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# GEOMETRIC FOKKER–PLANCK EQUATIONS

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*Recommended by J.P. Dias*

**Abstract:** We study the large deviation function and small time asymptotics near the diagonal for the heat equation associated to Geometric Fokker–Planck equations (GFK) on the cotangent bundle $\Sigma = T^*X$ of a Riemannian smooth compact connected variety X.

## Contents



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## 1 – Introduction

The purpose of this paper is to study the large deviation function and small time asymptotics near the diagonal for the heat equation associated to Geometric Fokker–Planck equations (GFK) on the cotangent bundle $\Sigma = T^*X$ of a Riemannian smooth compact connected variety $X$. These equations are the kinetic version on $T^*X$ of Laplacian operators defined on $X$. They can be view as generalized Kolmogorov equations, and are hypoelliptic operators on $T^*X$. The basic example of GFK is the hypoelliptic Laplacian introduce by J.-M. Bismut in [Bis04b], and applications to Ray–Singer metrics are given in [BL05]. There exist many papers devoted to the study of Fokker–Planck equations acting on functions (0-forms) in the case of flat metric on the Euclidean space $\mathbb{R}^n$ and with a potential, both from the PDE and probabilistic point of view. We refer mainly to the recent PDE study by B. Helffer, F. Herau, F. Nier, J. Sjostrand and C. Stolk [HN03], [HN04], [HSS04] and the references therein.

In this introductive paper to the subject, we give some basic PDE analysis results on GFK operators acting on differential forms on a compact Riemannian variety $X$ with coefficients in a fiber bundle $F$ over $X$. This auxiliary fiber bundle $F$ is a potential term in the equation, but will play no role in our study. In the definition 1.3 of GFK, we have introduce a real positive constant $\hbar$ in front of the harmonic oscillator part of the GFK. This constant $\hbar$ is a characteristic frequency. The reader may always take $\hbar = 1$, since we will not here study the operator limits $\hbar \to 0$ or $\hbar \to +\infty$. However, the introduction of the constant $\hbar$ makes the discussion of the Hamilton–Jacobi equation more clear in section 3. The reader will find the study of the limit $\hbar \to +\infty$ in the case of the the hypoelliptic Laplacian in [BL05].

The paper is organized as follows. In the first section, we define what are Geometric Fokker–Planck operators. In the second section, we give weighted estimates of Agmon type on the resolvant of GFK operators (see theorem 2.4). These estimates are far to be the best possible: first we just follow the Kohn proof of Hörmander theorem in suitable Sobolev spaces, so we do not make a precise analysis of the hypoellipticity of the GFK, and secondly, we use only the real part of the spectral parameter in the estimate. However, these estimates show that the property of finite speed of propagation for the resolvant of a GFK is related to an Hamilton–Jacobi equation on the cotangent $T^*(T^*X)$ of the phase space $T^*X$ (see (2.14)). In the third section, we study some basic properties of this Hamilton–Jacobi equation which describe the propagation property for the GFK. This Hamilton–Jacobi equation is the counterpart for GFK operators of the geodesic flow on $T^*X$ which describe the propagation property for Laplace equations, and is associated with the following action for trajectories $s \to x(s) \in X$

$$\mathcal{I}_t = \int_0^t \frac{|a|^2}{2\hbar} + \frac{\hbar|v|^2}{2}\, ds$$

where $v$, $a$ are the velocity and acceleration. This action has been introduced in the study of the hypoelliptic Laplacian by J.-M. Bismut in [Bis04b]. The corresponding Euler–Lagrange equations are given in theorem 3.2. The associated Hamiltonian function on $T^*(T^*X)$ is

$$H(z,\zeta) = \frac{\hbar}{2}\big(|\zeta^V|^2 - |p|^2\big) + (p|\zeta^H)$$

with $z = (x,p) \in T^*X$ and where $\zeta = (\zeta^V, \zeta^H) \in T^*_z(T^*X)$ is the canonical decomposition of $\zeta$ in horizontal and vertical components. The function $-H(z,\zeta)$ is a principal symbol of the GFK operator $A(\hbar, z, \partial_z)$ evaluated at $\zeta = -\partial_z$. The correspondence between critical trajectories for the action $\mathcal{I}_t$ and integral curves of the Hamiltonian of $H$ is given in theorem 3.3. The structure of the large deviation function $\mathcal{D}(t,z,z')$ which describe the decay of the heat kernel is given in theorem 3.14, and its relations to the solution of the Hamilton–Jacobi equation for the resolvant is given in theorem 3.17. In subsection 3.6, the reader will find the behavior of the geometry of the phase space $T^*X$ associated to the action $\mathcal{I}$, in the simplest but non trivial case of flat metric. Finally, in the last section we give the asymptotic, near the diagonal, of the kernel of the heat equation in theorem 4.1.

### 1.1. Geometry

Let $(X,g)$ be a Riemannian compact connected manifold of dimension $n$. Let $TX$ and $T^*X$ the tangent and cotangent bundles to $X$. We shall denote by $(x,p)$ points in $T^*X$, by $\pi$ the projection of $T^*X$ on $X$, by $g$ the canonical isomorphism from $TX$ to $T^*X$ given by the metric, and by $(.|.)$ the scalar metric product on $TX$ or $T^*X$.

If $(x_1,...,x_n)$ are local coordinates defined in an open subset $U$ of $X$, we shall denote by $(x_1,...,x_n;p_1,...,p_n)$ the coordinates on $T^*X|_U$ such that $p_j = \langle p, \frac{\partial}{\partial x_j}\rangle$. For $u = \Sigma u^j \frac{\partial}{\partial x_j} \in T_xX$ one has

$$|u|^2 = (u|u) = \Sigma g_{i,j}(x)\, u^i u^j$$

and the square of the length of the covector $p = \Sigma p_j dx_j \in T^*_x X$ at a given point $x$ is

$$|p|^2 = (p|p) = \Sigma g^{i,j}(x)\, p_i p_j$$

where $(g^{i,j}) = g^{-1}$. We shall often use the Einstein convention relative to indices summation.

Let $T\Sigma$ be the tangent bundle on $\Sigma = T^*X$. We introduce the following splitting of $T\Sigma$

$$T\Sigma = T^H\Sigma \oplus T^V\Sigma \tag{1.1}$$

where the vertical tangent space $T^V\Sigma$ is the tangent space to the fibration $T^*X \to X$, thus is span by the vector fields

$$\hat{e}^j = \frac{\partial}{\partial p_j}$$

and the horizontal space $T^H\Sigma$ is span by the vector fields

$$e_i = \frac{\partial}{\partial x_i} + \Gamma^\alpha_{\beta,i}\, p_\alpha \frac{\partial}{\partial p_\beta}$$

where $\Gamma^\alpha_{\beta,i}$ denotes the Christoffel symbols

$$\Gamma^\alpha_{\beta,i} = \frac{1}{2}\, g^{\alpha,\mu}\left[\frac{\partial g_{\mu,\beta}}{\partial x_i} + \frac{\partial g_{i,\mu}}{\partial x_\beta} - \frac{\partial g_{i,\beta}}{\partial x_\mu}\right]$$

so that the Levi–Civita connection $\nabla^{LC}$ on $TX$ is

$$\nabla^{LC}_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right) = \Gamma^k_{i,j}\, \frac{\partial}{\partial x_k}\ .$$

If $u(x) = \Sigma u^j(x)\frac{\partial}{\partial x_j}$ is a section of the tangent bundle $TX$, then $\langle p, u\rangle = \Sigma u^j p_j$ is a function on $\Sigma$, and one has the identity

$$e_i\big(\langle p, u\rangle\big) = \big\langle p, \nabla^{LC}_{\frac{\partial}{\partial x_i}} u\big\rangle\ .$$

From the identity

$$g^{i,k}\,\Gamma^j_{k,l}\, p_i p_j = -\frac{1}{2}\,\frac{\partial g^{i,j}}{\partial x_l}\, p_i p_j$$

one gets that the vector fields $e_i$ are tangent to the subvarieties $|p|^2 = \text{Cte}$, and that the Hamiltonian field of the function $|p|^2/2$ on the symplectic manifold $\Sigma$ is equal to

$$H_{|p|^2/2} = \Sigma g^{i,j} p_j e_i\, \in\, T^H\Sigma\ . \tag{1.2}$$

Set $\Gamma = \Sigma\Gamma_j\, dx_j$. Let us recall that the Riemann curvature tensor $R$ is the 2-form with values $End(TX)$ given by

$$R = \Sigma R_{j,k}\, dx^j \wedge dx^k$$
$$R_{j,k} = \frac{\partial\Gamma_k}{\partial x_j} - \frac{\partial\Gamma_j}{\partial x_k} + [\Gamma_j, \Gamma_k]\ .$$

For $u(x) = \Sigma u^j(x)\frac{\partial}{\partial x_j}$, $v(x) = \Sigma v^j(x)\frac{\partial}{\partial x_j}$, one has

$$R(u,v) \ = \ \Sigma R_{j,k}\, u^j v^k \ = \ \nabla^{LC}_u \nabla^{LC}_v - \nabla^{LC}_v \nabla^{LC}_u - \nabla^{LC}_{[u,v]} \ \in \ End(TX)$$

and the Riemann tensor $R$ satisfies the symmetries identities

$$\tag{1.3} \begin{aligned} &R(u,v) = -R(v,u) \\ &\big(R(u,v)w|z\big) = -\big(w|R(u,v)z\big) \end{aligned}$$

where the second identity is consequence of $(uv - vu - [u,v])\,(w|z) = 0$ and $u(w|z) = (\nabla^{LC}_u w|z) + (w|\nabla^{LC}_u z)$. Then the commutator $[e_j, e_k] = e_j e_k - e_k e_j$ of the vector fields $e_j, e_k$ is the vertical vector field on $\Sigma$ given by the formula

$$[e_j, e_k] = \Sigma R^{\alpha}_{j,k,\beta}\, p_\alpha \frac{\partial}{\partial p_\beta} \ \in \ T^V\Sigma$$

where $p_\alpha \frac{\partial}{\partial p_\beta} \in End(T^*X)$ acts as $p_\alpha \frac{\partial}{\partial p_\beta}(\Sigma p_j \hat{e}^j) = p_\alpha \hat{e}^\beta$, so that

$$\tag{1.4} [e_j, e_k] = {}^t R_{j,k}\ .$$

We shall equip $T\Sigma$ with the Riemannian metric such that the splitting (1.1) is orthogonal, the induces metrics on $T^H\Sigma, T^V\Sigma$ being the canonical ones defined by the isomorphisms $T^H_{x,p}\Sigma \simeq T_x X$ and $T^V_{x,p}\Sigma \simeq T^*_x X$.

Let $F \to X$ be an hermitian bundle on $X$ and $\nabla^F$ a connection on $F$. Let $Y^F$ be the first order operator acting on sections of the fiber bundle $\Sigma \times_X F$ on $\Sigma$ given in local coordinates by the formula, where $(f^i(x))_i$ is a local basis of $F$

$$\tag{1.5} Y^F\big(\Sigma a_i(x,p) f^i(x)\big) \ = \ \Sigma\big\{|p|^2/2, a_i\big\} f^i + \Sigma g^{l,j} p_j a_i \nabla^F_{\frac{\partial}{\partial x_l}}(f^i)$$

where $\{.\}$ denotes the Poisson bracket on the symplectic manifold $\Sigma$. Then the principal symbol of $Y^F$ is the vector field $H_{|p|^2/2}$. Let

$$\Lambda^\bullet(T^*\Sigma) \otimes \pi^* F$$

be the fiber bundle on $\Sigma = T^*X$ of differential forms on the cotangent bundle $T^*X$ with coefficients in $F$. For any real $s$, $\exp(sY^F)$ is an isomorphism of $\Sigma \times_X F$, and for any section $\omega$ of $\Lambda^\bullet(T^*\Sigma) \otimes \pi^* F$, the Lie derivative $\mathcal{L}ie(Y^F)(\omega)$ is defined by the formula

$$\mathcal{L}ie(Y^F)(\omega) = \frac{d}{ds} \exp(sY^F)^*(\omega)|_{s=0} \ . \tag{1.6}$$

If $e^i, \hat{e}_j$ is the dual basis of $e_i, \hat{e}^j$, given by

$$e^i = dx_i \ , \qquad \hat{e}_j = dp_j - \Gamma^\alpha_{j,k}\, p_\alpha\, dx_k$$

then one has the formula

$$\mathcal{L}ie(Y^F)\,(\Sigma \omega_I^J e^I \hat{e}_J) \ = \ \Sigma Y^F(\omega_I^J)\, e^I \hat{e}_J + \Sigma \omega_I^J \mathcal{L}ie(H_{|p|^2/2})\,(e^I \hat{e}_J) \ . \tag{1.7}$$

Let $h$ be a function on $\Sigma$, and $Z = H_h$ the Hamiltonian vector field of $h$. The action of the Lie derivative $\mathcal{L}ie(H_h)$ on 1-forms is given by the formula

$$\begin{aligned} &\mathcal{L}ie(H_h)\,(\alpha_j\, dx_j + \beta_j\, dp_j) \ = \ \alpha'_j\, dx_j + \beta'_j\, dp_j \\ &\alpha'_j \ = \ \{h, \alpha_j\} + \frac{\partial^2 h}{\partial x_j\, \partial p_k}\, \alpha_k - \frac{\partial^2 h}{\partial x_j\, \partial x_k}\, \beta_k \\ &\beta'_j \ = \ \{h, \beta_j\} + \frac{\partial^2 h}{\partial p_j\, \partial p_k}\, \alpha_k - \frac{\partial^2 h}{\partial p_j\, \partial x_k}\, \beta_k \ . \end{aligned} \tag{1.8}$$

The matrix

$$\mathcal{N}_h = \begin{pmatrix} \dfrac{\partial^2 h}{\partial x\, \partial p} & -\dfrac{\partial^2 h}{\partial x\, \partial x} \\ \dfrac{\partial^2 h}{\partial p\, \partial p} & -\dfrac{\partial^2 h}{\partial p\, \partial x} \end{pmatrix}$$

is skew adjoint for the symplectic structure. With the choice $h = |p|^2/2$ and our choice of basis $e^i = dx_i$, $\hat{e}_j = dp_j - \Gamma^\alpha_{j,k}\, p_\alpha\, dx_k$, which preserves the fact that $e^i$ is homogeneous of degree 0 in $p$, and $\hat{e}_j$ homogeneous of degree 1, on gets that $\mathcal{N} = \mathcal{N}_{|p|^2/2}$ has the following homogeneity relative to $p$

$$\begin{aligned} &\mathcal{N}(e^i) = \mathcal{N}^{i,\alpha}_{1,l}(x)\, p_\alpha e^l + \mathcal{N}^{i}_{2,l}(x)\, \hat{e}_l \\ &\mathcal{N}(\hat{e}_i) = \mathcal{N}^{i,\alpha,\beta}_{3,l}(x)\, p_\alpha p_\beta e^l + \mathcal{N}^{i,\alpha}_{4,l}(x)\, p_\alpha \hat{e}_l \ . \end{aligned} \tag{1.9}$$

We equip the fiber bundle $\Lambda^\bullet(T^*\Sigma) \otimes \pi^* F$ over $\Sigma$ with the hermitian metric induced by the one's of $\Lambda^1(T^*\Sigma)$ and $F$. We define the connection $\nabla$ on

$\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$ as the connection induced by the relations

$$\begin{aligned}\nabla_{e_k}(f^i e^j) &= \nabla^F_{\partial_{x_k}}(f^i)\, e^j + f^i\, \nabla^{LC}_{\partial_{x_k}}(e^j)\\ \nabla_{e_k}(f^i \hat{e}_j) &= \nabla^F_{\partial_{x_k}}(f^i)\, \hat{e}_j + f^i\, \nabla^{LC}_{\partial_{x_k}}(\hat{e}_j)\\ \nabla_{\hat{e}^k}(f^i e^j) &= 0\\ \nabla_{\hat{e}^k}(f^i \hat{e}_j) &= 0\end{aligned}\tag{1.10}$$

where $\nabla^{LC}_{\partial_{x_k}}(e^j)$ is the Levi–Civita connection on $T^*X$, and $\nabla^{LC}_{\partial_{x_k}}(\hat{e}_j)$ is the Levi–Civita connection on $TX$ with the identification $\hat{e}_j = \partial_{x_j}$.

### 1.2. Spaces and operators

Let $\omega$ be a section of $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$. We write in local coordinates

$$\omega = \Sigma\omega_I^J e^I \hat{e}_J$$

where $\omega_I^J(x,p)$ are sections of the fiber bundle $\Sigma\times_X F$ on $\Sigma$. We define the vertical number operator $N_V$ by the formula

$$N_V(\Sigma\omega_I^J e^I\hat{e}_J) = \Sigma\omega_I^J\,|J|\,e^I\hat{e}_J\ .\tag{1.11}$$

We define the following $L^2$ structures on the space of sections of $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$.

**Definition 1.1.** Let $dx\,dp$ be the canonical volume form on $\Sigma$. Let $\langle p\rangle$ be the function on $\Sigma$, $\langle p\rangle = (1+|p|^2)^{1/2}$. For $\omega(x,p) = \Sigma_{0\le j\le n}\omega_j(x,p)$, $N_V(\omega_j) = j\,\omega_j$ we define the weighted norm

$$\big|\omega(x,p)\big|_w^2 \;=\; \Sigma|\omega_j|^2(x,p)\,\langle p\rangle^{2j}\ .$$

Then we define the spaces $L^2$ and $L^2_w$ of sections of $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$, as the set of $\omega$ such that

$$\|\omega\|^2 \;=\; \int|\omega|^2(x,p)\;dx\,dp \;<\;\infty\tag{1.12}$$

$$\|\omega\|_w^2 \;=\; \Sigma\int|\omega_j|_w^2(x,p)\;dx\,dp\;<\;\infty\ .\ \square\tag{1.13}$$

Let us remark that by (1.5), (1.6), and (1.9) we get that the first order operator $\mathcal{L}ie(Y^F)$ can be written on the form, with $N(x,p)\in End(\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F)$

$$\begin{aligned}&\mathcal{L}ie(Y^F) = \nabla_{\{|p|^2/2,.\}} + N\\ &\|\langle p\rangle^{-1}N(\omega)\|_w \le C\,\|\omega\|_w\ .\end{aligned}\tag{1.14}$$

**Definition 1.2.** Let $M(x,p)$ be a smooth section of $End(\Lambda^{\bullet}(T^*\Sigma)\otimes\pi^*F)$, and $d\in\mathbb{R}$. Then $M(x,p)$ is a symbol of degree $d$ (resp. a weighted symbol of degree p) if for all $\alpha,\beta$, there exist $C_{\alpha,\beta}$ such that

$$\|\nabla^{\alpha}_{e_i}\nabla^{\beta}_{\hat{e}^j}M\| \leq C_{\alpha,\beta}\,\langle p\rangle^{d-|\beta|} \tag{1.15}$$

and respectively in the weighted case

$$\|\nabla^{\alpha}_{e_i}\nabla^{\beta}_{\hat{e}^j}M\|_w \leq C_{\alpha,\beta}\,\langle p\rangle^{d-|\beta|} \tag{1.16}$$

where $\|M\|_w$ is the norm of $M$ relative to the weighted norm $|\,.\,|_w$. □

Let us recall that the vertical derivative $\partial_{p_j}$ is defined by the formula

$$\partial_{p_j}(\Sigma\omega_I^J e^I\hat{e}_J) = \Sigma\partial_{p_j}(\omega_I^J)\, e^I\hat{e}_J\ . \tag{1.17}$$

We define the vertical harmonic oscillator $\mathcal{O}$ by the formula

$$\begin{aligned}
&\mathcal{O}(\Sigma\omega_I^J e^I\hat{e}_J) = \Sigma\mathcal{O}(\omega_I^J)\, e^I\hat{e}_J\\
&\mathcal{O} = \frac{1}{2}\big[-\Delta_p + |p|^2 + (2N_V - n)\big]\\
&\Delta_p = \Sigma g_{i,j}(x)\,\frac{\partial^2}{\partial p_i\,\partial p_j}\ .
\end{aligned} \tag{1.18}$$

Let $M^j_{0,1}(x,p), M(x,p)\in End(\Lambda^{\bullet}(T^*\Sigma)\otimes\pi^*F)$, be weighted symbols of degree 0 and $\hbar>0$ a real constant.

**Definition 1.3.** A Geometric Fokker–Planck operator is an operator $A$ acting on sections of $\Lambda^{\bullet}(T^*\Sigma)\otimes\pi^*F$ of the following form

$$\begin{aligned}
&A = \hbar\mathcal{O} + \mathcal{L}ie(Y^F) + \mathcal{M}\\
&\mathcal{M} = \Sigma\partial_{p_j}M_0^j + \Sigma p_j M_1^j + M\ .\ \square
\end{aligned} \tag{1.19}$$

A fundamental example of Geometric Fokker–Planck operator is the hypoelliptic Laplacian introduced by J.-M. Bismut in [Bis04b], and we refer to [Bis04a] for an introduction to the subject.

Observe that a GFK operator $A$ is a second order differential operator, partially elliptic in $p$, involving only first order derivatives in $x$. Observe also that the vertical derivatives $\partial_{p_j}$ and the commutators $[\partial_{p_k},\{|p|^2/2,.\}]$ span the tangent space to $\Sigma$, and thus, by Hörmander theorem, a GFK operator $A$ is always hypoelliptic, and the heat operator $\partial_t + A$ is also hypoelliptic.

Let $\rho$ be the linear map, defined by the formula, where $N_V(\omega_j) = j\omega_j$,

$$\rho\big(\Sigma_{0\le j\le n}\,\omega_j(x,p)\big) = \Sigma_{0\le j\le n}\,\omega_j(x,p)\,\langle p\rangle^j \ .$$

Then, if $A$ is a GFK operator, the conjugate operator

$$\rho^{-1}A\rho = A_\rho$$

is of the form

$$\begin{aligned} A_\rho &= \hbar\mathcal{O} + \nabla_{\{|p|^2/2,.\}} + \mathcal{M} \\ \mathcal{M} &= \Sigma\partial_{p_j}M_0^j + \Sigma p_j M_1^j + M \end{aligned} \tag{1.20}$$

where now the matrices $M_{0,1}^j(x,p)$, $M(x,p)$ are symbols of degree 0. Moreover, on has the obvious

$$\|\rho(\omega)\| = \|\omega\|_w \ .$$

In the rest of the paper, we will always work with conjugate GFK $A_\rho$, with the notation $A = A_\rho$, and therefore, we will only use the standard $L^2$ structure on sections of $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$.

In the analysis of GFK operators, we will use the following rules to evaluate the degree of an operator acting on sections of $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$

$$\begin{aligned} \partial_{x_j} &\quad \text{is of order } 1 \\ p_j,\, \partial_{p_j} &\quad \text{are of order } 1/2\ . \end{aligned}$$

Observe that with this definition, for any GFK operator $A$ of the form (1.20), one has obviously

$$\begin{aligned} &\mathcal{O} \text{ is self adjoint on } L^2 \text{ and of order } 1 \\ &\nabla_{\{|p|^2/2,.\}} = \Sigma g^{i,j}p_j\nabla_{e_i} \text{ is of order } 3/2 \\ &\quad \text{and has skew adjoint principal part on } L^2 \\ &\nabla_{\{|p|^2/2,.\}} + \nabla^*_{\{|p|^2/2,.\}} \ \text{ and } \mathcal{M} \ \text{ are of order at most } 1/2\ . \end{aligned} \tag{1.21}$$

Observe that a modification of the connection $\nabla^F$ on the fiber bundle $F$ will just modify the lower order term $\mathcal{M}$ in (1.20).

