the tangent space $T_z L$, that is, if and only if L is an orbit. ∎

Hence L is an orbit and thus, $L = S^3/\Gamma$ for some cyclic group Γ of order, say, m (notice that Γ is the principal stabilizer in W). Moreover, on a neighborhood of L, the symplectic manifold (with action) is isomorphic with

$$T^\star L \cong T^\star(S^3/\Gamma) \cong S^3/\Gamma \times \mathbf{R}^3$$

with momentum mapping

$$\langle \mu(z, \psi), X \rangle = \psi(\underline{X}^L_z)$$

and, since we are using an invariant inner product on S^3,

$$f(z, \psi) = \|\mu(z, \psi)\|^2 = \|\psi\|^2 .$$

Notice that, although this is only a semi-local model in the neighborhood of L in W, this describes almost all W and more precisely the complement of the maximum V in W. In particular, the regular level hypersurface $\mathcal{E}$ is, as a hypersurface in the symplectic manifold W is isomorphic with the sphere bundle in the symplectic manifold $T^\star L$.

The maximum is diffeomorphic with $S^2 \times S^2$

The fundamental group of the regular level $\mathcal{E}$ is the cyclic group Γ as well, and this surjects to the fundamental group of $V \cong S^2 \times \Sigma$, hence Σ is a sphere as well.

Description of the regular level

The only thing which is left to prove is that the cyclic group Γ can only have order 1 or 2. For this, we need to understand the regular level hypersurface $\mathcal{E}$ from the point of view of the symplectic submanifold V.

As a symplectic manifold, a neighborhood of an orbit contained in V has the form $G \times_{S^1} \mathbf{C}^2$, a complex bundle of rank 2 over the orbit $G/S^1 = S^2$. The circle acts on $\mathbf{C}^2$ by

$$u \cdot (x, y) = (\overline{u}^m x, \overline{u}^n y).$$

The symplectic submanifold V is a union of orbits which are all spheres so that one of the weights, n say, must be zero (and the other one, m, is the order of the principal stabilizer in W). A neighborhood of our orbit thus has the form

$$(G \times_{S^1} \mathbf{C}) \times \mathbf{C}$$

where S^1 acts on $\mathbf{C}$ with weight m. As in the case of SU(2)-manifolds of dimension 4, the momentum mapping for the SU(2)-action is simply given by

$$\mu([v, x], y) = (1 - m\,|x|^2)[v],$$

for $v \in S^3 \subset \mathbf{C}^2$, $x, y \in \mathbf{C}$, $[v] \in \mathbf{P}^1(\mathbf{C}) = S^2 \subset \mathbf{R}^3$. The normal bundle of the symplectic 4-manifold V in W is thus a complex line bundle E and the regular levels of the mapping are the circle bundles of this complex bundle.

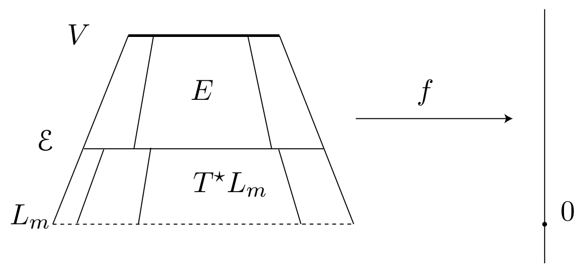

Figure 5

Hence the regular level hypersurface $\mathcal{E}$, as a hypersurface in the symplectic manifold W, is isomorphic both with

- the sphere bundle $S(T^\star L)$
- the circle bundle $S(E)$.

We conclude the proof by proving that these two hypersurfaces can only be isomorphic when Γ has order 1 or 2. For this, we look at the space of characteristics — that is, of trajectories of the Hamiltonian vector field X_f — of the symplectic form on these two hypersurfaces.

The space of characteristics, V side

In the case of the circle bundle $S(E)$, all the trajectories of X_f have the same period, the characteristics are simply the fibers (circles), the space of characteristics is isomorphic with the symplectic manifold V.

The space of characteristics, L side

On the other side, the situation can be very different. The Lagrangian L is a lens space $L_m = S^3/(\mathbf{Z}/m)$. In general, the space of characteristics of $S(T^\star M)$ for a Riemannian manifold M is the space of oriented geodesics of M. When $M = S^n$ (the sphere with the standard Euclidean metric) for instance, the geodesics are the intersections of S^n with the 2-planes through the origin in $\mathbf{R}^{n+1}$, so that the space of oriented geodesics is our old friend the Grassmannian $\widetilde{G}_2(\mathbf{R}^{n+1})$ — in particular, this is $\widetilde{G}_2(\mathbf{R}^4) = S^2 \times S^2$ in our 3-dimensional case.

Let us look at the space of geodesics in the quotient space L_m. The geodesics of L_m lift to geodesics on S^3, so that we only need to look at the effect of the $\mathbf{Z}/m$-action on the geodesics of S^3.

Notice first that, for $m = 2$, the oriented geodesic through x with tangent unit vector $y \in T_x S^3 = x^\perp$ is the same as the oriented geodesic through $-x$ with tangent unit vector $-y$, so that the space of oriented geodesics in $L_2 = \mathbf{P}^3(\mathbf{R})$ is the same as that of oriented geodesics in S^3.

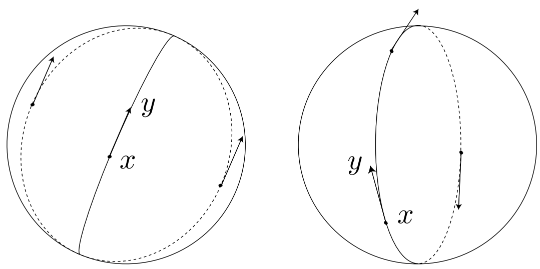

Figure 6 – A long and a short geodesic in L_m

Assume now that $m \geq 3$. There are two possibilities for the geodesic through x with unit tangent vector $y \in x^\perp$:

- either the 2-plane (x, y) is a complex line in $\mathbf{C}^2$
- or it is not.

