PORTUGALIAE MATHEMATICA
Vol. 62 Fasc. 4 – 2005
Nova Série

MATROID POLYTOPES, NESTED SETS AND BERGMAN FANS

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Abstract: The tropical variety defined by linear equations with constant coefficients is the Bergman fan of the corresponding matroid. Building on a self-contained introduction to matroid polytopes, we present a geometric construction of the Bergman fan, and we discuss its relationship with the simplicial complex of nested sets in the lattice of flats. The Bergman complex is triangulated by the nested set complex, and the two complexes coincide if and only if every connected flat remains connected after contracting along any subflat. This sharpens a result of Ardila–Klivans who showed that the Bergman complex is triangulated by the order complex of the lattice of flats. The nested sets specify the De Concini–Procesi compactification of the complement of a hyperplane arrangement, while the Bergman fan specifies the tropical compactification. These two compactifications are almost equal, and we highlight the subtle differences.

1 – Introduction

Let V be an r-dimensional linear subspace of the n-dimensional vector space $\mathbb{C}^n$ over the field $\mathbb{C}$ of complex numbers. The *amoeba* of V is the set of all vectors of the form

$$\big(\log|v_1|, \log|v_2|, ..., \log|v_n|\big) \ \in\ \mathbb{R}^n \ ,$$

where $(v_1, ..., v_n)$ runs over all vectors in V whose coordinates are non-zero. The asymptotic behavior of the amoeba is given by an r-dimensional polyhedral fan in $\mathbb{R}^n$. This fan was introduced for arbitrary algebraic varieties V in $\mathbb{C}^n$ in George Bergman's seminal paper [3]. He called it the *logarithmic limit set of V*;

Received: November 12, 2004; *Revised*: March 15, 2005.
AMS Subject Classification: Primary 52B20; Secondary 05B35, 14D99, 52B40, 52C35.

we shall use the term *Bergman fan* of V. The study of such polyhedral spaces is now an active area of research, known as *tropical geometry* [14, 18, 19, 21]. This paper is concerned with the *tropical variety* defined by a system of linear equations with constant coefficients. The case of linear equations with coefficients in a power series field is treated in [18]. Bergman fans are the local building blocks of Speyer's *tropical linear spaces*.

Our starting point is the observation of [20, §9.3] that the Bergman fan of a linear space V depends only on the associated *matroid*. One way to specify this matroid is by its collection of circuits C. These are the minimal sets arising as supports of linear forms $\sum_{i\in C} a_i x_i$ which vanish on V. Introductory references on matroids include [5, 16, 24].

Let M be any matroid of rank r on the ground set $[n] = \{1, 2, ..., n\}$. The *Bergman fan* $\widetilde{\mathcal{B}}(M)$ is the set of all vectors $w = (w_1, ..., w_n) \in \mathbb{R}^n$ such that, for every circuit C of M, the minimum of the set $\{\, w_i \mid i \in C \,\}$ is attained at least twice. Note that $\widetilde{\mathcal{B}}(M)$ is invariant under translation along the line $\mathbb{R}(1, ..., 1)$ in $\mathbb{R}^n$ and under positive scaling. Hence we lose no information restricting our attention to the intersection of the fan $\widetilde{\mathcal{B}}(M)$ with the unit sphere in the hyperplane orthogonal to the line $\mathbb{R}(1, ..., 1)$ in $\mathbb{R}^n$:

$$(1.1)\quad \mathcal{B}(M) := \widetilde{\mathcal{B}}(M) \cap \mathbb{S} \quad \text{with } \mathbb{S} = \left\{ w \in \mathbb{R}^n \mid \sum_{i=1}^{n} w_i = 0 \text{ and } \sum_{i=1}^{n} w_i^2 = 1 \right\}.$$

The spherical set $\mathcal{B}(M)$ is called the *Bergman complex* of the matroid M. The terms "fan" and "complex" are justified by our discussion of matroid polytopes in Section 2. In fact, $\widetilde{\mathcal{B}}(M)$ has a canonical decomposition as a subfan of the normal fan of the matroid polytope, and, accordingly, $\mathcal{B}(M)$ is a complex of spherical polytopes. Example 2.8 reveals that the faces of the Bergman complex are not always simplices.

In Section 3 we introduce the nested set complexes of an arbitrary lattice. If the lattice is the Boolean lattice $2^{[r]}$ of all subsets of $[r] = \{1, 2, ..., r\}$ then each nested set complex arises as the boundary of a simplicial $(r-1)$-polytope. The simple polytope dual to that simplicial polytope is constructed as the Minkowski sum of faces of the $(r-1)$-simplex; we use this to develop a polyhedral theory of local Bergman complexes.

In Section 4 we apply the local theory of Section 3 to the lattice of flats $\mathcal{L}_M$ of a rank r matroid M, and we derive the theorem that every nested set complex of $\mathcal{L}_M$ is a unimodular triangulation of the Bergman complex $\mathcal{B}(M)$. We examine the local structure of this triangulation in matroid-theoretic terms, thus refining the results in [1].

Among all nested set complexes of a matroid M, there is always a minimal one, whose vertices are indexed by the connected flats of M. Section 5 is concerned with this minimal nested set complex. It is generally quite close to the Bergman complex $\mathcal{B}(M)$, and we show that they are equal if and only if every contraction of a connected flat remains connected. We also discuss algorithmic tools for computing the Bergman complex along with its triangulation by minimal nested sets, and we discuss some non-trivial examples.

In Section 6 we relate our combinatorial results to algebraic geometry. The space $X = V \cap (\mathbb{C}^*)^n$ is the complement of an arrangement of n hyperplanes in complex affine r-space. The nested set complex specifies the *wonderful compactification* of X, due to De Concini and Procesi [7], while the Bergman complex specifies the *tropical compactification* of X, due to Tevelev [21]. The subdivision of Section 4 induces a canonical morphism from the former onto the latter. We describe this morphism geometrically.

We close the introduction with two examples where our complexes are one-dimensional.

Example 1.1 ($n = 6,\ r = 3$)**.** Let M be the graphical matroid of the complete graph K_4 on four vertices. Here the nested set complex coincides with the Bergman complex, and it equals the *Petersen graph*, as depicted in Figure 2 below. When passing to the order complex of $\mathcal{L}_M$, three of the edges are subdivided into two, so the order complex is a graph with 13 vertices and 18 edges. For connections to phylogenetics see [1, §3]. □

Example 1.2 ($n = 5,\ r = 3$)**.** Consider the graph gotten from K_4 by removing one edge, and let M' be the corresponding graphic matroid. The Bergman complex $\mathcal{B}(M')$ is the *complete bipartite graph* $K_{3,3}$. Its six vertices are the two 3-cycles and the four edges adjacent to the missing edge. One of the nine edges connects the two vertices indexed by the two 3-cycles. In the nested set complex of M' that edge is further subdivided by one vertex, corresponding to the edge of K_4 which is disjoint from the missing edge. This example appeared in [20, Example 9.14] and we shall return to it in Example 3.3. □

2 – The matroid polytope and the Bergman complex

We start with a brief introduction to matroid theory with an emphasis on polyhedral aspects. Let M be a family of r-element subsets of the ground set

$[n] = \{1, 2, ..., n\}$. We represent each subset $\sigma = \{\sigma_1, ..., \sigma_r\}$ by the corresponding sum of r unit vectors

$$e_\sigma = \sum_{i=1}^{r} e_{\sigma_i} \in \mathbb{R}^n .$$

The set family M is represented by the convex hull of these points

$$P_M := \mathrm{conv}\{e_\sigma : \ \sigma \in M\} \subset \mathbb{R}^n .$$

This is a convex polytope of dimension $\leq n-1$. It is a subset of the $(n-1)$-simplex

$$\Delta = \Big\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \ x_1 \geq 0, \ ..., \ x_n \geq 0, \ \ x_1 + x_2 + \cdots + x_n = r\Big\} .$$

Definition 2.1. A *matroid of rank* r is a family M of r-element subsets of $[n]$ such that every edge of the polytope P_M is parallel to an edge of the simplex Δ. $\square$

Experts in matroid theory may be surprised to see this unusual definition, but it is in fact equivalent to the many other definitions familiar to combinatorialists. This equivalence was first proved by Gel'fand, Goresky, MacPherson and Serganova [13]. It forms the point of departure for the theory of *Coxeter matroids* in [5]. This suggests that it would be worthwhile to extend the results in this paper to root systems other than A_n.

The basic idea behind Definition 2.1 is as follows. Every edge of the simplex Δ has the form $\mathrm{conv}(re_i, re_j)$, so it is parallel to a difference $e_i - e_j$ of two unit vectors. The elements $\sigma \in M$ are the *bases* of the matroid, and two bases σ, τ are connected by an edge $\mathrm{conv}(e_\sigma, e_\tau)$ if and only if $e_\sigma - e_\tau = e_i - e_j$. The latter condition is equivalent to $\sigma \backslash \tau = \{i\}$ and $\tau \backslash \sigma = \{j\}$, so the edges of P_M represent the basis exchange axiom.

Here is a brief summary of matroid terminology. Fix a matroid M on $[n]$. A subset $I \subseteq [n]$ is *independent* in M if $I \subset \sigma$ for some basis σ. Otherwise I is *dependent*. The *rank* of a subset $F \subseteq [n]$ is the cardinality of the largest independent subset of F. A *circuit* is a dependent set which is minimal with respect to inclusion. A subset $F \subseteq [n]$ is a *flat* of M if there is no circuit C such that $C \backslash F$ consists of exactly one element. The intersection of two flats is again a flat. The span of $G \subseteq [n]$ is the smallest flat F with $G \subseteq F$. The collection of all flats is partially ordered by inclusion. The resulting poset $\mathcal{L}_M$ is a *geometric lattice*, where $G_1 \wedge G_2 = G_1 \cap G_2$ and $G_1 \vee G_2 =$ the span of $G_1 \cup G_2$.

The polytope P_M is called the *matroid polytope* of M. What we are interested in here is the following natural question concerning the inclusion $P_M \subset \Delta$:

How does the boundary ∂P_M of the matroid polytope intersect the boundary $\partial\Delta$ of the ambient simplex? The objects of study in this paper are polyhedral complexes which represent the topological space $\partial P_M \backslash \partial\Delta$. As we shall see, their combinatorial structure is truly "wonderful" [7].

Example 2.2. Let $r = 2$ and $n = 4$ and consider the *uniform matroid* $M = \big\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\big\}$. Its matroid polytope is the regular octahedron

$$P_M = \operatorname{conv}\Big\{(1,1,0,0),\ (1,0,1,0),\ (1,0,0,1),\ (0,1,1,0),\ (0,1,0,1),\ (0,0,1,1)\Big\} \subset \mathbb{R}^4 .$$

Here $\partial P_M \backslash \partial\Delta$ consists of the relative interiors of four of the eight triangles in ∂P_M. Here the Bergman complex consists of four points, which is the homotopy type of $\partial P_M \backslash \partial\Delta$. $\square$

In order to understand the combinatorics of $\partial P_M \backslash \partial\Delta$ for general matroids M, we need to represent the matroid polytope P_M by a system of linear inequalities.