## 2 – Resolvant estimates

In all this section, we denote by $A$ a GFK operator of the form (1.20). Let $V$ be an open subset of $\mathbb{C}$. We shall denote by $\lambda \in V$ a spectral parameter. We decompose $\lambda$ according to

$$\lambda = -\mu + i\beta \,, \qquad \mu, \beta \in \mathbb{R} \,.$$

Let $\Phi(x,p,\lambda)$ be a smooth function on $\Sigma \times V$. We introduce the conjugate operator

$$\begin{aligned} A_\Phi &= e^{\Phi} A e^{-\Phi} = A + B_\Phi \\ B_\Phi &= \hbar\Big[-\frac{(\partial_p\Phi)^2}{2} + (\partial_p\Phi|\partial_p) + \frac{1}{2}\Delta_p\Phi\Big] - \big\{|p|^2/2, \Phi\big\} - \Sigma M_0^j \frac{\partial\Phi}{\partial p_j} \end{aligned} \tag{2.1}$$

where we use the notations

$$\begin{aligned} (\partial_p\Phi)^2 &= \Sigma g_{i,j} \frac{\partial\Phi}{\partial p_i} \frac{\partial\Phi}{\partial p_j} \\ (\partial_p\Phi|\partial_p) &= \Sigma g_{i,j} \frac{\partial\Phi}{\partial p_i} \partial_{p_j} \,. \end{aligned} \tag{2.2}$$

Then we have the following integration by part formula, with $\omega = e^{\Phi} u$

$$\begin{aligned} \int \big((A-\lambda)\omega_1|\omega_2\big)\, e^{2\Phi}\, dx\, dp &= \int \big((A_\Phi-\lambda)u_1|u_2\big)\, dx\, dp \\ &= \frac{\hbar}{2}\int (\partial_p u_1|\partial_p u_2)\, dx\, dp + \int (C_\Phi u_1|u_2)\, dx\, dp \end{aligned} \tag{2.3}$$

where the operator $C_\Phi$ is given by

$$\begin{aligned} C_\Phi &= C_\Phi^+ + C_\Phi^- + C^0 \\ C_\Phi^+ &= \mu + \Big[\frac{\hbar}{2}\Big(|p|^2 + 2N_V - n - (\partial_p\Phi)^2\Big) - \big\{|p|^2/2, \Phi\big\}\Big] \\ C_\Phi^- &= -i\beta + \hbar\Big[(\partial_p\Phi|\partial_p) + \frac{1}{2}\Delta_p\Phi\Big] + \nabla_{\{|p|^2/2,.\}} - \Sigma M_0^j \frac{\partial\Phi}{\partial p_j} \\ C^0 &= \Sigma \partial_{p_j} M_0^j + \Sigma p_j M_1^j + M \,. \end{aligned} \tag{2.4}$$

The matrices $M_{0,1}^j(x,p), M(x,p)$ are symbols of degree 0 independent of $\Phi, \lambda$.

The purpose of this section is to get $L^2$ and Sobolev estimates on the resolvant $(A_\Phi - \lambda)^{-1}$.

### 2.1. Pseudodifferential calculus

In this subsection, we recall the basic facts on pseudodifferential calculus, and associated Sobolev spaces that we shall use in the study of the conjugate operator $A_\Phi$.

In view of (2.4), it is natural to attribute the following degree to the real and imaginary part of the spectral parameter $\lambda$

$$\mu \text{ is of order } 1$$
$$\beta \text{ is of order } 3/2$$

and also to assume the following property for the first order derivatives of the phase function $\Phi$: for any $\alpha, \gamma$, there exist $C_{\alpha,\gamma}$ such that for all $k$ and all $\lambda \in V$ one has

$$\begin{aligned} \left|{e_i}^{\alpha}\, \partial_p^{\gamma} e_k\big(\Phi(x,p,\lambda)\big)\right| &\leq C_{\alpha,\gamma}\, \langle p\rangle^{-|\gamma|}\big(\langle p\rangle^2+|\mu|\big)^{1/2} \\ \left|{e_i}^{\alpha}\, \partial_p^{\gamma}\partial_{p_k}\big(\Phi(x,p,\lambda)\big)\right| &\leq C_{\alpha,\gamma}\, \langle p\rangle^{-|\gamma|}\big(\langle p\rangle^2+|\mu|\big)^{1/2}\ . \end{aligned} \tag{2.5}$$

Observe that the vector fields $e_k$ are of order 1, while the vector fields $\partial_{p_k}$ are of order 1/2, so the 2 lines in (2.5) are not at the same level of homogeneity. We shall denote by $\mathcal{S}(\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F)$ the Schwartz space of smooth sections $u$ of $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$ such that for any $k,\alpha,\gamma$, there exist $C_{k,\alpha,\gamma}$ such that for any $x,p$ one has

$$\left\|\langle p\rangle^k\, \nabla_{e_i}^{\alpha}\partial_p^{\gamma} u(x,p)\right\| \leq C_{k,\alpha,\gamma}\ .$$

We denote by $\mathcal{S}'(\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F)$ the Schwartz space of tempered distributions sections of $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$.

We first introduce a Littlewood–Paley decomposition in the radial variable $|p|$ for sections of the fiber bundle $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$. Let $r_0\in\,]1,2[$ and $\phi(r)\in C_0^\infty(\mathbb{R})$ such that $\phi(r)\equiv 1$ for $|r|\leq\frac{1}{r_0}$ and $\phi(r)\equiv 0$ for $|r|\geq 1$. Let $\chi(r)=\phi(r/2)-\phi(r)$; then $\chi$ has support in $[\frac{1}{r_0},2]$. For $j\in\mathbb{N}$, let $\chi_j(r)=\chi(2^{-j}r)$; then one has

$$1 = \phi(r)+\sum_{j=0}^{\infty}\chi_j(r)\ .$$

One has $\langle p\rangle\geq 1$ so we have for $u\in\mathcal{S}'$, $\phi(\langle p\rangle)u=0$ and we get the Littlewood–Paley type decomposition of $u\in\mathcal{S}'$ in the form

$$\begin{aligned} u &= \sum_{j=0}^{\infty}\delta_j(u) \\ \delta_j(u) &= \chi\big(2^{-j}\langle p\rangle\big)\, u\ . \end{aligned} \tag{2.6}$$

For $j \geq 1$ the tempered distribution $\delta_j(u)$ has support in the annulus

$$C_j = \Big\{ \langle p \rangle \in [2^j/r_0\,,\, 2^{j+1}] \Big\} \tag{2.7}$$

and $\delta_0(u)$ has support in the ball $\{|p|^2 \leq 3\}$. One has $C_{j+2} \cap C_j = \phi$.

**Definition 2.1.** For any $u \in \mathcal{S}'$, we define $U = \bigoplus_j U_j$ by the formula

$$U_j(x,q) = \delta_j(u)(x, 2^j q)\ . \ \square \tag{2.8}$$

Then for any $j \in \mathbb{N}$, one has $U_j \in \mathcal{S}'$, $U_0$ has support in the ball $B = \{|q|^2 \leq 3\}$ and for $j \geq 1$, all the $U_j$ have support in the fixed annulus $\mathcal{R} = \big\{|q|^2 \in [\frac{1}{r_0^2} - \frac{1}{4}, 4]\big\}$, and one recover $u$ by the formula

$$u(x,p) \ = \sum_{j=0}^{\infty} U_j(x, 2^{-j}p)\ . \tag{2.9}$$

Let $B_0 = \{|q|^2 \leq 5\}$ so that all $U_j$ are supported in $B_0$. Let $Y$ be the projective compactification of $T^*X$. We denote by $y$ the coordinates on $Y$, and by $dy$ a volume form on $Y$ equal to $dx\,dq$ on $B_0$. Let $(.|.)$ a smooth extension to the fiber bundle $\Lambda^\bullet(T^*Y) \otimes \pi^*F$ on $Y$, of the $L^2$ scalar product on the restriction to $B_0$ of $\Lambda^\bullet(T^*\Sigma) \otimes \pi^*F$. We denote by $\mathcal{S}_Y$ the space of smooth section of $\Lambda^\bullet(T^*Y) \otimes \pi^*F$.

**Definition 2.2.** Let $D_Y$ be a second order selfadjoint positive operator on $Y$ acting on $\mathcal{S}_Y$. For $j \in \mathbb{N}$, set

$$\Lambda_j = \Big(2^{4j} + |\mu|^2 + 2^{-2j}|\beta|^2 + D_Y\Big)^{1/2}. \tag{2.10}$$

For $s \in \mathbb{R}$, and $U(y) \in \mathcal{S}_Y$ we define the Sobolev norm $|U|_{j,\lambda,s}$ by

$$|U|_{j,\lambda,s} = \ 2^{jn/2}\,|\Lambda_j^s U|_{L^2(Y)}\ . \tag{2.11}$$

We have introduce the normalisation factor $2^{jn/2}$ so that we have

$$\big|U_j(x,q)\big|_{j,\lambda,0} = \big|\delta_j(u)(x,p)\big|_{L^2(x,p)}\ .$$

For $u(x,p) \in \mathcal{S}$ we define the Sobolev norm $\|u\|_{\lambda,s}$ by

$$\|u\|_{\lambda,s}^2 = \sum_{j=0}^{\infty} |U_j|_{j,\lambda,s}^2\ . \tag{2.12}$$

We denote by $\mathcal{H}^s_\lambda$ the completion of $\mathcal{S}$ for the $\|.\|_{\lambda,s}$ norm. Remark that $|U|_{j,\lambda,s}$ depends on the choice of the operator $D_Y$, but that $|U|_{D^1_Y,j,\lambda,s}$ and $|U|_{D^2_Y,j,\lambda,s}$ are uniformly in $j,\lambda$ equivalent. In particular, the Hilbert spaces $\mathcal{H}^s_\lambda$ are independent of $D_Y$. Remark also that, as a vector space, $\mathcal{H}^s_\lambda$ is independent of $\lambda$, and that the norms $\|u\|_{\lambda,s}$ are uniformly equivalent for fixed $s$, and $\lambda$ in a fixed compact subset of $\mathbb{C}$. ▫

**Remark 2.3.** $\mathcal{H}^0_\lambda = \mathcal{H}$ is the space of $L^2$ sections of $\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F$, and $u \in \mathcal{H}^1_\lambda$ iff one has $u, \nabla_{e_i}u, \langle p\rangle\nabla_{p_i}u$ and $\big(\langle p\rangle^2 + |\mu| + \frac{|\beta|}{\langle p\rangle}\big)u \in \mathcal{H}$. Remark also that the $\mathcal{H}^s_\lambda$ are the usual Sobolev spaces for $p$ and $|\lambda|$ bounded, and that for any given $\lambda$, the injection of $\mathcal{H}^{s'}_\lambda$ in $\mathcal{H}^s_\lambda$ is compact for $s' > s$. ▫

We denote by $\tau \in ]0,1]$ a parameter, linked to $j$ by the relation $\tau = 2^{-j}$. We will work with classical pseudodifferential operators with weight $\Lambda$ on $Y$. A symbol of degree $d$ is a smooth function with $\tau,\lambda$ as parameter on $T^*Y$, $a(y,\zeta,\tau,\lambda)$, with values in $End(\Lambda^\bullet(T^*Y)\otimes\pi^*F)$, such that for any $\alpha,\gamma$, there exist $C_{\alpha,\gamma}$ (independent of $\tau,\lambda$) such that for any $(y,\zeta)$ and $\tau\in]0,1]$, $\lambda \in V$ one has

$$\left|\partial_y^\alpha\partial_\zeta^\gamma a(y,\zeta,\tau,\lambda)\right| \le C_{\alpha,\gamma}\Big(\tau^{-4}+|\mu|^2+\tau^2|\beta|^2+|\zeta|^2\Big)^{\frac{d-|\gamma|}{2}} .$$

We denote by $S^d$ the set of symbols of degree d. A smoothing operator on $Y$ is a family of operators $B(\tau,\lambda)$ with $\tau,\lambda$ as parameters, such that for any $s,t$ there exist $C_{s,t}$ independent of $\tau,\lambda$ such that one has

$$\begin{aligned} |B(\tau,\lambda)U|_s &\le C_{s,t}\,|U|_t \\ |U|_t &= \left|\big(\tau^{-4}+|\mu|^2+\tau^2|\beta|^2+D_Y\big)^{t/2}\,U\right| . \end{aligned}$$

We associate to a symbol $a$ in local coordinates and trivialisation an operator $A(\tau,\lambda) = Op(a)$ by the usual formula

$$A\Big(y,\frac{1}{i}\partial_y,\tau,\lambda\Big)\,U(y) = (2\pi)^{-2n}\int e^{iy\zeta}\,a(y,\zeta,\tau,\lambda)\,\hat{U}(\zeta)\,d\zeta .$$

We denote by $\mathcal{E}^d$ the associated set of pdo's of degree $d$ on $Y$. One has $A\in\mathcal{E}^d$ iff for any small compact subset $K$ of $Y$, any cutoff function $\theta(y)$ near $K$, there exist a cutoff $\theta'$ equal to 1 near the support of $\theta$, and $a \in S^d$ such that one can write in local coordinates and trivialisation $A(\tau,\lambda)\theta = \theta' OP(a)\theta + B(\tau,\lambda)$ with $B$ smoothing. For $A\in\mathcal{E}^d$,we denote by $\sigma(A)$ the principal symbol of $A$. If $A = Op(a)$, $\sigma(A)$ is the class of $a$ in the quotient space $S^d/S^{d-1}$.

For $E_d \in \mathcal{E}^d$ and $E_{d'} \in \mathcal{E}^{d'}$ one has $E_d E_{d'} \in \mathcal{E}^{d+d'}$, $\sigma(E_d E_{d'}) = \sigma(E_d)\,\sigma(E_{d'})$. Moreover, if $E_d = Op(e),\ E_{d'} = Op(e')$, one has $[E_d, E_{d'}] - OP([e,e'] + \frac{1}{i}\{e,e'\}) \in \mathcal{E}^{d-2}$, where $\{e,e'\} \in S^{d-1}$ denotes the Poisson bracket. Operators in $\mathcal{E}^0$ are uniformly in $\tau, \lambda$ bounded on $L^2$. We obviously have

$$\Lambda = \big(\tau^{-4} + |\mu|^2 + \tau^2|\beta|^2 + D_Y\big)^{1/2} \ , \qquad \mu,\, \tau^{-2},\, \tau\beta \,\in\, \mathcal{E}^1$$

and if $A(y, \partial_y)$ is a differential operator on $Y$ of degree $d$, with smooth coefficients independent of $\tau, \lambda$, then $A(y, \partial_y) \in \mathcal{E}^d$ since all derivatives in $\zeta$ of order $|\gamma| > d$ of its symbol vanish identically.

### 2.2. Hypoelliptic estimates

In this subsection, we assume that there exist $c_0 > 0$ such that

$$V \subset \big\{\mathrm{Re}(\lambda) < -c_0\big\} \tag{2.13}$$

and that the phase function $\Phi(x,p,\lambda)$ is real and satisfies the following elliptic estimate: there exist $\varepsilon_0 > 0$ such that for all $(x,p)$ and all $\lambda \in V$, the following inequality holds true

$$\mu + \frac{\hbar}{2}\big(|p|^2 - (\partial_p\Phi)^2\big) - \big\{|p|^2/2, \Phi\big\} \,\geq\, \varepsilon_0\big(\langle p\rangle^2 + \mu\big) \ . \tag{2.14}$$

For $s \in \mathbb{R},\ \lambda \in V$, let

$$D_{s,\lambda}(A_\Phi) \,=\, \big\{u \in \mathcal{H}^s_\lambda,\ A_\Phi(u) \in \mathcal{H}^s_\lambda\big\} \ .$$

The following theorem is the main hypoelliptic estimate that we shall use in this paper.

**Theorem 2.4.** *Let $\Phi$ be a real phase function such that (2.5) and (2.14) hold true. Then, if $c_0$ is large enough, for any $s \in \mathbb{R}$, any $\lambda \in V$, and any $u \in \mathcal{S}'$, $(A_\Phi - \lambda)u \in \mathcal{H}^s_\lambda$ implies $u \in \mathcal{H}^{s+1/4}_\lambda$, and there exist a constant $C_s$ independent of $\lambda \in V$ such that the following inequality holds true*

$$\|u\|_{\lambda,s+1/4} \,\leq\, C_s\,\big\|(A_\Phi - \lambda)(u)\big\|_{\lambda,s} \ . \tag{2.15}$$

*Moreover, $\mathcal{S}$ is dense in $D_{s,\lambda}(A_\Phi)$, i.e for all $u \in D_{s,\lambda}(A_\Phi)$, there exist a sequence $u_k \in \mathcal{S}$ such that $\lim\big(\|u - u_k\|_s + \|A_\Phi(u - u_k)\|_s\big) = 0$.*

*Finally, let $\theta_1, \theta_2$ be two functions on $\Sigma$ which are symbols of degree $0$ in the sense of definition 1.2, such that $F_1 = support(\theta_1)$, $F_2 = support(\theta_2)$ are disjoints,* *and such that for $r_0$ large enough, there exist 2 closed disjoints subset of $\Sigma$, $G_1, G_2$, conic in $p$, such that $F_j \cap \{|p| \geq r_0\} \subset G_j$. Let $u \in \mathcal{S}'$ be such that $(A_\Phi - \lambda)(u) = v \in \mathcal{H}^s_\lambda$, satisfies $\theta_1 v = v$. Then for any $t$, there exist $C_t$ such that the following inequality holds true for all $\lambda \in V$*

$$\|\theta_2 u\|_{\lambda,t} \leq C_t \, \|v\|_{\lambda,s} \ . \tag{2.16}$$

**Proof:** This result is proved in a slightly different context in [BL05]; we will just recall the main steps of the arguments used in [BL05] and the modifications necessary to handle the dependence in the phase function $\Phi$. The proof of theorem 2.4 will be complete at the end of this subsection.

The main step is to get estimates on each term of the Littlewood–Paley decomposition. Let $\Phi_\tau$ be the rescaled phase function

$$\Phi_\tau(x, q, \lambda) = \Phi(x, \tau^{-1} q, \lambda) \ .$$

Then $\Phi_\tau$ satisfies the following estimates, with a constant $\varepsilon_0$ equivalent to the one of (2.14), uniformly in $y = (x, q) \in B_\tau$, $\lambda \in V$ and $\tau \in \,]0, 1]$, where $B_\tau$ is equal to the ball $B_0$ when $\tau \in \,]1/2, 1]$, and is equal to the annulus $\mathcal{R}$ when $\tau \in \,]0, 1/2]$

$$\begin{aligned}
&\left|\partial_y^\alpha e_k(\Phi_\tau(y, \lambda))\right| \leq C_\alpha (\tau^{-2} + \mu)^{1/2} \\
&\left|\partial_y^\alpha \partial_{q_k}(\Phi_\tau(y, \lambda))\right| \leq C_\alpha \, \tau^{-1} (\tau^{-2} + \mu)^{1/2} \\
&\mu + \frac{\hbar}{2}\big(\tau^{-2}|q|^2 - (\tau \partial_q \Phi_\tau)^2\big) - \tau^{-1}\big\{|q|^2/2, \Phi_\tau\big\} \ \geq \ \varepsilon_0 (\tau^{-2} + \mu)
\end{aligned} \tag{2.17}$$

where we still denote by $e_k$ the vector field

$$e_k = \frac{\partial}{\partial x_k} + \Gamma^\alpha_{\beta,k} \, q_\alpha \frac{\partial}{\partial q_\beta} \ .$$

In particular, one has

$$e_k(\Phi_\tau)|_{B_\tau} \in \mathcal{E}^{1/2} \ , \qquad \tau \nabla_q(\Phi_\tau)|_{B_\tau} \in \mathcal{E}^{1/2} \ . \tag{2.18}$$

Let $Q_{\Phi,\tau}$ be the operator deduce from $A_\Phi$ by the change of variables $(x, p) \to (x, q = \tau p)$

$$Q_{\Phi,\tau}\big[U(x, q)\big] = A_\Phi\big[U(x, \tau p)\big](x, q/\tau) \ .$$

Then we have

$$
\begin{aligned}
&Q_{\Phi,\tau} = A_\tau + B_{\Phi,\tau}\\
&A_\tau = \hbar\mathcal{O}_\tau + \tau^{-1}\nabla_{\{|q|^2/2,.\}} + \mathcal{M}_\tau\\
&\mathcal{O}_\tau = \frac{1}{2}\Big[-\tau^2\Delta_q + \tau^{-2}|q|^2 + (2N_V - n)\Big]\\
&\mathcal{M}_\tau = \tau\,\Sigma\partial_{q_j}M_0^j + \tau^{-1}\Sigma q_j M_1^j + M \qquad (2.19)\\
&B_{\Phi,\tau} = \hbar\tau^2\Big[-\frac{(\partial_q\Phi_\tau)^2}{2} + (\partial_q\Phi_\tau|\partial_q) + \frac{1}{2}\Delta_q\Phi_\tau\Big]\\
&\qquad\qquad - \tau^{-1}\big\{|q|^2/2, \Phi_\tau\big\} - \tau\,\Sigma M_0^j\frac{\partial\Phi_\tau}{\partial q_j}\ .
\end{aligned}
$$

Let $\theta_0(x,q,j)$ be a smooth function with compact support in $B_0$, equal to 1 in a neighborhood of: the ball $B$ if $j=0$, the ring $\mathcal{R}$ if $j\geq 1$, vanishing near $q=0$ if $j\geq 1$, and independent of $j\geq 1$. Let $R_{\Phi,\tau} = R$ the operator on $Y$

$$R \,=\, \theta_0(Q_{\Phi,\tau} - \lambda)\,\theta_0\ .$$

Then the decomposition of $R$ in its self adjoint part $R'$ and its skew adjoint part $R''$ is of the form, where $E_d$ denotes any element in $\mathcal{E}^d$

$$
\begin{aligned}
&R \,=\, R' + R''\\
&R' = R'_0 + R'_\Phi\ , \qquad R'' = R''_0 + R''_\Phi
\end{aligned}
\qquad (2.20)
$$

with

$$
\begin{aligned}
&R'_0 =\, \theta_0\Big[\frac{\hbar}{2}\big[-\tau^2\Delta_q + \tau^{-2}|q|^2\big] + \mu + \tau^{-1}E_0 + \tau E_0\partial_q\Big]\theta_0\\
&R'_\Phi =\, \theta_0\Big[-\hbar\tau^2\frac{(\partial_q\Phi_\tau)^2}{2} - \tau^{-1}\{|q|^2/2,\Phi_\tau\} + \tau\frac{\partial\Phi_\tau}{\partial q_j}E_0\Big]\theta_0\\
&R''_0 =\, \theta_0\Big[\tau^{-1}\{|q|^2/2, .\} - i\beta + \tau^{-1}E_0 + \tau E_0\partial_q\Big]\theta_0\\
&R''_\Phi =\, \theta_0\Big[\hbar\tau^2\Big((\partial_q\Phi_\tau|\partial_q) + \frac{1}{2}\Delta_q\Phi_\tau\Big) + \tau\frac{\partial\Phi_\tau}{\partial q_j}E_0\Big]\theta_0\ .
\end{aligned}
\qquad (2.21)
$$

From (2.18), (2.21), and $\big[\{|q|^2/2,.\},E_d\big]\big|_{B_0}\in\mathcal{E}^d$ we deduce

$$
\begin{aligned}
&[R,E_d] \,\in\, \tau\mathcal{E}^d\nabla_q + (\tau^{-2}+\mu)^{1/2}\,\mathcal{E}^d\\
&R'_\Phi,\ \tau R''_0 \,\in\, \mathcal{E}^1\\
&R''_\Phi \,\in\, (\tau^{-2}+\mu)^{1/2}\,[\mathcal{E}^0\tau\nabla_q + \mathcal{E}^0]\ .
\end{aligned}
\qquad (2.22)
$$

We will use the notation

$$\tau^{-2}+\mu=m\ ,\qquad \mathcal{A}=\tau^{1/4}m^{-1/4}\big(\tau^{-1}m^{-1/2}+m^{-1/4}\big)^{1/2}$$
$$2^{2j}+\mu=m_j\ ,\qquad \mathcal{A}_j=(2^j m_j)^{-1/4}\big(2^j m_j^{-1/2}+m_j^{-1/4}\big)^{1/2}\ . \tag{2.23}$$

Observe that one has

$$m_j\geq 1+c_0\ ,\qquad \mathcal{A}_j\leq\sqrt{2}(1+c_0)^{-1/4}\ . \tag{2.24}$$

**Theorem 2.5.** *Let $c_0$ large enough. Then for any $s$, there exist a constant $C_s$ such that for any $j\in\mathbb{N}$ and any $U(y)\in\mathcal{S}_Y$ (with support in the ball $B$ if $j=0$ and in the annulus $\mathcal{R}$ otherwise) the following inequality holds true for all $\lambda\in V$*

$$\begin{aligned} m_j|2^{-j}\nabla_q U|^2_{j,\lambda,s}+m_j^2|U|^2_{j,\lambda,s}+\mathcal{A}_j^{-2}|U|^2_{j,\lambda,s+1/4}&\\ +\mathcal{A}_j^{-1}|2^{-j}\nabla_q U|^2_{j,\lambda,s+1/8}\ &\leq\ C_s|RU|^2_{j,\lambda,s}\ .\end{aligned} \tag{2.25}$$

**Proof:** We will denote by $C$ (resp. $C_s$) various constant independent of $\tau\in\,]0,1]$, $\lambda\in V, s$ (resp. $\tau\in\,]0,1]$, $\lambda\in V$). We denote by $\Lambda$ the positive self adjoint operator

$$\Lambda=\big(\tau^{-4}+|\mu|^2+\tau^2|\beta|^2+D_Y\big)^{1/2}$$

and we set

$$|U|_t=\|\Lambda^t U\|_{L^2}\ ,\qquad |U|=|U|_0\ .$$

Observe that the factor $2^{jn/2}$ in the definition 2.2 of the norm $|\,.\,|_{j,\lambda,s}$ is both on the left and on the right hand side of (2.25), so we can forget it.