In the first case, the geodesic is invariant under the $\mathbf{Z}/m$-action, while in the latter it is mapped to another geodesic. Hence, if the period of a geodesic in S^3

is T, there are, in L_m, geodesics of period T and others of period T/m. Thus $S(T^\star L)$ cannot be identified with $S(E)$ in this case. And this ends the proof of our proposition. ∎

2.5. Back to Chiang's example

Notice that Proposition 2.4.1 implies that, if a symplectic SU(2)-action on a 6-dimensional manifold has a stabilizer of order $m \geq 3$, then the square of the norm of the momentum mapping must have at least three critical values.

Let us check that Chiang's example (§ 1.3) does not contradict Proposition 2.4.1. The principal stabilizer has order 3 (this is Remark 1.3.2). The Lagrangian is indeed the minimal submanifold for the square of the norm of the momentum mapping of an SU(2)-action, however, it has a fundamental group which is not cyclic. Moreover, the maximum of the function is reached on a symplectic 2-sphere, in apparent contradiction with Remark 2.1.1 this time. Fortunately, the function has an intermediate critical value.

Proposition 2.5.1. *The square of the norm of the momentum mapping*

$$\mu : \mathbf{P}^3(\mathbf{C}) \longrightarrow \mathbf{R}^3$$

for the "diagonal" SO(3)*-action on*

$$(S^2 \times S^2 \times S^2)/\mathfrak{S}_3 = \mathbf{P}^3(\mathbf{C})$$

has exactly three critical values

- *the minimum,* 0*, obtained along the Lagrangian submanifold* $\mu^{-1}(0) = S^3/\Gamma$*,*
- *the value* 1*, obtained along a symplectic* 2*-sphere, which is a non degenerate critical submanifold of index* 2*,*
- *the maximum,* 9*, also obtained along a symplectic non degenerate* 2*-sphere.*

Proof: Of course, we rather look at the function $\|\widetilde{\mu}\|^2$

$$\begin{aligned} S^2 \times S^2 \times S^2 &\longrightarrow \mathbf{R} \\ (x, y, z) &\longmapsto \|x + y + z\|^2 . \end{aligned}$$

Its differential is the linear map

$$\begin{aligned} T_x\mu : x^\perp \times y^\perp \times z^\perp &\longrightarrow \mathbf{R} \\ (\xi, \eta, \zeta) &\longmapsto 2(x + y + z) \cdot (\xi + \eta + \zeta). \end{aligned}$$

The map $T_x\mu$ vanishes at (x, y, z) when the rank of the 9×12-matrix

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \\ y+z & z+x & x+y \end{bmatrix}$$

is not maximal, that is, if there exists real numbers α, β, γ such that

$$y + z = \alpha x, \qquad z + x = \beta y, \qquad x + y = \gamma z.$$

Since x, y and z are unit vectors, this gives

- either $\alpha = \beta = \gamma = -1$, $x + y + z = 0$, this being the minimum, of $\|\widetilde{\mu}\|^2$,
- or α, β and γ are all different from -1, we find
  - three spheres $(x, x, -x)$, $(x, -x, x)$ and $(-x, x, x)$ at which $\|\widetilde{\mu}\|^2 = 1$ and which are exchanged by the $\mathfrak{S}_3$-action, giving a single 2-sphere in $\mathbf{P}^3(\mathbf{C})$
  - a sphere (x, x, x), on which $\|\widetilde{\mu}\|^2 = 9$, which is invariant under the $\mathfrak{S}_3$-action and at which the maximum of $\|\widetilde{\mu}\|^2$ is obtained.

The only thing which is still left to do is to compute the indices. Let us do this at a point, for instance, of the intermediate 2-sphere (and of course, in $S^2 \times S^2 \times S^2$ rather than in the quotient). We choose the point $(e_1, e_1, -e_1)$, near which we have local coordinates $(x_2, x_3, y_2, y_3, z_2, z_3)$, mapped to

$$\left(\left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3\right), \left(\sqrt{1 - y_2^2 - y_3^2}, y_2, y_3\right), \left(-\sqrt{1 - z_2^2 - z_3^2}, z_2, z_3\right)\right).$$

We just have to compute $\|x + y + z\|^2$, up to the order 2, in terms of these coordinates. We find

$$\|\widetilde{\mu}\|^2 \sim 1 + 2\left(z_2 + \frac{x_2 + y_2}{2}\right)^2 - \frac{1}{2}(x_2 - y_2)^2 + 2\left(z_3 + \frac{x_3 + y_3}{2}\right)^2 - \frac{1}{2}(x_3 - y_3)^2,$$

so that these points do indeed have index 2. The other assertion is proved in exactly the same way. ∎

3 – Graded Lagrangians (after Seidel)

The results on Lagrangian embeddings stated in Seidel's paper [22] turn out to be special cases of stronger results([5]) obtained shortly later by different methods

([5]) Typically, Seidel's technique works in a monotone symplectic manifold while Biran-Cieliebak's gives analogous results for Lagrangian submanifolds of products of a monotone manifold and a torus.

by Biran and Cieliebak [11]. This is nevertheless Seidel's methods that I will present here: his grading is a clever additional structure on Floer homology which deserves to be better known, and which can probably be extended to produce new results.

3.1. The results

The results of Oh [19] and the method of Seidel [22] allow to prove the following theorem (implicit although not explicitly stated in [22]).

Theorem 3.1.1. *Assume S^n is a Lagrangian submanifold of a compact monotone symplectic manifold W with $H_1(W;\mathbf{Z}) = 0$. Assume moreover that W is endowed with a non constant periodic Hamiltonian. Let N_W be the generator of the subgroup*

$$\Big\{\langle c_1(W), A\rangle \mid A \in H_2(W;\mathbf{Z})\Big\} \subset \mathbf{Z}.$$

Assume that $N_W \geq \frac{n}{2} + 1$. Then $n \equiv 0 \mod N_W$ and the sum of the weights of the linearized S^1-action at any point where the Hamiltonian is minimal is also congruent to n modulo N_W.

