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DYNAMICS IN THE MODULI SPACE OF ABELIAN DIFFERENTIALS

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Abstract: We announce the proof of the Zorich–Kontsevich conjecture: the non-trivial Lyapunov exponents of the Teichmüller flow on (any connected component of a stratum of) the moduli space of Abelian differentials on compact Riemann surfaces are all distinct. By previous work of those authors, this implies the existence of the complete asymptotic Lagrangian flag describing the behavior in homology of the vertical foliation in a typical translation surface.

1 – Abelian differentials

1.1. An *Abelian differential* on a compact Riemann surface M is a holomorphic complex 1-form ω on the surface. In local coordinates z, it may be written

$$\omega_z = \varphi(z)\,dz$$

where the coefficient φ is a holomorphic function. Given another local coordinate w, the corresponding local expression $\omega_w = \psi(w)dw$ is determined by

$$\psi(w) = \varphi(z)\,\frac{dz}{dw}$$

on the intersection of the coordinate domains.

1.2. We assume that the Abelian differential is not identically zero. Then its zeros are isolated and, hence, finitely many. Let them be $z_1, ..., z_\kappa$, with $\kappa \geq 0$.

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Near any point p such that ω_p is non-zero, we may always find *adapted local coordinates*

$$\zeta = \int_p^z \varphi(w)\,dw \tag{1}$$

for which the local expression of the Abelian differential is particularly simple: $\omega_\zeta = d\zeta$. Moreover, near a zero z_i, of order $m_i \geq 1$, we may choose adapted local coordinates

$$\zeta = (m_i+1)\left(\int_{z_i}^z \varphi(w)\,dw\right)^{\frac{1}{m_i+1}} \tag{2}$$

relative to which $\omega_\zeta = \zeta^{m_i} d\zeta$.

2 – Translation surfaces

2.1. Abelian differentials carry a very rich geometric structure. To begin with, adapted local coordinates as in (1) form a *translation atlas* on the complement of the zeros: changes between two such local coordinates ζ and ζ' are given by translations

$$\zeta' = \zeta + \text{const} .$$

Such an atlas permits to transport from the complex plane to the complement $M \backslash \{z_1, ..., z_\kappa\}$ of the zeros

- a *flat Riemann metric* and
- a parallel unit *upward vector field*.

Conversely, the flat metric and the upward vector field characterize the translation structure completely.

2.2. Using the local coordinates (2), one can also describe the flat metric near each of the zeros z_i: in polar coordinates, it takes the form

$$ds^2 = d\rho^2 + (c_i \rho\, d\theta)^2, \quad \text{with } c_i = m_i + 1 .$$

In other words, z_i is a *conical singularity* for the metric, with conical angle equal to $2\pi(m_i+1)$. The upward vector field extends to the singularities, in a multivalued fashion: there are exactly m_i+1 values at each z_i.

The presence of these singularities implies that the geodesic flow is not complete: countably many geodesics leaving from any point hit some singularity in finite time.

2.3. Let us describe a geometric construction of translation surfaces (Figure 1). Consider any planar polygon with even number of sides, organized in pairs of parallel sides with the same length. Identifying the two sides in the same pair, by translation, one obtains a translation surface. The corresponding Abelian differential comes, simply, from the canonical differential dz on the complex plane.

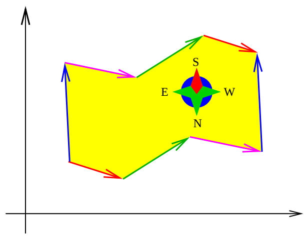


Figure 1

Every translation surface may be represented in this way. Yet, it is important to keep in mind that such representations are far from being unique.

3 – Geodesic flows

A major problem we are interested in is to describe the behavior of the *trajectories* of the Abelian differential or, in other words, the geodesics of the corresponding flat metric. Locally, in adapted coordinates, they are given by straight lines. We want to describe their global behavior (see Zorich [34]):

- When are geodesics closed? When are they dense?
- Quantitatively, how do they wrap around the surface?

These questions admit notably precise answers, as we are going to see. We take a Dynamics approach, based on analyzing the behavior of certain dynamical systems acting on the space of all translation surfaces (or Abelian differentials), especially the Teichmüller flow and certain renormalization operators.

4 – Measured foliations

4.1. One important motivation for raising these questions comes from Thurston's theory [27] of measured foliations. The (oriented) *measured foliation* defined by a real closed 1-form β on a smooth surface M is the foliation of the complement of the zeros whose leaves are tangent to the kernel of β at every point.