We start observing that for $c_0$ large enough, we have the following continuity and coercivity estimates.

**Lemma 2.6.** *Let $c_0$ large enough. There exist $C$ such that for all $U$ with support in $B_\tau$ the following inequalities hold true*

$$\big|(R'U|U)\big|\ \leq\ C\Big[|\tau\,\nabla_q U|^2+(\tau^{-2}+\mu)\,|U|^2\Big]$$
$$\frac{\hbar\tau^2}{2}|\nabla_q U|^2+\frac{\varepsilon_0}{2}\,(\tau^{-2}+\mu)\,|U|^2\ \leq\ C(R'U|U)\ . \tag{2.26}$$

**Proof:** This is an easy consequence of (2.21), and of the hypothesis (2.17) on the phase function $\Phi$: by the integration by part formula (2.4), we get

$$\mathrm{Re}(RU|U) \geq \frac{\hbar\tau^2}{2}\,|\nabla_q U|^2 + \varepsilon_0(\tau^{-2}+\mu)\,|U|^2 - C\Big[(\tau^{-2}+\mu)^{1/2}\,|U|^2 + \tau\,|\nabla_q U|\,|U|\Big]$$

and we conclude for the lower bound on $(R'U|U)$, using (2.24). We obtain the upper bound on $|(R'U|U)|$ by the integration by part formula (2.4) and the a priori estimates (2.5) on the phase function $\Phi$. The proof of lemma 2.6 is complete. ∎

In the sequel, we will always assume that $c_0$ is such that lemma 2.6 holds true. In particular, lemma 2.6 gives for any $U$ supported in $B_\tau$

$$\begin{aligned} |\tau\,\nabla_q U| &\leq C\,m^{-1/2}\,|RU| \\ |U| &\leq C\,m^{-1}\,|RU|\ . \end{aligned} \tag{2.27}$$

**Lemma 2.7.** *There exist $C$ such that for all $U$ with support in $B_\tau$ the following inequality holds true*

$$|R''U|^2_{-1/2} \leq C\big(\tau^{-1}m^{-1} + \tau\,m^{-1/2}\big)\,|RU|^2\ .$$

**Proof:** Let $E = \Lambda^{-1}R''$. From (2.22), one has

$$E \in \tau^{-1}\mathcal{E}^0 + m^{1/2}[\tau\,\mathcal{E}^{-1}\nabla_q + \mathcal{E}^{-1}]$$

and

$$|R''U|^2_{-1/2} = (\Lambda^{-1/2}R''U\,|\,\Lambda^{-1/2}R''U) = (R''U|EU)\ .$$

One has

$$(R''U|EU) = (RU|EU) - (R'U|EU)$$

and from (2.26) and $|V|_{-1} \leq Cm^{-1}|V|$ we get

$$|EU| \leq C\big(\tau^{-1}m^{-1} + m^{-1} + m^{-3/2}\big)|RU| \leq C\,\tau^{-1}m^{-1}|RU|\ .$$

Now, observe that $R'$ is selfadjoint and non negative by (2.26), so by Cauchy–Schwarz we get

$$2\,|(R'U|EU)| \leq \tau^{-1}|(R'U|U)| + \tau|(R'EU|EU)|\ .$$

From (2.26), we have

$$\tau^{-1}|(R'U|U)| = \tau^{-1}|\,\mathrm{Re}(RU|U)| \leq \tau^{-1}|RU|\,|U| \leq \tau^{-1}m^{-1}|RU|^2\ .$$

Finally, we write

$$\tau(R'EU|EU) = \mathrm{Re}(\tau ERU|EU) + \mathrm{Re}([R,\tau E]U|EU) \ .$$

By (2.21) we obtain

$$\tau E = \tau\Lambda^{-1}R'' \in \mathcal{E}^0 + m^{1/2}\tau^2\mathcal{E}^{-1}\nabla_q \subset (1+\tau^2 m^{1/2})\mathcal{E}^0$$

and thus we get

$$\left|\mathrm{Re}(\tau ERU|EU)\right| \le C\,|RU|\,(1+\tau^2 m^{1/2})\,|EU| \le C(\tau^{-1}m^{-1}+\tau m^{-1/2})\,|RU|^2$$

and from (2.22), we get

$$[R,\tau E] \in (1+\tau^2 m^{1/2})\,(\tau\mathcal{E}^0\nabla_q + m^{1/2}\mathcal{E}^0)$$

which implies

$$\left|\big([R,\tau E]U\,|\,EU\big)\right| \le C(\tau^{-1}m^{-3/2}+\tau m^{-1})\,|RU|^2 \ .$$

The proof of lemma 2.7 is complete. ∎

**Lemma 2.8.** *There exist $C$ such that for all $U$ with support in $B_\tau$ the following inequality holds true*

$$|U|_{1/4} \le C\mathcal{A}\,|RU| \ .$$

**Proof:** Let $R''_0$ be the skew adjoint operator defined in (2.21). Our lemma will be consequence of the main algebraic commutation relation which is the core of the hypoellipticity, namely that we have near $B_\tau$

$$[\partial_{q_k},\tau R''_0] - \Sigma g^{i,k}\partial_{x_i} \in \mathcal{E}^0\nabla_q + \mathcal{E}^0 \ . \tag{2.28}$$

By lemmas 2.6, 2.7, and (2.22) one has

$$\begin{aligned}\tau|R''_0U|_{-1/2} &\le \tau|R''U|_{-1/2} + \tau|R''_\Phi U|_{-1/2}\\ &\le C\Big(\tau^{3/2}(\tau^{-2}m^{-1}+m^{-1/2})^{1/2}+\tau m^{-1/2}\Big)\,|RU|\\ &\le C\,\tau^{3/2}(\tau^{-1}m^{-1/2}+m^{-1/4})\,|RU| \ .\end{aligned} \tag{2.29}$$

We first verify that the following inequality holds true

$$\left|[\partial_{q_k},\tau R''_0]U\right|_{-3/4} \le C\,\tau^{1/4}m^{-1/4}(\tau^{-1}m^{-1/2}+m^{-1/4})^{1/2}\,|RU| = C\mathcal{A}\,|RU| \, . \tag{2.30}$$

We have, with $E_{-1/2} = \Lambda^{-3/2}[\partial_{q_k}, \tau R_0''] \in \mathcal{E}^{-1/2}$

$$\big|[\partial_{q_k}, \tau R_0'']U\big|^2_{-3/4} = \big([\partial_{q_k}, \tau R_0'']U \,|\, E_{-1/2}U\big)$$

and

$$(2.31)\qquad \begin{aligned} -\big(\partial_{q_k}\tau R_0''U \,|\, E_{-1/2}U\big) &= \big(\tau R_0''U \,|\, \partial_{q_k}E_{-1/2}U\big) \\ &= \big(E^*_{-1/2}\tau R_0''U \,|\, \partial_{q_k}U\big) + \big(\tau R_0''U \,|\, [\partial_{q_k}, E_{-1/2}]U\big) \\ &\quad - \big(\tau R_0''\partial_{q_k}U \,|\, E_{-1/2}U\big) \\ &= \big(\partial_{q_k}U \,|\, \tau R_0''E_{-1/2}U\big) \\ &= \big(\partial_{q_k}U \,|\, E_{-1/2}\tau R_0''U\big) + \big(\partial_{q_k}U \,|\, [\tau R_0'', E_{-1/2}]U\big) \end{aligned}$$

and we conclude that (2.30) holds true using (2.29) and $[\partial_{q_k}, E_{-1/2}] \in \mathcal{E}^{-1/2}$, $[\tau R_0'', E_{-1/2}] \in \mathcal{E}^{-1/2} + \tau^2\mathcal{E}^0\nabla_q$. We next observe that lemma 2.8 is equivalent to

$$(m + \tau|\beta|)\,|U|_{-3/4} + |\partial_x U|_{-3/4} + |\partial_q U|_{-3/4} \,\leq\, C\mathcal{A}\,|RU|\ .$$

From the obvious $|\Lambda^{-3/4}U| \leq m^{-3/4}|U|$, we get by (2.27)

$$m|U|_{-3/4} + |\partial_q U|_{-3/4} \,\leq\, Cm^{-3/4}(1 + \tau^{-1}m^{-1/2})|RU| \,\leq\, C\mathcal{A}\,|RU|\ .$$

From (2.28) and (2.30) we get

$$|\partial_x U|_{-3/4} \leq C\mathcal{A}\,|RU|\ .$$

Finally, using (2.21) we get

$$\theta_0\, i\tau\beta\,\theta_0 \,=\, -\tau R'' + \theta_0\Big[\{|q|^2/2, .\} + E_0 + \tau^2 E_0\partial_q\Big]\,\theta_0$$

and therefore from lemma 2.7 we get

$$\tau\,|\beta|\,|U|_{-3/4} \,\leq\, C\Big(m^{-1/4}(\tau^{-1}m^{-1} + \tau m^{-1/2})^{1/2} + \mathcal{A}\Big)|RU| \,\leq\, C\mathcal{A}\,|RU|\ .$$

The proof of lemma 2.8 is complete. ∎

From (2.26) we get, for any $\delta > 0$ and $U$ with support in $B_\tau$ that the following inequality holds true

$$(2.32)\qquad |\tau\nabla_q U|^2 \,\leq\, C\,\mathrm{Re}\big(\Lambda^{-1/8}RU \,|\, \Lambda^{1/8}U\big) \,\leq\, C\bigg(\delta|\Lambda^{-1/8}RU|^2 + \frac{1}{\delta}|\Lambda^{1/8}U|^2\bigg)\,.$$

Let $\theta(x, q, j)$ be a smooth function with compact support in $B_0$, equal to 1 in a neighborhood of: the ball $B$ if $j = 0$, the ring $\mathcal{R}$ if $j \geq 1$, vanishing near $q = 0$ if

$j \geq 1$, and independent of $j \geq 1$. We can apply the previous estimates to $\theta\Lambda^s U$. From lemma 2.8 we get for any $s$

$$|\Lambda^{1/4}\theta\Lambda^s U| \leq C\mathcal{A}\,|R\,\theta\Lambda^s U|\ .$$

One has

$$\Lambda^{1/4}\theta\Lambda^s U = \Lambda^{s+1/4}\theta U + \Lambda^{1/4}[\theta,\Lambda^s]U$$

$$R\,\theta\Lambda^s U = \Lambda^s R\,\theta U + R[\theta,\Lambda^s]U + [R,\Lambda^s]\theta U\ .$$

The essential support of $[\theta,\Lambda^s]$ doesn't intersect the support of $U$, so for any $s,\sigma$, there exist $C_{s,\sigma}$ such that

$$\Big|[\theta,\Lambda^s]U\Big|_\sigma \leq C_{s,\sigma}|U|_s\ .$$

From $\theta U = U$, $|V|_{-\alpha} \leq m^\alpha |V|$ for $\alpha \geq 0$, and $[R,\Lambda^s] \in \mathcal{E}^s\tau\nabla_q + m^{1/2}\mathcal{E}^s$ we thus get that for any $s$, there exist $C_s$ such that

$$|U|^2_{s+1/4} \ \leq\ C\mathcal{A}^2\Big(|RU|^2_s + C_s|\tau\nabla_q U|^2_s + C_s m|U|^2_s\Big)\ . \tag{2.33}$$

If we apply now (2.32) to $\theta\Lambda^{s+1/8}U$, using the same commutation argument, we get

$$\begin{aligned}|\tau\nabla_q U|^2_{s+1/8} \ \leq\ & C\,\delta\Big(|RU|^2_s + C_s|\tau\nabla_q U|^2_s + C_s m|U|^2_s\Big)\\ &+ C\,\frac{1}{\delta}\Big(|U|^2_{s+1/4} + C_s\tau^2\mathcal{A}|U|^2_s\Big) + C_s\tau^2|U|^2_{s+1/8}\ .\end{aligned} \tag{2.34}$$

Adding $\mathcal{A}^{-2}$ (2.33) and $\delta^{-1}$ (2.34), with the choice $\delta = C_1\mathcal{A}$ with $C_1$ large, we get

$$\begin{aligned}&\mathcal{A}^{-2}|U|^2_{s+1/4} + \mathcal{A}^{-1}|\tau\nabla_q U|^2_{s+1/8} \ \leq\\ &\qquad\leq\ C|RU|^2_s + C_s|\tau\nabla_q U|^2_s + C_s m|U|^2_s + C_s\mathcal{A}^{-1}\tau^2|U|^2_{s+1/8}\ .\end{aligned} \tag{2.35}$$

Applying (2.27) to $\theta\Lambda^s U$ we obtain

$$m|\tau\nabla_q U|^2_s + m^2|U|^2_s \ \leq\ C\Big(|RU|^2_s + C_s|\tau\nabla_q U|^2_s + C_s m|U|^2_s\Big)\ . \tag{2.36}$$

Adding (2.35) and (2.36) we get

$$\begin{aligned}&m|\tau\nabla_q U|^2_s + m^2|U|^2_s + \mathcal{A}^{-2}|U|^2_{s+1/4} + \mathcal{A}^{-1}|\tau\nabla_q U|^2_{s+1/8} \ \leq\\ &\qquad\leq\ C|RU|^2_s + C_s|\tau\nabla_q U|^2_s + C_s m|U|^2_s + C_s\mathcal{A}^{-1}\tau^2|U|^2_{s+1/8}\ .\end{aligned} \tag{2.37}$$

We can now end the proof of theorem 2.5 by a simple contradiction argument; suppose that for some value of $s$, (2.25) is untrue. Then there exist sequences $\tau_k = 2^{-j_k}, U_k, \lambda_k$ such that the left hand-side of (2.25) (with $j = j_k$ and $U = U_k$) is equal to 1 and $\lim |R(U_k)|_s = 0$. By (2.37), the sequence $j_k$ is necessarily bounded, so we may suppose that $j_k = j$ is constant, and the sequence $\lambda_k$ is necessarily bounded, so we may forget the $\lambda$ dependence. Then all the $U_k$ have their support in $B_\tau$, $\tau$ is fix, the sequence $U_k$ is bounded in the space $W^{s+1/8} = \{u \in \mathcal{H}^{s+1/4}, \nabla_q u \in \mathcal{H}^{s+1/8}\}$, thus we may suppose that $U_k$ is strongly convergent in $W^s$, and from the inequality (2.37), we get that its limit satisfy $U_\infty \neq 0$. By (2.37), we get for any $U \in \mathcal{S}_Y$ with support in $B_\tau$ and any $\sigma$

$$|U|^2_{\sigma+1/4} + |\nabla_q U|^2_{\sigma+1/8} \leq C_\sigma \Big( |RU|^2_\sigma + |\nabla_q U|^2_\sigma + |U|^2_{\sigma+1/8} \Big) \tag{2.38}$$

and by iteration of (2.38)

$$|U|^2_{\sigma+1/4} + |\nabla_q U|^2_{\sigma+1/8} \leq C_{\sigma,N} \Big( |RU|^2_\sigma + |U|^2_{\sigma-N} \Big) \; . \tag{2.39}$$

**Lemma 2.9.** *Let $\tau$ be fixed, and $K$ a compact subset of the interior of $B_\tau$. For any $\sigma$, and any distribution $U \in \mathcal{D}'_Y$ with support in $K$, $RU \in \mathcal{H}^\sigma$ imply $U \in H^{\sigma+1/4}, \nabla_q U \in H^{\sigma+1/8}$, and (2.39) is valid.*

**Proof:** This is a standard consequence of the inequality (2.39) Let $\Psi_\varepsilon \in \mathcal{E}^{-\infty}$ be bounded in $\mathcal{E}^0$ with scalar principal symbol, such that for $U \in \mathcal{D}'_Y$ with support in $K$, $\Psi_\varepsilon(u)$ has support in $B_\tau$ and $\lim_{\varepsilon\to 0} \Psi_\varepsilon(u) = u$. From $[R, \Psi_\varepsilon] = A_\varepsilon \nabla_q + B_\varepsilon$ with $A_\varepsilon, B_\varepsilon$ bounded in $\mathcal{E}^0$, and using (2.39) we get with $U_\varepsilon = \Psi_\varepsilon U$ for all $t, N$

$$|U_\varepsilon|^2_{t+1/4} + |\nabla_q U_\varepsilon|^2_{t+1/8} \leq C_{t,N} \Big( |RU|^2_t + |U|^2_{t-N} + |A_\varepsilon \nabla_q U|^2_t + |B_\varepsilon U|^2_t \Big) \; .$$

By induction, we conclude that the set of $t$ such that $U \in H^{t+1/4}$, $\nabla_q U \in H^{t+1/8}$ and (2.39) is valid for $\sigma = t$ contains $]-\infty, \sigma]$ ∎

One has $RU_\infty = 0$, so lemma 2.9 imply that $U_\infty$ is smooth, and we get by (2.26) $U_\infty = 0$: contradiction. The proof of theorem 2.5 is complete. ∎

Let us recall that by (2.6)

$$u = \sum_{j=0}^{\infty} \delta_j(u) \; , \qquad \delta_j(u) = \chi(2^{-j}\langle p \rangle)\, u \; .$$

Let $v = (A_\Phi - \lambda)u$, and $U_j, V_j$ associated to $u, v$ by (2.8).

The operator $\chi_j = \chi(2^{-j}\langle p\rangle)$ commutes with $\{|p|^2/2, .\}$, so we have

$$[A_\Phi - \lambda, \chi_j] \;=\; M_0\,\phi(2^{-j}\langle p\rangle)\,2^{-j}\nabla_p + 2^{-j}\phi(2^{-j}\langle p\rangle)M_0 + 2^{-j}\phi(2^{-j}\langle p\rangle)\,\hbar\,\partial_p\Phi$$

where $M_0$ are symbols of degree 0 in p, and $\phi(r)$ is compactly supported. We thus get for some constant $C, M,$ if $u, \nabla_p u, (A_\Phi - \lambda)u \in \mathcal{H}^t_\lambda$

$$\begin{aligned} |RU_j|_t \;&\le\; C\big(|V_j|_t + w_{t,j}\big) \\ w_{t,j} \;&=\; \Sigma_{|j-k|\le M}\Big(2^{-k}m_k^{1/2}|U_k|_t + 2^{-2k}|\nabla_q U_k|_t\Big)\;. \end{aligned} \tag{2.40}$$

Let $\alpha_{t,\lambda,j} = |V_j|_t$; under the hypothesis of theorem 2.4, one has $\alpha_{s,\lambda,j} \in l^2$. Let

$$\beta_{t,\lambda,j} \;=\; m_j^{1/2}|2^{-j}\nabla_q U_j|_t + m_j|U_j|_t + \mathcal{A}_j^{-1}|U_j|_{t+1/4} + \mathcal{A}_j^{-1/2}|2^{-j}\nabla_q U_j|_{t+1/8}\,. \tag{2.41}$$

From theorem 2.5 and (2.40) we get with

$$\sigma_{\mu,j} \;=\; 2^{-j}\big(m_j^{3/8}\mathcal{A}_j + \mathcal{A}_j^{1/2}\big) \;\le\; C\,2^{-j}$$

$$\beta_{t,\lambda,j} \;\le\; C_t|RU_j|_t \;\le\; C_t\Big(\alpha_{t,\lambda,j} + \Sigma_{|j-k|\le M}\,\sigma_{\mu,k}\,\beta_{t-1/8,\lambda,k}\Big)\;. \tag{2.42}$$

We conclude that the set of $t \in \mathbb{R}$ such that $\beta_{t,\lambda,j} \in l^2$ contains $]-\infty, s]$, and in particular $u \in \mathcal{H}^{s+1/4}_\lambda$, so $A_\Phi$ is hypoelliptic in the scale $\mathcal{H}^s_\lambda$. Let

$$\mathcal{W}^s_\lambda = \big\{u, \beta_{s,\lambda,j} \in l^2\big\}\;, \qquad \|u\|_{\mathcal{W}^s_\lambda} = \|\beta_{s,\lambda,j}\|_{l^2}\;.$$

From (2.42), we get

$$\|u\|_{\mathcal{W}^s_\lambda} \;\le\; C_s\Big(\|(A_\Phi - \lambda)u\|_{\lambda,s} + \|u\|_{\mathcal{W}^{s-1/8}_\lambda}\Big)$$

and by iteration for any $N$,

$$\|u\|_{\mathcal{W}^s_\lambda} \;\le\; C_{s,N}\Big(\|(A_\Phi - \lambda)u\|_{\lambda,s} + \|u\|_{\mathcal{W}^{s-N}_\lambda}\Big)\;.$$

By a contradiction argument like in the proof of theorem 2.5, using $(A_\Phi - \lambda)u = 0 \Rightarrow u \in \mathcal{S}$ and the fact that the integration by part formula (2.3) and the hypothesis (2.14) gives $(A_\Phi - \lambda)u = 0 \Rightarrow u = 0$ if $c_0$ is large enough, we get

$$\|u\|_{\mathcal{W}^s} \;\le\; C_s\|(A_\Phi - \lambda)u\|_{\lambda,s} \tag{2.43}$$

which implies (2.15). Observe that (2.43) is much more precise that the estimate (2.15) of theorem 2.4. The assertion on the density of $\mathcal{S}$ in $D_{s,\lambda}(A_\Phi)$ can be shown by the same argument that we use in the proof of lemma 2.9.

