PORTUGALIAE MATHEMATICA
Vol. 62 Fasc. 4 – 2005
Nova Série

A GENERAL EXISTENCE THEOREM FOR DIFFERENTIAL INCLUSIONS IN THE VECTOR VALUED CASE

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Abstract: We discuss the existence of solutions, $u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$, for differential inclusions of the form

$$Du(x) \in E, \quad \text{a.e. in } \Omega .$$

1 – Introduction

In this article we discuss the existence of solutions, $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$, for the Dirichlet problem involving differential inclusions of the form

$$\begin{cases} Du(x) \in E , & \text{a.e. in } \Omega \\ u(x) = \varphi(x) , & x \in \partial\Omega \end{cases}$$

where φ is a given function and $E \subset \mathbb{R}^{m\times n}$ is a given set.

In the scalar case ($n = 1$ or $m = 1$) a sufficient condition for solving the problem is

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E, \quad \text{a.e. in } \Omega$$

where $\operatorname{int}\operatorname{co} E$ stands for the interior of the convex hull of E. This fact was observed by several authors, with different proofs and different levels of generality; notably in [1], [2], [5], [6], [8], [12] or [13]. It should be noted that this sufficient condition is very close from the necessary one.

When turning to the vectorial case ($n, m \geq 2$) the problem becomes considerably harder and no result with such a degree of elegancy and generality is available. The first general results were obtained by Dacorogna and Marcellini

Received: December 3, 2004; *Revised*: January 5, 2005.

(see the bibliography, in particular [8]). At the same time Müller and Sverak [16] introduced the method of convex integration of Gromov in this framework, obtaining comparable results.

The present paper is, in part, a review article of results by Dacorogna–Marcellini [8]. It however provides a sharp theorem generalizing their results. The main theorem below was first proved by Dacorogna–Marcellini in [7] (cf. also [8]) under a further hypothesis (see below for details). This hypothesis was later removed by Sychev in [20] (see also Müller and Sychev [17]), using the theory of convex integration. Kirchheim in [14] pointed out that using a classical result in function theory (Theorem 17) then the proof of Dacorogna–Marcellini was still valid without the extra hypothesis on E.

2 – Preliminaries

We recall the main notations that we will use throughout the article and we refer, if necessary, for more details to Dacorogna–Marcellini [8].

In the sequel we will always assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set, however the boundedness is not a real restriction, since all the constructions are local.

Notation 1. We will denote by

- $W^{1,\infty}(\Omega;\mathbb{R}^m)$ the space of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ such that

$$u \in L^\infty(\Omega;\mathbb{R}^m) \quad \text{and} \quad Du = \left(\frac{\partial u^i}{\partial x_j}\right)_{1\le j\le n}^{1\le i\le m} \in L^\infty(\Omega;\mathbb{R}^{m\times n}) \ ;$$

- $W_0^{1,\infty}(\Omega;\mathbb{R}^m) = W^{1,\infty}(\Omega;\mathbb{R}^m) \cap W_0^{1,1}(\Omega;\mathbb{R}^m)$;
- $\mathit{Aff}_{\text{piec}}(\overline{\Omega};\mathbb{R}^m)$ will stand for the subset of $W^{1,\infty}(\Omega;\mathbb{R}^m)$ consisting of piecewise affine maps;
- $C^1_{\text{piec}}(\overline{\Omega};\mathbb{R}^m)$ will denote the subset of $W^{1,\infty}(\Omega;\mathbb{R}^m)$ consisting of piecewise C^1 maps. □

For higher derivatives we will adopt the following notations.

Notation 2.

- Let $N, n, m \geq 1$ be integers. For $u : \mathbb{R}^n \to \mathbb{R}^m$ we write

  $$D^N u = \left(\frac{\partial^N u^i}{\partial x_{j_1} ... \partial x_{j_N}} \right)_{1 \leq j_1, ..., j_N \leq n}^{1 \leq i \leq m} \in \mathbb{R}_s^{m \times n^N}.$$

  (The index s stands here for all the natural symmetries implied by the interchange of the order of differentiation). When $N = 1$ we have

  $$\mathbb{R}_s^{m \times n} = \mathbb{R}^{m \times n}$$

  while if $m = 1$ and $N = 2$ we obtain

  $$\mathbb{R}_s^{n^2} = \mathbb{R}_s^{n \times n}$$

  i.e., the usual set of symmetric matrices.