Proposition 2.3. *The matroid polytope equals the following subset of the simplex Δ:*

$$P_M = \Big\{(x_1,\dots,x_n) \in \Delta : \sum_{i\in F} x_i \leq \operatorname{rank}(F) \ \text{ for all flats } F \subseteq [n]\Big\}.$$

Proof: Consider any facet of the polytope P_M and let $\sum_{i=1}^n a_i x_i \leq b$ be an inequality defining this facet. The normal vector $(a_1, a_2, \dots, a_n)$ is perpendicular to the edges of that facet. But each edge of that facet is parallel to some difference of unit vectors $e_i - e_j$. Hence the only constraints on the coordinates of the normal vector are of the form $a_i = a_j$. Using the equation $\sum_{i=1}^n x_i = r$ and scaling the right hand side b, we can therefore assume that $(a_1, a_2, \dots, a_n)$ is a vector in $\{0,1\}^n$. Hence the polytope P_M is characterized by the inequalities of the form $\sum_{i\in G} x_i \leq b_G$ for some $G \subseteq [n]$. The right hand side b_G equals

$$b_G = \max\Big\{|\sigma \cap G| : \sigma \text{ basis of } M\Big\} = \operatorname{rank}(G) .$$

The second equality holds because every independent subset of G can be completed to a basis σ. Let F be the flat spanned by G. Then $G \subseteq F$ and $\operatorname{rank}(G) = \operatorname{rank}(F)$, and hence the inequality $\sum_{i\in F} x_i \leq \operatorname{rank}(F)$ implies the inequality $\sum_{i\in G} x_i \leq \operatorname{rank}(G)$. ∎

The circuit exchange axiom gives rise to the following equivalence relation on the ground set $[n]$ of the matroid M: i and j are *equivalent* if there exists a circuit C with $\{i, j\} \subseteq C$. The equivalence classes are the *connected components* of M. Let $c(M)$ denote the number of connected components of M. We say that M is connected if $c(M) = 1$.

Proposition 2.4. *The dimension of the matroid polytope P_M equals $n - c(M)$.*

Proof: Two elements i and j are equivalent if and only if there exist bases σ and τ with $i \in \sigma$ and $\tau = (\sigma \backslash \{i\}) \cup \{j\}$. The linear space parallel to the affine span of P_M is spanned by the vectors $e_i - e_j$ arising in this manner. The dimension of this space equals $n - c(M)$. ∎

Definition 2.1 implies that every face of a matroid polytope is a matroid polytope. Consider the face P_{M_w} of P_M at which the linear form $\sum_{i=1}^{n} w_i x_i$ attains its maximum. The bases of the matroid M_w are precisely the bases σ of M of maximal w-cost $\sum_{i \in \sigma} w_i$.

Two vectors $w, w' \in \mathbb{R}^n$ are considered *equivalent* for the matroid M if $M_w = M_{w'}$. The equivalence classes are relatively open convex polyhedral cones. These cones form a complete fan in $\mathbb{R}^n$. This fan is the *normal fan* of P_M. If Γ is a cone in the normal fan of P_M and $w \in \Gamma$ then we write $M_\Gamma = M_w$. The following proposition shows that the Bergman fan $\widetilde{\mathcal{B}}(M)$ is a subfan of the normal fan of the matroid polytope P_M.

Proposition 2.5. *The following are equivalent for a vector $w \in \mathbb{R}^n$:*

(**1**) *The vector w lies in the Bergman fan $\widetilde{\mathcal{B}}(M)$.*

(**2**) *The matroid M_w has no loops.*

(**3**) *Every element $i \in [n]$ lies in some basis of M_w.*

(**4**) *The face P_{M_w} has non-empty intersection with the interior of the simplex Δ.*

(**5**) *The linear functional $\sum_{i=1}^{n} w_i x_i$ attains its maximum over P_M in $\partial P_M \backslash \partial \Delta$.*

Proof: The equivalence of (2) and (3) is the definition of loops in matroids, namely, $i \in [n]$ is a *loop* of M_w if it lies in no basis of M_w. To see that (3) and (4) are equivalent, we note that the polytope P_{M_w} is the convex hull of the vectors e_σ where σ runs over all bases of M_w. This convex hull contains a point with full support (i.e. intersects the interior of Δ) if and only if condition (3) holds.

Conditions (4) and (5) are equivalent because $\sum_{i=1}^{n} w_i x_i$ attains its maximum over P_M in P_{M_w}. Finally, condition (1) fails if and only if there exists a circuit C of M and an element $i \in C$ such that $w_i < \min\{w_j : j \in C \backslash \{i\}\}$. This shows that if i appears in any basis σ of M then we can replace σ by another basis $(\sigma \backslash \{i\}) \cup \{j\}$ of higher w-weight, i.e., i is a loop in M_w. Conversely, if (2) fails for i then M has a circuit C with $i \in C$ and $w_i < \min\{w_j : j \in C\backslash\{i\}\}$, i.e., (1) fails. ∎

A question left open in Proposition 2.3 is to identify the facet-defining inequalities for the matroid polytope P_M other than the facets given by the ambient simplex Δ. Let us assume that M is connected, so that P_M is $(n-1)$-dimensional. We identify a flat F with its 0/1-incidence vector. Then P_{M_F} is the face of P_M at which the inequality $\sum_{i \in F} x_i \leq \operatorname{rank}(F)$ is attained with equality. A flat F of M is called a *flacet* if $\dim(P_{M_F}) = n - 2$. Thus we can replace the word "flat" by the word "flacet" in Proposition 2.3.

In order to characterize the flacets of a matroid M combinatorially, we note that every flat F of rank s defines two new matroids. The *restriction* to F is the rank s matroid

$$M[\emptyset, F] = \Big\{ \sigma \cap F : \ \sigma \in M \ \text{ and } \ |\sigma \cap F| = s \Big\} .$$

The *contraction* of M at F is the rank $r - s$ matroid

$$M\big[F, [n]\big] = \Big\{ \sigma \backslash F : \ \sigma \in M \ \text{ and } \ |\sigma \cap F| = s \Big\} .$$

Proposition 2.6. *A flat F of a connected matroid M is a flacet (i.e. it defines a facet of the polytope P_M) if and only if both of the matroids $M[\emptyset, F]$ and $M[F,[n]]$ are connected.*

Proof: Let $s = \operatorname{rank}(F)$. The matroid indexed by the incidence vector of the flat F equals

$$M_F = M[\emptyset, F] \oplus M[F,[n]] = \Big\{ \sigma \in M : \ |\sigma \cap F| = s \Big\} . \tag{2.1}$$

The corresponding face of P_M is the direct product of matroid polytopes

$$P_{M_F} = P_{M[\emptyset,F]} \times P_{M[F,[n]]} \ .$$

By Proposition 2.4, the dimension of this polytope equals

$$n - c(M_F) \;=\; |F| - c(M[\emptyset, F]) \;+\; n - |F| - c(M[F, [n]])\;.$$

This number is at most $n-2$. It equals $n-2$ if and only if F is a flacet. This happens if and only if both positive integers $c(M[\emptyset, F])$ and $c(M[F, [n]])$ are equal to one. ∎

Assuming that M is connected, the normal fan of P_M defines a subdivision of the $(n-2)$-sphere $\mathbb{S}$ specified in (1.1). This subdivision is isomorphic to the polar dual P_M^* of the matroid polytope. (If M is not connected then we replace $\mathbb{S}$ by an appropriate sphere of dimension $n - c(M) - 1$.) Each cone Γ in the normal fan of P_M is identified with a face of the *dual matroid polytope* P_M^*. Hence every face Γ of P_M^* has an associated rank r matroid M_Γ. We can now recast the definition of the Bergman complex as follows:

Theorem 2.7. *The Bergman complex $\mathcal{B}(M)$ of a matroid M is the subcomplex in the boundary ∂P_M^* of the dual matroid polytope P_M^* which consists of all faces Γ such that M_Γ has no loop. The vertices of $\mathcal{B}(M)$ are indexed by the flacets of M, provided M is connected. The Bergman complex $\mathcal{B}(M)$ is homotopy equivalent to the space $\partial P_M \backslash \partial \Delta$.*

Proof: Proposition (2.5) implies that a vector $w \in \partial P_M^*$ lies in a face of the Bergman complex $\mathcal{B}(M)$ if and only if M_Γ has no loop. This proves the first assertion. The second assertion follows from (the discussion prior to) Proposition 2.6. The third assertion follows from the general fact that a subcomplex in the boundary of a polytope is homotopy equivalent to the corresponding co-complex in the boundary of the dual polytope. This version of Alexander duality is proved using barycentric subdivisions. ∎

For instance, in Example 2.2, the dual matroid polytope P_M^* is a regular 3-cube, and the Bergman complex $\mathcal{B}(M)$ consists of four of the eight vertices of the cube P_M^*. Here is an example which shows that the Bergman complex $\mathcal{B}(M)$ is generally not simplicial.

Example 2.8. Let M be the rank 4 matroid on $[6] = \{1, 2, ..., 6\}$ whose non-bases are $A = 1234$, $B = 1356$ and $C = 2456$. This matroid is realized by the six vertices of a triangular prism, or by the six vertices of an octahedron, or by the six facet planes of a regular 3-cube. Figure 1 shows a picture of this affine hyperplane arrangement. The three non-bases correspond to the three special points A, B, C on the "plane at infinity".

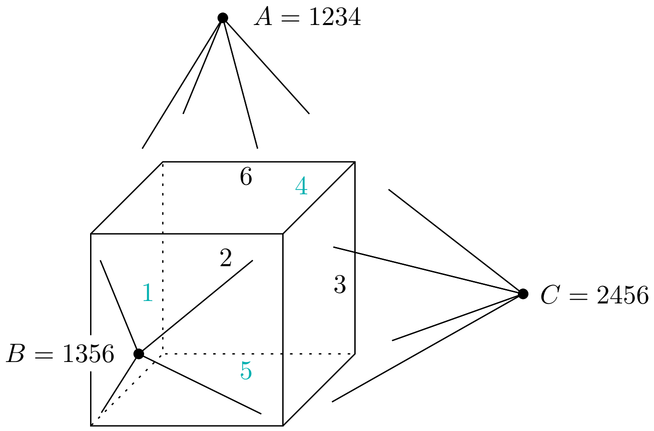

Figure 1 – The cube realization of M

The matroid M has six flacets of rank one, namely $1, 2, 3, 4, 5, 6$, and three flacets of rank three, namely A, B, C. The Bergman complex $\mathcal{B}(M)$ is two-dimensional. It has 9 vertices (the flacets), 24 edges, and 23 two-dimensional faces. Twenty faces are triangles:

$$125 \quad 126 \quad 145 \quad 146 \quad 235 \quad 236 \quad 345 \quad 346$$

$$12A \;\; 14A \;\; 23A \;\; 34A \;\; 15B \;\; 16B \;\; 35B \;\; 36B \;\; 25C \;\; 26C \;\; 45C \;\; 46C$$

The remaining three two-dimensional faces are squares:

$$1A3B \qquad 2A4C \qquad 5B6C$$

The nested set complex, to be introduced in the next section, will subdivide these squares by adding the diagonals 13, 24 and 56. We invite the reader to compute the five-dimensional matroid polytope P_M and to visualize the cocomplex $\partial P_M \backslash \partial\Delta$. Here $\mathcal{B}(M)$ and $\partial P_M \backslash \partial\Delta$ have the homotopy type of a bouquet of seven two-dimensional spheres. □

3 – Complexes of nested sets

Let $\mathcal{L}$ be a finite lattice. We review the construction of a family of simplicial complexes associated with $\mathcal{L}$ due to Feichtner and Kozlov [10], who generalized earlier work by De Concini and Procesi [7] on the special case (of interest here) when $\mathcal{L} = \mathcal{L}_M$ is the geometric lattice of a (realizable) matroid M. Intervals in $\mathcal{L}$ are denoted $[X, Y] = \{Z \in \mathcal{L} : X \leq Z \leq Y\}$. If $\mathcal{S} \subseteq \mathcal{L}$ and $X \in \mathcal{L}$, then we write $\mathcal{S}_{\leq X} := \{Y \in \mathcal{S} : Y \leq X\}$, and similarly for $\mathcal{S}_{<X}$, $\mathcal{S}_{\geq X}$, and $\mathcal{S}_{>X}$ The set of maximal elements in $\mathcal{S}$ is denoted $\max \mathcal{S}$.