Finally in order to prove (2.16), let $S=(A_\Phi-\lambda)^{-1}$. Let $\theta=(\theta_1(z),...,\theta_N(z))$ be a collection of smooth functions $\theta_j(z)$ on $\Sigma$ which are symbols of degree 0 in the sense of definition 1.2 and such that $\theta_j=1$ on the support of $\theta_{j+1}$. We denote by $Ad_\theta^N(S)$ the iterated commutator

$$Ad_\theta^N(S)=\Big[\theta_N,...\big[\theta_2,[\theta_1,S]\big]...\Big] \ .$$

One has

$$\begin{aligned}(2.44)\qquad &[\theta_1,S] \;=\; -S[\theta_1,A_\Phi-\lambda]S \;=\; S[A+B_\Phi,\theta_1]S\\ &[A+B_\Phi,\theta_1] \;=\; \hbar\Big(\Big[\frac{-\Delta_p}{2},\theta_1\Big]+(\partial_p\Phi|\partial_p\theta_1)\Big)+\big\{|p|^2/2,\theta_1\big\}+\Sigma M_0^j\,\partial_{p_j}\theta_1 \ .\end{aligned}$$

Therefore, we get

$$(2.45)\qquad D_j \;=\; [A+B_\Phi,\theta_j] \;=\; \hbar(\partial_p\Phi|\partial_p\theta_j)+\hbar M_{-1,j}\nabla_p+M_{1,j}$$

where the $M_{k,j}$'s are symbols of degree $k$ with support include in the support of $d\theta_j$. From $[D_j,\theta_m]=0$ for $j\neq m$, one gets

$$(2.46)\qquad Ad_\theta^N(S) \;=\; \Sigma\; SD_{j_1}SD_{j_2}\cdots SD_{j_N}\,S \ .$$

**Lemma 2.10.** *The operator $(A_\Phi-\lambda)Ad_\theta^N(S)$ is bounded from $\mathcal{H}_\lambda^s$ to $\mathcal{H}_\lambda^{s+N/8}$ and the following inequality holds true*

$$(2.47)\qquad \big\|(A_\Phi-\lambda)\,Ad_\theta^N(S)\,v\big\|_{\lambda,s+N/8} \;\leq\; C\,\|v\|_{\lambda,s} \ .$$

**Proof:** In view of (2.46), it is enough to verify

$$(2.48)\qquad \|DSv\|_{\lambda,s+1/8} \;\leq\; C\,\|v\|_{\lambda,s} \ .$$

Let $u=Sv$ and $w=Du$. One has $v=(A_\Phi-\lambda)u$. From $w=Du$, and (2.45) we get

$$|W_j|_t \;\leq\; C\Big(m_j^{1/2}|U_j|_t+2^{-j}|2^{-j}\nabla_qU_j|_t\Big) \ .$$

From (2.24), one gets obviously

$$\Lambda^{1/8}\,m_j^{1/2} \;\leq\; C\big(m_j+\mathcal{A}_j^{-1}\Lambda^{1/4}\big)$$

$$\Lambda^{1/8}\,2^{-j} \;\leq\; C\big(m_j^{1/2}+\mathcal{A}_j^{-1/2}\Lambda^{1/8}\big)$$

so we get

$$\|w\|_{\lambda,s+1/8} \le C\|u\|_{\mathcal{W}^s_\lambda}$$

and (2.43) implies

$$\|DSv\|_{\lambda,s+1/8} \le C\|v\|_{\lambda,s} \ .$$

Therefore, (2.47) holds true. The proof of lemma 2.10 is complete. ∎

The proof of (2.16) is now obvious: one has $u = Sv, \theta_2 v = 0$, and for any $N$, with a convenient choice of $\theta$, one has $\theta_2 u = \theta_2 Ad^N_\theta(S)v = \theta_2 S(A_\Phi - \lambda)Ad^N_\theta(S)v$. The proof of theorem 2.4 is complete. ∎

By theorem 2.4, (2.16), applied with the phase function $\Phi = 0$, the distribution kernel $(A-\lambda)^{-1}(z,z')$ of the resolvant $(A-\lambda)^{-1}$ is well defined for $\mu = -\operatorname{Re}(\lambda) > c_0$, and is smooth outside the diagonal $z = z'$. Let us give an elementary result of finite speed of propagation for the resolvant $(A-\lambda)^{-1}$, with $-\operatorname{Re}(\lambda) = \mu$ large, which shows that a GFK is almost local in $|p|$, and satisfies with respect to $|p| = y$ the same bounds that the resolvant of the 1-d harmonic oscillator

$$\frac{\hbar}{2}(-\partial_y^2 + y^2) \ .$$

This was already clear in the strategy of proof of theorem 2.4 and is consequence of the fact that the Hamiltonian field $H_{|p|^2/2}$ is tangent to the level surface $|p| = \mathrm{Cte}$.

The condition (2.14) for a phase function $\Phi = f(|p|^2/2, \mu)$ depending only on $|p|, \mu = -\operatorname{Re}(\lambda)$ is

$$u\Big(\frac{df}{du}\Big)^2 - u - \frac{\mu}{\hbar} \ \le\ -\varepsilon(\mu + u) \ .$$

Let $J = [u_0, u_1]$, $0 \le u_0 < u_1 < \infty$. For $\varepsilon \in ]0, \min(1, \frac{1}{\hbar})[$ the solution $f$ of the Hamilton–Jacobi equation

$$u\Big(\frac{df}{du}\Big)^2 \ =\ u + \frac{\mu}{\hbar} - \varepsilon(\mu + u) \tag{2.49}$$

vanishing for $u \in J$ is equal to

$$\begin{aligned} f_{J,\varepsilon}(y^2/2,\mu) \ &=\ \int_y^{y_0} \sqrt{(1-\varepsilon)s^2 + 2\mu\Big(\frac{1}{\hbar} - \varepsilon\Big)}\ ds \quad \text{for } y^2/2 \le y_0^2/2 = u_0 \\ f_{J,\varepsilon}(y^2/2,\mu) \ &=\ \int_{y_1}^{y} \sqrt{(1-\varepsilon)s^2 + 2\mu\Big(\frac{1}{\hbar} - \varepsilon\Big)}\ ds \quad \text{for } y^2/2 \ge y_1^2/2 = u_1 \ . \end{aligned} \tag{2.50}$$

**Lemma 2.11.** *Let $J=[y_0,y_1]\subset[0,\infty[$, $r>0$ be given. There exist $\mu_0$, and for any $\delta>0$, and any $\alpha,\beta,\alpha',\beta'$, there exist $C$ such that the following inequality holds true for $\mu\geq\mu_0$ and*

$$z'=(x',p'),\quad y_0\leq|p'|\leq y_1\quad ;\quad z=(x,p),\quad \mathrm{dist}(|p|,J)\geq r$$

$$\begin{aligned}\big\|\nabla^{\alpha}_{e_i,z}\nabla^{\beta}_{\hat{e}^j,z}\nabla^{\alpha'}_{e_i,z'}\nabla^{\beta'}_{\hat{e}^j,z'}(A-\lambda)^{-1}(z,z')\big\| &\leq \\ &\leq C\exp\Big(-\Big[f_{J,0}(|p|^2/2,\mu)-\delta\sqrt{\mu+|p|^2/2}\Big]\Big)\,.\end{aligned}\tag{2.51}$$

**Proof:** By theorem 2.4, using (2.16), we have just to verify that for any $\delta>0$, there exist a phase function $\Phi=f(|p|^2/2,\mu)$ such that (2.14) holds true, with $f(u,\mu)$ such that

$$\left|(u\partial_u)^k\left(\sqrt{u}\,\frac{\partial f}{\partial u}\right)\right|\leq C_k(u+\mu)^{1/2}$$

and such that

$$\begin{aligned}&f(y^2/2,\mu)\leq 0\quad\text{for }\ |y|\in J\\ &f(y^2/2,\mu)\ \geq\ f_{J,0}(y^2/2,\mu)-\delta\sqrt{\mu+y^2/2}\qquad\text{for }\ \mathrm{dist}(|y|,J)\geq r/2\,.\end{aligned}\tag{2.52}$$

This can be done easily by regularization of the function $y\in\mathbb{R}\to f_{J,\varepsilon}(y^2/2,\mu)$ near its singular points $y=0$, $y=y_0$, $y=y_1$, with $\varepsilon>0$ small enough with respect to $\delta$. One has

$$\lim_{\mu\to+\infty}\sqrt{\frac{\hbar}{2\mu}}\,f_{J,0}(|p|^2/2,\mu)\ =\ \mathrm{dist}(|p|,J)\ .\ \blacksquare\tag{2.53}$$

## 3 – Hamilton–Jacobi

Let $s\in[0,1]\to x(s)\in X$ be a trajectory on $X$. The velocity at a given $s$ is the tangent vector $v(s)\in T_{x(s)}X$

$$v(s)=\frac{d}{ds}x(s)$$

and the acceleration $a(s)$ is the tangent vector $a(s)\in T_{x(s)}X$

$$a(s)=\nabla_{v(s)}v(s)\ .$$

In a coordinate system $x = (x_1, ..., x_n)$, one has

$$\begin{aligned} v(s) &= \Sigma \, \frac{dx_i}{ds} \frac{\partial}{\partial x_i} \\ a(s) &= \Sigma \left( \frac{d^2 x_i}{ds^2} + \Gamma^i_{j,k} \frac{dx_j}{ds} \frac{dx_k}{ds} \right) \frac{\partial}{\partial x_i} \ . \end{aligned} \tag{3.1}$$

With the identification of $TX$ and $T^*X$ given by the metric, then $(x(s), v(s))$ is identified with $(x(s), p(s) = g(v(s)))$ and one has

$$\begin{aligned} g(a(s)) &= b(s) = \Sigma \, b_i(s) \frac{\partial}{\partial p_i} \\ b_i(s) &= \frac{dp_i}{ds} + \frac{1}{2} \frac{\partial g^{j,k}}{\partial x_i} p_j \, p_k \end{aligned} \tag{3.2}$$

and the identity

$$\frac{d}{ds}\big(x(s), p(s)\big) = \Sigma \frac{dx_i}{ds} \, e_i + \Sigma b_i(s) \, \hat{e}^i = H_{|p|^2/2} + b(s) \tag{3.3}$$

where $H_{|p|^2/2} \in T^H\Sigma$ is the Hamiltonian vector field of $|p|^2/2$ and $b(s) \in T^V\Sigma$. Let $\mathcal{L}(x(s))$ be the Lagrangian function

$$\mathcal{L}(x(s)) = \frac{|a(s)|^2}{2\hbar} + \frac{\hbar |v(s)|^2}{2} \ . \tag{3.4}$$

Then $\mathcal{L}$ defines a function $L$ on the affine subvariety $\mathcal{T}\Sigma = H_{|p|^2/2} + T^V\Sigma$ of $T\Sigma = T^H\Sigma \oplus T^V\Sigma$

$$L(x, p; b) = \frac{|b|^2}{2\hbar} + \frac{\hbar |p|^2}{2} \ , \qquad b \in T^V\Sigma \ . \tag{3.5}$$

The Legendre transform of $L$ is the Hamiltonian function $H(x, p; \zeta)$ on $T^*\Sigma$ given by, with $\zeta^V = \zeta|_{T^V\Sigma}$

$$\begin{aligned} H(x, p; \zeta) &= \sup_{b \in T^V\Sigma} \Big[ \zeta(H_{|p|^2/2} + b) - L(x, p; b) \Big] \\ &= \frac{\hbar}{2} \big( |\zeta^V|^2 - |p|^2 \big) + \zeta(H_{|p|^2/2}) \ . \end{aligned} \tag{3.6}$$

In local coordinates $(x, p; \xi, \eta)$ with $\zeta = \xi \, dx + \eta \, dp$, one has

$$H(x, p; \xi, \eta) = \frac{\hbar}{2} \Sigma \big( g_{i,j} \, \eta_i \, \eta_j - g^{i,j} \, p_i \, p_j \big) + \Sigma \left( g^{i,j} \, \xi_i \, p_j - \frac{1}{2} \frac{\partial g^{i,j}}{\partial x_l} \, p_i \, p_j \, \eta_l \right) . \tag{3.7}$$

Observe that the Hamilton–Jacobi equation associated to (2.14) with $\varepsilon_0 = 0$ is precisely

$$H(x, p; \partial_x \Phi, \partial_p \Phi) = \frac{\hbar}{2} \big( (\partial_p \Phi)^2 - |p|^2 \big) + \{|p|^2/2, \Phi\} = \mu \ . \tag{3.8}$$

### 3.1. Symplectic Geometry

Let $z=(x,p)\in\Sigma$, $(z,\zeta)=(x,p,\xi,\eta)\in T^*\Sigma$, and $\zeta\,dz=\xi\,dx+\eta\,dp$ the canonical 1-form on $T^*\Sigma$. We shall denote by $\pi_*$ the projection $T^*\Sigma\to\Sigma$, and by $\{.,.\}_*$ the Poisson bracket on the symplectic manifold $T^*\Sigma$. One has

$$T^*\Sigma \;=\; T^{*,H}\Sigma\oplus T^{*,V}\Sigma \tag{3.9}$$

and we shall denote by $e$ the canonical map from $T^*_{x,p}\Sigma$ to $T^*_xX$ given by

$$e(z,\zeta)(u)=\zeta(v)\ ,\qquad v\in T^H_z\Sigma,\quad u=d\pi(v)\ .$$

In local coordinates, one has $e(z,\zeta)=\Sigma e_i(z,\zeta)dx_i$, where the $e_i$' are the functions on $T^*\Sigma$

$$e_i(x,p,\xi,\eta)\;=\;\xi_i+\Gamma^{\alpha}_{\beta,i}\,p_\alpha\,\eta_\beta\ .$$

Let $|\eta|^2$ be the square of the length of the vertical component $\eta=\zeta^V=\zeta|_{T^V\Sigma}$ of $\zeta$

$$|\eta|^2\;=\;\Sigma g_{i,j}\,\eta^i\,\eta^j\ .$$

Observe that we have the following identities

$$\{|\eta|^2,e_i\}_*=0\ ,\qquad\{|p|^2,e_i\}_*=0\ . \tag{3.10}$$

We still denote by $(.|.)$ the scalar product on $T^*X$ and $TX$

$$(p|q)=\Sigma g^{i,j}\,p_i\,q_j\ ,\qquad(u|v)=\Sigma g_{i,j}\,u^i\,v^j$$

and by $\langle .|.\rangle$ the pairing between $T^*X$ and $TX$, so that

$$\langle e|\eta\rangle=e_i\,\eta^i\ ,\qquad\langle p|\eta\rangle=p_i\,\eta^i\ .$$

Then the Hamiltonian $H$ is the function on $T^*\Sigma$

$$H\;=\;\frac{\hbar}{2}\big(|\eta|^2-|p|^2\big)+(p|e)$$

or equivalently if $\zeta=(\zeta^H,\zeta^V)$ is the decomposition of $\zeta$ in the splitting (3.9)

$$H(z,\zeta)\;=\;\frac{\hbar}{2}\big(|\zeta^V|^2-|p|^2\big)+(p|\zeta^H)\ . \tag{3.11}$$

Let

$$s\;\to\;\big(Z(s,z,\zeta),\,\Xi(s,z,\zeta)\big)$$

the flow of the Hamiltonian of $H(z,\zeta)$ on $T^*\Sigma$. For any function $F(z,\zeta)$, one has

$$\frac{d}{ds}F\big(Z(s,z,\zeta),\,\Xi(s,z,\zeta)\big)\;=\;\{H,F\}_*\big(Z(s,z,\zeta),\,\Xi(s,z,\zeta)\big)\ .$$

**Lemma 3.1.** *The following identities hold true*

$$
\begin{aligned}
&\{H, |p|^2/2\}_* = \hbar\langle p|\eta\rangle \\
&\{H, \langle p|\eta\rangle\}_* = -H + \frac{\hbar}{2}|\eta|^2 \\
&\{H, |\eta|^2/2\}_* = \hbar\langle p|\eta\rangle - \langle e|\eta\rangle \\
&\{H, (p|e)\}_* = \hbar\langle e|\eta\rangle
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&\{H, \langle e|\eta\rangle\}_* = \hbar(p|e) - |e|^2 + \Big\langle p \,|\, R\big(g^{-1}(p), \eta\big)\eta\Big\rangle \\
&\{H, |e|^2/2\}_* = \Big\langle p \,|\, R\big(g^{-1}(p), g^{-1}(e)\big)\eta\Big\rangle \;.
\end{aligned}
\tag{3.13}
$$

**Proof:** These identities are obtained by computation in local coordinates, using in particular (1.4) and (3.10). ∎

We shall now investigate the correspondence between the critical points for the action defined by the Lagrangian function $L(z, b)$ on $\mathcal{T}\Sigma$ and the integral curves of the Hamiltonian function $H(z, \zeta)$ on $T^*\Sigma$. If $F(z, \zeta)$ is a function on $T^*\Sigma$, we will denote by $H_F$ the Hamiltonian vector field of the function $F$, and by $\exp(tH_F)$ the associated flow.

Set $z_0 = (x_0, p_0)$, let $z = (x, p)$ be a given point in $\Sigma$, and take $T > 0$. Let $t \in [0, T] \to x(t)$ be a trajectory on $X$ such that $z_0 = (x_0, p_0) = (x(0), g(v(0)))$ and $z = (x, p) = (x(T), g(v(T)))$. Then the Euler–Lagrange equations for the action $\mathcal{I}_T(z_0, z)$

$$
\mathcal{I}_T(z_0, z) = \int_0^T \mathcal{L}(x(t))\, dt = \int_0^T \frac{|a(t)|^2}{2\hbar} + \frac{\hbar|v(t)|^2}{2}\, dt
\tag{3.14}
$$

that is, the equations satisfied by critical points of the functional $x(t) \to \mathcal{I}_T(z_0, z)$, are given in local coordinates by

$$
\begin{aligned}
&-\frac{d^2}{\hbar\, dt^2}(g_{i,l}a^i) + \frac{d}{\hbar\, dt}(2\, g_{i,m}\, \Gamma^i_{j,l}\, a^m v^j) + \frac{\hbar d}{dt}(g_{i,l}v^i) \;= \\
&\quad = \frac{1}{2\hbar}\left(\frac{\partial g_{i,j}}{\partial x_l}\, a^i a^j + 2\, g_{i,m}\frac{\partial \Gamma^i_{j,k}}{\partial x_l}\, a^m v^j v^k\right) + \frac{\hbar}{2}\left(\frac{\partial g_{i,j}}{\partial x_l}\, v^i v^j\right).
\end{aligned}
\tag{3.15}
$$

Observe that (3.15) is a fourth order ordinary differential equation on $x(t)$, and that any solution of these equation with $a \in L^2[0, T]$ and $v \in L^\infty[0, T]$ is smooth.

Let $x(t)$ be a critical point for the action $\mathcal{I}_T(z_0,z)$. Let $t\to z(t)=(x(t),p(t))=(x(t),g(v(t)))$ be the corresponding trajectory in $\Sigma$, and $\big(z(t),\frac{dz}{dt}=H_{|p|^2/2}+b(t)\big)$ the section of $\mathcal{T}\Sigma$ over $z(t)$ associated to these trajectory. Then $z(t)$ is a critical point connecting $z_0$ to $z$ for the action

$$\mathcal{J}_T(z_0,z)\,=\int_0^T L\big(z(t),b(t)\big)\,dt\,=\int_0^T\frac{|b(t)|^2}{2\,\hbar}+\frac{\hbar|p(t)|^2}{2}\,dt \tag{3.16}$$

and from (3.15) we get the corresponding Euler–Lagrange equations in local coordinates

$$\begin{aligned}&-\frac{d^2}{\hbar\,dt^2}(b_l)+2\,\frac{d}{\hbar\,dt}\langle b|\Gamma_l(v)\rangle+\frac{\hbar d}{dt}(p_l)\;=\\ &\qquad=\;\frac{1}{\hbar}\Big\langle b\,\big|\,\frac{\partial\Gamma_k}{\partial x_l}(v)\Big\rangle v^k-\Big(\frac{1}{2\hbar}\frac{\partial g^{i,j}}{\partial x_l}\,b_i\,b_j+\frac{\hbar}{2}\frac{\partial g^{i,j}}{\partial x_l}\,p_i\,p_j\Big)\;.\end{aligned}\tag{3.17}$$

For $c(t)\in T^V_{z(t)}\Sigma\simeq T^*_{x(t)}X$, and $u(t)\in T^H_{z(t)}\Sigma\simeq T_{x(t)}X$ we shall denote by $\frac{Dc}{Dt}\in T^V_{z(t)}\Sigma$ and $\frac{Du}{Dt}\in T^H_{z(t)}\Sigma$ the covariant derivatives

$$\frac{Dc}{Dt}=\nabla^{LC}_{v(t)}c\;,\qquad\frac{Du}{Dt}=\nabla^{LC}_{v(t)}u$$

so that we have

$$\begin{aligned}\frac{d}{dt}(c|c')\;&=\Big(\frac{Dc}{Dt}|c'\Big)+\Big(c|\frac{Dc'}{Dt}\Big)\\ \frac{d}{dt}(u|u')\;&=\Big(\frac{Du}{Dt}|u'\Big)+\Big(u|\frac{Du'}{Dt}\Big)\\ \frac{d}{dt}\langle c|u\rangle\;&=\Big\langle\frac{Dc}{Dt}|u\Big\rangle+\Big\langle c|\frac{Du}{Dt}\Big\rangle\;.\end{aligned}$$

In local coordinates, with $c=\Sigma c_i\frac{\partial}{\partial p_i}$ and $u=\Sigma u^ie_i$, one has

$$\begin{aligned}\Big(\frac{Dc}{Dt}\Big)_i&=\;\frac{dc_i}{dt}-\Gamma^j_{i,k}\,c_j\,v^k\\ \Big(\frac{Du}{Dt}\Big)_i&=\;\frac{du^i}{dt}+\Gamma^i_{j,k}\,u^j\,v^k\;.\end{aligned}$$

We have

$$b=\frac{Dp}{Dt}\;.$$

**Theorem 3.2.** *Let $z(t)$ be a critical point for the action $\int_0^T L(z(t),b(t))\,dt$. The Euler–Lagrange equations for $z(t)$ in terms of $v(t)=g^{-1}(p(t))$, covariant derivatives of $b$ and of the Riemann curvature tensor $R$ are*

$$-\frac{D}{Dt}\frac{Db}{Dt}+\big\langle b\,|\,R(v,.)v\big\rangle+\hbar^2 b=0\;,\qquad v=\frac{dx}{dt}\;. \tag{3.18}$$

**Proof:** (3.18) is an obvious consequence of (3.17) and $b=\frac{Dp}{Dt}$. ∎

**Theorem 3.3.** *Let $z(t)$ be a critical point for the action $\int_0^T L(z(t),b(t))\,dt$. Let*

$$t\to\big(z(t),\zeta(t)\big)\;,\qquad \zeta(t)=\big(\zeta^H(t),\zeta^V(t)\big)$$

*be the curve on $T^*\Sigma$ such that*

$$\begin{aligned}\zeta^H(t)&=\hbar p(t)-\frac{1}{\hbar}\frac{Db}{Dt}\;\in\;T^*_{x(t)}X\\ \zeta^V(t)&=\frac{1}{\hbar}g^{-1}(b(t))\;\in\;T_{x(t)}X\;.\end{aligned} \tag{3.19}$$

*Then $t\to(z(t),\zeta(t))$ is an integral curve of the Hamiltonian of $H$, that is $(z(t),\zeta(t))$ satisfies the Hamilton–Jacobi equations*

$$\begin{aligned}\frac{dz}{dt}&=\frac{\partial H}{\partial\zeta}\big(z(t),\zeta(t)\big)\\ \frac{d\zeta}{dt}&=-\frac{\partial H}{\partial z}\big(z(t),\zeta(t)\big)\;.\end{aligned} \tag{3.20}$$

**Proof:** This result is a special case of Pontryagin duality. Observe that due to the constraint $\frac{dz}{dt}\in\mathcal{T}\Sigma$, the correspondence (3.19) involves first order derivatives of $b$.