- For $u : \mathbb{R}^n \to \mathbb{R}^m$ we let

  $$D^{[N]} u = \left(u, Du, ..., D^N u \right)$$

  stand for the matrix of all partial derivatives of u up to the order N. Note that

  $$D^{[N-1]} u \in \mathbb{R}_s^{m \times M_N} = \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathbb{R}_s^{m \times n^2} \times ... \times \mathbb{R}_s^{m \times n^{(N-1)}},$$

  where

  $$M_N = 1 + n + ... + n^{(N-1)} = \frac{n^N - 1}{n - 1}.$$

  Hence

  $$D^{[N]} u = \left(D^{[N-1]} u, D^N u \right) \in \mathbb{R}_s^{m \times M_N} \times \mathbb{R}_s^{m \times n^N}. \ \square$$

We therefore have the following

Notation 3. We will denote by

- $W^{N,\infty}(\Omega; \mathbb{R}^m)$ the space of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ such that $D^{[N]} u \in L^\infty$;
- $W_0^{N,\infty}(\Omega; \mathbb{R}^m) = W^{N,\infty}(\Omega; \mathbb{R}^m) \cap W_0^{N,1}(\Omega; \mathbb{R}^m)$;
- $Aff_{\text{piec}}^N(\overline{\Omega}; \mathbb{R}^m)$ will stand for the subset of $W^{N,\infty}(\Omega; \mathbb{R}^m)$ so that $D^N u$ is piecewise constant;
- $C_{\text{piec}}^N(\overline{\Omega}; \mathbb{R}^m)$ will denote the subset of $W^{N,\infty}(\Omega; \mathbb{R}^m)$ so that $D^N u$ is piecewise continuous. $\square$

We finally recall the notations for various convex hulls of sets.

Notation 4. We let, for $E \subset \mathbb{R}^{m\times n}$,

$$\overline{\mathcal{F}}_E = \left\{ f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} : f|_E \le 0 \right\} ,$$
$$\mathcal{F}_E = \left\{ f : \mathbb{R}^{m\times n} \to \mathbb{R} : f|_E \le 0 \right\} .$$

We then have respectively, the *convex*, *polyconvex*, *rank one convex* and (closure of the) *quasiconvex hull* defined by

$$\operatorname{co} E = \left\{ \xi \in \mathbb{R}^{m\times n} : f(\xi) \le 0, \text{ for every convex } f \in \overline{\mathcal{F}}_E \right\} ,$$
$$\operatorname{Pco} E = \left\{ \xi \in \mathbb{R}^{m\times n} : f(\xi) \le 0, \text{ for every polyconvex } f \in \overline{\mathcal{F}}_E \right\} ,$$
$$\operatorname{Rco} E = \left\{ \xi \in \mathbb{R}^{m\times n} : f(\xi) \le 0, \text{ for every rank one convex } f \in \overline{\mathcal{F}}_E \right\} ,$$
$$\overline{\operatorname{Qco}} E = \left\{ \xi \in \mathbb{R}^{m\times n} : f(\xi) \le 0, \text{ for every quasiconvex } f \in \mathcal{F}_E \right\} . \square$$

We should point out that by replacing $\overline{\mathcal{F}}_E$ by $\mathcal{F}_E$ in the definitions of $\operatorname{co} E$ and $\operatorname{Pco} E$ we get their closures denoted by $\overline{\operatorname{co}} E$ and $\overline{\operatorname{Pco}} E$. However if we do so in the definition of $\operatorname{Rco} E$ we get a larger set than the closure of $\operatorname{Rco} E$.

3 – The main theorem

We start with the following definition introduced by Dacorogna–Marcellini in [7] (cf. also [8]), which is the key condition to get existence of solutions.

Definition 5 (Relaxation property)**.** Let $E, K \subset \mathbb{R}^{m\times n}$. We say that K has the *relaxation property* with respect to E if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every affine function u_ξ satisfying

$$Du_\xi(x) = \xi \in K ,$$

there exist a sequence $u_\nu \in \mathit{Aff}_{\text{piec}}(\overline{\Omega}; \mathbb{R}^m)$

$$u_\nu \in u_\xi + W_0^{1,\infty}(\Omega; \mathbb{R}^m) , \quad Du_\nu(x) \in E \cup K, \text{ a.e. in } \Omega ,$$

$$u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty} , \quad \int_\Omega \operatorname{dist}(Du_\nu(x); E)\, dx \to 0 \text{ as } \nu \to \infty . \square$$

Remark 6.

(**i**) It is interesting to note that in the scalar case ($n = 1$ or $m = 1$) then $K = \operatorname{int}\operatorname{co} E$ has the relaxation property with respect to E.