It is assumed that β has finitely many zeros and near each one of the zeros it is given by

$$\beta_z = \Im(z^{m_i} dz)$$

for some appropriate choice of *smooth* coordinates. One calls *saddle-connection* any leaf that connects two zeros; if the two zeros coincide, the saddle-connection is called a *homoclinic loop*.

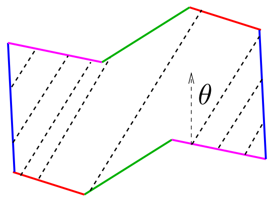


Figure 2

Maier [21] described the global structure of any measured foliation: there exists a finite decomposition of the surface into periodic regions, where all leaves are closed (homeomorphic to the circle), and minimal regions, where all leaves are dense; these regions are separated by saddle-connections and homoclinic loops.

4.2. Geodesics in a given fixed direction on a translation surface (Figure 2) are a special case of a *measured foliation*: consider the real closed 1-form

$$\beta = \Re(e^{i\theta}\omega)$$

where ω is the Abelian differential and θ is the angle between the chosen direction and the upward vector field. Results of Calabi [7], Katok [16], Hubbard, Masur [15], and Kontsevich, Zorich [19] show that this case is actually not so special: every measured foliation with no saddle-connections can be realized as the vertical geodesic foliation of some translation surface.

5 – Moduli spaces

5.1. The numbers and orders of the singularities of an Abelian differential are linked to the topology of the ambient surface through the Gauss–Bonnet relation

$$\sum_{i=1}^{\kappa} m_i \;=\; 2\,g(M) - 2 \;=\; -\mathcal{X}(M)\;, \tag{3}$$

where $g(M)$ is the genus and $\mathcal{X}(M)$ is the Euler characteristic of M. In particular, the set of zeros is non-empty if and only if the genus is larger than 1. We focus on that case in what follows.

Let the genus $g \geq 2$ be fixed. We denote by $\mathcal{M}_g$ the moduli space of compact Riemann surfaces of genus g, that is, the space of all complex structures on the compact surface of genus g, modulo conformal equivalence. Similarly, let $\mathcal{A}_g$ be the moduli space of Abelian differentials on compact Riemann surfaces of genus g. Both moduli spaces are complex orbifolds, with complex dimensions

$$\dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3 \quad \text{and} \quad \dim_{\mathbb{C}} \mathcal{A}_g = 4g - 3 \ .$$

In addition, $\mathcal{A}_g$ is a fiber bundle over $\mathcal{M}_g$ (see [19]): the fiber bundle projection assigns to each (non-identically zero) Abelian differential the unique Riemann surface structure that is compatible with it.

5.2. Consider any $m_1, ..., m_\kappa \geq 1$ satisfying the relation (3). We denote by $\mathcal{A}_g(m_1, ..., m_\kappa)$ the subset of Abelian differentials having κ zeros, with multiplicities $m_1, ..., m_\sigma$. Each of these *strata* of the moduli space $\mathcal{A}_g$ is also a complex orbifold, with complex dimension

$$\dim_{\mathbb{C}} \mathcal{A}_g(m_1, ..., m_\kappa) = 2g + \kappa - 1 \ .$$

This is largest, and coincides with the dimension of the whole moduli space $\mathcal{A}_g$, when $\kappa = 2g - 2$ and $m_i = 1$ for all i: we call $\mathcal{A}_g(1, ..., 1)$ the *principal stratum*. On the other hand, the dimension is smallest when $\kappa = 1$ and $m_1 = 2g - 2$: the dimension of this stratum $\mathcal{A}_g(2g - 2)$ is equal to $4g$. In general, strata are not fiber bundles over $\mathcal{M}_g$.

5.3. Local coordinates on each stratum may be constructed as follows. Let $S = \{z_1, ..., z_\kappa\}$ be the singular set and $\{\gamma_j, j = 1, ..., d\}$ be a basis of the relative homology $H_1(M, S, \mathbb{Z})$: each γ_j is a relative homology class of paths joining two elements of S. Then the *relative period mapping*

$$\omega \ \mapsto \left(\int_{\gamma_j} \omega \right)_{j=1,...,d}$$

defines a local chart on the stratum. Thus, the stratum is locally identified with the relative cohomology $\mathbb{C}^d = H^1(M, S, \mathbb{C})$, where

$$d \ = \ \dim_{\mathbb{C}} H_1(M, S, \mathbb{C}) \ = \ \dim_{\mathbb{C}} H^1(M, S, \mathbb{C}) \ = \ 2g + \kappa - 1 \ .$$

This falls short of a manifold structure because ramifications may arise at points (Abelian differentials) with non-trivial symmetries. On the other hand, coordinate changes are complex affine maps, and so this atlas endows the stratum with a complex affine structure.