We get from (3.18) and (3.19)

$$\begin{aligned}&\frac{D\zeta^H}{Dt}=\frac{-1}{\hbar}\,\big\langle b\,|\,R(v,.)v\big\rangle\\ &\left(p\,\middle|\,\frac{D\zeta^H}{Dt}\right)=0\;.\end{aligned} \tag{3.21}$$

In formula (3.6), the supremum in $b\in T^V\Sigma$ is reached for $b=\hbar g(\zeta^V)$, and one has

$$\begin{aligned}\frac{\partial H}{\partial z}(z,\zeta)&=\frac{\partial}{\partial z}\big(\zeta(H_{|p|^2/2})\big)-\frac{\partial L}{\partial z}\big(z,\hbar g(\zeta^V)\big)\\ \frac{\partial H}{\partial\zeta}(z,\zeta)&=H_{|p|^2/2}+\hbar g(\zeta^V)\;.\end{aligned} \tag{3.22}$$

From (3.22) and (3.19), the first line of (3.20) is obvious. We will get that the second line holds true by a computation in local coordinates. One has $(z(t),\zeta(t)) = (x(t),p(t),\xi(t),\eta(t))$ and the decomposition of $\zeta(t)$ in its horizontal and vertical components

$$\xi_i\, dx^i + \eta^j dp_j \;=\; \zeta^H_i e^i + \zeta^{V,j}\,\hat{e}_j$$

is

$$\begin{aligned} \zeta^H_i \;&=\; \xi_i + \Gamma^\alpha_{j,i}\, p_\alpha\, \eta^j \\ \zeta^{V,j} &= \eta^j \end{aligned} \tag{3.23}$$

and thus we get

$$\begin{aligned} \Big(\frac{d\zeta}{dt}\Big)^H &= \frac{D\zeta^H}{Dt} + \Big[\big\langle \zeta^H|\Gamma_l(v)\big\rangle - \Big\langle \frac{Dp}{Dt}|\Gamma_l(\eta)\Big\rangle - \Big\langle p\,|\Big(\frac{\partial \Gamma_l}{\partial x_i} + \Gamma_i\Gamma_l\Big)(\eta)\Big\rangle\, v^i\Big]\,\frac{\partial}{\partial p_l} \\ \Big(\frac{d\zeta}{dt}\Big)^V &= \frac{d\eta}{dt}\ . \end{aligned} \tag{3.24}$$

One has

$$-\big\langle \zeta^H|\Gamma_l(v)\big\rangle \;=\; \frac{\partial g^{i,j}}{\partial x_l}\, p_j\, \xi^H_i + \big\langle p\,|\,\Gamma_l g^{-1}(\xi^H)\big\rangle$$

and from (3.7) we get that the decomposition of $\frac{\partial H}{\partial z}(z,\zeta)$ in his horizontal and vertical components is

$$\begin{aligned} \Big(\frac{\partial H}{\partial z}(z,\zeta)\Big)^H \;&=\; \frac{\hbar}{2}\,\frac{\partial g_{i,j}}{\partial x_l}\,\eta^i\eta^j\,\frac{\partial}{\partial p_l} \\ &\qquad + \Big\langle p\,|\Big(\frac{\partial \Gamma_i}{\partial x_l} + \Gamma_l\Gamma_i\Big)(\eta)\Big\rangle\, v^i\,\frac{\partial}{\partial p_l} \;-\; \big\langle \zeta^H|\Gamma_l(v)\big\rangle\,\frac{\partial}{\partial p_l} \\ \Big(\frac{\partial H}{\partial z}(z,\zeta)\Big)^V \;&=\; -\,\hbar v + g^{-1}(\zeta^H) + \Gamma_i(\eta)v^i\ . \end{aligned} \tag{3.25}$$

If (3.19) is satisfied, one has

$$\hbar v - g^{-1}(\zeta^H) - \Gamma_i(\eta)v^i \;=\; \frac{1}{\hbar}\Big[g^{-1}\Big(\frac{Db}{Dt}\Big) - \Gamma_i\big(g^{-1}(b)\big)v^i\Big] \;=\; \frac{1}{\hbar}\,\frac{d}{dt}g^{-1}(b)$$

and thus the vertical component of the second line of (3.20) holds true, and we get that the horizontal component holds true using (1.3), (3.18), (3.21) and the identities

$$\begin{aligned} \big\langle b|R_{i,l}(v)\big\rangle \;&=\; \big\langle p|R_{l,i}(g^{-1}(b))\big\rangle \\ \big\langle b|\Gamma_l(\eta)\big\rangle \;&=\; \frac{\hbar}{2}\,\frac{\partial g_{i,j}}{\partial x_l}\,\eta^i\eta^j\ . \end{aligned}$$

The proof of theorem 3.3 is complete. ∎

**Lemma 3.4.** *Let $\delta > 0$ small. Let $z(t) = (x(t), p(t))$ be the solution of the differential equation (3.18) with data $x_0, p_0, b_0, b_1 = \frac{Db}{Dt}|_{t=0}$. Let $(x^0(t), p^0(t)) = \exp tH_{|p|^2/2}(x_0, p_0)$. Then, in geodesics coordinates centered at $x_0$ one has the following formulas, where $\mathcal{O}$ is uniform in the set $|\hbar t|^2 + |tp_0| + |t^2 b_0| + |t^3 b_1| \leq \delta$*

$$\begin{aligned} x^0(t) &= t\, p_0 + \mathcal{O}\big(t^3 |p_0|^3\big) \\ p^0(t) &= p_0 + \mathcal{O}\big(t^2 |p_0|^3\big) \end{aligned} \tag{3.26}$$

$$\begin{aligned} x(t) - x^0(t) &= \mathcal{O}\big(t^2 |b_0| + t^3 |b_1|\big) \\ p(t) - p^0(t) &= \mathcal{O}\big(t |b_0| + t^2 |b_1|\big) \end{aligned} \tag{3.27}$$

*and with*

$$\mathcal{R} = \big(|tb_0| + |t^2 b_1|\big)\,\big(\hbar^2 t + |p_0| + |tb_0| + |t^2 b_1|\big)^2$$

$$\begin{aligned} x(t) &= x_0 + p_0 t + b_0 t^2/2 + b_1 t^3/6 + \hbar^2 \left(b_0 \frac{t^4}{24} + b_1 \frac{t^5}{120}\right) + \mathcal{O}\big(t^3(\mathcal{R} + |p_0|^3)\big) \\ p(t) &= p_0 + b_0 t + b_1 t^2/2 + \hbar^2 \left(b_0 \frac{t^3}{6} + b_1 \frac{t^4}{24}\right) + \mathcal{O}\big(t^2(\mathcal{R} + |p_0|^3)\big) \\ b(t) &= b_0 + b_1 t + \hbar^2 (b_0 t^2/2 + b_1 t^3/6) + \mathcal{O}(t\mathcal{R}) \\ \frac{Db}{Dt}(t) &= b_1 + \hbar^2 (b_0 t + b_1 t^2/2) + \mathcal{O}(\mathcal{R})\ . \end{aligned} \tag{3.28}$$

**Proof:** In geodesic coordinates centered at $x_0$, one has $g(x_0) = Id$, $\nabla g(x_0) = 0$, so (3.26) and (3.27) are obvious. Using

$$\big|\Gamma^i_{j,k}(x(t))\, v^k(t)\big| \leq C\, |x(t) - x_0|\, |p(t)| = \mathcal{O}\Big(t\big(|p_0| + |tb_0| + |t^2 b_1|\big)^2\Big)$$

one gets (3.28) by an easy computation, writing (3.18) on the form, with $b_1(t) = \frac{Db}{Dt}$

$$\begin{aligned} \frac{\partial b}{\partial t} &= b_1(t) + \mathcal{O}\Big(t\, |b(t)|\, \big(|p_0| + |tb_0| + |t^2 b_1|\big)^2\Big) \\ \frac{\partial b_1}{\partial t} &= \hbar^2 b(t) + \mathcal{O}\Big(t\, |b_1(t)|\, \big(|p_0| + |tb_0| + |t^2 b_1|\big)^2 + |b(t)|\, \big(|p_0| + |tb_0| + |t^2 b_1|\big)^2\Big)\ . \ \blacksquare \end{aligned} \tag{3.29}$$

**Remark 3.5.** Let $M, T_0$ be given. Let $z_0 = (x_0, p_0)$ such that $|p_0| \leq M$. For $t \in [-T_0, T_0]$ set as above $(x^0(t), p^0(t)) = \exp tH_{|p|^2/2}(x_0, p_0)$. Identify $T_{x_0}X$

and $T_{x(t)}X$ by parallel transport along the geodesic $x^0(t)$, and for any $t$ write $x = x^0(t) + y, y \in T_{x_0}X \simeq T_{x(t)}X$ the geodesics coordinates centered at $x^0(t)$. Then, by the same proof as the one of lemma 3.4, we get that there exist $\delta > 0$ depending only on $M, T_0$, such that the solution $z(t) = (x(t), p(t))$ of the differential equation (3.18) with data $x_0, p_0, b_0, b_1 = \frac{Db}{Dt}|_{t=0}$ satisfies in the above coordinates the following estimates, where $\mathcal{O}$ is uniform in $|tb_0| + |t^2 b_1| \le \delta$, keeping the notation

$$\mathcal{R} = \left(|tb_0| + |t^2b_1|\right)\left(\hbar^2 t + |p_0| + |tb_0| + |t^2b_1|\right)^2$$

$$\begin{aligned}
x(t) &= x^0(t) + b_0t^2/2 + b_1t^3/6 + \hbar^2\left(b_0\frac{t^4}{24} + b_1\frac{t^5}{120}\right) + \mathcal{O}(t^3\mathcal{R}) \\
p(t) &= p^0(t) + b_0t + b_1t^2/2 + \hbar^2\left(b_0\frac{t^3}{6} + b_1\frac{t^4}{24}\right) + \mathcal{O}(t^2\mathcal{R}) \\
b(t) &= b_0 + b_1t + \hbar^2(b_0t^2/2 + b_1t^3/6) + \mathcal{O}(t\mathcal{R}) \\
\frac{Db}{Dt}(t) &= b_1 + \hbar^2(b_0t + b_1t^2/2) + \mathcal{O}(\mathcal{R}) \ . \ \square
\end{aligned} \tag{3.30}$$

**Lemma 3.6.** *There exist $t_0 > 0, \delta_0 > 0$ and two smooth functions $B_{0,1}(t, z_0)$ defined for $t \in [-t_0, t_0]$, $z_0 = (x_0, p_0) \in T^*_{x_0}X, |p_0| \le \delta_0$, with values in $T^*_{x_0}X$, such that $B_0(0, z_0) = 0$, $B_1(0, z_0) = \hbar^2 p_0$, and such that if $z(t) = (x(t), p(t))$ is the solution of the differential equation (3.18) with data $x_0, p_0, b_0, b_1 = \frac{Db}{Dt}|_{t=0}$, then the equation in $b_0, b_1$*

$$\begin{aligned}
&b(t) = 0 \\
&\frac{Db}{Dt}(t) = \hbar^2 p(t)
\end{aligned} \tag{3.31}$$

*admits the unique solution $(b_0, b_1)$ closed to $(0, \hbar^2 p_0)$*

$$(b_0, b_1) = \left(B_0(t, z_0),\, B_1(t, z_0)\right) . \tag{3.32}$$

*Moreover, one has*

$$\begin{aligned}
B_0(t, z_0) &= -\hbar^2 p_0 t + \mathcal{O}\left(t^3|p_0|\hbar^4\right) \\
B_1(t, z_0) &= \hbar^2 p_0 + \mathcal{O}\left(t^2|p_0|\hbar^2\left(\hbar^2|t| + |p_0|\right)^2\right) .
\end{aligned} \tag{3.33}$$

**Proof:** This lemma is an obvious consequence of (3.28). ∎

**Lemma 3.7.** *There exist $t_0 > 0$, $\delta_0 > 0$, $\delta_1 > 0$ and two smooth functions $\mathcal{B}_{0,1}(t, z, z_0)$ defined for $t \in ]0, t_0]$, $z = (x, p)$, $z_0 = (x_0, p_0)$, $\mathrm{dist}(x, x_0) + |\hbar t|^2 + |tp| + |tp_0| \le \delta_1$, with values in $T^*_{x_0}X$, such that if $z(t) = (x(t), p(t))$ is the solution of the differential equation (3.18) with data $x_0, p_0, b_0, b_1 = \frac{Db}{Dt}|_{t=0}$, then the equation in $b_0, b_1$*

$$z(t) = z \tag{3.34}$$

*admits the unique solution $(b_0, b_1)$ in the set $|t^2 b_0| + |t^3 b_1| \le \delta_0$*

$$(b_0, b_1) \;=\; \big(\mathcal{B}_0(t, z, z_0), \mathcal{B}_1(t, z, z_0)\big)\ . \tag{3.35}$$

*Moreover, in geodesic coordinates centered at $x_0$, one has with*

$$M = \begin{pmatrix} 1/2 & 1/6 \\ 1 & 1/2 \end{pmatrix}\ , \qquad M^{-1} = \begin{pmatrix} 6 & -2 \\ -12 & 6 \end{pmatrix}$$

$$\begin{aligned} &\begin{pmatrix} t^2 \mathcal{B}_0(t, z, z_0) \\ t^3 \mathcal{B}_1(t, z, z_0) \end{pmatrix} \;=\; M^{-1} \begin{pmatrix} x - x_0 - tp_0 \\ t(p - p_0) \end{pmatrix} \\ &\quad + \mathcal{O}\Big(\Big(|x - x_0 - tp_0| + |t(p - p_0)|\Big) \Big(|\hbar t|^2 + \mathrm{dist}(x, x_0) + |tp_0| + |tp|\Big)\Big)\ . \end{aligned} \tag{3.36}$$

**Proof:** This lemma is an obvious consequence of (3.28). ∎

**Theorem 3.8.** *Let $M$ be a upper bound for the $L^\infty$ norm of the Riemann tensor $R$. There exist a universal constant $C$ such that the solution of the differential equation (3.18) with data $x_0, p_0, b_0, b_1 = \frac{Db}{Dt}|_{t=0}$ exists for $t \in [-T_*, T_*]$ with*

$$T_* \;\ge\; C \int_0^\infty \frac{ds}{\sqrt{|b_0|^2 + 2s|b_1|} + \hbar s + \sqrt{M}\, s(|p_0| + s)}\ . \tag{3.37}$$

**Proof:** We first observe that when $b_0 = b_1 = 0$, the solution of the differential equation (3.18) is

$$z(t) = \exp(tH_{|p|^2/2})(x_0, p_0)\ , \qquad b(t) = 0$$

and thus is defined for all $t$. Set $u = |p|^2/2$, $v = |b|^2/2$, $w = |\frac{Db}{Dt}|^2/2$. Then we get using the rule $(a|b)' = (\frac{Da}{Dt}|b) + (a|\frac{Db}{Dt})$

$$\begin{aligned} u' \;&\le\; 2\sqrt{u}\,\sqrt{v} \\ v' \;&\le\; 2\sqrt{v}\,\sqrt{w} \\ w' \;&\le\; 2\,\hbar^2 \sqrt{v}\,\sqrt{w} + 4\,Mu\,\sqrt{v}\,\sqrt{w}\ . \end{aligned}$$

Thus we can estimate the life span $T_*$ by the life span of the following differential equation on $\mathbb{R}_+$

$$y^{(3)} = (My^2 + \hbar^2)y' \\ \big(y(0), y'(0), y''(0)\big) = \big(|p_0|, |b_0|, |b_1|\big) \ . \tag{3.38}$$

With $A(y) = My^3/3 + \hbar^2 y$ and $B(y) = My^4/6 + \hbar^2 y^2 + 2Cy$, integration of (3.38) leads to

$$y'' = A(y) + C \ , \qquad y'^2 = G(y) = B(y) - B(y(0)) + y'(0)^2 \ .$$

One has $B'(y) = 2(A(y) - A(y(0)) + y''(0))$, and we get the lower bound for the life span

$$T_* \ \geq \int_{y_0}^{\infty} \frac{dy}{\sqrt{G(y)}} \ .$$

With $y = y(0) + s$, we thus get (3.37) from

$$\sqrt{G(y)} \ \leq \ C\Big(\sqrt{|b_0|^2 + 2s|b_1|} + \hbar s + \sqrt{M} s(|p_0| + s)\Big) \ . \ \blacksquare$$

**Remark 3.9.** Observe that in the above theorem, we have not take in account the sign of the curvature tensor. □

### 3.2. Homogeneity

In the sequel, we shall often use scaling arguments linked with the homogeneity of the phase space $\Sigma = T^*X$. For $\varepsilon > 0$, let $\varepsilon(x,p) = (x, \varepsilon p)$ be the natural action of homogeneity on $\Sigma$. We will denote by $j_\varepsilon$ the corresponding action on the cotangent space which is the canonical transformation of the symplectic variety $T^*\Sigma$ given by

$$j_\varepsilon(x, p, \zeta^H, \zeta^V) \ = \ (x, \varepsilon p, \zeta^H, \varepsilon^{-1}\zeta^V) \ .$$

For $\lambda > 0$, let $\lambda(z,\zeta) = (z, \lambda\zeta)$ be the natural action of homogeneity on $T^*\Sigma$. If $F_\lambda(z,\zeta) = F(z, \lambda\zeta)$, one has $\lambda \exp(tH_{F_\lambda})(z_0, \zeta_0) = \exp(\lambda t H_F)(z_0, \lambda\zeta_0)$. Let $z(t) = (x(t), p(t))$ be a solution of the differential equation (3.18).

Let $(x(t), p(t), \zeta^H(t), \zeta^V(t))$ be the corresponding integral curve of the Hamiltonian vector field of $H$. Set $t' = t/\varepsilon$ and $z'(t') = (x(t'), p'(t') = \varepsilon p(t'))$. Then, with $b' = \varepsilon^2 b$ and $v' = \varepsilon v = g^{-1}(p')$, $z'(t')$ satisfies the differential equation

$$-\frac{D}{Dt'}\frac{Db'}{Dt'} + \big\langle b' \,|\, R(v', .)v' \big\rangle + (\hbar\varepsilon)^2\, b' \ = \ 0 \ , \qquad v' = \frac{dx'}{dt'} \ . \tag{3.39}$$

By theorem 3.3, the transformation

$$\begin{aligned} \zeta'^H(t') &= \hbar\,\varepsilon\, p'(t') - \frac{1}{\hbar\varepsilon}\frac{Db'}{Dt'} \\ \zeta'^V(t') &= \frac{1}{\hbar\varepsilon}\, g^{-1}\big(b'(t')\big) \end{aligned}$$

defines an integral curve parametrized by $t'$ of the Hamiltonian vector field of $H'$ with

$$H'(z',\zeta') = \frac{\hbar\varepsilon}{2}\big(|\zeta'^V|^2 - |p'|^2\big) + (p'|\zeta'^H)\ . \tag{3.40}$$

One has

$$\begin{aligned} (x',p',\zeta'^H,\zeta'^V) &= \varepsilon^2\, j_\varepsilon(x,p,\zeta^H,\zeta^V) \\ H'(\varepsilon^2 j_\varepsilon(z,\zeta)) &= \varepsilon^3 H(z,\zeta)\ . \end{aligned} \tag{3.41}$$

Let $H_\varepsilon(z,\zeta)$ be the Hamiltonian

$$H_\varepsilon(z,\zeta) = \frac{\hbar}{2}\big(|\zeta^V|^2 - \varepsilon^2|p|^2\big) + (p|\zeta^H)\ . \tag{3.42}$$

Observe that one has

$$\begin{aligned} \varepsilon\,\exp(tH_{H'})(z,\zeta) &= \exp(tH_{H_\varepsilon})(z,\varepsilon\zeta) \\ \varepsilon^3 j_\varepsilon \exp(\varepsilon t H_H)(z_0,\zeta_0) &= \exp(tH_{H_\varepsilon})(\varepsilon^3 j_\varepsilon(z_0,\zeta_0))\ . \end{aligned} \tag{3.43}$$

We will denote by

$$\begin{aligned} s &\to \big(Z_\varepsilon(s,z_0,\zeta_0),\ \Xi_\varepsilon(s,z_0,\zeta_0)\big) \\ s &\to \big(Z'(s,z_0,\zeta_0),\ \Xi'(s,z_0,\zeta_0)\big) \end{aligned} \tag{3.44}$$

the flows of the Hamiltonian of $H_\varepsilon(z,\zeta)$, $H'(z,\zeta)$ on $T^*\Sigma$ starting from $(z_0,\zeta_0)$. By (3.43), one has

$$\begin{aligned} Z'(s,z_0,\zeta_0) &= Z_\varepsilon(s,z_0,\varepsilon\zeta_0) \\ \varepsilon\Xi'(s,z_0,\zeta_0) &= \Xi_\varepsilon(s,z_0,\varepsilon\zeta_0)\ . \end{aligned} \tag{3.45}$$

### 3.3. Time dependent eikonal

In this subsection, we shall study the following eikonal equation related to the decay of the kernel of the heat equation.

$$\partial_t\psi + \frac{\hbar}{2}\big((\partial_p\psi)^2 - |p|^2\big) + \big\{|p|^2/2,\psi\big\} = 0\ . \tag{3.46}$$

For $t_0 > 0$ and $\delta = (\delta_0, \delta_1, \delta_2)$ with $\delta_0 > 0, \delta_1 > 0, \delta_2 > 0$, we denote by $U_{t_0,\delta}$ the subset of $\mathbb{R}_+^* \times \Sigma \times \Sigma$

$$U_{t_0,\delta} = \Big\{(t, z, z_0),\ t \in ]0, t_0],\ \mathrm{dist}(x, x_0) < \delta_0,\ |tp| < \delta_2,\ |tp_0| < \delta_1\Big\} \tag{3.47}$$

where $z = (x, p)$, and $z_0 = (x_0, p_0)$.

**Lemma 3.10.** *There exist $t_0 > 0$ and $\delta_1 > 0$, $\delta_2$ such that the equation in $\zeta_0$*

$$\Xi(t, z_0, \zeta_0) = 0$$

*admits an unique smooth solution $\zeta_0 = \Xi_*(t, z_0)$ defined in the subset*

$$V_{t_0,\delta_1} = \Big\{t \in [-t_0, t_0],\ x_0 \in X,\ |tp_0| \le \delta_1\Big\}$$

*of $[-t_0, t_0] \times \Sigma$, and such that $\Xi_*(0, z_0) = 0$. The smooth functions $Z_*(t, z_0), \gamma(t, z_0)$ defined in the set $(t, z_0) \in V_{t_0,\delta_1}$ by*

$$\begin{aligned} &Z_*(t, z_0) = Z(t, z_0, \Xi_*(t, z_0)) \ \in\ \Sigma \\ &\gamma(t, z_0) = \int_0^t \Big(\zeta \frac{\partial H}{\partial \zeta} - H\Big)\big(Z_*(s, z_0),\, \Xi_*(s, z_0)\big)\, ds \ \in\ \mathbb{R} \end{aligned} \tag{3.48}$$

*are such that*

$$\begin{aligned} \gamma(t, z_0) &= \min \int_0^t \mathcal{L}(x(s))\, ds \\ &= \frac{\hbar t}{2}|p_0|^2 - \frac{(\hbar t)^3}{6}|p_0|^2 + \mathcal{O}\big(t^3 \hbar |p_0|^4 + t^5 \hbar^5 |p_0|^2\big) \end{aligned} \tag{3.49}$$

*where the minimum is taken over all trajectories $s \in [0, t] \to x(s)$ such that $(x(0), g(v(0))) = z_0$ and data $(b_0, b_1)$ such that $t^2|b_0| + t^3|b_1| \le \delta_2$. Moreover, in geodesic coordinates centered at $x_0$, one has*

$$\begin{aligned} Z_*(t, z_0) &= \exp(tH_{|p|^2/2})(z_0) \\ &\quad + \Big(-\hbar^2 p_0 t^3/6 + \mathcal{O}(t^5\hbar^4|p_0|),\, -\hbar^2 p_0 t^2/2 + \mathcal{O}(t^4\hbar^4|p_0|)\Big)\ . \end{aligned} \tag{3.50}$$

**Proof:** By (3.19), the equation $\Xi(t, z_0, \zeta_0) = 0$ is equivalent to

$$b(t) = 0\ , \qquad \frac{Db}{Dt}(t) = \hbar p(t)\ . \tag{3.51}$$

Therefore, if one sees $\Xi_*$ as a function of the parameter $\hbar$, one has obviously

$$\Xi_*|_{\hbar=0} = 0 \ , \quad Z_*(t,z_0)|_{\hbar=0} = \exp(tH_{|p|^2/2})(z_0) \ , \quad \gamma(t,z_0)|_{\hbar=0} = 0 \ .$$

We shall use the scaling transformation, where $\varepsilon \in ]0,1]$

$$(x_0', p_0', b_0', b_1';\, t', \hbar') \;=\; (x_0, \varepsilon p_0, \varepsilon^2 b_0, \varepsilon^3 b_1;\, t/\varepsilon, \hbar\varepsilon) \ .$$

Let $t_0', \delta_0' \leq 1$ such that lemma 3.6 holds true for solutions of the rescaled differential equation (3.39), uniformly with respect to $\hbar' \in [0,\hbar]$. Let us denote by $B_{0,1}'(\varepsilon, t', z_0')$ the associated functions, where the dependence in $\varepsilon$ comes from the dependence of $B_{0,1}'$ in $\hbar' = \hbar\varepsilon$. Then we have by (3.33)

$$\begin{aligned} B_0'(\varepsilon, t', z_0') \;&=\; -\hbar'^2 p_0' t' + \mathcal{O}(t'^3 |p_0'| \hbar'^4) \\ B_1'(\varepsilon, t', z_0') \;&=\; \hbar'^2 p_0' + \mathcal{O}\Big(t'^2 |p_0'| \hbar'^2 \big(\hbar'^2 |t'| + |p_0'|\big)^2\Big) \ . \end{aligned} \tag{3.52}$$