(**ii**) In the vectorial case we have that, if K has the relaxation property with respect to E, then necessarily

$$K \subset \overline{\mathrm{Qco}} E\ .$$

Indeed first recall that the definition of quasiconvexity implies that, for every quasiconvex $f \in \mathcal{F}_E$,

$$f(\xi) \operatorname{meas} \Omega \ \le \int_\Omega f\big(Du_\nu(x)\big)\, dx\ .$$

Combining this last result with the fact that $\{Du_\nu\}$ is uniformly bounded, the fact that any quasiconvex function is continuous and the last property in the definition of the relaxation property, we get the inclusion $K \subset \overline{\mathrm{Qco}} E$. □

The main theorem is then.

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^{m\times n}$ be such that E is compact and K is bounded. Assume that K has the relaxation property with respect to E. Let $\varphi \in \mathit{Aff}_{\mathrm{piec}}(\overline{\Omega}; \mathbb{R}^m)$ be such that*

$$D\varphi(x) \in E \cup K, \qquad \text{a.e. in } \ \Omega\ .$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$Du(x) \in E, \qquad \text{a.e. in } \ \Omega\ .$$

Remark 8.

(**i**) According to Chapter 10 in [8], the boundary datum φ can be more general if we make the following extra hypotheses:

- in the scalar case, if K is open, φ can be even taken in $W^{1,\infty}(\Omega; \mathbb{R}^m)$, with $D\varphi(x) \in E \cup K$ (cf. Corollary 10.11 in [8]);
- in the vectorial case, if the set K is open, φ can be taken in $C^1_{\mathrm{piec}}(\overline{\Omega}; \mathbb{R}^m)$ (cf. Corollary 10.15 or Theorem 10.16 in [8]), with $D\varphi(x) \in E \cup K$. While if K is open and convex, φ can be taken in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ provided

$$D\varphi(x) \in C, \qquad \text{a.e. in } \ \Omega\ ,$$

where $C \subset K$ is compact (cf. Corollary 10.21 in [8]).

(**ii**) As already mentioned this theorem was first proved by Dacorogna–Marcellini in [7] (cf. also Theorem 6.3 in [8]) under the further hypothesis that

$$E = \left\{\xi \in \mathbb{R}^{m\times n} : F_i(\xi) = 0,\ i = 1, 2, ..., I\right\}$$

where $F_i : \mathbb{R}^{m\times n} \to \mathbb{R}$, $i = 1, 2, ..., I$, are quasiconvex. This hypothesis was later removed by Sychev in [20] using the theory of convex integration (see also Müller and Sychev [17]). Kirchheim in [14] and [15] also showed, using a classical result (Theorem 17) applied to the gradient operator (Corollary 19), that the extra hypothesis on E of Dacorogna–Marcellini [8] can be removed. The proof that we provide here is a combination of the original one of Dacorogna–Marcellini with the one of Kirchheim. More precisely we replace the density argument in [8], which was based on weak lower semicontinuity and quasiconvexity, by Corollary 19.

(**iii**) It should be pointed out that in the scalar case the theorem is in fact more general, since then no restriction on E has to be imposed and we can choose the largest possible K namely $\operatorname{int}\operatorname{co} E$.

(**iv**) For recent applications of this theorem see Croce [3], Dacorogna–Pisante–Ribeiro [9] or Dacorogna–Ribeiro [10]. □

Proof: We let $\overline{V}$ be the closure in $L^\infty(\Omega;\mathbb{R}^m)$ of

$$V = \left\{u \in \mathit{Aff}_{\text{piec}}(\overline{\Omega};\mathbb{R}^m) :\ u = \varphi \text{ on } \partial\Omega \text{ and } Du(x) \in E \cup K\right\}.$$

V is non empty since $\varphi \in V$. Let, for $k \in \mathbb{N}$,

$$V^k = \operatorname{int}\left\{u \in \overline{V} : \int_\Omega \operatorname{dist}\big(Du(x); E\big)\, dx \le \frac{1}{k}\right\}$$

where int stands for the interior of the set. We claim that V^k, in addition to be open, is dense in the complete metric space $\overline{V}$. Postponing the proof of the last fact for the end of the proof, we conclude by Baire category theorem that

$$\bigcap_{k=1}^{\infty} V^k \subset \left\{u \in \overline{V} :\ \operatorname{dist}\big(Du(x), E\big) = 0,\ \text{a.e. in } \Omega\right\} \subset \overline{V}$$

is dense, and hence non empty, in $\overline{V}$. The result then follows, since E is compact.