Corollary 3.1.2 (Seidel [22])**.** *The sphere S^n cannot be embedded as a Lagrangian submanifold of $\mathbf{P}^n(\mathbf{C})$.*

The method gives a more precise result in the case where the ambient manifold is $\mathbf{P}^n(\mathbf{C})$.

Theorem 3.1.3 (Seidel [22])**.** *If L is a Lagrangian submanifold of $\mathbf{P}^n(\mathbf{C})$, then $H^1(L;\mathbf{Z}/2n+2) \neq 0$.*

I will show below that the assumptions in the statement of Theorem 3.1.1 are, in some sense sharp. Note that Biran and Cieliebak have stronger results for manifolds which are not necessarily monotone. However, Seidel's method can be used to give one of their results.

Proposition 3.1.4 (Biran and Cieliebak [11])**.** *Let L be a compact submanifold of dimension $2n$ with $H_1(L;\mathbf{Z}) = 0$ and $H_2(L;\mathbf{Z}) = 0$. Then L cannot be embedded as a Lagrangian manifold in $\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})$ endowed with the product symplectic form $\omega \oplus \omega$.*

In this case the assumption $H_2(L;\mathbf{Z})=0$ is necessary, since we have seen that $\mathbf{P}^n(\mathbf{C})$ can be embedded as a Lagrangian submanifold of $\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})$.

Ideas of proofs

To a Lagrangian submanifold $L\subset W$ is associated a number N_L (depending on the embedding, not only on the manifold L) analogous to the minimal Chern number N_W: this is the generator of the subgroup

$$\Big\{\langle c_1(W,L),A\rangle \mid A\in H_2(W,L;\mathbf{Z})\Big\}\subset\mathbf{Z}$$

(see also § 3.2.c). Assume now that L is a closed manifold of dimension n with $H^1(L;\mathbf{R})=0$ and that the ambient symplectic manifold W is *monotone*(6).

- The first ingredient of the proof is a theorem of Oh [19], that asserts that, under these assumptions, the Floer cohomology of the pair (L,L) is well-defined and, if $N_L\geq n+2$, that this is
$$HF(L,L)\cong\bigoplus_i H^i(L;\mathbf{Z}/2).$$
Notice that (under the assumptions made), this depends only on the manifold L.
- The second ingredient is an addition to the structure of Floer cohomology, due to Seidel [22]. This is a $\mathbf{Z}/2N_W$-grading on this cohomology group. Notice that this depends only on the symplectic manifold.
- The last ingredient is a subtle use of a Hamiltonian circle action, which gives a periodicity (modulo $2N_W$) on this Floer cohomology.

3.2. The grading

A Lagrangian immersion f of a manifold L in a symplectic manifold W defines a Gauss mapping

$$\begin{aligned}\gamma(f): L &\longrightarrow \Lambda(W)\\ x &\longmapsto T_xf(T_xW),\end{aligned}$$

here $\Lambda(W)$ denotes Lagrangian Grassmannian bundle over W associated with the tangent bundle TW. The fiber of $\Lambda(W)$ at $x\in W$ is the Grassmannian of all Lagrangian subspaces in the symplectic vector space T_xW.

The grading of Seidel is a lift of this Gauss mapping to a certain covering of $\Lambda(W)$, when it exists. I describe this covering now.

(6) That is, such that $[\omega]=\lambda c_1(W)\in H^2(W;\mathbf{R})$ for some $\lambda>0$.

3.2.a. The Lagrangian Grassmannian and the covering $\Lambda^N(W)$

Let us fix an integer N.

The Lagrangian subspaces of $\mathbf{R}^{2n} \cong \mathbf{C}^n$ form the Lagrangian Grassmannian $\Lambda_n = \mathrm{U}(n)/\mathrm{O}(n)$ (see *e.g.* [17, 3]). Our main tool will be the mapping

$$\mathrm{d\acute{e}t}^2 : \mathrm{U}(n) \longrightarrow \Lambda_n \longrightarrow S^1$$

which induces an isomorphism $\pi_1\Lambda_n \to \pi_1 S^1$. If we pull back the N-fold connected covering of S^1

$$\begin{array}{ccccc} \mathrm{U}^N(n) & \longrightarrow & \Lambda_n^N & \longrightarrow & S^1 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{U}(n) & \longrightarrow & \Lambda_n & \longrightarrow & S^1 \end{array}$$

we get the two coverings

$$\mathrm{U}^N(n) = \left\{(A, z) \in \mathrm{U}(n) \times S^1 \mid \mathrm{d\acute{e}t}(A)^2 = z^N\right\} \longrightarrow \mathrm{U}(n)$$

and

$$\Lambda_n^N = \left\{(P, z) \in \Lambda_n \times S^1 \mid \mathrm{d\acute{e}t}(P)^2 = z^N\right\} \longrightarrow \Lambda_n.$$

Notice that $\mathrm{U}^N(n)$ is connected if and only if N is odd, from what it is easily deduced that Λ_n^N is always connected. These coverings correspond, as they should, to elements of

$$H^1(\mathrm{U}(n); \mathbf{Z}/N) \cong \mathrm{Hom}(\pi_1\mathrm{U}(n), \mathbf{Z}/N), \quad \text{resp. } H^1(\Lambda_n; \mathbf{Z}/N) \cong \mathrm{Hom}(\pi_1\Lambda_n, \mathbf{Z}/N),$$

namely to the homomorphism

$$\begin{aligned} \mathbf{Z} &\longrightarrow \mathbf{Z}/N \\ m &\longmapsto 2m \mod N. \end{aligned}$$

3.2.b. On a symplectic manifold W

We try now to copy this construction to construct an N-fold covering of $\Lambda(W)$ which restricts, at each point of W, to $\Lambda_n^N \to \Lambda_n$. The mapping $\mathrm{d\acute{e}t}^2 : \Lambda_n \to S^1$ globalizes into a mapping

$$\Lambda(W) \longrightarrow \mathrm{d\acute{e}t}(W)^{\otimes 2}$$

where $\mathrm{d\acute{e}t}(W)$ denotes the complex line bundle $\bigwedge^n(TW)$, the tangent bundle TW being considered as a complex n-plane bundle, using any almost complex structure compatible with the symplectic form on W. To give the expected N-fold covering amounts to giving an N-th root of the complex line bundle $\text{dét}(W)^{\otimes 2}$. The latter can exist if and only if

$$c_1(\text{dét}(W)^{\otimes 2}) = 0 \in H^2(W; \mathbf{Z}/N),$$

that is, if and only if

$$2c_1(W) = 0 \in H^2(W; \mathbf{Z}/N).$$

This is the case if and only if the integer N we are dealing with is a divisor of $2N_W$.