For a given $z_0$, set

$$\varepsilon = \frac{\delta_0'}{\langle p_0 \rangle} \;\in\; ]0,1] \ .$$

Then $p_0' = \varepsilon p$ satisfies $|p_0'| \leq \delta_0'$. Set

$$t_0 \;=\; \delta_1 = \frac{\delta_0' t_0'}{\sqrt{2}}$$

Then for $(t, z_0) \in V_{t_0,\delta_1}$ one has $|t'| = |t/\varepsilon| \leq t_0'$. Therefore, by lemma 3.6, the unique solution in $b_0, b_1$ of the equation (3.51) closed to $(0, \hbar^2 p_0)$ is given by

$$\begin{aligned} b_0 \;&=\; B_0(t, z_0) \;=\; \varepsilon^{-2} B_0'(\varepsilon, t', z_0') \\ b_1 \;&=\; B_1(t, z_0) \;=\; \varepsilon^{-3} B_1'(\varepsilon, t', z_0') \end{aligned} \tag{3.53}$$

and the expansions (3.33) holds true. Therefore, if $\zeta_0 = \Xi_*(t, z_0)$ is a smooth solution of $\Xi(t, z_0, \zeta_0) = 0$ defined in the subset $V_{t_0,\delta_1}$ of $[-t_0, t_0] \times \Sigma$, such that $\Xi_*(0, z_0) = 0$, one has

$$\begin{aligned} \Xi_*^H(t, z_0) \;&=\; \hbar p_0 - \frac{B_1(t, z_0)}{\hbar} \;=\; \mathcal{O}\Big(t^2\, |p_0|\, \hbar \big(\hbar^2 |t| + |p_0|\big)^2\Big) \\ \Xi_*^V(t, z_0) \;&=\; \frac{g^{-1}\big(B_0(t, z_0)\big)}{\hbar} \;=\; -\hbar t g^{-1}(p_0) + \mathcal{O}\big(t^3 |p_0| \hbar^3\big) \ . \end{aligned} \tag{3.54}$$

Let $\Xi_*'(t', z_0')$ the associated rescaled functions, so that

$$\Xi_*'^H(t', z_0') = \varepsilon^2\, \Xi_*^H(t, z_0) \ , \quad \Xi_*'^V(t', z_0') = \varepsilon\, \Xi_*^V(t, z_0) \ .$$

Let $Z'_*(t',z'_0), \gamma'(t',z'_0)$ the associated functions defined by

$$
(3.55)\qquad
\begin{aligned}
Z'_*(t',z'_0) &= Z'\big(t',z'_0,\Xi'_*(t',z'_0)\big)\\
\gamma'(t',z'_0) &= \int_0^{t'} \Big(\zeta'\frac{\partial H'}{\partial \zeta'} - H'\Big)\big(Z'_*(s,z'_0),\Xi'_*(s,z'_0)\big)\,ds
\end{aligned}
$$

where the Hamiltonian $H'$ is defined in (3.40). Then one has

$$
(3.56)\qquad
\begin{aligned}
Z_*(t,z_0) &= \varepsilon^{-1} Z'_*(t',z'_0)\\
\gamma(t,z_0) &= \varepsilon^{-2}\,\gamma'(t',z'_0)\ .
\end{aligned}
$$

Let

$$
(3.57)\qquad \mathcal{L}'(x'(s)) = \frac{|a'(s)|^2}{2\hbar'} + \frac{\hbar'|v'(s)|^2}{2}\ .
$$

Then one has

$$
\gamma'(t',z'_0) = \int_0^{t'} \mathcal{L}'(x'(s))\,ds
$$

where $s \to x'(s)$ is the trajectory of the rescaled equation (3.39) with data at $s=0$ depending on the final time $t'$ equal to $(x'_0,p'_0,B'_0(\varepsilon,t',z'_0),B'_1(\varepsilon,t',z'_0))$. From (3.52), and lemma 3.4, we get

$$
(3.58)\qquad
\begin{aligned}
Z'_*(t',z'_0) &= \big(X'_*(t',z'_0),P'_*(t',z'_0)\big)\\
X'_*(t',z'_0) &= x'_0 + t'p'_0 - \hbar'^2 p'_0 t'^3/6 + \mathcal{O}\big(t'^3|p'_0|^3 + t'^5\hbar'^4|p'_0|\big)\\
P'_*(t',z'_0) &= p'_0 - \hbar'^2 p'_0 t'^2/2 + \mathcal{O}\big(t'^2|p'_0|^3 + t'^4\hbar'^4|p'_0|\big)
\end{aligned}
$$

and

$$
(3.59)\qquad \gamma'(t',z'_0) = \frac{\hbar' t'}{2}|p'_0|^2 - \frac{(\hbar' t')^3}{6}|p'_0|^2 + \mathcal{O}\big(t'^3\hbar'|p'_0|^4 + t'^5\hbar'^5|p'_0|^2 + t'^7\hbar'^3|p'_0|^6\big)\ .
$$

Observe that formula's (3.58) and (3.59) have the correct scaling invariance compatible with (3.56), and therefore, (3.50) and (3.49) holds true. The proof of lemma 3.10 is complete. ∎

**Lemma 3.11.** *There exist $t_0, \delta$ with $\delta_1 << \delta_2$ and a unique smooth function $\psi(t,z,z_0)$ defined on $U_{t_0,\delta}$, which is a solution of the eikonal equation (3.46), and such that (3.60), (3.61) hold true*

$$
(3.60)\qquad
\begin{aligned}
&(z,d_z\psi)(t,z,z_0) \in \exp(tH_H)(T^*_{z=z_0})\\
&d_z\psi(t,z,z_0) = 0 \iff z = Z_*(t,z_0)\\
&\psi(t,z,z_0) \geq \psi\big(t,Z_*(t,z_0),z_0\big) \geq 0
\end{aligned}
$$

$$
(3.61)\qquad \psi\big(t,Z_*(t,z_0),z_0\big) = \gamma(t,z_0)\ .
$$

**Proof:** We first observe that if $\psi(t,z,z_0)$ exists, it is unique by (3.60) up to a function $f(t,z_0)$ depending only on $t,z_0$, and that the function $f(t,z_0)$ is determined by (3.61). In order to prove the existence of $\psi(t,z,z_0)$, we shall first use a scaling argument to reduce the problem in a situation where $(z,z_0)\in\Sigma\times\Sigma$ is in a small neighborhood of the diagonal $X\simeq X\times_X X\subset\Sigma\times\Sigma$.

Let $\psi$ a solution of the eikonal equation (3.46). Then the function

$$\psi_\varepsilon(t,z,z_0) \;=\; \varepsilon^3\,\psi(\varepsilon t,\varepsilon^{-1}z,\varepsilon^{-1}z_0)$$

satisfies the eikonal equation

$$\partial_t\psi_\varepsilon+\frac{\hbar}{2}\big((\partial_p\psi_\varepsilon)^2-\varepsilon^2|p|^2\big)+\big\{|p|^2/2,\psi_\varepsilon\big\} \;=\; 0 \tag{3.62}$$

and if (3.60) holds true for $\psi$, then from (3.43) we get that $\psi_\varepsilon$ is such that

$$\begin{aligned}&(z,d_z\psi_\varepsilon)(t,z,z_0)\;\in\;\exp(tH_{H_\varepsilon})(T^*_{z=z_0})\\&d_z\psi_\varepsilon\big(t,\varepsilon Z_*(\varepsilon t,\varepsilon^{-1}z_0),z_0\big)\;=\;0\;.\end{aligned} \tag{3.63}$$

Now, let $\psi_\varepsilon(t,z,z_0)$ be a smooth function defined for $\varepsilon\in\,]0,1]$, $t\in[t_0/2,2t_0]$, $\mathrm{dist}(x,x_0)<\delta_0$, $|p|<\delta_4$, $|p_0|<\delta_3$, such that (3.62) and (3.63) holds true. The function

$$\psi^\varepsilon(t,z,z_0)\;=\;\varepsilon^{-3}\,\psi_\varepsilon\Big(\frac{t}{\varepsilon},\varepsilon z,\varepsilon z_0\Big)$$

satisfies (3.46) and (3.60) and is define in the set

$$t\in[\varepsilon t_0/2,2\varepsilon t_0]\;,\qquad \mathrm{dist}(x,x_0)<\delta_0\;,\qquad \varepsilon|p|<\delta_4\;,\qquad \varepsilon|p_0|<\delta_3\;.$$

If moreover $\psi^\varepsilon$ satisfies

$$\psi^\varepsilon\big(t,Z_*(t,z_0),z_0\big)\;=\;\gamma(t,z_0)$$

then by (3.60), $\psi^\varepsilon$ will be independent of $\varepsilon$, and therefore the function $\psi=\psi^\varepsilon$ will be given by the formula

$$\psi(t,z,z_0)\;=\;\Big(\frac{t}{t_0}\Big)^{-3}\,\psi_{\frac{t}{t_0}}\Big(t_0,\frac{t}{t_0}z,\frac{t}{t_0}z_0\Big) \tag{3.64}$$

and will be defined and smooth on $U_{t_0,\delta}$ if one takes

$$\delta_1=t_0\delta_3\;,\qquad \delta_2=t_0\delta_4\;.$$

Therefore, lemma 3.11 will be consequence of the next lemma. ∎

**Lemma 3.12.** *There exist $t_0 > 0$, $\delta_0 > 0$ and $0 < \delta_3 \ll \delta_4$ and a unique smooth function $\psi_\varepsilon(t, z, z_0)$ defined for $\varepsilon \in ]0, 1]$, $t \in [t_0/2, 2t_0]$, $\mathrm{dist}(x, x_0) < \delta_0$, $|p| < \delta_4$, $|p_0| < \delta_3$, solution of (3.62) and (3.63), and such that, with*

$$Z_{*,\varepsilon}(t, z_0) \;=\; \varepsilon Z_*(\varepsilon t, \varepsilon^{-1} z_0) \;=\; Z'_*(t, z_0)$$

*one has*

$$\begin{aligned}
&\psi_\varepsilon(t, z, z_0) \;\geq\; \psi_\varepsilon\big(t, Z_{*,\varepsilon}(t, z_0), z_0\big) \;\geq\; 0 \\
&d_z\psi_\varepsilon(t, z, z_0) = 0 \quad\Leftrightarrow\quad z = Z_{*,\varepsilon}(t, z_0) \\
&\psi_\varepsilon\big(t, Z_{*,\varepsilon}(t, z_0), z_0\big) \;=\; \varepsilon^3\gamma(\varepsilon t, \varepsilon^{-1} z_0) \;=\; \varepsilon\gamma'(t, z_0)\ .
\end{aligned} \tag{3.65}$$

*Moreover, in geodesic coordinates centered at $x_0$ one has*

$$\begin{aligned}
\hbar\psi_\varepsilon\Big(t_0, Z_{*,\varepsilon}(t_0, z_0) + (t_0 X, P), z_0\Big) \;&=\; \hbar\varepsilon^3\,\gamma(\varepsilon t_0, \varepsilon^{-1} z_0) \\
&+ \frac{2}{t_0}(3X^2 - 3PX + P^2) + \mathcal{O}\Big(t_0\big(|p_0|^2 + \hbar'^2 + (X, P)^2\big)(X, P)^2\Big)
\end{aligned} \tag{3.66}$$

*where the $\mathcal{O}$ is uniform with respect to $\varepsilon$.*

**Proof:** The smooth dependence in $z_0 = (x_0, p_0)$ of the function $\psi_\varepsilon$ will be clear by the proof, and we will work in geodesic coordinates centered at $x_0$ for $x$. Using (3.43), one gets that (3.63) is equivalent to

$$\begin{aligned}
&(z, d_z\psi_\varepsilon)(t, z, z_0) \;\in\; \varepsilon\exp(tH_{H'})(T^*_{z=z_0}) \\
&d_z\psi_\varepsilon\big(t, Z'_*(t, z_0), z_0\big) \;=\; 0\ .
\end{aligned} \tag{3.67}$$

Let $\Phi^\varepsilon_{t,z_0}$ the map from $T^*_{x_0} \times T^*_{x_0}$ to $\Sigma$,

$$(b_0, b_1) \to \Phi^\varepsilon_{t,z_0}(b_0, b_1) = z(t)$$

where $z(s)$ is the solution of the differential equation (3.39) with data $x_0, p_0, b_0, b_1$ at $s = 0$. We first show that there exist $t_0, \delta_3$, such that the differential of $\Phi^\varepsilon_{t_0,z_0}$ at $(b_0, b_1) = (0, 0)$ is non singular for all $z_0$ such that $|p_0| \leq \delta_3$ and all $\varepsilon \in [0, 1]$. It is obviously sufficient to show that there exist $t_0$ such that the differential of $\Phi^\varepsilon_{t_0,(x_0,0)}$ at $(b_0, b_1) = (0, 0)$ is non singular for all $\varepsilon \in [0, 1]$. But when $p_0 = 0$, we get by lemma 3.4 in geodesics coordinates centered at $x_0$

$$
\begin{aligned}
z(s) &= \big(x(s), p(s)\big)\\
x(s) &= b_0 s^2/2 + b_1 s^3/6 + \hbar'^2\Big(b_0\frac{s^4}{24} + b_1\frac{s^5}{120}\Big)\\
&\quad + \mathcal{O}\Big(s^6\big(|b_0| + |sb_1|\big)\big(\hbar'^2 + |b_0| + |sb_1|\big)^2\Big)\\
p(s) &= b_0 s + b_1 s^2/2 + \hbar'^2\Big(b_0\frac{s^3}{6} + b_1\frac{s^4}{24}\Big)\\
&\quad + \mathcal{O}\Big(s^5\big(|b_0| + |sb_1|\big)\big(\hbar'^2 + |b_0| + |sb_1|\big)^2\Big)\,.
\end{aligned}
\tag{3.68}
$$

For $s \neq 0$, the matrix

$$
\begin{pmatrix} s^2/2 & s^3/6 \\ s & s^2/2 \end{pmatrix}
$$

is non singular, and therefore the differential of $\Phi^{\varepsilon}_{s,(x_0,0)}$ at $(b_0, b_1) = (0,0)$ is non singular for any $s \in \,]0, 4t_0]$ and all $\varepsilon \in [0,1]$ if $t_0$ is small enough. We next choose $\delta_3$ small enough. Then by theorems 3.2 and 3.3, there exist $\delta_0, \delta_4$, and for any $t \in [t_0/2, 2t_0]$, a smooth function $\psi_{1,\varepsilon}(t, z, z_0)$ defined for $\varepsilon \in \,]0,1]$, $\mathrm{dist}(x, x_0) < \delta_0$, $|p| < \delta_4$, $|p_0| < \delta_3$, which is solution of (3.63). Adding to $\psi_{1,\varepsilon}(t, z, z_0)$ a function independent of $z$, we get a function $\psi_\varepsilon(t, z, z_0)$ such that the third line of (3.65), (3.62), (3.63) hold true, and also the second line of (3.65) by lemma 3.10.

To get the first line of (3.65) and the formula (3.66), it remains to prove that the hessian of $\psi_\varepsilon(t, z, z_0)$ is non degenerate and positive at is only critical point which is precisely $Z_{*,\varepsilon}(t, z_0)$ by lemma 3.10. As above, we may restrict the verification to the case $p_0 = 0$; we then have

$$
\begin{aligned}
&Z'_*(s, z_0) = z_0 = (x_0, 0)\\
&\gamma'(s, z_0) = 0
\end{aligned}
\tag{3.69}
$$

and by theorems 3.2, 3.3, and making use of (3.67), (3.28) and $\hbar'\varepsilon^{-1} = \hbar$, we get that the parametrization of $(z, d_z\psi_\varepsilon(s, z, z_0)$ in terms of $(b_0, b_1)$ has the form, with

$$
\mathcal{R} = s^3\big(|b_0| + |sb_1|\big)\big(\hbar'^2 + |b_0| + |sb_1|\big)^2
$$

$$
(3.70)\qquad
\begin{aligned}
x &= b_0 s^2/2 + b_1 s^3/6 + \hbar'^2\Big(b_0\frac{s^4}{24} + b_1\frac{s^5}{120}\Big) + \mathcal{O}(s^3\mathcal{R})\\
p &= b_0 s + b_1 s^2/2 + \hbar'^2\Big(b_0\frac{s^3}{6} + b_1\frac{s^4}{24}\Big) + \mathcal{O}(s^2\mathcal{R})\\
\hbar(d_z\psi_\varepsilon)^H &= -b_1 + \mathcal{O}(\mathcal{R})\\
\hbar(d_z\psi_\varepsilon)^V &= b_0 + b_1 s + \hbar'^2\Big(b_0\frac{s^2}{2} + b_1\frac{s^3}{6}\Big) + \mathcal{O}(s\mathcal{R})
\end{aligned}
$$

we thus get from the two first lines of (3.70)

$$
(3.71)\qquad
\begin{aligned}
x - sp/2 &= \frac{-b_1 s^3}{12} + \mathcal{O}\Big(\hbar'^2 s^4\big(|b_0| + |sb_1|\big) + s^6\big(|b_0| + |sb_1|\big)^3\Big)\\
x - sp/3 &= \frac{b_0 s^2}{6} + \mathcal{O}\Big(\hbar'^2 s^4\big(|b_0| + |sb_1|\big) + s^6\big(|b_0| + |sb_1|\big)^3\Big)
\end{aligned}
$$

and we deduce from the two last lines of (3.70)

$$
(3.72)\qquad \hbar\psi_\varepsilon\big(s, sX, P, (x_0, 0)\big) = \frac{2}{s}(3X^2 - 3XP + P^2) + \mathcal{O}\Big(\hbar'^2 s(X,P)^2 + s(X,P)^4\Big)\,.
$$

The proof of lemma 3.12 is complete. ∎

**Definition 3.13.** We define the large deviation function on $\mathbb{R}_+^* \times \Sigma \times \Sigma$ by

$$
(3.73)\qquad \mathcal{D}(t, z, z_0) \;=\; \min \int_0^t \mathcal{L}(x(s))\,ds
$$

where the minimum is taken over all trajectories $s \in [0, t] \to x(s)$ such that

$$
\big(x(0), g(v(0))\big) = z_0\ , \qquad \big(x(t), g(v(t))\big) = z\ . \ \square
$$

Remark that the function $\mathcal{D}(t, z, z_0)$ depends on the constant $\hbar$, and satisfies the following scaling invariance

$$
(3.74)\qquad \mathcal{D}(\hbar\varepsilon, t/\varepsilon, \varepsilon z, \varepsilon z_0) \;=\; \varepsilon^2 \mathcal{D}(\hbar, t, z, z_0)\ .
$$

**Theorem 3.14.** **i**) *For all $t > 0$, $z_0, z$, there exist a solution $s \in [0, t] \to z_{\mathrm{opt}}(s)$ of the differential equation (3.18) connecting $z_0$ to $z$ such that*

$$
(3.75)\qquad \mathcal{D}(t, z, z_0) \;=\; \int_0^t \mathcal{L}(x_{\mathrm{opt}}(s))\,ds\ .
$$

*The function $t \to \mathcal{D}(t, z, z_0)$ is continuous on $\mathbb{R}_+^*$.*

**ii**) *Let $t_0$ small enough. There exist $0 < \delta_0 = \delta_1 \leq \delta_2$ such that the following equality of functions on $U_{t_0,\delta}$ holds true*

$$\psi(t, z, z_0) = \mathcal{D}(t, z, z_0)|_{U_{t_0,\delta}} \ . \tag{3.76}$$

*Moreover, in geodesic coordinates centered at $x_0$, one has for $(t, z, z_0) \in U_{t_0,\delta}$, with $z = Z_*(t, z_0) + (tX, P)$, and $|(X, P)|$ small*

$$\begin{aligned} \mathcal{D}(t, z, z_0) \ &= \ \gamma(t, z_0) + \frac{2}{\hbar t}(3X^2 - 3PX + P^2) \\ &\quad + \mathcal{O}\Big(\frac{t}{\hbar}\Big(|p_0|^2 + \hbar^2 + (X, P)^2\Big)(X, P)^2\Big) \ . \end{aligned} \tag{3.77}$$

**iii**) *There exist a universal constant $C$ such that for all $\hbar, t, z, z_0$, one has*

$$\mathcal{D}(t, z, z_0) \ \leq \ \frac{\hbar d^2}{t} + C\Big(1 + \frac{1}{\hbar t}\Big)\ \Big(|p| + |p_0| + \frac{d}{t}\Big)^2 \tag{3.78}$$

*where $d = \mathrm{dist}_X(x, x_0)$ is the Riemannian distance between $x$ and $x_0$.*

*If $K_0, K$ are two disjoint compact subsets of $\Sigma$, there exist $C_\hbar > 0$ such that for $z_0 \in K_0$, $z \in K$, $t \in ]0, 1]$ one has*

$$\mathcal{D}(t, z, z_0) \ \geq \ C_\hbar / t \ . \tag{3.79}$$

**Proof:** **i**) The action $\mathcal{I}_t(z_0, z)$ being non-negative, one has $\mathcal{D}(t, z, z_0) \geq 0$. On a minimizing sequence $x_k(s)$ of the action, the $L^2$ norm of $p_k$ and $b_k = \frac{Dp_k}{Ds}$ are bounded by $\sqrt{\frac{2}{\hbar}(1 + \mathcal{D}(t, z, z_0))}$ and $\sqrt{2\hbar(1 + \mathcal{D}(t, z, z_0))}$; Using $\frac{d}{ds}|p|^2/2 = (p|b)$, we thus get that

$$|p_k(s)|^2_{L^\infty} \leq \ |p_0|^2 + 4 + 4\,\mathcal{D}(t, z, z_0)$$

thus all the sequence remains in a compact subset of $\Sigma$, the sequence $b_k$ is bounded in $L^2$, and therefore a sub-sequence will converge to a solution $z_{\mathrm{opt}}(s)$ of the differential equation (3.18). By the previous estimates, the sup-norm of the velocity on this optimal trajectory satisfies

$$|p_{\mathrm{opt}}(s)|^2_{L^\infty} \leq \ |p_0|^2 + 4\,\mathcal{D}(t, z, z_0) \ . \tag{3.80}$$

The continuity of $t \to \mathcal{D}(t, z, z_0)$ on $\mathbb{R}_+^*$ is now obvious since for $t \in K$, with $K$ compact subset of $\mathbb{R}_+^*$, the data of an optimal trajectory connecting $z_0$ to $z$ remains in a compact subset of $T^*_{z_0}X$.

**ii**) Set $\varepsilon = t/t_0$, $z' = \varepsilon z$, $z'_0 = \varepsilon z_0$. Using (3.64) and (3.74), we are reduce to prove that for $t_0$ small, there exist $0 \le \delta_0 = \delta_1 \le \delta_2$, such that for all $\varepsilon \in ]0,1]$, one has

$$\psi_\varepsilon(t_0, z', z'_0) \;=\; \varepsilon\,\mathcal{D}(\hbar\varepsilon, t_0, z', z'_0) \tag{3.81}$$

for $\mathrm{dist}(x, x_0) \le \delta_0$, $t_0|p'| \le \delta_2$, $t_0|p'_0| \le \delta_1$. By lemma 3.12 and i) one has

$$\psi_\varepsilon(t_0, z', z'_0) \;\ge\; \varepsilon\,\mathcal{D}(\hbar\varepsilon, t_0, z', z'_0)\ . \tag{3.82}$$

Set as before $\hbar' = \hbar\varepsilon$. Choose $t_0$, $\delta_3 \le \delta_4$, $\delta_5$ as in lemma 3.12 such that for all $\varepsilon \in [0,1]$, and $z', z'_0$ such that $\mathrm{dist}(z', \Phi^\varepsilon_{t_0,z'_0}(0,0)) \le \delta_4$, $\mathrm{dist}(x, x_0) \le t_0\delta_3$, $|p'_0| \le \delta_3$, the equation

$$\Phi^\varepsilon_{t_0,z'_0}(b'_0, b'_1) \;=\; z'$$

admits in the set $|b'_0| + |b'_1| \le \delta_5$ a unique solution. This solution satisfies for some $C$ independent of $\varepsilon$

$$|b'_0| + |b'_1| \;\le\; C\,\mathrm{dist}(z', \Phi^\varepsilon_{t_0,z'_0}(0,0))\ .$$

Set $\delta_0 = t_0\beta\delta_3$, $\delta_1 = t_0\beta\delta_3$, $\delta_2 = t_0\beta\delta_4$ with $\beta > 0$ small. We claim that if $b'_0, b'_1$ are the data of an optimal trajectory $x'_{\mathrm{opt}}$ connecting $z'_0 = (x_0, p'_0)$ to $z' = (x, p')$ with $\mathrm{dist}(x, x_0) \le t_0\beta\delta_3$, $|p'_0| \le \beta\delta_3$, $|p'|\beta \le \delta_4$ for the rescaled action

$$\varepsilon\int_0^{t_0} \frac{|b'|^2}{2\hbar'} + \frac{\hbar'|p'|^2}{2} \;=\; \int_0^{t_0} \frac{|b'|^2}{2\hbar} + \frac{\varepsilon^2\hbar|p'|^2}{2}$$

then, there exist a constant $C$, such that for $\beta$ small enough, one has

$$|b'_0| + |b'_1| \;\le\; \delta_5\ . \tag{3.83}$$

Clearly, this fact will imply (3.81). Set $z' = Z'_*(t_0, z'_0) + (t_0X, P)$. Then using (3.82) and (3.66) we get

$$\begin{aligned}\varepsilon\mathcal{D}(\hbar', t_0, z', z'_0) \;&\le \varepsilon\,\gamma'(t_0, z'_0) + \frac{2}{\hbar t_0}(3X^2 - 3PX + P^2)\\ &\quad+\; \mathcal{O}\Big(\frac{t_0}{\hbar}\Big(|p'_0|^2 + \hbar'^2 + (X,P)^2\Big)(X,P)^2\Big)\ .\end{aligned} \tag{3.84}$$

From (3.58) one has

$$|X| \le C\Big(\frac{\mathrm{dist}(x, x_0)}{t_0} + |p'_0|\Big) \le C\beta\delta_3\ , \qquad |P| \le C\big(|p'| + |p'_0|\big) \le C\beta\delta_4$$

and thus we get using (3.59)

$$\varepsilon\,\mathcal{D}(\hbar', t_0, z', z_0') \;\le\; C\beta^2\delta_4^2\left(\frac{1}{\hbar t_0} + \varepsilon^2\hbar t_0\right) \;=\; M\ . \tag{3.85}$$

We thus get $|b'|_{L^2(0,t_0)} \le \sqrt{2\hbar M}$, and from $\frac{d}{ds}|p'|^2 = 2(p'|b')$ we get for $s \in [0, t_0]$ the $L^\infty$ bound on the velocity

$$|p'(s)| \;\le\; \sqrt{s}\,|b'|_{L^2} + \sqrt{s|b'|^2_{L^2} + |p_0'|^2} \;\le\; C\beta\delta_4(1 + \hbar' t_0)\ . \tag{3.86}$$

Using the fact that $z'(s)$ satisfies the equation (3.39), we get with $b_1'(s) = \frac{Db'}{Ds}$

$$\begin{aligned} |b'|_{L^2} \;&\le\; C\,\frac{\beta\delta_4}{\sqrt{t_0}}(1 + \hbar' t_0) \\ |\frac{Db_1'}{Ds}|_{L^2} \;&\le\; C|b'|_{L^2}\big(\hbar'^2 + \delta_4^2(1 + \hbar' t_0)^2\big) \end{aligned} \tag{3.87}$$

and therefore (3.83) holds true if $\beta$ is small enough. The formula (3.77) follows easily from (3.66).