We now show that V^k is dense in $\overline{V}$. So let $u \in \overline{V}$ and $\epsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{L^\infty} \le \epsilon \ .$$

We recall (cf. Appendix) that

$$\omega_D(\alpha) \ = \ \lim_{\delta \to 0} \ \sup_{v,w \in B_\infty(\alpha,\delta)} \ \|Dv - Dw\|_{L^1(\Omega)}$$

where

$$B_\infty(\alpha, \delta) = \Big\{ u \in \overline{V} \colon \ \|u - \alpha\|_{L^\infty} < \delta \Big\} \ .$$

- We start by finding $\alpha \in \overline{V}$ a point of continuity of the operator D so that

$$\|u - \alpha\|_{L^\infty} \le \frac{\epsilon}{3} \ .$$

This is always possible by virtue of Corollary 19. In particular we have that the oscillation $\omega_D(\alpha)$ of the gradient operator at α is zero.

- We next approximate $\alpha \in \overline{V}$ by $\beta \in V$ so that

$$\|\beta - \alpha\|_{L^\infty} \le \frac{\epsilon}{3} \quad \text{ and } \quad \omega_D(\beta) < \frac{1}{2k} \ .$$

This is possible since by Proposition 16 we know that for every $\varepsilon > 0$ the set

$$\Omega_D^\varepsilon := \Big\{ u \in \overline{V} \colon \ \omega_D(u) < \varepsilon \Big\}$$

is open in $\overline{V}$.

- Finally we use the relaxation property on every piece where $D\beta$ is constant and we then construct $v \in V$, by patching all the pieces together, such that

$$\|\beta - v\|_{L^\infty} \le \frac{\epsilon}{3} \ , \qquad \omega_D(v) < \frac{1}{2k} \quad \text{ and } \quad \int_\Omega \operatorname{dist}\big(Dv(x); E\big)\, dx < \frac{1}{2k} \ .$$

Moreover since $\omega_D(v) < \frac{1}{2k}$ we can find $\delta = \delta(k, v) > 0$ so that

$$\|v - \psi\|_{L^\infty} \le \delta \implies \|Dv - D\psi\|_{L^1} \le \frac{1}{2k}$$

and hence

$$\int_\Omega \operatorname{dist}\big(D\psi(x); E\big)\, dx \ \le \int_\Omega \operatorname{dist}\big(Dv(x); E\big)\, dx \ + \ \|Dv - D\psi\|_{L^1} < \ \frac{1}{k}$$

for every $\psi \in B_\infty(v, \delta)$; which implies that $v \in V^k$.

Combining these three facts we have indeed obtained the desired density result. ∎

To conclude this section we give a sufficient condition that ensures the relaxation property. In concrete examples this condition is usually much easier to check than the relaxation property. We start with a definition.

Definition 9 (Approximation property)**.** Let $E \subset K(E) \subset \mathbb{R}^{m\times n}$. The sets E and $K(E)$ are said to have the *approximation property* if there exists a family of closed sets E_δ and $K(E_\delta)$, $\delta > 0$, such that

(**1**) $E_\delta \subset K(E_\delta) \subset \operatorname{int} K(E)$ for every $\delta > 0$;

(**2**) for every $\epsilon > 0$ there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that $\operatorname{dist}(\eta; E) \leq \epsilon$ for every $\eta \in E_\delta$ and $\delta \in [0, \delta_0]$;

(**3**) if $\eta \in \operatorname{int} K(E)$ then $\eta \in K(E_\delta)$ for every $\delta > 0$ sufficiently small. □

We therefore have the following theorem (cf. Theorem 6.14 in [8] and for a slightly more flexible one see Theorem 6.15).

Theorem 10. *Let $E \subset \mathbb{R}^{m\times n}$ be compact and $\operatorname{Rco} E$ has the approximation property with $K(E_\delta) = \operatorname{Rco} E_\delta$, then $\operatorname{int}\operatorname{Rco} E$ has the relaxation property with respect to E.* ∎

4 – Some extensions

In the present section we will extend the results of the preceding section. We first define the relaxation property in a more general context.