Definition 3.1. Let $\mathcal{L}$ be a finite lattice. A subset $\mathcal{G}$ in $\mathcal{L}_{>\hat{0}}$ is a *building set* if for any $X \in \mathcal{L}_{>\hat{0}}$ and $\max \mathcal{G}_{\leq X} = \{G_1, ..., G_k\}$ there is an isomorphism of partially ordered sets

$$\varphi_X : \prod_{j=1}^{k} [\hat{0}, G_j] \xrightarrow{\cong} [\hat{0}, X] , \tag{3.1}$$

where the j-th component of the map φ_X is the inclusion of intervals $[\hat{0}, G_j] \subset [\hat{0}, X]$ in $\mathcal{L}$. □

The full lattice $\mathcal{L}_{>\hat{0}}$ is the simplest example of a building set for $\mathcal{L}$. Besides this maximal building set (which we denote by $\mathcal{L}$ for simplicity), there is always a minimal building set $\mathcal{G}_{\min}$ consisting of all elements X in $\mathcal{L}_{>\hat{0}}$ which do not allow for a product decomposition of the lower interval $[\hat{0}, X]$. We call these elements the *connected elements* of $\mathcal{L}$. They have been termed *irreducible elements* at other places; we here borrow the term *connected* from matroid terminology (see the beginning of Section 5).

Definition 3.2. Let $\mathcal{L}$ be a finite lattice and $\mathcal{G}$ a building set in $\mathcal{L}$ containing the maximal element $\hat{1}$ of $\mathcal{L}$. A subset $\mathcal{S}$ in $\mathcal{G}$ is called *nested* if for any set of incomparable elements $X_1, ..., X_t$ in $\mathcal{S}$ of cardinality at least two, the join $X_1 \vee ... \vee X_t$ does not belong to $\mathcal{G}$. The nested sets form an abstract simplicial complex $\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{G})$. Topologically, $\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{G})$ is a cone with apex $\hat{1}$, its link $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is the *nested set complex* of $\mathcal{L}$ with respect to $\mathcal{G}$. □

If a building set $\mathcal{G}$ does not contain the maximal element $\hat{1}$, it can always be extended to a building set $\widehat{\mathcal{G}} = \mathcal{G} \cup \{\hat{1}\}$. We can define nested sets with respect to $\mathcal{G}$ as above, and we then find the resulting abstract simplicial complex being equal to the base of $\widetilde{\mathcal{N}}(\mathcal{L}, \widehat{\mathcal{G}})$.

For the maximal building set $\mathcal{L}$, the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{L})$ coincides with the order complex $\Delta(\mathcal{L})$ of $\mathcal{L}$, i.e., the abstract simplicial complex of totally ordered subsets in the proper part of the lattice, $\mathcal{L} \setminus \{\hat{0}, \hat{1}\}$. As we pointed out above, there is always the unique minimal building set $\mathcal{G}_{\min}$ of connected elements in $\mathcal{L}$, hence a simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G}_{\min})$ with the least number of vertices among the nested set complexes of $\mathcal{L}$.

Example 3.3. Let M be the matroid of the complete graph K_4 as in Example 1.1 and consider its lattice of flats $\mathcal{L}_M$. We depict its nested set complex with respect to the minimal building set $\mathcal{G}_{\min}$ in Figure 2. The minimal building set is given by the atoms, the 3-point lines and the top element $\hat{1}$ in $\mathcal{L}_M$. Larger building sets are obtained by adding some of the three 2-point lines. This results in a subdivision of edges in $\mathcal{N}(\mathcal{L}_M, \mathcal{G}_{\min})$. □

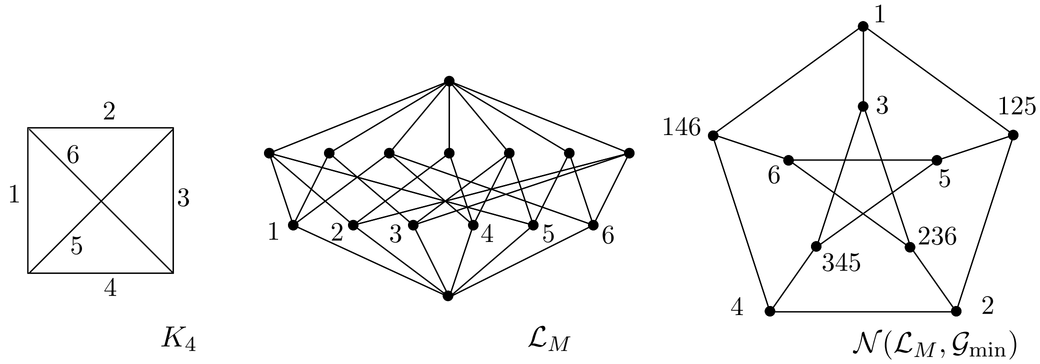

Figure 2 – The lattice of flats $\mathcal{L}_M$ and its minimal nested set complex $\mathcal{N}(\mathcal{L}_M, \mathcal{G}_{\min})$.

Example 3.4. Remove one edge from the graph K_4 to get the graphic matroid M' of Example 1.2. We depict its lattice of flats $\mathcal{L}_{M'}$ in Figure 3. Again, the minimal building set $\mathcal{G}_{\min}$ is given by the atoms, the two 3-point lines and the top element. Any other building set is obtained by adding some of the four 2-point lines. We depict the nested set complex with respect to $\mathcal{G}_{\min}$: it is a $K_{3,3}$ which is subdivided in one edge. Nested set complexes for larger building sets are obtained by subdividing up to four further edges. □

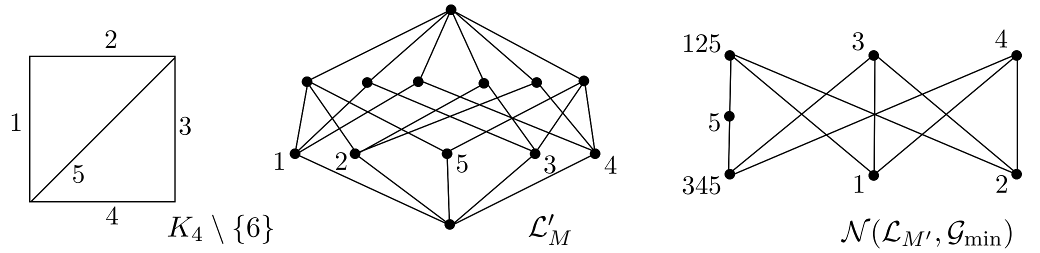

Figure 3 – The lattice of flats $\mathcal{L}_{M'}$ and its nested set complex $\mathcal{N}(\mathcal{L}_{M'}, \mathcal{G}_{\min})$.

A lattice $\mathcal{L}$ is *atomic* if every element is a join of atoms. The lattice of flats $\mathcal{L}_M$ of a matroid is an atomic lattice. For arbitrary atomic lattices $\mathcal{L}$, Feichtner and Yuzvinsky [12] proposed the following polyhedral realization of the nested set complexes $\mathcal{N}(\mathcal{L},\mathcal{G})$.

Definition 3.5. Let $\mathcal{L}$ be an atomic lattice and $\mathfrak{A}=\{A_1,\dots,A_n\}$ its set of atoms; $\mathcal{G}$ a building set in $\mathcal{L}$ containing $\hat{1}$. For any $G\in\mathcal{G}$, let $\lfloor G\rfloor := \{A\in\mathfrak{A}\,|\,A\leq G\}$, the subset of atoms below G. We define the characteristic vector v_G in $\mathbb{R}^n$ by

$$(v_G)_i := \begin{cases} 1 & \text{if } A_i\in\lfloor G\rfloor \\ 0 & \text{otherwise}\end{cases} \qquad \text{for } \ i=1,\dots,n\ .$$

For any nested set $\mathcal{S}\in\widetilde{\mathcal{N}}(\mathcal{L},\mathcal{G})$, the set of vectors $\{v_G\,|\,G\in\mathcal{S}\}$ is linearly independent, and it hence spans a simplicial cone $V_{\mathcal{S}}=\mathbb{R}_{\geq 0}\{v_G\,|\,G\in\mathcal{S}\}$. These cones intersect along faces, namely $V_{\mathcal{S}}\cap V_{\mathcal{S}'} = V_{\mathcal{S}\cap\mathcal{S}'}$, and hence they form a simplicial fan $\widetilde{\Sigma}(\mathcal{L},\mathcal{G})$ in $\mathbb{R}^n$. □

As we did before with the facet normals to the matroid polytope P_M, we can replace the vectors v_G by equivalent vectors which lie on the $(n-2)$-dimensional sphere $\mathbb{S}=\{\,w\in\mathbb{R}^n\,|\,\sum_{i=1}^n w_i=0,\ \sum_{i=1}^n w_i^2=1\}$. This is accomplished by translating v_G along the line $\mathbb{R}(1,\dots,1)$ and then scaling it to have unit length. We lose no information by replacing $\widetilde{\Sigma}(\mathcal{L},\mathcal{G})$ with its restriction to the $(n-2)$-sphere $\mathbb{S}$. The resulting complex $\Sigma(\mathcal{L},\mathcal{G})$ is a geometric realization of the abstract simplicial complex $\mathcal{N}(\mathcal{L},\mathcal{G})$.

We recall some results concerning the geometry and topology of nested set complexes:

Proposition 3.6 ([12, Prop. 2], [11, Thm. 4.2, Cor. 4.3])**.**

(**1**) *For any atomic lattice $\mathcal{L}$ and any building set $\mathcal{G}$ in $\mathcal{L}$, the fan $\widetilde{\Sigma}(\mathcal{L},\mathcal{G})$ is unimodular.*

(**2**) *For building sets $\mathcal{H}\subseteq\mathcal{G}$ in $\mathcal{L}$, the simplicial fan $\widetilde{\Sigma}(\mathcal{L},\mathcal{G})$ can be obtained from $\widetilde{\Sigma}(\mathcal{L},\mathcal{H})$ by a sequence of stellar subdivisions. In particular, the support sets of the fans $\widetilde{\Sigma}(\mathcal{L},\mathcal{G})$ coincide for all building sets $\mathcal{G}$ in $\mathcal{L}$.*

(**3**) *For an atomic lattice $\mathcal{L}$ and any building set $\mathcal{G}$ in $\mathcal{L}$, the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$ is homeomorphic to the order complex $\Delta(\mathcal{L})$.*

We shall see in Theorem 4.1 that, for any matroid M and any building set $\mathcal{G}$ in $\mathcal{L}_M$, the fan $\widetilde{\Sigma}(\mathcal{L}_M, \mathcal{G})$ is a refinement of the Bergman fan $\widetilde{\mathcal{B}}(M)$. In particular, the fans have the same support sets in $\mathbb{R}^n$. Also, the nested set complex $\mathcal{N}(\mathcal{L}_M, \mathcal{G})$ is a triangulation of the Bergman complex $\mathcal{B}(M)$ for any building set $\mathcal{G}$ in $\mathcal{L}_M$. The special case of this result when $\mathcal{G} = \mathcal{L}_M$ and the nested set complex equals the order complex of $\mathcal{L}_M$ is due to Ardila and Klivans [1, Sect. 2, Thm. 1]. It is possible to derive Theorem 4.1 from their result using the techniques of combinatorial blow-ups developed in [10]. However, we have chosen a different route which will keep this paper self-contained.