The isomorphism $H^1(M,S,\mathbb{C}) = \mathbb{C}^d$ induces a natural Lebesgue measure on the cohomology space, relative to which the lattice $H^1(M,S,\mathbb{Z}\oplus i\mathbb{Z})$ has co-volume 1. This measure does not depend on the choice of the basis $\{\gamma_j\}$, although the isomorphism does. In this way we get that each stratum carries a canonical volume measure. Masur [23] and Veech [28] proved that the volume of every stratum is finite. The volumes of all strata have been computed recently by Eskin, Okounkov, Pandharipande [8, 9].

5.4. Let us also mention that strata need not be connected, in fact some have up to 3 connected components (Arnoux [1], Veech [29]). Kontsevich, Zorich [19] gave the complete classification of the connected components of all strata.

6 – Teichmüller flow

6.1. The *Teichmüller flow* $\mathcal{T}^t$ is the natural action of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$ on the fiber bundle $\mathcal{A}_g$, by postcomposition with local adapted charts. In terms of Abelian differentials:

$$\mathcal{T}^t(\omega)_z = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \omega_z \equiv \big[e^t \Re\omega_z\big] + i\big[e^{-t}\Im\omega_z\big] .$$

It leaves invariant every connected component of strata, as well as the corresponding canonical volume measure. In addition, it preserves the area of the translation surface M.

6.2. Figure 3 gives a geometric illustration of the Teichmüller flow in terms of the action on planar polygons. Recall however that different polygons may define the same translation surface. Indeed, while the action on polygons is rather uninteresting, because there is no recurrence, the Teichmüller flow in the space of Abelian differentials has very rich dynamical behavior, as we shall comment upon in a while.

Most of what follows is guided by the general principle that *properties of the Teichmüller flow reflect upon dynamical properties of almost all Abelian differentials.*

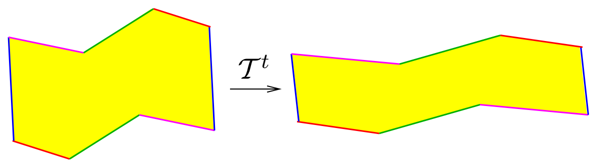


Figure 3

7 – Ergodicity

7.1. An important manifestation of this principle is the proof, by Masur [23], Veech [28], that the geodesic flow g^t of almost every translation surface in almost every direction is uniquely ergodic: given any continuous function $\varphi : M \to \mathbb{R}$,

$$\frac{1}{T}\int_0^T \varphi(g^t(z))\,dt \quad \text{converges uniformly to} \quad \int_M \varphi \; d(area)$$

when $T \to \infty$. Closely related, is their proof of the Keane conjecture [17]: almost all interval exchange maps are uniquely ergodic.

Indeed, the crucial ingredient in the proofs is the following statement about the Teichmüller flow:

Theorem 1 (Masur [23], Veech [28])**.** *The Teichmüller flow is ergodic on every connected component of every stratum, restricted to any constant area hypersurface.*

The previous conclusion was much improved by

Theorem 2 (Kerckhoff, Masur, Smillie [18])**.** *For every translation surface, the geodesic flow g^t in almost every direction is uniquely ergodic.*

7.2. The *asymptotic cycle* (Schwartzman [26]) of the geodesic leaving a point z in a given direction is defined as follows: Let γ be a (long) geodesic segment starting from z. Denote by $[\gamma] \in H_1(M, \mathbb{Z})$ the cycle represented by the closed curve obtained by connecting the endpoint of γ to the starting point z by some segment with bounded length. Then let

$$c_1 = \lim \frac{1}{|\gamma|}[\gamma]$$

where $|\gamma|$ denotes the length. Unique ergodicity implies that the limit exists, uniformly, and does not depend on the point z: it depends only on the translation surface and the choice of the direction.