Definition 11 (Relaxation property)**.** Let $E, K \subset \mathbb{R}^n \times \mathbb{R}_s^{m\times M_N} \times \mathbb{R}_s^{m\times n^N}$. We say that K has the *relaxation property* with respect to E if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every $u_\xi \in \mathit{Aff}^N(\overline{\Omega}; \mathbb{R}^m)$ with $D^N u_\xi(x) = \xi$, satisfying

$$\big(x, D^{[N-1]}u_\xi(x), D^N u_\xi(x)\big) \in K\ ,$$

there exists a sequence $u_\nu \in \mathit{Aff}^N_{\text{piec}}(\overline{\Omega}; \mathbb{R}^m)$ such that

$$\begin{aligned}
&u_\nu \in u_\xi + W_0^{N,\infty}(\Omega;\mathbb{R}^m)\ , \qquad u_\nu \overset{*}{\rightharpoonup} u_\xi \ \text{ in } W^{N,\infty}\ ,\\
&\big(x, D^{[N-1]}u_\nu(x), D^N u_\nu(x)\big) \in E \cup K, \qquad \text{a.e. in } \ \Omega\ ,\\
&\int_\Omega \operatorname{dist}\Big(\big(x, D^{[N-1]}u_\nu(x), D^N u_\nu(x)\big); E\Big)dx \ \to 0 \qquad \text{as } \ \nu \to \infty\ .
\end{aligned}$$ □

In the sequel we will denote points of E by (x, s, ξ) with $x \in \mathbb{R}^n$, $s \in \mathbb{R}_s^{m\times M_N}$ and $\xi \in \mathbb{R}_s^{m\times n^N}$.

The following theorem is the main abstract existence theorem. The proof will be done essentially following the same argument of the proof of Theorem 7 and using the standard procedure of freezing the lower order terms as in [8] Theorem 6.3.

Theorem 12. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^n \times \mathbb{R}_s^{m\times M_N} \times \mathbb{R}_s^{m\times n^N}$ be such that E is closed, and both E and K are bounded uniformly for $x \in \Omega$ and whenever s vary on a bounded set of $\mathbb{R}_s^{m\times M_N}$. Assume that K has the relaxation property with respect to E. Let $\varphi \in \mathit{Aff}^N_{\text{piec}}(\overline{\Omega};\mathbb{R}^m)$ be such that*

$$\big(x, D^{[N-1]}\varphi(x), D^N\varphi(x)\big) \in E\cup K, \qquad \text{a.e. in } \ \Omega \ ;$$

then there exists (a dense set of) $u \in \varphi + W_0^{N,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\big(x, D^{[N-1]}u(x), D^N u(x)\big) \in E, \qquad \text{a.e. in } \ \Omega \ .$$

Remark 13.

(i) The boundedness of E (or of K) stated in the theorem should be understood as follows. For every $R > 0$, there exists $\gamma = \gamma(R)$ so that

$$(x,s,\xi)\in E, \ \ x\in\Omega \ \text{ and } \ |x|+|s|\le R \quad \Longrightarrow \quad |\xi|\le\gamma \ .$$

(ii) In this theorem if K is open (in the relative topology of $\mathbb{R}^n \times \mathbb{R}_s^{m\times M_N} \times \mathbb{R}_s^{m\times n^N}$) we can also take $\varphi \in C^N_{\text{piec}}(\overline{\Omega};\mathbb{R}^m)$ according to Corollary 10.18 in [8] (for a detailed proof of this statement see [19]).

(iii) As in the previous section, a theorem such as Theorem 10 is also available in the present context, but we do not discuss the details and we refer to Theorem 6.14 and Theorem 6.15 in [8]. $\square$

Proof: Since $\varphi \in W^{N,\infty}(\Omega;\mathbb{R}^n)$ we can find $R > 0$ so that

$$\big|D^{[N-1]}\varphi(x)\big| < R \ .$$

We let $\overline{V}$ be the closure in $C^{N-1}(\Omega;\mathbb{R}^m)$ of

$$V = \Big\{u \in \mathit{Aff}^N_{\text{piec}}(\overline{\Omega};\mathbb{R}^m) : \ u\in\varphi+W_0^{N,\infty}(\Omega;\mathbb{R}^m), \ \big|D^{[N-1]}u(x)\big|<R \\ \text{and } \ \big(x, D^{[N-1]}u(x), D^N u(x)\big)\in E\cup K\Big\} \ .$$

V is non empty since $\varphi \in V$ and $\overline{V}$ is a complete metric space when endowed with the C^{N-1} norm.

Let, for $k \in \mathbb{N}$,

$$V^k = \operatorname{int}\left\{u \in \overline{V} \colon \int_\Omega \operatorname{dist}\Big(\big(x, D^{[N-1]}u(x), D^N u(x)\big); E\Big) dx \le \frac{1}{k}\right\}.$$

The result will follow as in the proof of Theorem 7 once we have proved that V^k is dense in the complete metric space $\overline{V}$.