In what follows we take $\mathcal{L}$ to be the Boolean lattice $2^{[r]}$ whose elements are the subsets of $[r] = \{1, 2, ..., r\}$. Clearly, $2^{[r]}$ is an atomic lattice, in fact, it is the lattice of flats of the free rank r matroid $M = \{\{1, 2, ..., r\}\}$. We will show (in Theorem 3.14) that, for any building set $\mathcal{G}$ in $2^{[r]}$, the fan $\widetilde{\Sigma}(2^{[r]}, \mathcal{G})$, when regarded modulo the line $\mathbb{R}(1, ..., 1)$ as always, is the normal fan to a simple $(r-1)$-polytope $\Delta_{\mathcal{G}}$. Equivalently, $\mathcal{N}(2^{[r]}, \mathcal{G})$ is the boundary of a simplicial $(r-1)$-polytope $\Delta^*_{\mathcal{G}}$. This dual pair of polytopes should be of independent interest for further study of the toric manifolds introduced in [12].

Remark 3.7. The minimal building set in the Boolean lattice $2^{[r]}$ is the set of atoms, and, following our convention when defining nested set complexes, we include the maximal lattice element $[r]$; hence $\mathcal{G}_{\min} = \{\{1\}, \{2\}, ..., \{r\}, [r]\}$. The fan $\widetilde{\Sigma}(2^{[r]}, \mathcal{G}_{\min})$ is the normal fan to the $(r-1)$-simplex $\Delta_{\mathcal{G}_{\min}}$, and $\mathcal{N}(2^{[r]}, \mathcal{G}_{\min})$ is the boundary complex of the dual simplex $\Delta^*_{\mathcal{G}_{\min}}$. On the other extreme, the maximal nested set complex $\mathcal{N}(2^{[r]}, \mathcal{G}_{\max})$, $\mathcal{G}_{\max} = 2^{[r]}$, is the barycentric subdivision of the boundary of the $(r-1)$-simplex. The corresponding fan $\widetilde{\Sigma}(2^{[r]}, \mathcal{G}_{\max})$ is the braid arrangement $\{x_i = x_j\}$, and the simple polytope $\Delta_{\mathcal{G}_{\max}}$ is the *permutohedron*. Hence the polytopes $\Delta_{\mathcal{G}}$ for $\mathcal{G}$ building sets in $2^{[r]}$ interpolate between the $(r-1)$-simplex $\Delta_{\mathcal{G}_{\min}}$ and the $(r-1)$-dimensional permutohedron $\Delta_{\mathcal{G}_{\max}}$. This class of polytopes includes many interesting polytopes, such as the *associahedron*, where $\mathcal{G}$ is the set of all segments $\{i, i+1, ..., j-1, j\}$ for $1 \le i < j \le r$, and the *cyclohedron*, where $\mathcal{G}$ is the set of all cyclic segments. Both of these polytopes are special cases of the *graph-associahedra* of Carr and Devadoss [6]. □

Remark 3.8. After completion of this paper, we learned that most of the results in the remainder of Section 3 had been obtained independently by A. Postnikov [17]. The main focus of Postnikov's work is the study of the Ehrhart polynomials of the polytopes $\Delta_{\mathcal{G}}$. □

The following lemma characterizes arbitrary building sets in a Boolean lattice.

Lemma 3.9. *A family $\mathcal{F}$ of subsets of $[r]$ is a building set in the Boolean lattice $2^{[r]}$ if and only if $\mathcal{F}$ contains all singletons $\{i\}$, $i \in [r]$, and the following condition holds: if $F, F' \in \mathcal{F}$ and $F \cap F' \neq \emptyset$ then $F \cup F' \in \mathcal{F}$.*

Proof: If $\mathcal{F}$ is a subset of $2^{[r]}$ and $X \in 2^{[r]}$ then $\mathcal{F}_{\leq X}$ consists of all subsets of X that are in the family $\mathcal{F}$. The poset map in (3.1) is an isomorphism if the factors $G_1, ..., G_k$ are pairwise disjoint and their union is X. We conclude that $\mathcal{F}$ is a building set for $2^{[r]}$ if and only if all singletons are in $\mathcal{F}$ and, for every $X \in 2^{[r]}$, the maximal elements in $\mathcal{F}_{\leq X}$ are pairwise disjoint. This condition is equivalent to the one stated in the lemma. ∎

Lemma 3.10. *Any family $\mathcal{F}$ of subsets of $[r]$ can be enlarged to a unique minimal family $\widehat{\mathcal{F}}$ such that $\widehat{\mathcal{F}}$ is a building set in $2^{[r]}$. We call $\widehat{\mathcal{F}}$ the building closure of $\mathcal{F}$.*

Proof: Fix a subset X of $[r]$. We regard the set family $\max \mathcal{F}_{\leq X}$ as (the set of facets of) a simplicial complex of $[r]$. Let $\widehat{\mathcal{F}}$ be the set of all subsets X of $[r]$ such that X is a singleton or the simplicial complex $\max \mathcal{F}_{\leq X}$ is connected. It follows from Lemma 3.9 that $\widehat{\mathcal{F}}$ is a building set, and every other building set containing $\mathcal{F}$ also contains $\widehat{\mathcal{F}}$. ∎

We consider the standard simplex of dimension $r-1$. It is here denoted

$$\Delta_{[r]} \;=\; \Big\{ (x_1, ..., x_r) \in \mathbb{R}^r : \text{ all } x_i \geq 0 \text{ and } x_1 + x_2 + \cdots + x_r = 1 \Big\} .$$

Every subset F of $[r]$ defines a face of $\Delta_{[r]}$ which is a simplex of dimension $|F|-1$:

$$\Delta_F \;=\; \Big\{ (x_1, ..., x_r) \in \Delta_{[r]} : \; x_i = 0 \text{ for } i \notin F \Big\} .$$

With a family $\mathcal{F}$ of subsets of $[r]$ we associate the following Minkowski sum of simplices

$$\Delta_{\mathcal{F}} \;=\; \sum_{F \in \mathcal{F}} \Delta_F \, .$$

The dimension of the convex polytope $\Delta_{\mathcal{F}}$ is given by the following formula:

Remark 3.11. Suppose $\mathcal{F}$ contains all singletons $\{i\}$, $i \in [r]$. The dimension of the polytope $\Delta_{\mathcal{F}}$ equals $r-c$, where c is the number of connected components of the simplicial complex with facets $\max \mathcal{F}$. □

Each edge of the polytope $\Delta_{\mathcal{F}}$ is parallel to the difference of unit vectors $e_i - e_j$ in $\mathbb{R}^r$. This means that $\Delta_{\mathcal{F}}$ is a Minkowski summand of the $(r-1)$-dimensional permutohedron. In fact, $\Delta_{\mathcal{F}}$ is a permutohedron whenever $\mathcal{F}$ contains all the two-element subset of $[r]$. This observation implies the following facet description of $\Delta_{\mathcal{F}}$.

Proposition 3.12. *The polytope $\Delta_{\mathcal{F}}$ consists of all non-negative vectors $(x_1, ..., x_r)$ such that $x_1 + \cdots + x_r = |\mathcal{F}|$, and the following inequality holds for all subsets G of $[r]$:*

$$\sum_{i\in G} x_i \geq \left| \left\{ F \in \mathcal{F} : \; F \subseteq G \right\} \right| . \tag{3.2}$$

Here it suffices to take those subsets G which lie in the building closure $\widehat{\mathcal{F}}$ of $\mathcal{F}$.

Proof: Since $\Delta_{\mathcal{F}}$ is a Minkowski summand of the permutohedron, it is defined by inequalities of the form $\sum_{i\in G} x_i \; \geq \; \delta_G$ for some parameters δ_G. The minimum value of the linear form $\sum_{i\in G} x_i$ on the simplex equals one if $F \subseteq G$, and it equals zero otherwise. This shows that $\delta_G = |\{ F \in \mathcal{F} \, : \, F \subseteq G \}|$ as desired. The face of $\Delta_{\mathcal{F}}$ at which the linear form $\sum_{i\in G} x_i$ attains its minimum can be expressed as follows:

$$\Delta_{\{ F\in\mathcal{F} : F\subseteq G \}} + \Delta_{\{F\backslash G : F\in\mathcal{F}, F\backslash G\neq\emptyset\}} \, . \tag{3.3}$$

In order for this face to have codimension one in $\Delta_{\mathcal{F}}$, it is necessary, by Remark 3.11, that the set family $\{F \in \mathcal{F} : \; F \subseteq G\}$ represents a connected simplicial complex. This condition is equivalent to the set G being in the building closure $\widehat{\mathcal{F}}$. ∎

Corollary 3.13. *If $[r] \in \mathcal{F}$ then $\Delta_{\mathcal{F}}$ is $(r-1)$-dimensional and its facets are indexed by the building closure $\widehat{\mathcal{F}}$, i.e., the inequality presentation in Proposition 3.12 is irredundant.*

Proof: We have $\dim(\Delta_{\mathcal{F}}) = r - 1$ by Remark 3.11. Let $G \in \widehat{\mathcal{F}}$. The left polytope in (3.3) has dimension $|G| - 1$ as argued above. The right polytope in (3.3) contains $\Delta_{[r]\backslash G}$ as a Minkowski summand, so it has dimension $r - |G| - 1$. Hence the dimension of the face (3.3) is $(|G| - 1) + (r - |G| - 1) \, = \, r - 2$, which means it is a facet. ∎

We are interested in conditions under which the polytope $\Delta_{\mathcal{F}}$ is simple, or equivalently, the normal fan of $\Delta_{\mathcal{F}}$ is simplicial. The next theorem says that this happens if $\mathcal{F} = \widehat{\mathcal{F}}$.

Theorem 3.14. *Let $\mathcal{F}$ be a building set in the Boolean lattice $2^{[r]}$ such that $[r] \in \mathcal{F}$. Then $\Delta_{\mathcal{F}}$ is an $(r-1)$-dimensional simple polytope, and its normal fan is a unimodular simplicial fan which is combinatorially isomorphic to the nested set complex $\mathcal{N}(2^{[r]}, \mathcal{F})$.*

Proof: Our assumptions say that $\widehat{\mathcal{F}} = \mathcal{F}$ and $[r] \in \mathcal{F}$. By Corollary 3.13, the polytope $\Delta_{\mathcal{F}}$ has dimension $r-1$. The facets of $\Delta_{\mathcal{F}}$ are indexed by the elements G of $\mathcal{F}\backslash\{[r]\}$. The facet-defining inequality indexed by G is given in (3.2). We need to show that precisely $r-1$ of these inequalities are attained with equality at any vertex of $\Delta_{\mathcal{F}}$.