Proposition 3.2.1 (Seidel [22]). *There exists a covering $\Lambda^N(W) \to \Lambda(W)$ which restricts, at any fiber, to the covering $\Lambda_n^N \to \Lambda_n$, if and only if $N \mid 2N_W$. This covering is unique if and only if $H^1(W; \mathbf{Z}/N) = 0$.* ∎

Seidel calls such a covering *an N-fold Maslov covering* of W. Notice that we will use only simply connected symplectic manifolds, so that we will not need to care about uniqueness. Notice also that $\Lambda^N(W)$ is connected if W is.

3.2.c. Back to the "Chern numbers" N_W and N_L

Let us now assume for simplicity that our symplectic manifold W is simply connected. We use a compatible almost complex structure and consider the principal $\mathrm{U}(n)$-bundle $\mathrm{U}(W) \to W$ associated with the complex vector bundle $TW \to W$. The exact homotopy sequence gives

$$\pi_2 W \longrightarrow \pi_1 \mathrm{U}(n) \longrightarrow \pi_1 \mathrm{U}(W) \longrightarrow 0$$

and the mapping $\pi_2 W \to \pi_1 \mathrm{U}(n) = \mathbf{Z}$, considered as an element of $\mathrm{Hom}(\pi_2 W; \mathbf{Z}) = H^2(W; \mathbf{Z})$, is the first Chern class of W. Hence N_W is the index of the image of $\pi_2 W$ in $\pi_1 \mathrm{U}(n)$ (or the order of the fundamental group $\pi_1 \mathrm{U}(W)$).

We can do the same thing with the bundle $\Lambda(W) \to W$, getting

$$\pi_2 W \longrightarrow \pi_1 \Lambda_n \longrightarrow \pi_1 \Lambda(W) \longrightarrow 0,$$

the image of $\pi_2 W$ being now $2N_W \mathbf{Z} \subset \mathbf{Z} = \pi_1 \Lambda_n$.

If $f : L \to W$ is the inclusion of a Lagrangian submanifold, the Gauss mapping of which is $\gamma(f) : L \to \Lambda(W)$, we consider now the induced map $\gamma(f)_\star$ at the

π_1 level. Recall that we are assuming W to be simply connected. We can put together our exact sequence with the homology sequence of the pair (W, L)

$$\begin{tikzcd} H_2(W;\mathbf{Z}) \arrow[r] \arrow[d, "="] & H_2(W,L;\mathbf{Z}) \arrow[r] \arrow[d, dotted] & H_1(L;\mathbf{Z}) \arrow[r] \arrow[d, "\gamma(f)_\star"] & 0 \\ \pi_2 W \arrow[r] & \pi_1\Lambda_n \arrow[r] & \pi_1\Lambda(W) \arrow[r] & 0 \end{tikzcd}$$

a diagram in which the dotted arrow is the map $H_2(W, L; \mathbf{Z}) \to \mathbf{Z}$ that defines the relative "Chern number" N_L.

A relative homology class in $H_2(W, L; \mathbf{Z})$ can be represented by a map

$$(D^2, S^1) \longrightarrow (W, L).$$

We choose a trivialization of the complex vector bundle $TW\mid_{D^2}$, so that, at each point of the boundary S^1, we have a Lagrangian subspace of $\mathbf{C}^n$. This defines a map $S^1 \to \Lambda_n$ and the expected mapping $H_2(W, L; \mathbf{Z}) \to \pi_1\Lambda_n = \mathbf{Z}$.

Notice that the subgroup

$$\gamma(f)_\star\pi_1 L = \gamma(f)_\star H_1(L;\mathbf{Z}) \subset \pi_1\Lambda(W) = H_1(\Lambda(W);\mathbf{Z})$$

consists of the elements in $\pi_1(\Lambda_n) \cong \mathbf{Z}$ that are Maslov classes in this sense, that is, by the very definition of N_L, of the multiples of N_L.

Example 3.2.2. Assume that $H_1(L; \mathbf{Z}) = 0$; then the diagram above shows that

$$N_L = 2N_W. \square$$

3.2.d. On a Lagrangian submanifold

Let us go back now to our Lagrangian immersion $f : L \longrightarrow W$ and to its Gauss mapping $\gamma(f) : L \longrightarrow \Lambda(W)$. Assume there exists an N-fold Maslov covering on W. The question we investigate now if whether the Gauss mapping can be lifted to a mapping

$$\begin{tikzcd} & \Lambda^N(W) \arrow[d, "p"] \\ L \arrow[ur, dashed] \arrow[r, "\gamma(f)"'] & \Lambda(W) \end{tikzcd}$$

to $\Lambda^N(W)$. Such a lift will be called an N-*grading* of the immersion f (or the submanifold L).