So let $u \in \overline{V}$ and $\epsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{N-1,\infty} \le \epsilon \ .$$

We recall (cf. the Appendix) that

$$\omega_{D^N}(u) = \lim_{\delta \to 0} \sup_{\varphi,\psi \in B_{N-1,\infty}(u,\delta)} \|D^N\varphi - D^N\psi\|_{L^1} \ ,$$

where $B_{N-1,\infty}(u,\delta) = \{v \in \overline{V} \colon \|u - v\|_{N-1,\infty} < \delta\}$.

- We start by finding $\alpha \in \overline{V}$ a point of continuity of the operator D^N (in particular $\omega_{D^N}(\alpha) = 0$) so that

$$\|u - \alpha\|_{N-1,\infty} \le \frac{\epsilon}{3} \ .$$

- We next approximate $\alpha \in \overline{V}$ by $\beta \in V$ so that,

$$\|\beta - \alpha\|_{N-1,\infty} \le \frac{\epsilon}{3} \quad \text{and} \quad \omega_{D^N}(\beta) < \frac{1}{3k} \ .$$

Since $|D^{[N-1]}\beta(x)| < R$, from now on all the approximations can be supposed, without loss of generality, sufficiently small in order to work always under the hypothesis

$$|D^{[N-1]}u(x)| < R \ .$$

- By working on each piece where $D^N\beta$ is constant, without loss of generality, we can assume that $\beta \in C^N(\overline{\Omega}; \mathbb{R}^m)$ with $D^N\beta(x) = \text{constant in } \overline{\Omega}$ and $(x, D^{[N-1]}\beta(x), D^N\beta(x)) \in E \cup K$. Therefore let

$$\Omega_0 = \Big\{x \in \Omega : \big(x, D^{[N-1]}\beta(x), D^N\beta(x)\big) \in E\Big\} \ ,$$

$$\Omega_1 = \Omega \setminus \Omega_0 \ .$$

It is clear that Ω_0 is closed, since E is compact, hence Ω_1 is open.

• We can now use the relaxation property on Ω_1 to find $v_1 \in \mathit{Aff}^N_{\text{piec}}(\overline{\Omega}_1;\mathbb{R}^m)$ such that

$$\begin{cases} v_1 \in \beta + W_0^{N,\infty}(\Omega_1;\mathbb{R}^m) \\ \|v_1 - \beta\|_{N-1,\infty} \le \dfrac{\epsilon}{3}\,; \\ \big(x, D^{[N-1]}v_1(x), D^N v_1(x)\big) \in E \cup K \quad \text{a.e. } x \in \Omega_1 \\ \displaystyle\int_{\Omega_1} \operatorname{dist}\Big(\big(x, D^{[N-1]}v_1(x), D^N v_1(x)\big); E\Big)\,dx \le \frac{1}{3k}\,. \end{cases}$$

We can now define

$$v(x) = \begin{cases} \beta(x) & \text{if } x \in \Omega_0 \\ v_1(x) & \text{if } x \in \Omega_1\,. \end{cases}$$

Observe that v is $\mathit{Aff}^N_{\text{piec}}(\overline{\Omega};\mathbb{R}^m)$ and

$$\begin{cases} v \in \varphi + W_0^{N,\infty}(\Omega;\mathbb{R}^m) \\ \|v - \beta\|_{N-1,\infty} \le \dfrac{\epsilon}{3}\,; \\ \big(x, D^{[N-1]}v(x), D^N v(x)\big) \in E \cup K \quad \text{a.e. } x \in \Omega \\ \displaystyle\int_{\Omega} \operatorname{dist}\Big(\big(x, D^{[N-1]}v(x), D^N v(x)\big); E\Big)\,dx \le \frac{1}{3k}\,. \end{cases}$$

Moreover by taking a smaller ε if needed we can ensure also that

$$\omega_{D^N}(v) < \frac{1}{3k}\,,$$

then we can find $h = h(k, v)$ so that

$$\|v - \psi\|_{N-1,\infty} \le h \implies \|D^N\psi - D^N v\|_{L^1} \le \frac{1}{3k}\,.$$