Pick a generic vector $w = (w_1, ..., w_r)$ and let $v = (v_1, ..., v_r)$ be the vertex of $\Delta_{\mathcal{F}}$ at which $\sum_{i=1}^{r} w_i x_i$ attains its minimum. After relabeling, we may assume that $w_1 < w_2 < ... < w_r$. Then the i-th coordinate of the vertex v equals

$$v_i = \left| \left\{ F \in \mathcal{F} :\ \min(F) = i \right\} \right| .$$

The inequality (3.2) indexed by $G \in \mathcal{F}$ holds with equality at v if and only if

$$\left| \left\{ F \in \mathcal{F} :\ \min(F) \in G \right\} \right| = \left| \left\{ F \in \mathcal{F} :\ F \subseteq G \right\} \right| .$$

Given that $\mathcal{F}$ is a building set, a necessary and sufficient condition for this equality to hold is that the set G has the following specific form for some index $i \in [r]$:

$$G_i := \bigcup \left\{ F \in \mathcal{F} :\ \min(F) = i \right\} .$$

Here we are using the fact that $\mathcal{F}$ is a building set, which ensures that G_i is in $\mathcal{F}$. Since $[r] \in \mathcal{F}$, we have $G_1 = [r]$, which is excluded from the sets in (3.2). Hence the facets incident to v are precisely the facets defined by $G_2, G_3, ..., G_r$. In particular, v is a simple vertex, and, by the relabeling argument we conclude that $\Delta_{\mathcal{F}}$ is a simple polytope.

The family $\{G_2, ..., G_r\}$ is a simplex in the nested set complex $\mathcal{N}(2^{[r]}, \mathcal{F})$. It remains to be seen that this simplex is maximal and that all maximal simplices arise in this manner, after some permutation of $[r]$. Indeed, suppose that $\mathcal{S} \subset \mathcal{F}$ is a facet of $\mathcal{N}(2^{[r]}, \mathcal{F})$. The maximal elements of $\mathcal{S}$ are pairwise disjoint, and since $[r] \in \mathcal{F}$, their union has cardinality less than r. After relabeling we may assume that the element 1 is not in the union. Since $\mathcal{S}$ was assumed to be maximal, its union equals $\{2, ..., r\}$. After relabeling again, we can write the maximal sets as $G_2, G_3, ..., G_k$ with $\min(G_i) = i$. Since the G_i are pairwise disjoint, we can now apply this construction recursively by restricting $\mathcal{S}$ and $\mathcal{F}$ to each of the subsets G_i. This construction shows that $\mathcal{S}$ has cardinality $r-1$, and it arises precisely in the manner indicated above. (See also Proposition 3.17 below).

It remains to note that the simple polytope $\Delta_{\mathcal{F}}$ is "smooth" in the sense of toric geometry. The $r-1$ edges emanating from the vertex v have directions $e_i - e_j$, and since the configuration of all vectors $e_i - e_j$ is unimodular, it follows that these $r-1$ edges form a basis for the ambient lattice. It follows that the normal fan of $\Delta_{\mathcal{F}}$ is unimodular. ∎

Consider now an arbitrary family $\mathcal{F}$ of subsets of $[r]$ and let $\widehat{\mathcal{F}}$ be its building closure as before. Since $\mathcal{F} \subseteq \widehat{\mathcal{F}}$, the polytope $\Delta_{\mathcal{F}}$ is a Minkowski summand of the simple polytope $\Delta_{\widehat{\mathcal{F}}}$. The Minkowski summand relation of convex polytopes corresponds to refinement at the level of normal fans. Hence Theorem 3.14 implies

Corollary 3.15. *The normal fan of $\Delta_{\widehat{\mathcal{F}}}$ is a triangulation of the normal fan of $\Delta_{\mathcal{F}}$.*

Example 3.16. Let $r = 4$ and $\mathcal{F} = \big\{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\big\}$. Then

$$\widehat{\mathcal{F}} = \mathcal{F} \cup \Big\{\{1\},\{2\},\{3\},\{4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\Big\},$$

We consider orbits with respect to the action of the dihedral group D_4 on $\mathcal{F}$ and on $\widehat{\mathcal{F}}$. The polytope $\Delta_{\mathcal{F}}$ is a three-dimensional zonotope with four zones, known as the *rhombic dodecahedron*. It has 14 vertices, namely eight simple vertices like $(0,1,1,2)$, four four-valent vertices like $(0,1,2,1)$, and two four-valent vertices like $(0,2,0,2)$. The normal fan of $\Delta_{\mathcal{F}}$ is the subdivision of three-space by four general planes through the origin. Corollary 3.15 describes a triangulation where each of the square-based cones are subdivided into two triangular cones. For instance, the normal cone of $\Delta_{\mathcal{F}}$ at the vertex $(0,1,2,1)$ is subdivided into the normal cones of $\Delta_{\widehat{\mathcal{F}}}$ at the vertices $(1,2,7,3)$ and $(1,3,7,2)$. Likewise, the normal cone of $\Delta_{\mathcal{F}}$ at $(0,2,0,2)$ is subdivided into normal cones of $\Delta_{\widehat{\mathcal{F}}}$ at the vertices $(1,4,1,7)$ and $(1,7,1,4)$. The normal cone of $\Delta_{\mathcal{F}}$ at $(0,1,1,2)$ remains unsubdivided. It equals the normal cone of $\Delta_{\widehat{\mathcal{F}}}$ at $(1,2,3,7)$. □

We close this section by describing a convenient representation of nested sets in terms of labeled trees. This representation appeared implicitly in the proof of Theorem 3.14, and in a modified version for partition lattices it was used previously in [9]. Let T be a rooted tree whose nodes ν are labeled by non-empty pairwise disjoint subsets T_ν of $[r]$. For each node ν we write $T_{\leq\nu}$ (resp. $T_{<\nu}$) for the union of all sets T_μ where μ is any node in the subtree of T below the node ν and including ν (resp. excluding ν). We write $\mathrm{sets}(T)$ for the set family $\{T_{\leq\nu}\}_\nu$ where ν is a non-root node of T. Since T has at most r nodes, including the root, the cardinality of $\mathrm{sets}(T)$ is at most $r-1$,

We now fix a building set $\mathcal{F}$ in $2^{[r]}$ with $[r]\in\mathcal{F}$. A tree T whose nodes are labeled by non-empty pairwise disjoint subsets as above is called an *$\mathcal{F}$-tree* if $\mathrm{sets}(T)\subseteq\mathcal{F}$.

Proposition 3.17. *For any nested set $\mathcal{S}$ of $\mathcal{F}$ there exists a unique $\mathcal{F}$-tree T such that* $\mathrm{sets}(T)=\mathcal{S}$. *The nested set complex equals $\mathcal{N}(2^{[r]},\mathcal{F}) = \big\{\mathrm{sets}(T):$ T is an $\mathcal{F}$-tree$\big\}$.*

Proof: If T is an $\mathcal{F}$-tree then $\mathrm{sets}(T)$ is nested because $T_\nu = T_{\leq\nu}\backslash T_{<\nu}$ is non-empty. Conversely, suppose that $\mathcal{S}$ is nested with k elements and contains $G_1=[r]$. We build the tree T inductively, starting with the root. Let $G_2,...,G_k$ be the maximal elements of $\mathcal{S}\backslash\{[r]\}$. We label the root by the non-empty set $\rho=[r]\backslash(G_2\cup\cdots\cup G_k)$. If $\mathcal{S}=\{[r]\}$, then $\rho=[r]$ and we are done. Otherwise note that the restriction of $\mathcal{F}$ to G_i is a building set in 2^{G_i}, and the restriction of the nested set $\mathcal{S}$ to G_i is a nested set. By induction, it is represented by a labeled tree T_i whose root is labeled by G_i. Attaching the labeled trees $T_2,...,T_k$ to the root ρ, we obtain the unique $\mathcal{F}$-tree T with $\mathrm{sets}(T)=\mathcal{S}$. ∎

4 – Triangulations of the Bergman complex

In this section we prove the following theorem relating nested sets and Bergman fans.

Theorem 4.1. *For any matroid M and any building set $\mathcal{G}$ in its lattice of flats $\mathcal{L}_M$, the fan $\widetilde{\Sigma}(\mathcal{L}_M,\mathcal{G})$ refines the Bergman fan $\widetilde{\mathcal{B}}(M)$. The geometric realization $\Sigma(\mathcal{L}_M,\mathcal{G})$ of the nested set complex $\mathcal{N}(\mathcal{L}_M,\mathcal{G})$ is a triangulation of the Bergman complex $\mathcal{B}(M)$.*

We will first prove a local version of Theorem 4.1. The Bergman complex $\mathcal{B}(M)$ of a matroid M of rank r is an $(r-2)$-dimensional subcomplex in the boundary of the dual matroid polytope P_M^*. We fix a basis σ of M. The *local Bergman complex* $\mathcal{B}_\sigma(M)$ is defined as the intersection of $\mathcal{B}(M)$ with the facet of P_M^* dual to the vertex e_σ of the matroid polytope P_M. Equivalently, we can consider the *local Bergman fan* $\widetilde{\mathcal{B}}_\sigma(M)$, which is the restriction of the Bergman fan $\widetilde{\mathcal{B}}(M)$ to the maximal cone of the normal fan of P_M indexed by σ. Consider the sublattice $\mathcal{L}_M(\sigma)$ of the geometric lattice consisting of all flats of M that are spanned by subsets of the basis σ. Clearly, $\mathcal{L}_M(\sigma)$ is a Boolean lattice of rank r, i.e., it is isomorphic to the lattice of subsets of $\{1, 2, ..., r\}$.

Let $\mathcal{G}$ be any building set in $\mathcal{L}_M$. We write $\mathcal{G}_\sigma$ for the set of all flats in $\mathcal{G}$ which are spanned by subsets of the basis σ. Then $\mathcal{G}_\sigma$ is a building set in the Boolean lattice $\mathcal{L}_M(\sigma)$. The nested set complex $\mathcal{N}\big(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma\big)$ is called the *localization* of the big nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ at the basis σ. We assume here that the matroid M is connected.

Theorem 4.2. *The localization $\mathcal{N}\big(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma\big)$ is a triangulation of the local Bergman complex $\mathcal{B}_\sigma(M)$. Both complexes are homeomorphic to the $(r-2)$-sphere. Each of them is naturally realized as the boundary complex of an $(r-1)$-dimensional polytope.*

Proof: After relabeling we may assume that the basis σ of the matroid M equals $\sigma = \{1, 2, ..., r\}$. Every element $i \in \{r+1, r+2, ..., n\}$ lies in the span of a unique subset F_i of the basis σ. This specifies the following family of subsets of $[r]$:

$$\mathcal{F} \;=\; \big\{F_{r+1}, F_{r+2}, ..., F_n\big\} \;.$$

Let $w = (w_1, ..., w_r, w_{r+1}, ..., w_n)$ be a vector in the local Bergman fan $\widetilde{\mathcal{B}}_\sigma(M)$. Using the fact that $\sigma = \{1, ..., r\}$ is a basis of the matroid M_w, and applying the "minimum attained twice condition" to the *basic circuit* $C = F_i \cup \{i\}$ of M, we find that

$$w_i \;=\; \min\{w_j : \, j \in F_i\} \qquad \text{for} \;\; i = r+1, r+2, ..., n \;.$$

This defines a piecewise-linear map from $\mathbb{R}^r$ onto the support of the local Bergman fan:

$$\psi : \; \mathbb{R}^r \mapsto |\tilde{\mathcal{B}}_\sigma(M)| \;, \qquad (w_1, ..., w_r) \;\mapsto\; (w_1, ..., w_r, w_{r+1}, ..., w_n) \;. \tag{4.1}$$

This map is obviously a bijection. The domains of linearity of the i-th coordinate (for $i > r$) of the map ψ form the normal fan of the simplex Δ_{F_i}. The common refinement of these normal fans is the normal fan of the polytope $\Delta_{\mathcal{F}}$. Hence the domains of linearity of the map ψ are the cones in the normal fan of $\Delta_{\mathcal{F}}$.