Proposition 3.2.3 (Seidel [22]). *A Lagrangian immersion in a simply connected symplectic manifold admits an N-grading if and only if $N \mid N_L$.*

Proof: A lift in the diagram above exists if and only if, in $\pi_1(\Lambda(W))$, we have the inclusion of subgroups

$$\gamma(f)_\star \pi_1(L) \subset p_\star \pi_1(\Lambda^N(W)).$$

Let us thus consider a loop

$$g : S^1 \longrightarrow L.$$

As we have assumed that W is simply connected, this is the boundary of some disk

$$D^2 \longrightarrow W.$$

Along this disk, we choose a trivialization of the (complex) vector bundle TW,

$$TW\mid_{D^2} \cong D^2 \times \mathbf{C}^n,$$

this defines a map $S^1 \to \Lambda_n$, the image of which by

$$j_\star : \pi_1(\Lambda_n) \longrightarrow \pi_1 \Lambda(W)$$

is just $\gamma(f)_\star[g]$ (and two such maps differ by an element of $\pi_2(W)$). We are thinking of the diagram

$$\begin{tikzcd}
\pi_2(W) \arrow[r] \arrow[d, "="'] & \pi_1(\Lambda_n^N) \arrow[r] \arrow[d, "\times N"] & \pi_1(\Lambda^N(W)) \arrow[r] \arrow[d] & 0 \\
\pi_2(W) \arrow[r] & \pi_1(\Lambda_n) \arrow[r] & \pi_1(\Lambda(W)) \arrow[r] & 0
\end{tikzcd}$$

Recall that we have noticed above that the subgroup $\gamma(f)_\star \pi_1(L)$ consists of the images of those elements in $\pi_1(\Lambda_n) \cong \mathbf{Z}$ that are multiples of N_L. Hence, the Lagrangian L admits an N-grading if and only if the set of multiples of N_L is included in the set of multiples of N, that is, if and only if $N \mid N_L$. ∎

The same considerations give:

Proposition 3.2.4 (Seidel [22]). *A Lagrangian submanifold L which satisfies $H_1(L; \mathbf{Z}/N) = 0$ always admits an N-grading.*

Proof: To say that $H^1(L; \mathbf{Z}/N) = 0$ is to say that any morphism $\pi_1(L) \to \mathbf{Z}/N$ is trivial. Hence $\gamma(f)_\star \pi_1(L)$ is contained in the index-N subgroup

$$p_\star \pi_1(\Lambda^N(W)) \subset \pi_1(\Lambda(W)). \blacksquare$$

3.2.e. Maslov indices

Recall that, if λ_0 and $\lambda_1 : [a,b] \to \Lambda_n$ are two paths, there is a "Maslov index" $\mu(\lambda_0, \lambda_1) \in \frac{1}{2}\mathbf{Z}$ defined as follows (this description comes from [23] and [20]).

We start with a single smooth path $\gamma : [a,b] \to \Lambda_n$ and a fixed Lagrangian subspace $P \in \Lambda_n$. Recall that we can then write $\mathbf{C}^n = P \oplus JP$. The set of Lagrangian subspaces that are *not* transversal to P is a hypersurface $\Sigma(P)$ in Λ_n (actually dual to the generator of $H^1(\Lambda_n; \mathbf{Z}) = \mathbf{Z}$, namely to the Maslov class). We look at the way our path γ meets $\Sigma(P)$. Assume that $t_0 \in [a,b]$ is such that $\gamma(t_0) \in \Sigma(P)$. In a neighborhood of t_0, we lift $\gamma(t)$ to a map into $\mathrm{U}(n)$:

$$\widetilde{\gamma}(t) = X(t) + JY(t).$$

We consider $Y(t)$ as a linear map $\mathbf{R}^n \to \mathbf{R}^n$. To say that $\gamma(t_0) \in \Sigma(P)$ is to say that $Y(t_0)$ has a nontrivial kernel. Then the formula

$$Q(\gamma, P; t_0)(u) = \langle \dot{Y}(t_0) \cdot u, X(t_0) \cdot u \rangle$$

defines a quadratic form on the kernel of $Y(t_0)$ which does not depend on the lift $\widetilde{\gamma}$.

Let me give a simple example. Let $P = \mathbf{R}^n \subset \mathbf{C}^n$ and let γ be the path $t \mapsto (e^{it}e_1, ie_2, \ldots, ie_n)$ for, say, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Here

$$Y(t) = \begin{bmatrix} \sin t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

and the only intersection point with $\Sigma(P)$ is for $t = 0$, the kernel of $Y(0)$ is the line $\mathbf{R} \cdot e_1$ and the quadratic form is $Q(\gamma, \mathbf{R}^n; 0)(x) = x^2$. Notice that the signature is 1, this corresponding to the fact that our path intersects $\Sigma(P)$ transversally. As this path is a loop and generates $\pi_1\Lambda_n$, this shows that $\Sigma(\mathbf{R}^n)$ is indeed dual to the Maslov class.

Now we go back to defining a Maslov index for two paths λ_0 and λ_1. For any $t_0 \in [a,b]$, we can

- consider the quadratic form $Q(\lambda_0, \lambda_1(t_0); t_0)$ (here $\lambda_1(t_0)$ is a fixed Lagrangian and λ_0 is a path, t_0 is some parameter for which $\lambda_0(t_0)$ is not transversal to $\lambda_1(t_0)$ and our quadratic form is defined on their intersection),

– then exchange the roles of λ_0 and λ_1,

– then subtract the two quadratic forms, getting a quadratic form

$$R(\lambda_0, \lambda_1; t_0) = Q(\lambda_0, \lambda_1(t_0); t_0) - Q(\lambda_1, \lambda_0(t_0); t_0)$$

defined on $\lambda_0(t_0) \cap \lambda_1(t_0)$.

We define eventually

$$\mu(\lambda_0, \lambda_1) = \frac{1}{2} \operatorname{sign} R(\lambda_0, \lambda_1; a) + \sum_{a<t<b} \operatorname{sign} R(\lambda_0, \lambda_1; t) + \frac{1}{2} \operatorname{sign} R(\lambda_0, \lambda_1; b) \in \frac{1}{2}\mathbf{Z}\,.$$

Note that the two terms corresponding to the two ends of $[a, b]$ are necessary to have the additivity of μ by concatenation of paths, which is then obvious.