Hence choosing $h < 1/3k|\Omega|$, where $|\Omega| = \operatorname{meas}\Omega$, and writing for simplicity of notations

$$\eta_v(x) = \big(x, D^{[N-1]}v(x), D^N v(x)\big)\,, \qquad \eta_\psi(x) = \big(x, D^{[N-1]}\psi(x), D^N\psi(x)\big)$$

we have

$$\begin{aligned} \int_\Omega \operatorname{dist}(\eta_\psi(x); E)\,dx &\le \int_\Omega \operatorname{dist}(\eta_v(x); E)\,dx \\ &\quad + |\Omega|\,\big\|D^{[N-1]}\psi(x) - D^{[N-1]}v(x)\big\|_{N-1,\infty} \\ &\quad + \big\|D^N\psi(x) - D^N v(x)\big\|_{L^1} \\ &< \frac{1}{3k} + h|\Omega| + \frac{1}{3k} \le \frac{1}{k}\,, \end{aligned}$$

for every $\psi \in B_{N-1,\infty}(v, h)$; which implies that $v \in V^k$.

Combining these three facts we have indeed obtained the desired density result. ∎

5 – Appendix

In this appendix we recall some well known facts about the so called functions of first class in the sense of Baire, with particular interest in their application to the gradient operator.

We start recalling some definitions.

Definition 14. Let X, Y be metric spaces and $f : X \to Y$. We define the *oscillation* of f at $x_0 \in X$ as

$$\omega_f(x_0) \;=\; \lim_{\delta\to 0}\ \sup_{x,y\,\in B(x_0,\delta)} d_Y\big(f(y), f(x)\big)$$

where $B(x_0,\delta) := \{x \in X : d_X(x,x_0) < \delta\}$ is the open ball centered at x_0 and d_X, d_Y are the metric on the spaces X and Y respectively. □

Definition 15. A function f is said to be of *first class* (in the sense of Baire) if it can be represented as the pointwise limit of an everywhere convergent sequence of continuous functions. □

In the next proposition we recall some elementary properties of the oscillation function ω_f.

Proposition 16. *Let X, Y be metric spaces, and $f : X \to Y$.*

- **(i)** *f is continuous at $x_0 \in X$ if and only if $\omega_f(x_0) = 0$.*
- **(ii)** *The set $\Omega_f^\epsilon := \{x \in X : \omega_f(x) < \epsilon\}$ is an open set in X.* ∎

Using the notion of oscillation and Proposition 16 we can write the set $\mathcal{D}_f$ of all points at which a given function f is discontinuous as an F_σ set as follows

$$\mathcal{D}_f \;=\; \bigcup_{n=1}^{\infty}\left\{x \in X:\ \omega_f(x) \geq \frac{1}{n}\right\}. \tag{1}$$

We therefore have the following Baire theorem for functions of first class. For the convenience of the reader we will give a proof of this theorem (see also Theorem 7.3 in Oxtoby [18] or Yosida [21] page 12).

Theorem 17. *Let X, Y be metric spaces let X be complete and $f : X \to Y$. If f is a function of first class, then $\mathcal{D}_f$ is a set of first category.*

Proof: Using the representation (1) of $\mathcal{D}_f$ it suffices to show that, for each $\epsilon > 0$ the set $F = \{x \in X : \omega_f(x) \geq 5\epsilon\}$ is nowhere dense.

Let $f(x) = \lim_{n\to\infty} f_n(x)$, with f_n continuous and define the sets

$$E_n = \bigcap_{i,j \geq n} \Big\{x \in X \colon d_Y\big(f_i(x), f_j(x)\big) \leq \epsilon\Big\} , \qquad \forall\, n \in \mathbb{N} .$$

Then E_n is closed in X, by continuity of f_n, and $E_n \subset E_{n+1}$. Moreover $\bigcup_{n\in\mathbb{N}} E_n = X$, since for every $x \in X$ the sequence $\{f_n(x)\}$ is convergent and thus a Cauchy sequence in Y.

Consider any closed set with non-empty interior $I \subset X$. Since $I = \bigcup (E_n \cap I)$, the sets $E_n \cap I$ cannot all be nowhere dense, since (cf. Yosida [21] page 12) in this case the complement of I in X, I^c, should be a dense set as a complement of a set of first category by Baire theorem and this is a contradiction with the fact that I has non empty interior. Hence for some positive integer n, $E_n \cap I$ contains an open subset J, by definition (cf. Yosida [21] page 11) of a nowhere dense set.