We conclude that ψ induces a combinatorial isomorphism between the normal fan of $\Delta_{\mathcal{F}}$ and the local Bergman complex $B_\sigma(M)$.

The map $2^{[n]} \to 2^{[r]} = \mathcal{L}_M(\sigma)$, $F \mapsto [r] \cap F$ defines a bijection between flats spanned by subsets of $\sigma = [r]$ and subsets of $[r]$. Under this bijection, the flats in $\mathcal{G}_\sigma$ are identified with subsets of $[r]$, and we have

$$\mathcal{F} \subseteq \widehat{\mathcal{F}} \subseteq \mathcal{G}_\sigma \subset 2^{[r]} .$$

By Theorem 3.14, $\mathcal{N}(2^{[r]}, \mathcal{G}_\sigma)$ is the normal fan of the simple polytope $\Delta_{\mathcal{G}_\sigma}$.

We have shown that both the local Bergman complex $\mathcal{B}_\sigma(M)$ and the localization $\mathcal{N}\big(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma\big)$ arise as boundary complexes of $(r-1)$-dimensional polytopes, namely, the polytopes dual to $\Delta_{\mathcal{F}}$ and $\Delta_{\mathcal{G}_\sigma}$, respectively. Since the polytope $\Delta_{\mathcal{F}}$ is a Minkowski summand of $\Delta_{\mathcal{G}_\sigma}$, it follows that $\mathcal{N}\big(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma\big)$ is a triangulation of $\mathcal{B}_\sigma(M)$. ∎

We are now prepared to prove the theorem stated at the beginning of this section.

Proof of Theorem 4.1: Both the nested set complex $\mathcal{N}(\mathcal{L}_M, \mathcal{G})$ and the Bergman complex $\mathcal{B}(M)$ are regarded as polyhedral complexes in the sphere $\mathbb{S}$, so it suffices to prove the second assertion. Using the map ψ in (4.1), it is easy to see that these spherical complexes have the same support. Indeed, if $w \in \mathbb{S}$ and σ is any basis of M_w, then

$$\begin{aligned} w \in |\mathcal{N}(\mathcal{L}_M, \mathcal{G})| \iff w \in |\mathcal{N}(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma)| &\iff w \in |\mathcal{B}_\sigma(M)| \\ &\iff w \in |\mathcal{B}(M)| . \end{aligned}$$

Since $\mathcal{N}(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma)$ triangulates $\mathcal{B}_\sigma(\mathcal{G})$ by Theorem 4.2, taking the union over all bases σ of M, it follows that $\mathcal{N}(\mathcal{L}_M, \mathcal{G})$ triangulates $\mathcal{B}(M)$. ∎

Corollary 4.3 (Ardila–Klivans [1, Sect. 2, Thm. 1])**.** *The Bergman complex $\mathcal{B}(M)$ of a matroid M is homeomorphic to the order complex $\Delta(\mathcal{L}_M)$ of the lattice of flats $\mathcal{L}_M$. In particular, $\mathcal{B}(M)$ is homotopy equivalent to a wedge of spheres of dimension $r-2$; the number of spheres is given by the absolute value of the Möbius function on $\mathcal{L}_M$.*

Proof: The first assertion is Theorem 4.1 applied to the largest building set $\mathcal{G} = \mathcal{L}$. That the second assertion holds for the order complex $\Delta(L_M)$ is a well-known result in topological combinatorics (see e.g. [4]). Hence the first assertion implies the second. ∎

Theorem 4.1 raises the following combinatorial question: What is the matroid $M_{\mathcal{S}}$ corresponding to a particular nested set $\mathcal{S}$ of $\mathcal{G}$? Here $M_{\mathcal{S}} = M_w$, where w is any point on the sphere $\mathbb{S}$ that lies in the relative interior of the simplex corresponding to $\mathcal{S}$. To answer this question, we represent the nested set $\mathcal{S}$ by a labeled tree $T_{\mathcal{S}}$ as follows. Fix any basis σ such that $\mathcal{S} \in \mathcal{N}(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma)$. By Proposition 3.17, there exists a $\mathcal{G}_\sigma$-tree T with $\mathcal{S} = \mathrm{sets}(T)$. Consider the flats $F_{\leq\nu} = \mathrm{span}(T_{\leq\nu})$ and $F_{<\nu} = \mathrm{span}(T_{<\nu})$ of the matroid M. Then $[F_{<\nu}, F_{\leq\nu}]$ is an interval in the geometric lattice $\mathcal{L}_M$. We now replace the label T_ν of the node ν in the tree T by this interval. The resulting labeled tree $T_{\mathcal{S}}$ is independent of the choice of the basis σ. It only depends on the nested set $\mathcal{S}$. In the following description of the matroid $M_{\mathcal{S}}$ we will denote the matroid defined by an interval $[F, G]$ in the geometric lattice $\mathcal{L}_M$ by $M[F, G]$, extending our notation for restrictions and contractions of matroids introduced prior to Proposition 2.6.

Theorem 4.4. *The matroid $M_{\mathcal{S}}$ is the direct sum of the matroids which are defined by the geometric lattices that appear as labels of the nodes in the tree $T_{\mathcal{S}}$. In symbols,*

$$M_{\mathcal{S}} = \bigoplus_{\nu \text{ node of } T_{\mathcal{S}}} M[F_{<\nu}, F_{\leq\nu}]\ . \tag{4.2}$$

Remark 4.5. The special case of the formula (4.2) where $\mathcal{S} = \{F, [r]\}$ represents a vertex of $\mathcal{N}(\mathcal{L}_M, \mathcal{G})$ appears in equation (2.1) in the proof of Proposition 2.6. □

Remark 4.6. The case of Theorem 4.4 where $\mathcal{G} = \mathcal{L}_M$ is the maximal building set was proved by Ardila and Klivans [1, Section 2]. In their work, the tree $T_{\mathcal{S}}$ is always a chain. □

Proof of Theorem 4.4: A basis σ of M is a basis of $M_{\mathcal{S}}$ if and only if $\mathcal{S} \in \mathcal{N}(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma)$. We can use that basis to construct the tree $T_{\mathcal{S}}$. The subset T_ν of σ is a basis of the matroid $M[F_{<\nu}, F_{\leq\nu}]$ and hence σ is a basis of the matroid on the right hand side of (4.2). Conversely, suppose that σ is a basis of the matroid on the right hand side of (4.2). Then the set $\widetilde{T}_\nu = (\sigma \cap F_{\leq\nu}) \backslash F_{<\nu}$ is a basis of the matroid $M[F_{<\nu}, F_{\leq\nu}]$. If we take $\widetilde{T}$ to be the same tree as T but with each node labeled by $\widetilde{T}_\nu$, then $\widetilde{T}$ is a $\mathcal{G}_\sigma$-tree, and we conclude that $\mathcal{S} \in \mathcal{N}(\mathcal{L}_M(\sigma), \mathcal{G}_\sigma)$, i.e., σ is a basis of $M_{\mathcal{S}}$. ∎

Of special interest is the case when $\mathcal{S}$ is a facet of the nested set complex $\mathcal{N}(\mathcal{L}_M, \mathcal{G})$. In that case, $M_{\mathcal{S}}$ is a *transversal matroid*, which means that $M_{\mathcal{S}}$ is a direct sum of matroids of rank 1. Indeed, if $|\mathcal{S}| = r$ then $T_{\mathcal{S}}$ is a binary tree, and the total number of nodes ν is r. For each node ν, the set T_ν is a singleton, and the matroid $M[F_{<\nu}, F_{\leq\nu}]$ has rank 1. Since $M[F_{<\nu}, F_{\leq\nu}]$ has no loops, by construction, this rank one matroid is uniquely specified by the subset $F_{\leq\nu}\backslash F_{<\nu}$ of $[n]$. The collection of sets $F_{\leq\nu}\backslash F_{<\nu}$, as ν runs over the nodes of $T_{\mathcal{S}}$, is a partition of the set $[n]$ in r parts. The transversal matroid $M_{\mathcal{S}}$ is uniquely specified by this set partition.

Corollary 4.7. *The facets of the nested set complex $\mathcal{N}(\mathcal{L}_M, \mathcal{G})$ are indexed by partitions of the ground set $[n]$ into r parts.*

5 – Computing nested sets and the Bergman complex

In this section we consider a connected matroid M of rank r on n elements, and we fix $\mathcal{G} = \mathcal{G}_{\min}$ to be the *minimal* building set in its lattice of flats $\mathcal{L}_M$. We denote the corresponding nested set complex by

$$\mathcal{N}(M) \,=\, \mathcal{N}(\mathcal{L}_M, \mathcal{G}_{\min}) \;,$$

and, for the purpose of this section, we call $\mathcal{N}(M)$ simply **the** *nested set complex* of M.

The minimal building set $\mathcal{G}_{\min}$ consists of all connected flats F of M. Observe that this description coincides with the set of *connected* elements in $\mathcal{L}_M$ that we identified as the minimal building set for arbitrary lattices in the beginning of Section 3.

By our results in Section 4, the various nested set complexes triangulate the Bergman complex, and by Proposition 3.6, the minimal nested set complex $\mathcal{N}(M)$ is the coarsest among these triangulations. The vertices of the Bergman complex $\mathcal{B}(M)$ are the flacets of the matroid (by Theorem 2.7), and the vertices of the nested set complex $\mathcal{N}(M)$ are the connected flats of M (by definition). These notions do not coincide in general:

Remark 5.1. Every flacet is a connected flat but not conversely. The matroid M' in Examples 1.2 and 3.4 has a connected flat of rank one (i.e., a single element) which is not a flacet: the edge which is complementary to the edge that was removed from K_4. $\square$

This distinction between flacets and connected flats amounts to the fact that in general new vertices are added when passing from the Bergman to the nested set complex:

Corollary 5.2. *The Bergman complex $\mathcal{B}(M)$ is triangulated by the nested set complex $\mathcal{N}(M)$ without additional vertices if and only if every connected flat of M is a flacet.*

We shall now generalize this corollary to give a criterion for when $\mathcal{N}(M)$ equals $\mathcal{B}(M)$.