Assume now that $\widetilde{L}_0$ and $\widetilde{L}_1$ are two $\mathbf{Z}/N$-graded Lagrangian submanifolds of the symplectic manifold W that intersect transversally at some point x. We thus have two elements $\widetilde{L}_0(x)$ and $\widetilde{L}_1(x) \in \Lambda_n^N$. Let us fix a point of Λ_n^N and two paths $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_1$ starting at this point and ending respectively at $\widetilde{L}_0(x)$ and $\widetilde{L}_1(x)$ (see Figure 7). Call λ_0, λ_1 the images of these two paths in Λ_n and put

$$\widetilde{I}(\widetilde{L}_0, \widetilde{L}_1; x) = \frac{1}{2} n - \mu(\lambda_0, \lambda_1).$$

Proposition 3.2.5. *The number $\widetilde{I}(\widetilde{L}_0, \widetilde{L}_1; x)$ is an integer the class of which in $\mathbf{Z}/N$ does not depend on any choice.*

Proof: We parametrize the two paths by $[0, 1]$. Since $\lambda_0(1)$ and $\lambda_1(1)$ are transversal, this end does not contribute to the Maslov index. Now $\lambda_0(0) = \lambda_1(0)$. We can choose coordinates so that this Lagrangian subspace is $\mathbf{R}^n \times 0$. Then, for t small enough, $\lambda_i(t)$ is the graph of some symmetric matrix $A_i(t)$, such that $A_i(0) = 0$ and $A_i(t) = \widehat{A}_i t + O(t^2)$ for some symmetrix matrix $\widehat{A}_i$. This end contributes to the Maslov index by the signature of $\widehat{A}_0 - \widehat{A}_1$, a quadratic form on $\mathbf{R}^n$, the signature of which is congruent to the dimension n modulo 2. The formula thus gives an integer.

The latter does not depend on the choice of $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_1$: if we choose other paths, the concatenation is a loop in Λ_n^N and thus, in Λ_n, as a consequence of the additivity of μ, it contributes a multiple of N. ∎

Denote by $\widetilde{L}[k]$ the graded Lagrangian $\widetilde{L}$ with the N-grading shifted by $k \in \mathbf{Z}/N$.

Proposition 3.2.6. *The index of the shifted graded manifolds satisfy*

$$\widetilde{I}(\widetilde{L}_0[k], \widetilde{L}_1[\ell]; x) = \widetilde{I}(\widetilde{L}_0, \widetilde{L}_1; x) - k + \ell.$$

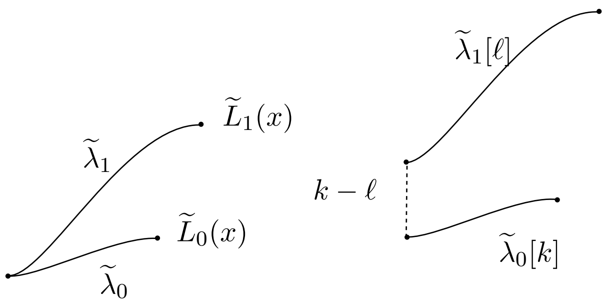

Figure 7

Proof: To compute $\widetilde{I}(\widetilde{L}_0[k], \widetilde{L}_1[\ell]; x)$, we can use the shifted paths $\widetilde{\lambda}_0[k]$, $\widetilde{\lambda}_1[\ell]$, the origins of which we must connect by some path connecting $\widetilde{\lambda}_0[k](0)$ and $\widetilde{\lambda}_1[\ell](0)$ (Figure 7). Of course, the shifted paths have the same images as their originals. But the image of the path connecting the two origins adds $k-\ell$ to the Maslov index (by definition of the covering $\Lambda_n^N \to \Lambda_n$. ∎

3.2.f. The grading of Floer cohomology

I will of course not make a review of Floer cohomology here. As Seidel says it in [22], the grading here is just an addition to Floer cohomology (I would add, a subtle addition). It means that if the ordinary Floer cohomology $HF(L_0, L_1)$ of a pair (L_0, L_1) is defined and if the two Lagrangian submanifolds L_0 and L_1 are $\mathbf{Z}/N$-graded (according to § 3.2.d), then $HF(L_0, L_1)$ is $\mathbf{Z}/N$-graded, in the sense that there is a decomposition

$$HF(L_0, L_1) = \bigoplus_{k \in \mathbf{Z}/N} HF^k(L_0, L_1).$$

We assume that L_0 and L_1 are two Lagrangian submanifolds of W intersecting transversally at a finite number of points. Then the Floer complex is defined as

– the $\mathbf{Z}/2$-vector space of basis the intersection points
– endowed with the differential

$$\partial x_+ = \sum n(x_-, x_+; J) x_-$$

for some number $n(x_-, x_+; J) \in \mathbf{Z}/2$ roughly defined as the "mod 2 number" of maps $u : \mathbf{R} \times [0,1] \to W$ such that

– $\mathbf{R} \times \{0\}$ is mapped to L_0,
– $\mathbf{R} \times \{1\}$ is mapped to L_1,
– $\lim_{t\to\pm\infty} u(x,t) = x_\pm$,
– and u satisfies the Floer equation $\partial_s u + J_t(u)\partial_t u = 0$.

See [14] or [19] for details. Notice that this uses a generic family of compatible complex structures J... and a theorem, the Floer index theorem, to ensure that the space of solutions is finite dimensional. Assume that u is a solution. Denote by u_i the restriction of u to $\mathbf{R} \times \{i\}$. Using a trivialization of $u^\star TW$, we can consider u_0 and u_1 as paths in Λ_n, so that they have a Maslov index $\mu(u_0, u_1)$. Since the two Lagrangian submanifolds intersect transversally, this is an integer and Floer's theorem asserts that the dimension of the component of the space of solutions to which u belongs is precisely $\mu(u_0, u_1)$.

The summation in the definition of ∂ will only take into account the points x_- such that this Maslov index is 1, so that the space of solutions has dimension 1 (but it can be quotiented by the $\mathbf{R}$-action, giving a finite number of points). Notice that this still takes some work to prove that $\partial \circ \partial = 0$. Then $\operatorname{Ker}\partial / \operatorname{Im}\partial$ is the Floer cohomology of (L_0, L_1).