We have $d_Y(f_j(x), f_i(x)) \leq \epsilon$ for all $x \in J$ and for all $i, j \geq n$. Putting $j = n$ and letting i tend to ∞, we find that $d_Y(f_n(x), f(x)) \leq \epsilon$ for all $x \in J$. By continuity of f_n for any $x_0 \in J$ there exists a neighborhood $I(x_0) \subset J$ such that $d_Y(f_n(x), f_n(x_0)) \leq \epsilon$ for all $x \in I(x_0)$ and hence

$$d_Y\big(f(x), f_n(x_0)\big) \leq 2\epsilon , \qquad \forall\, x \in I(x_0) .$$

Therefore

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f_n(x_0)\big) + d_Y\big(f(y), f_n(x_0)\big) \leq 4\epsilon , \qquad \forall\, x, y \in I(x_0) ,$$

then $\omega_f(x_0) \leq 4\epsilon$, and so no point of J belongs to F. Thus for every closed set I with non-empty interior there is an open set $J \subset I \cap F^c$. This shows that F is nowhere dense and therefore $\mathcal{D}_f$ is of first category. ∎

Remark 18. From Theorem 17 and the Baire category theorem follows in particular that the set of points of continuity of a function of first class from a complete metric space X to any metric space Y, i.e. the set $\mathcal{D}_f^c$ complement of $\mathcal{D}_f$, is a dense G_δ set. Indeed for any $\epsilon > 0$, the set

$$\Omega_f^\epsilon := \Big\{x \in X \colon \omega_f(x) < \epsilon\Big\}$$

is open and dense in X. □

In the proof of our main theorem we have used Theorem 17 applied to the following, quite surprising, special case of function of first class. This result was observed by Kirchheim in [14] (see also [15]) for complete sets of Lipschitz functions and the same argument gives in fact the result for general complete subsets $W^{1,\infty}(\Omega)$ functions.

Corollary 19. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $V \subset W^{1,\infty}(\Omega)$ be a non empty complete space with respect to the L^∞ metric. Then the gradient operator $D\colon V \to L^p(\Omega;\mathbb{R}^n)$ is a function of first class for any $1 \le p < \infty$.*

Proof: For $h \neq 0$, we let

$$D^h = (D_1^h, ..., D_n^h): \ V \to L^p(\Omega;\mathbb{R}^n)$$

be defined, for every $u \in V$ and $x \in \Omega$, by

$$D_i^h u(x) = \begin{cases} \dfrac{u(x+he_i)-u(x)}{h} & \text{if } \operatorname{dist}(x,\Omega^c) > |h| \\ \\ 0 & \text{elsewhere} \end{cases}$$

for $i = 1, ..., n$, where $e_1, ..., e_n$ stand for the vectors from the Euclidean basis.

The claim will follow once we will have proved that for any fixed h the operator D^h is continuous and that, for any sequence $h \to 0$,

$$\lim_{h\to 0} \|D_i^h u - D_i u\|_{L^p(\Omega)} = 0$$

for any $i = 1, ..., n$, $u \in V$.

The continuity of D^h follows easily by observing that for every $i = 1, ..., n$, $\epsilon > 0$ and $u, v \in V$ we have that

$$\begin{aligned} \|D_i^h u - D_i^h v\|_{L^p(\Omega)} &\le \frac{1}{|h|}\left(\int_{\Omega_h} \Big|u(x)-v(x)+u(x+he_i)-v(x+he_i)\Big|^p dx\right)^{\frac{1}{p}} \\ &\le \frac{2(\operatorname{meas}\Omega)^{\frac{1}{p}}}{|h|}\,\|u-v\|_{L^\infty(\Omega)} , \end{aligned}$$

where $\Omega_h = \{x \in \Omega\colon \operatorname{dist}(x,\Omega^c) > |h|\}$.

For the second claim we start observing that for any h and for any $u \in V$ we have

$$\|D_i^h u\|_{L^\infty(\Omega)} \le \left\|\frac{u(x+he_i)-u(x)}{h}\right\|_{L^\infty(\Omega_h)} \le \|D_i u\|_{L^\infty(\Omega)} < +\infty .$$

Moreover by Rademacher theorem, for any sequence $h \to 0$,

$$\lim_{h\to 0} D_i^h u(x) = D_i u(x) \quad \text{a.e.} \;\; x \in \Omega \; .$$

The result follows by Lebesgue dominated convergence theorem. ∎

ACKNOWLEDGEMENTS – We have benefitiated from the financial support of the Fonds National Suisse (grant 21-61390.00). The work of Giovanni Pisante was partially supported by the University of Naples "Federico II", by GNAMPA and was completed during his stay at the Section of Mathematics of EPFL. We would like to thank Kewei Zhang for pointing us a mistake in an earlier version of the article.

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