Theorem 5.3. *The nested set complex $\mathcal{N}(M)$ equals the Bergman complex $\mathcal{B}(M)$ if and only if the matroid $M[F,G]$ is connected for every pair of flats $F \subset G$ with G connected.*

Proof: Consider any simplex $\mathcal{S}$ of the nested set complex $\mathcal{N}(M)$ and let $\Gamma_{\mathcal{S}}$ be the smallest face of the Bergman complex $\mathcal{B}(M)$ which contains $\mathcal{S}$. A necessary and sufficient condition for $\mathcal{N}(M) = \mathcal{B}(M)$ to hold is that $\dim(\mathcal{S}) = \dim(\Gamma_{\mathcal{S}})$ for all such pairs $\mathcal{S} \subseteq \Gamma_{\mathcal{S}}$. Now, $\dim(\mathcal{S})$ is simply $|\mathcal{S}| - 1$, and $\dim(\Gamma_{\mathcal{S}})$ equals $c(M_{\mathcal{S}}) - 1$. From this we conclude

$$\mathcal{N}(M) = \mathcal{B}(M) \quad \text{if and only if} \quad c(M_{\mathcal{S}}) = |\mathcal{S}| \ \text{ for all } \mathcal{S} \in \mathcal{N}(M).$$

The matroid $M_{\mathcal{S}}$ was characterized in Theorem 4.4. We have $c(M_{\mathcal{S}}) = |\mathcal{S}|$ if and only if all the matroids $M[F_{<\nu}, F_{\leq\nu}]$ in the decomposition (4.2) are connected. Note that here $F_{\leq\nu}$ is always a connected flat and $F_{<\nu}$ is a subflat of $F_{\leq\nu}$. This establishes the if-direction of Theorem 5.3. For the only-if direction, we consider any pair of flats $F \subset G$ such that G is connected. Let $\mathcal{S}$ denote the nested set which consists of G and the connected components of F. If ν is the root of the tree $T_{\mathcal{S}}$ then we have $F_{<\nu} = F$ and $F_{\leq\nu} = G$. This shows that $\mathcal{N}(M) = \mathcal{B}(M)$ implies the connectedness of $M[F,G]$. ∎

Remark 5.4. Consider the graphic matroid $M = M(K_n)$ whose bases are the spanning trees in the complete graph K_n. Here the criterion of Theorem 5.3 is satisfied, and the nested set complex coincides with the Bergman complex. This was seen for $n = 4$ in Examples 1.1 and 3.3, and in [9, Rem. 3.4.(2)] for general n. Ardila and Klivans [1, Sect. 3, Prop.] showed that $\widetilde{\mathcal{N}}(M) = \widetilde{\mathcal{B}}(M)$ equals the *space of phylogenetic trees*. Theorem 4.4 states that **every** nested set complex can be interpreted as a certain complex of trees.

The hyperplane arrangement corresponding to M is the *braid arrangement* $\{x_i = x_j\}$. What we are discussing here is its *wonderful model* (in the sense of De Concini and Procesi [7], see Section 6). Ardila, Reiner and Williams [2] recently showed that nested set complexes and Bergman complexes coincide for any finite reflection arrangement. It might be interesting to classify all subarrangements of reflection arrangements (e.g., graphic matroids) for which the Bergman complex equals the nested set complex. □

We next present an algorithm for computing both the Bergman complex $\mathcal{B}(M)$ and its triangulation by the nested set complex $\mathcal{N}(M)$. We prepared a test implementation of this algorithm in `maple`. This code is available from Bernd Sturmfels upon request.

Algorithm 5.5 (Computing the Bergman complex and the nested set complex)**.**
Input: A rank r matroid M on $[n]$, given by its bases.
Output: All maximal faces of the Bergman complex $\mathcal{B}(M)$, represented by unordered partitions $\{B_1, ..., B_r\}$ of $[n]$ into r non-empty blocks, as described in Corollary 4.7.

1. Initialize $\Omega = \emptyset$. *(This will later be the set of all unordered partitions).*
2. Precompute all connected flats of M.
3. For every basis σ of M do the following:
  (**a**) For each $i \in [n]\backslash\sigma$ find the unique set $F_i \subseteq \sigma$ such that $F_i \cup \{i\}$ is a circuit. *(The local building set equals $\mathcal{G}_\sigma = \{\, F_i \,:\, i \in [n]\backslash\sigma \,\}$).*
  (**b**) For each permutation π of $[r]$ do
    • for each $j \in [r]$ set $\omega_j := \{j\} \cup \{\, i \in [n]\backslash\sigma \,:\, \min(F_i) = \pi_j \,\}$.
    • Set $\Omega := \Omega \cup \big\{\{\omega_1, \omega_2, ..., \omega_r\}\big\}$.
4. For each $\omega \in \Omega$ do:
  (**a**) Output: "The partition ω represents a facet of $\mathcal{B}(M)$".
  (**b**) Set $\Pi := \emptyset$. *(This will be the nested set triangulation of the facet ω).*
  (**c**) For each connected flat F of M do
    • Set $s = \mathrm{rank}(F)$.
    • If F is the union of s blocks $\omega_{i_1}, ..., \omega_{i_s}$ then $\Pi := \Pi \cup \{F\}$.
  (**d**) If the cardinality of Π is r then output
    "The simplex ω is not subdivided; it equals the simplex Π in $\mathcal{N}(M)$."
  (**e**) Otherwise compute and output the set of all maximal nested sets on Π. □

Discussion and Correctness. The loop in Step 3 computes the local Bergman complex $\mathcal{B}_\sigma(M)$ for every basis σ of M, and it saves all set partitions representing facets of $\mathcal{B}(M)$ (as in Corollary 4.7) in one big set Ω. Step (b) takes advantage of the many-to-one correspondence between permutations π of $[r]$ and vertices of $\Delta_{\mathcal{G}_\sigma}$, coming from the fact that $\Delta_{\mathcal{G}_\sigma}$ is a Minkowski summand of the permutohedron. In Step 4 we output each facet ω of $\mathcal{B}(M)$ along with the list of maximal simplices in its nested set triangulation. The crucial step is the second bullet • in Step 4 (c), which tests, for each connected flat F of M, whether or not the corresponding vertex of $\mathcal{N}(M)$ lies on the facet ω of $\mathcal{B}(M)$. □

Remark 5.6. We believe that an improved version of Algorithm 5.5 should be able to list the facets of both $\mathcal{B}(M)$ and $\mathcal{N}(M)$ in a *shelling order*, so as to reveal the topology of these spaces and to offer a practical tool for the computation of residues using the methods of [8]. The constructing of such shellings remains an open problem. □

We ran Algorithm 5.5 on a range of matroids of various ranks, including the following two examples which can serve as test cases for future tropical algebraic geometry software.

Example 5.7. We consider the famous self-dual unimodular matroid R_{10} of rank $r=5$ on $n=10$ elements. This matroid plays a special role in Seymour's decomposition theory for regular matroids (see [22]). The ground set for R_{10} is the set of the edges of the complete graph K_5, and its circuits are the four-cycles and their complements. Its geometric lattice has 45 lines, 75 planes and 30 three-dimensional subspaces. There are 40 connected flats:

- the ten points in the ground set,
- the 15 four-cycles (these are planes),
- the 5 copies of the complete graph K_4 (these are three-spaces),
- the 10 copies of the complete bipartite $K_{2,3}$ (also three-spaces).

Topologically, the Bergman complex of R_{10} is a bouquet of nine 3-dimensional spheres. It is constructed as follows. We first note that the nested set complex of R_{10} consists of 405 tetrahedra, which come in eight families:

Each of the five copies of K_4 contributes 27=1+18+4+4 tetrahedra:

(**a**) the four edges not in K_4,
(**b**) the K_4, a four-cycle in K_4 and any two of its edges,
(**c**) the K_4, and three of its edges that form a K_3,
(**d**) the K_4, and three of its edges that form a $K_{1,3}$.

Each of the ten copies of $K_{2,3}$ contributes 27=1+18+2+6 tetrahedra:

(**e**) the four edges not in $K_{2,3}$

(**f**) the $K_{2,3}$, a four-cycle in K_4 and any two of its edges

(**g**) the $K_{2,3}$, and three of its edges that form a $K_{1,3}$

(**h**) the $K_{2,3}$, and three edges that touch all six vertices.

The Bergman complex of R_{10} has 360 facets, namely 315 tetrahedra and 45 bipyramids. There are 15 bipyramids formed by pairs of tetrahedra of type (b), namely, two disjoint edges in a K_4 and the two four-cycles of K_4 containing them. The other 30 bipyramids are formed by pairs of tetrahedra of type (f), namely, two edges in $K_{2,3}$ complementary to a four-cycle and the two other four-cycles of in $K_{2,3}$ which contain them. □

Example 5.8. Let M be the *cographic matroid* $M(K_5)^*$. Here $r = 6$ and $n = 10$. The bases of M are the six-tuples of edges in the complete graph K_5 which are complementary to the spanning trees. The nested set complex $\mathcal{N}(M)$ is a four-dimensional simplicial complex with f-vector $(25, 185, 615, 955, 552)$. The Bergman complex $\mathcal{B}(M)$ has 447 facets, of which 105 are subdivided into pairs of 4-simplices when passing to $\mathcal{N}(M)$. □

Example 5.9. The equality $\mathcal{B}(M) = \mathcal{N}(M)$ holds when M is the rank $r = 4$ matroid specified by the $n = 8$ vertices of the three-dimensional unit cube. The simplicial complex $\mathcal{B}(M)=\mathcal{N}(M)$ has 20 vertices, 76 edges and 80 triangles. On the other hand, for the four-dimensional unit cube ($r = 5,\ n = 16$), the Bergman complex $\mathcal{B}(M)$ is not simplicial (it has 2600 facets). It is properly subdivided by the nested set complex $\mathcal{N}(M)$ which consists of 176 vertices, 1280 edges, 3360 triangles and 2720 tetrahedra. □

6 – Tropical compactification of arrangement complements

In this section we interpret our combinatorial results in terms of algebraic geometry. We consider a connected rank r matroid M on $[n]$, so its matroid polytope $P_M \subset \mathbb{R}^n$ has dimension $n-1$, and we assume that M is realized by an r-dimensional linear subspace V of the vector space $\mathbb{C}^n$, or equivalently, by a $(r-1)$-dimensional projective linear subspace $\overline{X}$ in the complex projective space $\mathbb{P}^{n-1}$. A subset D of $[n]$ is dependent in the matroid M if and only if there exists a non-zero linear form $\sum_{i\in D} c_i x_i$ that vanishes on $\overline{X}$.

We identify the algebraic torus $(\mathbb{C}^*)^{n-1}$ with the projective space $\mathbb{P}^{n-1}$ minus its n coordinate hyperplanes $\{x_i = 0\}$. We are interested in the non-compact variety $X = \overline{X} \cap (\mathbb{C}^*)^{n-1}$. This is the complement of an arrangement of n hyperplanes in $\overline{X} \simeq \mathbb{P}^{r-1}$.

A standard problem in algebraic geometry is to construct a smooth compactification of the arrangement complement X which has better properties than the ambient projective space $\overline{X}$. Ideally, one wants the complement of X in that compactification to be a *normal crossing divisor*. A solution to this problem was given by De Concini and Procesi [7].

We briefly describe the construction of their *wonderful compactification* X_{wond}. The geometric lattice $\mathcal{L}_M$ of the matroid M is the *intersection lattice* of the hyperplane arrangement. Each flat $F \subset [n]$ of rank s in M corresponds to a subspace $\{x_i = 0 \,:\, i \in F\}$ of codimension s in the arrangement. Let $\mathbb{P}^F = \mathbb{P}(\mathbb{C}^F)$ denote the coordinate subspace of $\mathbb{P}^{n-1}$ with coordinates x_i, $i \in F$, and consider the projection $\psi_F : \mathbb{P}^{n-1} \to \mathbb{P}^F$. The restriction of ψ_F to X is a regular map. Let $\mathcal{G} = \mathcal{G}_{min}$ be the minimal building set which consists of the connected flats, and consider the product of all of these regular maps

$$\psi_{\mathcal{G}} :\ X \to \prod_{G \in \mathcal{G}} \mathbb{P}^G\ , \qquad u \mapsto \big(\psi_G(u) :\ G \in \mathcal{G}\big)\ . \tag{6.1}$$

The *wonderful model* X_{wond} is the closure of $\psi_{\mathcal{G}}(X)$ in the compact variety $\prod_{G \in \mathcal{G}} \mathbb{P}^G$.