Even having skipped all the details and proofs, we can still notice that, although the difference of indices is well defines, the index of a single singular point is not defined. And this is precisely what the grading allows to do. We say that a transversal intersection point x of two graded Lagrangian submanifolds $\widetilde{L}_0$ and $\widetilde{L}_1$ has degree k if $\widetilde{I}(\widetilde{L}_0, \widetilde{L}_1; x) = k$.

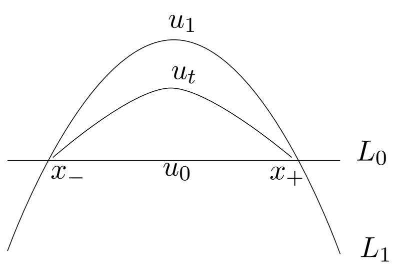

Figure 8

Consider now two points x_+, x_-, as on Figure 8, such that x_- contributes to ∂x_+. This means that $\mu(u_0, u_1) = 1$. We choose a trivialization of $u^\star TW$, so that we can consider u_0 and u_1 as two paths in Λ_n (which give transversal Lagrangians at both ends). We lift them to two paths

$$\widetilde{u}_0, \widetilde{u}_1 : [a,b] \longrightarrow \Lambda_n^N.$$

We choose two paths $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$, with the same origin, $\widetilde{\lambda}_0$ ending at $\widetilde{u}_0(a)$ and $\widetilde{\lambda}_1$ ending at $\widetilde{u}_1(a)$. We compute now

$$\begin{aligned}\widetilde{I}(\widetilde{L}_0,\widetilde{L}_1;x_+) &= \frac{1}{2}n-\mu(\lambda_0 u_0,\lambda_1 u_1)\\ &= \frac{1}{2}n-\mu(\lambda_0,\lambda_1)-\mu(u_0,u_1)\\ &= \widetilde{I}(\widetilde{L}_0,\widetilde{L}_1;x_-)-1.\end{aligned}$$

Hence, ∂ maps $CF^k(L_0,L_1)$ to $CF^{k+1}(L_0,L_1)$ and defines a grading on Floer cohomology.

It is not always the case that Floer cohomology is isomorphic with ordinary cohomology. However, when this is the case (see the assumptions of Oh's theorem quoted in § 3.1), the grading follows:

$$HF^k(L,L)\cong\bigoplus_{i\in\mathbf{Z}}H^{k+Ni}(L;\mathbf{Z}/2).$$

3.2.g. Using a Hamiltonian circle action

Let φ be any symplectic diffeomorphism of W. Its tangent map defines a map $\Lambda(W)\to\Lambda(W)$. An N-grading of φ is a $\mathbf{Z}/N$-equivariant mapping ψ making the diagram

$$\begin{array}{ccc}\Lambda^N(W) & \xrightarrow{\psi} & \Lambda^N(W)\\ \downarrow & & \downarrow\\ \Lambda(W) & \xrightarrow[T\varphi]{} & \Lambda(W)\end{array}$$

commute. The group of symplectic diffeomorphisms being denoted by $\mathrm{Sp}(W)$, the group of N-graded symplectic diffeomorphisms (consisting of pairs (φ, lift of $T\varphi$), with the obvious multiplication) will be denoted by $\mathrm{Sp}^N(W)$. Notice that the natural map $\mathrm{Sp}^N(W)\to\mathrm{Sp}(W)$ is a group homomorphism and that the fiber of the identity, its kernel, is identified with $\mathbf{Z}/N$.

Assume now that the manifold W is endowed with a Hamiltonian circle action. This is, in particular, a mapping

$$\sigma : S^1 \longrightarrow \mathrm{Sp}(W).$$

Let x be a fixed point of this action, so that there is a circle action on T_xW, that can be written, in suitable complex coordinates

$$u\cdot(z_1,\dots,z_n)=(u^{m_1}z_1,\dots,u^{m_n}z_n)$$

for some integers $m_1, \dots, m_n$. Notice that this is a mapping

$$\begin{aligned} S^1 &\longrightarrow \mathrm{U}(n) \\ u &\longmapsto \begin{pmatrix} u^{m_1} & & \\ & \ddots & \\ & & u^{m_n} \end{pmatrix}. \end{aligned}$$

We assume now that W is endowed with an N-fold Maslov covering. At the fixed point x, this is

$$\Lambda^N(T_xW) \cong \Lambda_n^N \longrightarrow \Lambda(T_xW) \cong \Lambda_n.$$

We have a map

$$\begin{aligned} h : S^1 &\longrightarrow \Lambda_n \\ e^{2i\pi t} &\longmapsto \text{class of } \begin{pmatrix} e^{2im_1\pi t} & & \\ & \ddots & \\ & & e^{2im_n\pi t} \end{pmatrix}. \end{aligned}$$

This map can be lifted to a map

$$\begin{tikzcd} \mathbf{R} \arrow[r, "\widetilde{h}"] \arrow[d, "\exp"'] & \Lambda_n^N \arrow[d, "p"] \\ S^1 \arrow[r, "h"'] & \Lambda_n \end{tikzcd}$$

namely

$$t \longmapsto \left(h(t), \exp\left(2i\pi \frac{2t(m_1 + \cdots + m_n)}{N}\right)\right) \in \Lambda_n^N \subset \Lambda_n \times S^1.$$

This means that the map $\sigma : S^1 \to \mathrm{Sp}(W)$ lifts to a map

$$\widetilde{\sigma} : [0,1] \longrightarrow \mathrm{Sp}^N(W)$$

with $\widetilde{\sigma}(0) = \mathrm{Id}$ and

$$\widetilde{\sigma}(1) = [2(m_1 + \cdots + m_n)] \in \mathbf{Z}/N.$$

The lifted map $\widetilde{\sigma}$ defines an isotopy between any N-graded Lagrangian submanifold and itself endowed with the grading shifted by $2(m_1 + \cdots + m_n)$. Then, looking at Proposition 3.2.6 and at its proof, we deduce:

Proposition 3.2.7. *Assume the symplectic manifold W is endowed with a periodic Hamiltonian. Let w be the sum of the weights at some fixed point. Then the Floer cohomology of any graded Lagrangian submanifold is periodic of period $2w$.* ∎

3.3. Proofs of the theorems

3.3.a. The sphere, proofs of Theorem 3.1.1 and Corollary 3.1.2

Consider firstly the case of the n-sphere S^n. We know (see Example 3.2.2) that $N_L = 2N_W$, so that Oh's theorem applies, provided $2N_W \geq \sup(3, n+2)$, which we have assumed.