**iii**) Let $z = (x, p), z_0 = (x_0, p_0)$. We split $[0, t]$ in $[0, t/4] \cup [t/4, 3t/4] \cup [3t/4, t]$, and we define a trajectory $x(s)$ in the following way: let $(x_0, p_0'), (x, p')$ such $p_0', p'$ are the data of a geodesic curve $s \in [t/4, 3t/4] \to x_0(s)$ connecting $x_0$ to $x$ in time $t/2$; one has $|p_0'| = |p'| = \frac{2d}{t}$. Take $x(s)$ such that $x(s) = x_0(s)$ for $s \in [t/4, 3t/4]$. Then the contribution of the interval $[t/4, 3t/4]$ to the action is equal to $\frac{\hbar d^2}{t}$. Thus we have reduce the problem in the geometric situation $z_0 = (x_0, 0)$, $z = (x_0, q)$, $|q| \le (|p| + |p_0| + \frac{2d}{t})$ where the trajectory $s \in [0, t] \to x(s)$ is a loop at $x_0$ starting with 0 velocity. Let $x_0(s)$ be the geodesic curve starting at $x_0$ with velocity $v = g^{-1}(q)$, and set $x(s) = x_0(tg(s/t))$ with $g(0) = g(1) = 0$, $g'(0) = 0$, $g'(1) = 1$. Then the action on the trajectory $x(s)$ is equal to

$$\frac{|q|^2}{2}\int_0^1 \frac{g''^2}{t\hbar} + t\hbar g'^2\, d\alpha \tag{3.88}$$

and we conclude that (3.78) holds true by

$$\min_g \int_0^1 \frac{g''^2}{N} + Ng'^2\, d\alpha \;=\; \frac{NchN - shN}{NshN - 2(chN - 1)} \;\le\; C\left(1 + \frac{1}{N}\right) \tag{3.89}$$

these last estimate being a particular case of the calculus for the Euclidean case given in the subsection 3.6.

Finally, let $z = (x, p) \in K$, $z_0 = (x_0, p_0) \in K_0$. From

$$\mathcal{I}_t(z_0, z) \;\geq\; \int_0^t \hbar |p|^2/2 \;\geq\; \frac{\hbar \operatorname{dist}(x, x_0)^2}{2t}$$

we may assume that $\pi(K) \cup \pi(K_0) \subset X$ is contained in a small neighborhood $U$ of a point $y_0 \in X$, that in geodesic coordinates centered at $y_0$, one has $|p - p_0| \geq c > 0$, and that the optimal trajectory $x(s)$ connecting $z_0$ to $z$ stays in $U$. Then from (3.80) we get

$$|p - p_0| \;\leq\; \sqrt{t}\, \|b\|_{L^2} + Ct\Big(|p_0|^2 + 4\mathcal{D}(t, z, z_0)\Big)$$

and (3.79) is obvious from

$$\mathcal{D}(t, z, z_0) \;\geq\; \frac{\|b\|_{L^2}^2}{2\hbar} \;.$$

The proof of theorem 3.14 is complete. ∎

Let $X \times_X X \simeq X$ be the subset of $\Sigma \times \Sigma$

$$X \times_X X \;=\; \Big\{(z, z);\; z = (x, 0)\Big\} \;. \tag{3.90}$$

**Definition 3.15.** For $(z, z_0) \in \Sigma \times \Sigma$, we define the subset $\mathcal{P}(z, z_0)$ of $\mathbb{R} \times T^*_{z_0}\Sigma$ by

$$\begin{aligned} \mathcal{P}(z, z_0) \;&=\; \Big\{(t, \zeta_0);\; t > 0,\; Z(t, z_0, \zeta_0) = z\Big\} \quad \text{if } \;(z, z_0) \notin X \times_X X \\ \mathcal{P}(z_0, z_0) \;&=\; \Big\{(t, \zeta_0);\; t \geq 0,\; Z(t, z_0, \zeta_0) = z_0\Big\} \quad \text{if } \;(z_0, z_0) \in X \times_X X \;. \end{aligned} \tag{3.91}$$

For $(z_0, \zeta_0) \in T^*_{z_0}\Sigma$, set

$$I(t, z_0, \zeta_0) \;=\; \int_0^t \frac{\hbar}{2}\big(|\zeta^V|^2 + |p|^2\big)\,\big(Z(s, z_0, \zeta_0),\, \Xi(s, z_0, \zeta_0)\big)\; ds \tag{3.92}$$

so that $I(t, z_0, \zeta_0)$ is the value of the action $\mathcal{I}_t(z_0, z)$ on the trajectory starting at $(z_0, \zeta_0)$. We define the subset $\mathcal{P}_0(z, z_0)$ of $\mathcal{P}(z, z_0)$ by

$$\mathcal{P}_0(z, z_0) \;=\; \Big\{(t, \zeta_0) \in \mathcal{P}(z, z_0),\; I(t, z_0, \zeta_0) = \mathcal{D}(t, z, z_0)\Big\} \;. \tag{3.93}$$

We define the subset $\mathcal{W}(z, z_0)$ of $\mathbb{R} \times \mathbb{R}^*_+$ by

$$\mathcal{W}(z, z_0) = \Big\{(\mu, t) \in \mathbb{R} \times \mathbb{R}^*_+,\; \exists (t, \zeta_0) \in \mathcal{P}_0(z, z_0) \text{ s.t } \mu = H(z_0, \zeta_0)\Big\} \,. \;\square \tag{3.94}$$

The sets $\mathcal{P}(z, z_0) \cap (t > 0)$ are closed in $\mathbb{R}_+^* \times T^*_{z_0}\Sigma$. By theorem 3.14 and his proof, for any $t_0 > 0$, $\mathcal{P}_0(z, z_0) \cap (t = t_0)$ is a compact non empty subset of $T^*_{z_0}\Sigma$, and is equal to the subset of $\mathcal{P}(z, z_0) \cap (t = t_0)$ where the continuous function $\zeta_0 \to I(t_0, z_0, \zeta_0)$ reach is minimum. For any $0 < t_0 \le t_1$, $\mathcal{P}_0(z, z_0) \cap \{t \in [t_0, t_1]\}$ is compact. The sets $\mathcal{W}(z, z_0) \cap (t > 0)$ are thus closed, and for any $0 < t_0 \le t_1$, $\mathcal{W}(z, z_0) \cap \{t \in [t_0, t_1]\}$ is compact. Moreover, if $t \in ]0, t_0]$, and $\mathrm{dist}(x, x_0) + |\hbar t|^2 + |tp| + |tp_0| \le \delta_1$, we get using lemma 3.7, that for $(t, \zeta_0) \in \mathcal{P}(z, z_0)$ and $\hbar t^2 |\zeta_0^V| + \hbar t^3 |\zeta_0^H - \hbar p_0| \le \delta_0$ one has

$$(t, \zeta_0) \in \mathcal{P}(z, z_0) \quad \Leftrightarrow \quad \zeta_0 = \zeta_0(t, z, z_0) \tag{3.95}$$

with

$$\begin{pmatrix} \zeta_0^H(t, z, z_0) \\ \zeta_0^V(t, z, z_0) \end{pmatrix} = \begin{pmatrix} \hbar p_0 - \frac{1}{\hbar}\mathcal{B}_1(t, z, z_0) \\ \frac{1}{\hbar} g^{-1} \mathcal{B}_0(t, z, z_0) \end{pmatrix} \tag{3.96}$$

and thus from the proof of ii) in theorem 3.14, we get

$$(t, \zeta_0) \in \mathcal{P}_0(z, z_0) \ \text{ and } \ (t, z, z_0) \in U_{t_0,\delta} \quad \Leftrightarrow \quad \zeta_0 = \zeta_0(t, z, z_0)\ . \tag{3.97}$$

### 3.4. Stationary eikonal

Let $\mu > 0$. In this subsection, we shall study the following Hamilton–Jacobi equation for the phase function $\Phi$

$$\frac{\hbar}{2}\big((\partial_p \Phi)^2 - |p|^2\big) + \big\{|p|^2/2, \Phi\big\} \ = \mu\ . \tag{3.98}$$

For $\mu > 0$ and $z_0 \in \Sigma$, set

$$T^*_{z_0,\mu} = T^*_{z_0}\Sigma \cap \big\{H(z, \zeta) = \mu\big\}\ .$$

Observe that $\mu \neq 0$ implies for $(z_0, \zeta_0) \in T^*_{z_0,\mu}$

$$\frac{dZ}{ds}(s = 0, z_0, \zeta_0) \ = \ \frac{\partial H}{\partial \zeta}(z_0, \zeta_0) \ \neq \ 0$$

so the Hamiltonian flow of $H$ restricted to $H(z, \zeta) = \mu$ is transversal to $T^*_{z_0}\Sigma$. Observe also that the set $H(z, \zeta) = \mu$ is not compact. Let $\Lambda_{z_0,\mu}$ be the imbedded subvariety of $T^*\Sigma$ equal to the union of maximal Hamiltonian curves $(Z(s, z_0, \zeta_0), \Xi(s, z_0, \zeta_0))$, with $(z_0, \zeta_0) \in T^*_{z_0,\mu}$. One has $\dim(\Lambda_{z_0,\mu}) = \dim(\Sigma)$ and $\Lambda_{z_0,\mu}$ is isotropic, hence is a Lagrangian imbedded subvariety of $T^*\Sigma$.

Let $G(t,z_0,\zeta_0)$ be the function on $\mathbb{R}_+\times T^*\Sigma$ defined by

$$G(t,z_0,\zeta_0) \;=\int_\gamma \zeta\,dz \tag{3.99}$$

where $\gamma$ is the path $s\in[0,t]\to(Z(s,z_0,\zeta_0),\Xi(s,z_0,\zeta_0))\in T^*\Sigma$. If $H=H_0+H_1+H_2$ is the decomposition of $H$ in homogeneous functions in $\zeta$ of degrees $0,1,2$, one has

$$\int_\gamma \zeta\,dz \;=\int_0^t 0H_0+1H_1+2H_2\,ds \;=\int_0^t H+H_2-H_0\,ds \;=\; \mu\,t+\frac{\hbar}{2}\int_0^t\big(|\zeta^V|^2+|p|^2\big)\,ds\,.$$

Thus we get the identity

$$G(t,z_0,\zeta_0) \;=\; I(t,z_0,\zeta_0)+tH(z_0,\zeta_0)\;. \tag{3.100}$$

Thus, for $\mu>0$ and $\zeta_0\in T^*_{z_0,\mu}$, the function $t\to G(t,z_0,\zeta_0)$ is strictly increasing, and one has

$$\frac{d}{dt}G(t,z_0,\zeta_0) \;=\; \mu+\frac{\hbar}{2}\big(|\zeta^V|^2+|p|^2\big)\;\geq\;\mu\;. \tag{3.101}$$

In particular, the inverse of the map $(\zeta_0,t)\to(\zeta_0,G(t,z_0,\zeta_0))$ defines for any $\mu>0$ an isomorphism

$$T^*_{z_0,\mu}\times\mathbb{R}\;\simeq\;\Lambda_{z_0,\mu}\;.$$

**Definition 3.16.** For $\mu>0$, we shall denote by $\Phi_\mu(z,z_0)$ the function on $\Sigma\times\Sigma$

$$\Phi_\mu(z,z_0)=\min\Big(G(t,z_0,\zeta_0);\;H(z_0,\zeta_0)=\mu,\;(t,\zeta_0)\in\mathcal{P}(z,z_0)\Big)\;.\;\square \tag{3.102}$$

We shall see in theorem 3.17 that $\Phi_\mu(z,z_0)$ is finite, i.e the projection of $\Lambda_{z_0,\mu}$ on $\Sigma$ is surjective for any $\mu>0$. Remark that the function $\Phi_\mu(z,z_0)$ depends on the constant $\hbar$, and satisfies the following scaling invariance

$$\Phi_{\varepsilon^3\mu}(\hbar\varepsilon,\varepsilon z,\varepsilon z_0)\;=\;\varepsilon^2\,\Phi_\mu(\hbar,z,z_0)\;. \tag{3.103}$$

In order to study the function $\Phi_\mu$, it will be convenient to reparametrized the set $\mathcal{P}(z,z_0)$ for $(z,z_0)\notin X\times_X X$ in the following way.

If $s\in[0,t]\to x(s)$ is a trajectory on $X$, set $s=t\alpha$. Then on the trajectory $\alpha\in[0,1]\to x(t\alpha)$, and with the notation

$$v'=\frac{dx}{d\alpha}\;,\qquad p'=g(v')\;,\qquad b'=\frac{Dp'}{D\alpha}\;,\qquad b'_1=\frac{Db'}{D\alpha}$$

the action is

$$\mathcal{I}_t = t^{-3}\int_0^1 \frac{|b'|^2}{2\hbar} + \frac{\hbar t^2 |p'|^2}{2}\, d\alpha \tag{3.104}$$

and the equation satisfied by critical trajectories is

$$\frac{Db'_1}{D\alpha} = \hbar^2 t^2 b' + \langle b' | R(v',.)v' \rangle \ , \qquad v' = \frac{dx}{d\alpha} \ . \tag{3.105}$$

Take as new parameters $(t, b', b'_1)$ such that

$$t^3 \zeta_0^H = \hbar t^2 p' - \frac{b'_1}{\hbar} \ , \qquad t^2 \zeta_0^V = \frac{g^{-1}(b')}{\hbar} \ .$$

Then one has

(3.106)

$$\mathcal{P}(z,z_0) = \left\{ (t,b',b'_1);\ t>0,\ \left(x, \frac{dx}{d\alpha}\right)\Big|_{\alpha=0} = (x_0, tp_0),\ \left(x, \frac{dx}{d\alpha}\right)\Big|_{\alpha=1} = (x, tp) \right\}$$

where $(x_0, tp_0, b', b'_1)$ are the initial conditions for the equation (3.105). In these parametrization, the function $H$ is given by

$$t^4 H = \frac{b'^2}{2\hbar} + \frac{\hbar t^2 |p'|^2}{2} - \frac{1}{\hbar}(p'|b'_1) \ . \tag{3.107}$$

**Theorem 3.17.** *For any $\mu > 0$ and $(z,z_0) \in \Sigma \times \Sigma$, $\Phi_\mu(z,z_0)$ is finite, non-negative, and the following identities hold true*

$$\begin{aligned} &\Phi_\mu(z,z_0) = \min_{t>0}\big(\mathcal{D}(t,z,z_0) + \mu t\big) \quad \text{if } (z,z_0) \notin X \times_X X \\ &\Phi_\mu(z_0,z_0) = 0 \quad \text{if } (z_0,z_0) \in X \times_X X \ . \end{aligned} \tag{3.108}$$

**Proof:** In the case $(z_0,z_0) \in X \times_X X$, one has $(0,\zeta_0) \in \mathcal{P}(z_0,z_0)$, for any $\zeta_0$, so $\Phi_\mu(z_0,z_0) = 0$. We next assume $(z,z_0) \notin X \times_X X$. Set

$$F_\mu(z,z_0) = \min_{t>0}\big(\mathcal{D}(t,z,z_0) + \mu t\big) \ . \tag{3.109}$$

Using (3.97) we get

$$F_\mu(z,z_0) = \min\Big( I(t,z_0,\zeta_0) + \mu t;\ (t,\zeta_0) \in \mathcal{P}(z,z_0) \Big) \ . \tag{3.110}$$

From (3.79) and (3.77) we get $\mathcal{D}(t,z,z_0) \geq Ct^{-1}$ for $t$ closed to 0, due to $(z,z_0) \notin X \times_X X$. Thus, the function $\mathcal{D}(t,z,z_0)$ being continuous in time, the minimum in (3.109) is reached on a compact non empty subset $\{t_{min}(\mu, z, z_0)\}$ of $]0, \infty[$, and for $\nu > 0$, $\mu > 0$ one has always

$$\max\Big(t \in \{t_{min}(\mu+\nu, z, z_0)\}\Big) \leq \min\Big(t \in \{t_{min}(\mu, z, z_0)\}\Big) .$$

One has obviously

$$\begin{aligned} F_\mu(z, z_0) &= \min_{t>0, x(\alpha)} \mathcal{F} \\ \mathcal{F} &= \mathcal{I}_t + \mu t \end{aligned} \tag{3.111}$$

where $\mathcal{I}_t$ is the functional given in (3.104) and $\alpha \in [0,1] \to x(\alpha)$ is any trajectory such that the constraints in (3.106) are satisfied. Set $\beta(\alpha) = \alpha + \varepsilon\delta(\alpha)$, with a function $\delta$ such that $\delta(0) = \delta(1) = 0$; $\delta'(0) = \delta'(1) = 1$. Then, using (3.104) and (3.107) we get that the derivative of the functional $\mathcal{F}$ at $\varepsilon = 0$ along the path

$$\varepsilon \ \rightarrow \ \Big(t_\varepsilon = (1{+}\varepsilon)t, \ \ x_\varepsilon(\alpha) = x\big(\alpha{+}\varepsilon\,\delta(\alpha)\big)\Big)$$

is equal to

$$\mu t^4 + t^4 \int_0^1 (\delta'{-}1) H \, d\alpha \, + \, \frac{1}{\hbar} \int_0^1 \Big[(\delta'{-}1)\big(|b'|^2 + (p'|b_1')\big) + \delta''(p'|b')\Big] \, d\alpha \ . \tag{3.112}$$

By integration by part, we get that the last term in (3.112) is equal to 0. Moreover, we already know by theorem 3.14 that the minimum of the functional $\mathcal{F}$ is reached at some $(t, x(\alpha))$ where $x(\alpha)$ is a solution of the differential equation (3.105). The Hamiltonian $H$ is constant on this trajectory, and therefore from (3.112), we get that at the minimum, one has

$$\mu = H(z_0, \zeta_0^{min}), \quad t \in \{t_{min}(\mu, z, z_0)\}, \quad (t, \zeta_0^{min}) \in \mathcal{P}_0(z, z_0) \ . \tag{3.113}$$

Thus $\Phi_\mu(z, z_0)$ is finite, and using (3.110) and (3.100) we get

$$\begin{aligned} F_\mu(z, z_0) &= \min\Big(I(t, z_0, \zeta_0) + t\mu; \ H(z_0, \zeta_0) = \mu, \ (t, \zeta_0) \in \mathcal{P}(z, z_0)\Big) \\ &= \min\Big(G(t, z_0, \zeta_0), \ H(z_0, \zeta_0) = \mu, \ (t, \zeta_0) \in \mathcal{P}(z, z_0)\Big) \end{aligned} \tag{3.114}$$

and thus

$$\Phi_\mu(z, z_0) = F_\mu(z, z_0) \ . \ \blacksquare \tag{3.115}$$

### 3.5. The Laplace operator case

Let $-\Delta_X$ be a Laplacian operator on $X$, that is a second order elliptic operator acting on sections of a fiber bundle over $X$, with scalar principal part equal to $-\Sigma g^{i,j}\frac{\partial^2}{\partial x_i \partial x_j}$. The action associated to $\frac{-\Delta_X}{2}$ is

$$\int_0^T |v|^2/2 \; dt \; .$$

The Legendre transform of $L(x,v) = |v|^2/2$ is $H(x,\xi) = |\xi|^2/2$, and the eikonal equations one gets on the weight function $\Phi(x)$ in order to evaluate the decay outside the diagonal of the kernel of the resolvant $\frac{-\Delta_X}{2} + \mu$ when $\mu \to +\infty$, or on the great deviation function $\Psi(t,x)$ in order to evaluate the decay of the heat kernel outside the diagonal are precisely

$$H(x, \partial_x \Phi) = \mu \; , \qquad \frac{\partial \Psi}{\partial t} + H(x, \partial_x \Psi) = 0 \; .$$

The solutions of these eikonal equations with data at a given point $x_0 \in X$ are equal to

$$\Phi(x) = \sqrt{2\mu}\; d_X(x,x_0) \; , \qquad \Psi(t,x) = \frac{d_X^2(x,x_0)}{2t}$$

where $d_X(x,x_0)$ is the Riemannian distance function on $X \times X$.

So, for any value of $t > 0$, $\mathcal{D}(t,z,z_0)$ play for GFK operators, the role of the square of the Riemannian distance $d_X$ for Laplacian operators. However, there are big differences. First, $\mathcal{D}(t,z_0,z_0)$ is not equal to 0, except in the case $z_0 = (x_0, 0)$. Moreover, for $z = (x,p) \in \Sigma$, and $-z = (x,-p)$, the symmetry relation $d_X(x,y) = d_X(y,x)$ has to be replaced by

$$\mathcal{D}(t,-z_1,-z_0) \;=\; \mathcal{D}(t,z_0,z_1) \; .$$

This is due to the fact that a GFK operator $A$ is not self adjoint, but the symmetry $p \to -p$ exchange $A$ and $A^*$ modulo lower order terms. Finally, the dependence in $t$ of $\mathcal{D}(t,z,z_0)$ is non trivial, since the geometric support of an optimal trajectory connecting $z_0$ to $z$ in time $t$ depends strongly on $t$, even in the Euclidean case, as we will see in the next subsection.