More recently, Tevelev [21] introduced an alternative compactification, called the *tropical compactification* X_{trop}. The advantage of this new construction is that X can now be an arbitrary subvariety of the algebraic torus $(\mathbb{C}^*)^{n-1}$. Let $\overline{X}$ be the closure of X in $\mathbb{P}^{n-1}$ and let $\mathrm{Hilb}_X(\mathbb{P}^{n-1})$ be the Hilbert scheme of all subschemes of $\mathbb{P}^n$ which have the same Hilbert polynomial as $\overline{X}$. Without loss of generality, we may assume that X is not fixed by any non-unit element $t \in (\mathbb{C}^*)^{n-1}$. Then the following map is an embedding:

$$\phi :\ X \to \mathrm{Hilb}_X(\mathbb{P}^{n-1})\ , \qquad t \mapsto t^{-1} \cdot X\ . \tag{6.2}$$

The tropical compactification X_{trop} is the closure of $\phi(X)$ in the compact scheme $\mathrm{Hilb}_X(\mathbb{P}^{n-1})$. The combinatorial structure of X_{trop} is governed by the *tropical variety* of X. This is a subfan of the Gröbner fan of the ideal I_X of X. This subfan consists of all cones such that the corresponding initial ideal contains no monomials. These cones correspond to the strata in the boundary $X_{trop} \backslash X$. For nice varieties X, the divisor $X_{trop} \backslash X$ is much better behaved than $\overline{X} \backslash X$, and it is expected that X_{trop} is smooth and its boundary $X_{trop}\backslash X$ can be made normal crossing using only toric blow-ups [14, 21].

Here we consider the nicest case, when $\overline{X} \subset \mathbb{P}^{n-1}$ is a linear space (with matroid M) and X is the complement of the arrangement of the n hyperplanes $\{x_i = 0\}$ in $\overline{X}$. The subspaces of $\mathbb{P}^{n-1}$ that can be gotten by intersecting a subset of the n hyperplanes correspond to the proper flats F of the matroid M. Note that a flat F of M is connected if and only if it corresponds to a divisor in the wonderful compactification X_{wond}.

Theorem 6.1. *For any hyperplane arrangement complement X, there is a canonical morphism from the wonderful compactification X_{wond} onto the tropical compactification X_{trop}. This morphism is an isomorphism whenever the following combinatorial condition holds: If $G \in \mathcal{L}_M$ corresponds to a divisor in X_{wond} then G also corresponds to a divisor in $(X \cap F)_{wond}$ where F is any intersection of hyperplanes which contains G.*

The combinatorial condition above is a translation of the condition in Theorem 5.3. However, as Remark 6.3 shows, we must apply the condition in Theorem 6.1 to the cone over the arrangement, because G might be the full set $[n]$ in Theorem 5.3

Example 6.2. Consider an arrangement of five planes in $\mathbb{P}^3$ which intersect in one point G and which represent the matroid M' in Examples 1.2 and 3.4. Let F be the plane indexed by the edge 5 in Figure 3. Then G does not correspond to a divisor in $(X \cap F)_{wond}$ because the restriction of the arrangement to F consists of two lines through G. Hence $X_{wond} \to X_{trop}$ is not an isomorphism. On the other hand, the recent work of Ardila, Reiner and Williams [2] implies that, for any finite reflection arrangement, the tropical compactification coincides with the wonderful compactification. □

Remark 6.3. The case when $\overline{X}$ is a two-dimensional plane in $\mathbb{P}^{n-1}$ is completely described in the work of Keel and Tevelev [15, Lemma 8.11, p. 43]. Here we are considering an arrangement of n lines in $\mathbb{P}^2$. The wonderful compactification is equal to the tropical compactification except when there is a line L in the arrangement and two points $a, b \in L$ such that each remaining line passes through a or b. In that case, X_{trop} is obtained by blowing up the points $a, b \in \mathbb{P}^2$, then contracting the strict transform of the line L. The tropical compactification is therefore isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and the strict transforms of the lines through a and b become fibers of the first and second projections, respectively. □

In what follows we shall give a more concrete description of the tropical compactification X_{trop} and we shall explain how the morphism $X_{wond} \to X_{trop}$ works. Since V is a linear space, the Hilbert scheme $\mathrm{Hilb}_X(\mathbb{P}^{n-1})$ is simply the Grassmannian $Gr_{r,n}$ of r-dimensional linear subspaces in $\mathbb{C}^n$. Let I_X denote the ideal in $\mathbb{C}[x_0, x_1, ..., x_n]$ generated by all linear forms that vanish on X. For $t = (t_1 : \cdots : t_n) \in (\mathbb{C}^*)^{n-1}$, we write $t \cdot I_X$ for the image of I_X under replacing x_i by $t_i x_i$ for all i. Clearly, $I_{t^{-1} \cdot X} = t \cdot I_X$ for any $t \in (\mathbb{C}^*)^{n-1}$, i.e., the variety of the ideal $t \cdot I_X$ is the translated subspace $t^{-1} \cdot X$.

The map $t \mapsto t \cdot I_X$ defines an embedding of the torus $(\mathbb{C}^*)^{n-1}$ into the Grassmannian $\mathrm{Gr}_{r,n}$. The closure of its image is a projective toric variety T_M. As an abstract toric variety T_M depends only on the matroid M, namely, it is the toric variety associated with the matroid polytope P_M. However, the specific embedding of T_M into $\mathrm{Gr}_{r,n}$ depends on the specific realization X of M. The coordinate ring of T_M is the *basis monomial ring* [23] of the matroid M, which is the subalgebra of $\mathbb{C}[t_1, ..., t_n]$ generated by the monomials $\prod_{i \in \sigma} t_i$ where σ runs over all bases of M. A result of Neil White [23] states that the basis monomial ring is a normal domain, hence the toric variety T_M is arithmetically normal.

The $(\mathbb{C}^*)^{n-1}$-action on the Grassmannian $\mathrm{Gr}_{r,n}$ restricts to the action of the dense torus on the toric variety T_M. (Note that $\dim(T_M) = n-1$ because M is connected). The $(\mathbb{C}^*)^{n-1}$-orbits on T_M correspond to the distinct *initial ideals* of the ideal I_X, i.e.,

$$\mathrm{in}_w(I_X) = \left\langle \mathrm{in}_w(f) : f \in I_X \right\rangle . \tag{6.3}$$

Two vectors $w, w' \in \mathbb{R}^n$ lie in the same cone of the fan of the toric variety T_M if and only if $\mathrm{in}_w(I_X) = \mathrm{in}_{w'}(I_X)$. This fan is the normal fan of the matroid polytope P_M.

We now restrict the map $t \mapsto t \cdot I_X$ from $(\mathbb{C}^*)^{n-1}$ to its subvariety X. The result is precisely the map ϕ defined in (6.2) above. We can now rewrite this map as follows:

$$\phi : X \to T_M , \quad t \mapsto t \cdot I_X . \tag{6.4}$$

Remark 6.4. The map ϕ defines an embedding of the hyperplane arrangement complement X into the projective toric variety T_M associated with the matroid polytope P_M. The tropical compactification X_{trop} is the closure of the image of this embedding. □

The following lemma characterizes the location of X_{trop} inside T_M combinatorially.

Lemma 6.5. *Let C be a cone in the normal fan of the matroid polytope P_M, and let w be any vector in the relative interior of C. Then the following are equivalent:*

(**1**) *C is a cone in the Bergman fan $\widetilde{\mathcal{B}}(M)$.*

(**2**) *The initial ideal $in_w(I_X)$ in (6.3) contains no monomial.*

(**3**) *The initial ideal $in_w(I_X)$ in (6.3) contains no variable x_i.*

(**4**) *X_{trop} does intersect the $(\mathbb{C}^*)^{n-1}$-orbit of T_M corresponding to C.*

Proof: Statements (2) and (3) are equivalent because I_X is generated by linear forms. The matroid associated with the linear ideal $\text{in}_w(I_X)$ is the matroid M_w, and (3) means that the element i is not a loop of M_w. Thus (3) is equivalent to (1). The equivalence of (2) and (4) holds in general for tropical compactifications of arbitrary varieties $X \subset (\mathbb{C}^*)^n$. It is seen by considering the universal family over the Hilbert scheme in (6.2). ∎

Proof of Theorem 6.1: Every projective space $\mathbb{P}^G$ in (6.1) corresponds to a face Δ_G of the $(n-1)$-dimensional simplex Δ. Let $\mathcal{G}$ be the collection of all connected flats and let $\Delta_{\mathcal{G}}$ denote the Minkowski sum of the simplices Δ_G, where $G \in \mathcal{G}$. Let $T_{\mathcal{G}}$ denote the projective toric variety associated with the polytope $\Delta_{\mathcal{G}}$. The toric variety $T_{\mathcal{G}}$ is gotten from projective space $\mathbb{P}^{n-1}$ by the sequence of blow-ups along the linear coordinate subspaces indexed by $\mathcal{G}$. The map $\psi_{\mathcal{G}}$ in (6.1) can therefore be replaced by the inclusion $X \subset (\mathbb{C}^*)^{n-1} \subset T_{\mathcal{G}}$, i.e., the wonderful compactification X_{wond} coincides with the closure of the arrangement complement X in the projective toric variety $T_{\mathcal{G}}$. Note that $T_{\mathcal{G}}$ is generally not smooth, but X_{wond} only meets smooth strata of $T_{\mathcal{G}}$, which explains all the wonderful properties of this compactification.

Our key observation in this proof is that the matroid polytope P_M is a Minkowski summand of the polytope $\Delta_{\mathcal{G}}$ constructed in the previous paragraph. Indeed, P_M is given by the inequalities in Proposition 2.3, where F runs over $\mathcal{G}$, and $\Delta_{\mathcal{G}}$ is defined the inequalities in Proposition 3.12 where G runs over $\mathcal{G}$. A careful examination of these inequality representations reveals that P_M is a Minkowski summand of $\Delta_{\mathcal{G}}$.

The relation "is Minkowski summand of" among lattice polytopes correspond to projective morphisms in toric geometry. Since P_M is a Minkowski summand of $\Delta_{\mathcal{G}}$, we get a projective morphism from the toric variety $T_{\mathcal{G}}$ onto the toric variety T_M. The inclusion of X in T_M given in (6.4) coincides with the composed map $X \subset (\mathbb{C}^*)^{n-1} \subset T_{\mathcal{G}} \to T_M$. Hence the closure of the image of X in $T_{\mathcal{G}}$ is mapped by a projective morphism onto the closure of the image of X in T_M. This is the desired morphism $X_{wond} \to X_{trop}$.

Consider the collection of cones C in the normal fan of $\Delta_{\mathcal{G}}$ such that X_{wond} intersects the orbit indexed by C. This subfan of the normal fan is precisely the geometric realization of the nested set complex $\mathcal{N}(M)$. Likewise, the collection of cones C in the normal fan of P_M such that X_{trop} intersects the orbit indexed by C is precisely the Bergman fan $\mathcal{B}(M)$. If these fans are equal then the projective morphism $X_{wond} \to X_{trop}$ is an isomorphism. Hence the second assertion of Theorem 6.1 follows from Theorem 5.3. ∎

ACKNOWLEDGEMENTS – We are grateful to Paul Hacking for his helpful comments on a draft of this paper, and to Lauren Williams and Alex Postnikov for pointing us to the references [2, 6, 17]. The IAS/Park City Mathematics Institute (PCMI, July 2004) and the Mathematical Sciences Institute in Berkeley (MSRI) provided the setting for us to work on this project. We are grateful to both institutions for their support. Bernd Sturmfels was supported in part by the U.S. National Science Foundation (DMS-0200729).

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