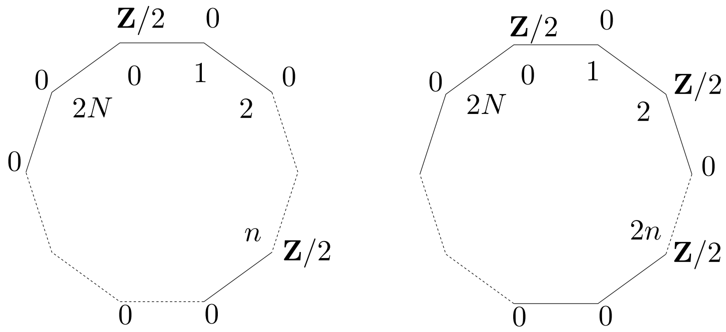

Figure 9 – Floer cohomology of a Lagrangian S^n (left) or $\mathbf{P}^n(\mathbf{C})$ (right)

The Floer cohomology is shown on Figure 9. The only possible way for this to be periodic is that $n \equiv 0$ or $2n \equiv 0$ (modulo $2N_W$), that is, $n \equiv 0$ modulo N_W. The period will then be n. Thus, if we want to find a symplectic manifold W (satisfying our assumptions) in which S^n is Lagrangian, it must have N_W (the minimal Chern number) a divisor of n (half the dimension) and a suitable S^1-action. This proves Theorem 3.1.1. ∎

Assume now that the symplectic manifold is the projective space $\mathbf{P}^n(\mathbf{C})$. The number N_W is $n+1$ here. In order that S^n be a Lagrangian submanifold of $\mathbf{P}^n(\mathbf{C})$, it is necessary that $n+1$ is a divisor of n, which is impossible. Recall that $\mathbf{P}^n(\mathbf{C})$ is endowed with a few Hamiltonian circle actions. We do not need to be more specific for this proof. This ends the proof of Corollary 3.1.2. ∎

This is very well illustrated by our example of a Lagrangian sphere $S^n \subset \widetilde{G}_2(\mathbf{R}^{n+2})$ in § 2.2.b. The Grassmannian $\widetilde{G}_2(\mathbf{R}^{n+2})$ is very similar to the projective space $\mathbf{P}^n(\mathbf{C})$. There are however a few differences. One of them is the minimal Chern number. As the Grassmannian is embedded as a quadric in $\mathbf{P}^{n+1}(\mathbf{C})$, its first Chern class is $n+2-2=n$. Hence Theorem 3.1.1 does not forbid it to have Lagrangian spheres!

3.3.b. In the projective space, proof of Theorem 3.1.3

Let L be a Lagrangian submanifold of $\mathbf{P}^n(\mathbf{C})$. Assume that $H^1(L;\mathbf{Z}/2n{+}2)=0$. According to Proposition 3.2.4, L admits a $2n{+}2$-grading and this in turn implies, according to Proposition 3.2.3, that $2n+2 \mid N_L$. Hence, Oh's theorem (see § 3.1) applies.

Here we need to specify a Hamiltonian circle action. We use, for instance

$$\begin{aligned} S^1 \times \mathbf{P}^n(\mathbf{C}) &\longrightarrow \mathbf{P}^n(\mathbf{C}) \\ (u, [z_0, z_1, \dots, z_n]) &\longmapsto [uz_0, z_1, \dots, z_n], \end{aligned}$$

corresponding to the Hamiltonian

$$H([z_0, z_1, \dots, z_n]) = \frac{|z_0^2|}{\sum |z_i^2|}.$$

The fixed points are $[1, 0, \dots, 0]$ (the maximum of H, weight 1) and the projective hyperplane $z_0 = 0$.

We have a contradiction as above: as the weight of one of the fixed points is 1, the period of the Floer homology of our graded Lagrangian submanifold is 2. Of course, this is incompatible with the fact that this homology is also, according to Oh's theorem,

$$\begin{cases} H^i(L;\mathbf{Z}/2) & \text{if } 0 \le i \le n \\ 0 & \text{if } n+1 \le i \le 2n+1. \end{cases}$$

This ends the proof of Theorem 3.1.3. ∎

The usual illustration of this theorem is the Lagrangian $\mathbf{P}^n(\mathbf{R}) \subset \mathbf{P}^n(\mathbf{C})$. Here

$$H^1(\mathbf{P}^n(\mathbf{R}); \mathbf{Z}/2n+2) = \operatorname{Hom}(\mathbf{Z}/2, \mathbf{Z}/2n+2) = \mathbf{Z}/2 \neq 0.$$

Once again, Chiang's example is more exotic. We have described its fundamental group in Remark 1.3.3. From what we deduce that the Lagrangian submanifold satisfies

$$H_1(L; \mathbf{Z}) = \mathbf{Z}/4 \text{ and } H^1(L; \mathbf{Z}/8) = \mathbf{Z}/4.$$

As far as I know, this is the only example of a Lagrangian in $\mathbf{P}^n(\mathbf{C})$ for which the first cohomology group is not $\mathbf{Z}/2$.

3.3.c. The Lagrangian projective space $\mathbf{P}^n(\mathbf{C})$, proof of Proposition 3.1.4

Look now at the right part of Figure 9. This can only be periodic if $N \leq n+1$. In $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$, $N = n+1$, in agreement with the fact that $\mathbf{P}^n(\mathbf{C})$ is a Lagrangian submanifold, as we have seen it above. Notice however that this applies only to the product symplectic forms on $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$, since these are the only monotone symplectic structures. ∎

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