### 3.6. The Euclidean case

Let us make the above calculations explicitly in the simple case of the constant metric on the Euclidean space $\mathbb{R}^n$. The Hamiltonian function $H(x,p,\xi,\eta)$ is

$$H(x,p;\xi,\eta) \;=\; \frac{\hbar}{2}(\eta^2 - p^2) + p\,\xi$$

and the associated flow on $T^*T^*(\mathbb{R}^n)$ is

(3.116)

$$x(s) = x_0 + s\frac{\xi_0}{\hbar} + \Big(p_0 - \frac{\xi_0}{\hbar}\Big)\frac{sh(\hbar s)}{\hbar} + \eta_0\frac{ch(\hbar s) - 1}{\hbar}\,, \quad \xi(s) = \xi_0$$
$$p(s) = \frac{\xi_0}{\hbar} + \Big(p_0 - \frac{\xi_0}{\hbar}\Big)\, ch(\hbar s) + \eta_0\, sh(\hbar s)\,, \quad \eta(s) = \eta_0\, ch(\hbar s) + \Big(p_0 - \frac{\xi_0}{\hbar}\Big)\, sh(\hbar s)\ .$$

Set $N = \hbar t$, so that $N$ is the number of cycles between 0 and $t$.
Let $u(N), v(N), w(N), \psi(N)$ be the functions

$$\begin{aligned} u(N) &= sh(N) - N \\ v(N) &= ch(N) - 1 \\ w(N) &= sh(N) \\ \psi(N) &= v^2(N) - u(N)\, w(N) \ = \ N sh(N) - 2\big(ch(N) - 1\big)\ . \end{aligned} \tag{3.117}$$

We may assume $x_0 = 0$. For $z = (x, p)$, $z_0 = (0, p_0)$, set $p' = tp$, $p'_0 = tp_0$. Then in the parametrization (3.104), $b'(\alpha)$ satisfies the differential equation $\frac{d^2b'}{d\alpha^2} = N^2 b'$, and (3.116) becomes

$$\begin{aligned} x(\alpha) &= p'_0\alpha + \frac{b'_0}{N^2}\, v(N\alpha) + \frac{b'_1}{N^3}\, u(N\alpha) \\ p'(\alpha) &= p'_0 + \frac{b'_0}{N}\, w(N\alpha) + \frac{b'_1}{N^2}\, v(N\alpha) \\ b'(\alpha) &= b'_0\, ch(N\alpha) + \frac{b'_1}{N}\, sh(N\alpha) \\ b'_1(\alpha) &= N b'_0\, sh(N\alpha) + b'_1\, ch(N\alpha)\ . \end{aligned} \tag{3.118}$$

We thus get that $(t, b'_0, b'_1) \in \mathcal{P}(z, z_0)$ is equivalent to

$$\begin{pmatrix} b'_0 \\ b'_1 \end{pmatrix} = \frac{N^4}{\psi(N)} \begin{pmatrix} v(N)/N^2 & -u(N)/N^3 \\ -w(N)/N & v(N)/N^2 \end{pmatrix} \begin{pmatrix} x - p'_0 \\ p' - p'_0 \end{pmatrix}. \tag{3.119}$$

From (3.118) and (3.119), we get that the optimal trajectory $s \in [0, t] \to x_t(s)$ connecting $z_0$ to $z$ in time $t$ is given, with $s = \alpha\, t$, $N = \hbar\, t$, by the following formula

$$\begin{aligned} x_t(t\alpha) &= t\, p_0\, \alpha + (x - tp_0)\, F(N, N\alpha) + (p - p_0)\, G(N, N\alpha) \\ F(N, N\alpha) &= \frac{v(N)\, v(N\alpha) - w(N)\, u(N\alpha)}{\psi(N)} \\ G(N, N\alpha) &= \frac{v(N)\, u(N\alpha) - u(N)\, v(N\alpha)}{\hbar\, \psi(N)}\ . \end{aligned} \tag{3.120}$$

Observe that when $t \to 0$, the curve (3.120) converge to the geodesic curve connecting $x_0$ to $x$ with the parametrization $\alpha \in [0,1] \to y(\alpha)$

$$y(\alpha) \;=\; x_0 + (3\alpha^2 - 2\alpha^3)\,(x - x_0) \tag{3.121}$$

and thus the direction of the velocity at the end points is lost. For $\alpha \in ]0,1[$, one has

$$\frac{d}{ds} x_t(s) \;=\; 6\,\frac{x - x_0}{t}\,(\alpha - \alpha^2) + \mathcal{O}(1)\ .$$

In particular, we see that the geometric support of the optimal trajectory $x_t$ depends on $t$, and that for $t$ small and $p$ or $p_0$ not closed to $\frac{x-x_0}{t}$, which is the initial velocity of the geodesic connecting $x_0$ to $x$ in time $t$, the minimal action is concentrated at the end points of the interval $[0,t]$, where the optimal trajectory connect the end points velocity to $\frac{x-x_0}{t}$.

From (3.118) and (3.119), we get

$$\begin{aligned} Z_*(t,z_0) \;&=\; \left(x_0 + \frac{p_0}{\hbar}\frac{sh(N)}{ch(N)},\, \frac{p_0}{ch(N)}\right) \\ \gamma(t,z_0) \;&=\; \frac{|p_0|^2}{2}\,\frac{sh(N)}{ch(N)}\ . \end{aligned} \tag{3.122}$$

From (3.104) one gets by integration by part

$$\begin{aligned} t^2\mathcal{D}(t,z,z_0) \;&=\; \frac{1}{2N}\int_0^1 |b'|^2\,d\alpha \,+\, \frac{N}{2}\int_0^1 |p'|^2\,d\alpha \\ &=\; \frac{1}{2N}[b'p']_0^1 - \frac{1}{2N}[b_1'x]_0^1 + \frac{N}{2}[xp']_0^1\ . \end{aligned} \tag{3.123}$$

In coordinates centered at $Z_*(t,z_0)$

$$(x,p) \;=\; Z_*(t,z_0) + (y,q)$$

we get from (3.119) (3.122) and (3.123) the value of the great deviation function

$$\begin{aligned} \mathcal{D}(t,z,z_0) \;&=\; \frac{|p_0|^2}{2}\,\frac{sh(N)}{ch(N)} \\ &+\, \frac{1}{2\,\psi(N)}\Big[sh(N)\,|\hbar y|^2 - 2\big(ch(N){-}1\big)\,(\hbar y|q) + \big(Nch(N){-}sh(N)\big)\,|q|^2\Big]\ . \end{aligned} \tag{3.124}$$

In the limit $\hbar \to 0$ one has

$$\begin{aligned} &\lim_{\hbar\to 0}\begin{pmatrix} Z_*(t,z_0) \\ \gamma(t,z_0)\end{pmatrix} = \begin{pmatrix} (x_0 + tp_0, p_0) \\ 0 \end{pmatrix} \\ &\mathcal{D}(t,z,z_0) \;\simeq_{\hbar\to 0}\; \frac{2}{\hbar t}\left(3\left|\frac{y}{t}\right|^2 - 3\Big(\frac{y}{t}|q\Big) + |q|^2\right) \end{aligned} \tag{3.125}$$

so we observe the convergence to the geodesic flow on $T^*X$, and in the limit $\hbar \to +\infty$ we get

$$\lim_{\hbar\to+\infty} \begin{pmatrix} Z_*(t,z_0) \\ \gamma(t,z_0) \end{pmatrix} = \begin{pmatrix} (x_0,0) \\ \frac{|p_0|^2}{2} \end{pmatrix}$$

$$\mathcal{D}(t,z,z_0) \simeq_{\hbar\to+\infty} \frac{\hbar d_X^2(x,x_0)}{2t} \tag{3.126}$$

so we observe the convergence to the Gaussian equilibrium.

The large time behavior of the great deviation function is given by

$$\lim_{t\to+\infty} \mathcal{D}(t,z,z_0) = \frac{|p|^2+|p_0|^2}{2} \tag{3.127}$$

which is compatible with the estimate (3.78) of theorem 3.14.

## 4 – The heat kernel

By theorem 2.4 applied with the phase function $\Phi=0$ and $s=0$, for $\mathrm{Re}(\lambda)<-c_0$, the resolvant $(A-\lambda)^{-1}$ exists as a uniformly bounded in $\lambda$ operator on $L^2$. Therefore, the Cauchy problem for the heat equation

$$(\partial_t + A)u = 0 \quad \text{in } \ t>0$$

$$u|_{t=0} = v \ \in \ L^2 \tag{4.1}$$

is well posed, and its solution $u(t) = e^{-tA}v$ is given by the Fourier integral, with $c > c_0$

$$u(t) = \frac{1}{2\pi}\int_{\mathrm{Im}(\tau)=-c} e^{i\tau t}(A+i\tau)^{-1}\, v\; d\tau\ . \tag{4.2}$$

We can rewrite formula (4.2) on the operator form

$$e^{-tA} = \frac{1}{2i\pi}\int_{\mathrm{Re}(\lambda)=-c} e^{-t\lambda}(A-\lambda)^{-1}\; d\lambda\ . \tag{4.3}$$

We shall denote by $P(t,z,z_0)$ the distribution kernel of $e^{-tA}$, so that we have

$$u(t,z) = \int P(t,z,z')\, v(z')\; dz' \tag{4.4}$$

where $dz' = dx'dp'$ is the canonical volume form on $\Sigma = T^*X$. By theorem 2.4 applied with the phase function $\Phi = 0$ and any $s$, and using

$$e^{-tA} = \frac{t^{-N} N!}{2i\pi} \int_{\mathrm{Re}(\lambda)=-c} e^{-t\lambda}(A-\lambda)^{-1-N}\, d\lambda$$

one gets from the definition of the norms $\|.\|_{\lambda,s}$, using in particular there dependence in $\lambda$, that $P(t,z,z')$ is smooth in $t>0$, and that all its derivatives in $t$ belong to the Schwartz class in $z,z'$. Moreover, computing $\frac{d}{dt}\|u(t)\|_{L^2}^2$ for $v \in \mathcal{S}$ which is dense in the domain of $A$, one gets that there exist $C,c>0$ such that for all $t>0$ one has

$$\|e^{-tA}\|_{L^2} \le C\, e^{ct} \ .$$

One recover for $\mathrm{Re}(\lambda) < -c$ the resolvant $(A-\lambda)^{-1}$ from the heat kernel by the formula

$$(A-\lambda)^{-1} = \int_0^\infty e^{-tA+t\lambda}\, dt \ . \tag{4.5}$$

Let $z_0 = (x_0, p_0) \in \Sigma$. In order to describe the asymptotic of the heat kernel $P(t,z,z_0)$ with $z$ closed to $z_0$ and $t \to 0$, we will work in geodesics coordinates centered at $x_0$. Set

$$Z_*(t,z_0) = \big(x_{z_0}(t), p_{z_0}(t)\big)$$

where $Z_*(t,z_0)$ is given by formula (3.50) of lemma 3.10. Let us introduce the rescaled coordinates $(y,q)$ centered at $z_0$

$$z = (x,p) = \big(x_{z_0}(t) + ty,\ p_{z_0}(t) + q\big) \ . \tag{4.6}$$

Observe that from (3.77), the large deviation function $\psi$ satisfies with a constant $C>0$ for $(y,q)$ in a neighborhood of $(0,0)$ and $t>0$ small

$$\psi \ge \gamma(t,z_0) + \frac{C}{t}(y^2+q^2) \ . \tag{4.7}$$

In these coordinates, we will denote by $\mathcal{C}^d(\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F)$ the space of smooth functions $f(t,y,q,z_0)$ with values $f(t,y,q,z_0) \in End(\Lambda^\bullet(T^*_{z_0}\Sigma)\otimes\pi^*F)$, defined for $(t,z_0) \in V_{t_0,\delta_1}$, $|(y,q)| \le \delta$, with $t_0>0$, $\delta_1>0$, $\delta>0$ small enough but fixed, $V_{t_0,\delta_1}$ being introduced in lemma 3.10, and such that for any $l,\alpha,\beta,\gamma$, there exist $C$ such that the following inequality holds true uniformly in $(t,z_0,y,q)$

$$\big\|\partial_t^l\, \partial_{y,q}^\gamma \nabla_{e_i,z_0}^\alpha \nabla_{\hat{e}^j,z_0}^\beta f\big\| \le C\, \langle p_0\rangle^{d+l-|\beta|} \ . \tag{4.8}$$

As an example, if $M(z_0)$ is a symbol of degree $d$ in the sense of definition 1.2, then

$$f(t,y,q,z_0) \;=\; M\Big(x_{z_0}(t)+ty,\, p_{z_0}(t)+q\Big)$$

belongs to $\mathcal{C}^d(\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F)$, due to (with obvious notations for $\mathcal{C}^d$)

$$x_{z_0}(t)+ty \,\in\, \mathcal{C}^0\;, \qquad p_{z_0}(t)+q \,\in\, \mathcal{C}^1\;. \tag{4.9}$$

Let $\theta(y,q)$ be a cutoff function with support in $|(y,q)|\le\delta$, and equal to 1 in a neighborhood of $(0,0)$. Let $\phi(u)$, $u\in\mathbb{R}$, be a cutoff function with support in $|u|\le\delta_1$, and equal to 1 in $|u|\le\delta_1/2$.

**Theorem 4.1.** *For all integer $j$, there exist*

$$c_j(t,y,q,z_0)\,\in\,\mathcal{C}^j\big(\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F\big)$$

*and for any $N$, an operator $R_N$ with distribution kernel $R_N(t,z,z_0)$ such that*

(4.10)

$$\begin{aligned} P(t,z,z_0) \;&=\; P_N(t,z,z_0)+R_N(t,z,z_0)\\ P_N(t,z,z_0) \;&=\; t^{-2n}\,\phi(t|p|)\,e^{-\psi(t,z,z_0)}\Big(\Sigma_{0\le j\le N}\,t^j\,c_j(t,y,q,z_0)\Big)\,\theta(y,q)\,\phi(t|p_0|)\end{aligned}$$

*where the sequence of operators $R_N$ is such that for any $s>0$, $M>0$ there exist $N$ and a constant $C$ such that for $t\in\,]0,t_0]$ one has*

$$\|R_N(u)\|_s \;\le\; C\,t^M\,\|u\|_{-s} \tag{4.11}$$

*where $\|u\|_t$ is the Sobolev norm (2.12) with $\lambda=0$, i.e $\|u\|_t=\|u\|_{0,t}$.*

**Remark 4.2.** Due to the presence of the cutoff functions $\phi,\theta$ in (4.11), the kernels $P_N(t,.,.)$ are globally defined on $\Sigma\times\Sigma$, and vanish identically outside a neighborhood of the diagonal of the form

$$\operatorname{dist}(x,x_{z_0}(t))\le Ct\;, \qquad |p-p_{z_0}(t)|\le C$$

and also for

$$t|p|\ge C' \quad\text{or}\quad t|p_0|\ge C'\;.$$

Observe that if (4.10) holds true, one has $\partial_t^l c_j(t,y,q,z_0)\in\mathcal{C}^{j+l}(\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F)$, and thus theorem 4.1 holds true if one replace the $c_j$' by the $d_j(y,q,z_0)\in\mathcal{C}^j(\Lambda^\bullet(T^*\Sigma)\otimes\pi^*F)$ independent of $t$ given by

$$d_n(y,q,z_0) \;=\; \Sigma_{j+l=n}\frac{1}{l!}\,\partial_t^l c_j(0,y,q,z_0)\;.\;\square$$

**Remark 4.3.** Using formula (4.5) and the asymptotic (4.10), one can write an asymptotic formula for the singularity of the kernel of the resolvant $(A-\lambda)^{-1}$ near the diagonal. □

**Proof:** Let u be a solution of $(\partial_t + A)u = 0$, and set

$$u \;=\; t^{-2n} e^{-\psi} v \;.$$

Then using the fact that $\psi$ satisfies the eikonal equation (3.46), we get that $v$ is solution of the conjugate equation

$$\partial_t + \left(A - \frac{\hbar|p|^2}{2}\right) - \frac{2n}{t} + \hbar\Sigma g_{i,j}\frac{\partial\psi}{\partial p_i}\frac{\partial}{\partial p_j} + \frac{\hbar}{2}\Delta_p\psi - \Sigma M_0^j\frac{\partial\psi}{\partial p_j}\;. \tag{4.12}$$

We first search a formal solution of (4.12) as a formal power series

$$v \;=\; \Sigma t^j\, c_j \tag{4.13}$$

such that

$$\lim_{t\to 0} t^{-2n} e^{-\psi} v \;=\; \delta_{z=z_0}\;. \tag{4.14}$$

In the coordinates $(t, y, q)$, partial derivatives are transformed according to

$$\begin{aligned}
\partial_t \;&\to\; \partial_t - \frac{y}{t}\partial_y - \frac{1}{t}\big(\partial_t x_{z_0}(t)\big)\partial_y - \big(\partial_t p_{z_0}(t)\big)\partial_q\\
\partial_x \;&\to\; \frac{1}{t}\partial_y\\
\partial_p \;&\to\; \partial_q
\end{aligned}$$

and using (3.66) and $t|p_0|^2 \in \mathcal{C}^1$ one gets

$$\psi(t,z,z_0) - \gamma(t,z_0) - \frac{2}{\hbar t}(3y^2 - 3yq + q^2) \;\in\; \mathcal{C}^1\;. \tag{4.15}$$

Thus we get

$$\begin{gathered}
\frac{\partial\psi}{\partial p} - \frac{1}{\hbar t}(-6y+4q) \;\in\; \mathcal{C}^1\\
\frac{\hbar}{2}\Delta_p\psi - \frac{2n}{t} \;\in\; \mathcal{C}^1\\
\Sigma M_0^j\frac{\partial\psi}{\partial p_j} \;\in\; \frac{1}{t}(y\mathcal{C}^0 + q\mathcal{C}^0) + \mathcal{C}^1\\
\big\{|p|^2/2, .\big\} - \frac{1}{t}\big(\partial_t x_{z_0}(t)\big)\partial_y - \big(\partial_t p_{z_0}(t)\big)\partial_q \;\in\; \frac{q}{t}\partial_y + \mathcal{C}^1\partial_y + \mathcal{C}^1\partial_q
\end{gathered}$$

$$\frac{-\hbar\Delta_p}{2} + \Sigma\partial_{p_j}M_0^j + \Sigma p_j M_1^j + M \ \in \ \mathcal{C}^0\partial_q^2 + \mathcal{C}^0\partial_q + \mathcal{C}^1$$
$$\hbar\Sigma g_{i,j}\frac{\partial\psi}{\partial p_i}\frac{\partial}{\partial p_j} \ \in \ \frac{1}{t}(-6y+4q)\partial_q + \mathcal{C}^1\partial_q$$

and therefore the conjugate operator (4.12) is of the form

$$\begin{aligned} &\partial_t + \frac{1}{t}(q-y)\partial_y + \frac{1}{t}(-6y+4q)\partial_q + \frac{g}{t} + B \\ &g \ \in \ y\mathcal{C}^0 + q\mathcal{C}^0 \\ &B \ \in \ \mathcal{C}^0\partial_q^2 + \mathcal{C}^1\partial_y + \mathcal{C}^1\partial_q + \mathcal{C}^1 \ . \end{aligned} \tag{4.16}$$

Let us introduce the first order operator $Z$

$$Z \ = \ (q-y)\partial_y + (-6y+4q)\partial_q + g \ . \tag{4.17}$$

Then the $c_j$'s are uniquely determined by the transport equations

$$\begin{aligned} &Zc_0 \ = \ 0 \\ &(Z+j)\,c_j \ = \ -\Big(\frac{\partial}{\partial t} + B\Big)c_{j-1} \ , \qquad j \geq 1 \end{aligned} \tag{4.18}$$

Observe that $\frac{\partial}{\partial t} + B$ maps $\mathcal{C}^m$ into $\mathcal{C}^{m+1}$, and thus it remains to verify

**i**) There exist an unique $c \in \mathcal{C}^0$ such that any solution of $Zc_0 = 0$ is of the form $c_0 = a(t, z_0)c$.

**ii**) For $j \geq 1$ and $h \in \mathcal{C}^m$, the equation $(Z+j)f = h$ admits an unique solution $f \in \mathcal{C}^m$.

The eigenvalues of the matrix

$$\begin{pmatrix} -1 & 1 \\ -6 & 4 \end{pmatrix}$$

are equal to $1, 2$, and therefore by a linear change of coordinates $(u,v) \to (y,q)$, $Z$ becomes

$$\begin{aligned} &Z \ = \ u\partial_u + 2v\partial_v + g \\ &g \ \in \ u\mathcal{C}^0 + v\mathcal{C}^0 \ . \end{aligned} \tag{4.19}$$

Set

$$G(t,u,v,z_0) = -\int_0^1 g(t, su, s^2v, z_0)\,\frac{ds}{s} \ \in \ \mathcal{C}^0 \ .$$

One has $\exp(G) \in \mathcal{C}^0$ and any solution of the equation $Zc = 0$ is of the form

$$c(t,y,q,z_0) \ = \ a(t,z_0)\exp(G)$$

and therefore i) holds true.

For $\lambda > 0$, the equation $(\lambda + Z)f = h \in \mathcal{C}^m$ admits the unique solution

$$f(t,u,v,z_0) \;=\; \exp(G) \int_0^1 s^\lambda \big[h\, \exp(-G)\big]\,(t, su, s^2 v, z_0)\, \frac{ds}{s} \;\;\in\;\; \mathcal{C}^m$$

and therefore ii) holds true.

We thus get $c_0 = a(t,z_0)\exp(G)$ and the function $a(t,z_0)$ is uniquely defined by the initial condition

$$a(t,z_0)\; \lim_{t\to 0} t^{-2n} \int e^{-\frac{2}{\hbar t}(3y^2-3yq+q^2)}\, l(y,q)\, t^n \; dy\, dq \;=\; l(0,0) \tag{4.20}$$

and thus we get

$$a(t,z_0) \;=\; \left(\frac{\sqrt{3}}{\hbar\pi}\right)^n . \tag{4.21}$$

It remains to estimate $R_N$. By the above construction, one has

$$(\partial_t + A)R_N \;=\; t^{-2n}\, \phi(t|p|)\, e^{-\psi(t,z,z_0)}(t^N f_N)\, \theta(y,q)\, \phi(t|p_0|) + \mathcal{T}_N^1 + \mathcal{T}_N^2 \tag{4.22}$$

with $f_N \in \mathcal{C}^{N+1}$, and where $\mathcal{T}_N^2$ contains error terms involving derivatives of the cutoff $\phi(t|p|)$, $\phi(t|p_0|)$, and $\mathcal{T}_N^1$ contains error terms involving derivatives of the cutoff $\theta(y,q)$. Set

$$R_N = R_N^1 + R_N^2$$

with

$$\begin{aligned} &(\partial_t + A)R_N^1 \;=\; K_N^1 \;=\; t^{-2n}\, \phi(t|p|)\, e^{-\psi(t,z,z_0)}(t^N f_N)\, \theta(y,q)\, \phi(t|p_0|) + \mathcal{T}_N^1 , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R_N^1|_{t=0} = 0 , \\ &(\partial_t + A)R_N^2 = K_N^2 = \mathcal{T}_N^2 , \qquad R_N^2|_{t=0} = 0 . \end{aligned} \tag{4.23}$$

From (3.49) and (3.77), one has $\psi \geq Ct|p_0|^2$, so we get for $f \in \mathcal{C}^{N+1}$

$$\|t^N f \theta(y,q)\, e^{-\psi}\|_{L^\infty} \;\leq\; C \sup \langle p_0\rangle^{N+1}\, t^N e^{-Ct|p_0|^2} \;\leq\; C_N\, t^{N/2-1/2}$$

and therefore the sequence of kernels $K_N^1$ satisfies (4.11). By Duhamel formula, one has

$$R_N^1 \;=\; \int_0^t e^{-(t-t')A}\, K_N^1(t',.)\; dt' \;.$$

Let $s > 0$ and $M > 0$; by theorem 2.4 there exist $L$ such that

$$\|u\|_s \;\leq\; \Sigma_{0\leq j\leq L} \|A^j u\|_0 \;.$$

Choose $N$ such that

$$\sup_{0\leq j\leq L} \|A^j K_N^1(t,.)v\|_0 \;\leq\; C\,t^M\,\|v\|_{-s}\;.$$

Since $e^{-tA}$ is bounded on $L^2$ for $t\in\,]0,1]$, one gets for $v\in\mathcal{H}^{-s}$

$$\|R_N^1 v\|_s \;\leq\; \int_0^t \Sigma_{0\leq j\leq L}\|A^j K_N^1(t',.)v\|_0\; dt' \;\;\leq\;\; Ct^{M+1}\|v\|_{-s}$$

and thus the sequence of kernels $R_N^1$ satisfies (4.11). Finally, one has $tp=tp_{z_0}(t)+tq$ and the support of $\phi'(t|p|)$ is contained in $\delta_1/2\leq|tp|\leq\delta_1$. Using (3.26) and (3.50) we get that on the support of $\mathcal{T}_N^2$, one has $d_0\leq|tp|\leq d_1$ and $d_0\leq|tp_0|\leq d_1$ for some $0<d_0<d_1$. Therefore, $\psi\geq Ct|p_0|^2$ implies that the sequence of kernels $\mathcal{T}_N^2$ satisfies (4.11), and the same argument as above shows that the sequence of kernels $R_N^2$ satisfies (4.11). The proof of theorem 4.1 is complete. ∎

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