8 – Asymptotic flag conjecture

8.1. If the genus $g(M)=1$ then c_1 provides a good approximation to the direction of the geodesic: the deviation of $[\gamma]$ from the line $L_1 \subset H_1(M,\mathbb{R})$ spanned by the asymptotic cycle is bounded.

For $g(M)=2$, one gets a richer picture (Figure 4 represents the component of $[\gamma]$ orthogonal to c_1, for various values of $|\gamma|$): the cycle $[\gamma]$ oscillates around L_1 with amplitude roughly $|\gamma|^{\nu_2}$, where $0<\nu_2<1$ and $|\gamma|$ denotes the length of the geodesic segment. Moreover, there is an asymptotic isotropic 2-plane $L_2 \supset L_1$ such that deviations from L_2 are bounded.

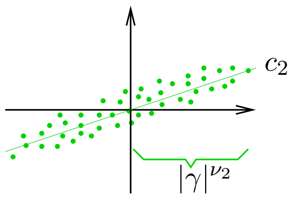


Figure 4

8.2. More generally (see Figure 5), in genus $g>1$ we have

Conjecture 1 (Zorich, Kontsevich)**.** *There are* $1>\nu_2>\cdots>\nu_g>0$ *and isotropic subspaces* $L_1 \subset L_2 \subset \cdots \subset L_g$ *of the homology* $H_1(M,\mathbb{R})$ *with* $\dim L_i = i$ *for every* $1 \le i \le g$ *such that*

- $[\gamma]$ *oscillates around* L_i *with amplitude* $|\gamma|^{\nu_{i+1}}$*:*

$$\limsup_{|\gamma|\to\infty} \frac{\log \operatorname{dist}([\gamma],L_i)}{\log|\gamma|} = \nu_{i+1} \qquad \text{for every} \quad 1 \le i \le g-1 \ ;$$

- *the deviation of* $[\gamma]$ *from* L_g *is bounded:* $\sup \operatorname{dist}([\gamma],L_g) < \infty$.

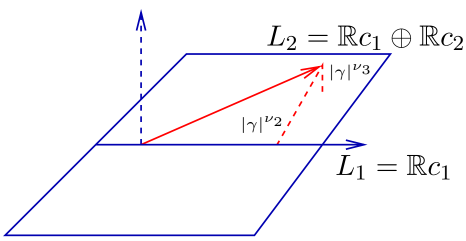


Figure 5

Moreover, the deviation spectrum $\nu_2 > \cdots > \nu_g$ is universal: it depends only on the connected component of the stratum to which the translation surface belongs.

8.3. The picture we just described was discovered empirically by Zorich [31]. Together with Kontsevich [19, 31, 32], he showed that this picture would follow from a statement about the Lyapunov exponents of the Teichmüller flow. Indeed, the Lyapunov spectrum of the Teichmüller flow (restricted to a hypersurface of constant area) has the form

$$\begin{aligned} 2 \geq 1+\nu_2 \geq \cdots \geq 1+\nu_g \geq 1 = \cdots = 1 \geq 1-\nu_g \geq \cdots \geq 1-\nu_2 \geq 0 \geq \\ -1+\nu_2 \geq \cdots \geq -1+\nu_g \geq -1 = \cdots = -1 \geq -1-\nu_g \geq \cdots \geq -1-\nu_2 \geq -2\,. \end{aligned} \tag{4}$$

Zero is a simple exponent, corresponding to the flow direction. There are $\kappa - 1$ exponents equal to 1 and to -1 arising from the action on relative cycles joining the σ singularities. Zorich and Kontsevich proved that the previous conjecture would follow from showing that all the inequalities in (4) are strict or, in other words, that apart from ± 1 all Lyapunov exponents of the Teichmüller flow are distinct; in that case the exponents ν_i in the description of the Zorich phenomenon are just the same numbers that appear in (4).

8.4. Veech [29] proved that the Teichmüller flow is non-uniformly hyperbolic. This means that the two middle inequalities in (4) are strict or, equivalently, that $\nu_2 < 1$. Then the fundamental work of Forni [10] proved that $\nu_g > 0$. This established the conjecture for $g = 2$, as well as the existence of the subspace L_g in the general case. Moreover, his conclusions have been used by Avila, Forni [2] to obtain other dynamical properties of translation flows. Finally, here we announce

Main Result. *The Zorich–Kontsevich conjecture is true, on every connected component of any stratum.*

The connection between the Zorich phenomenon and the Teichmüller flow can be understood by means of another dynamical system, the Zorich cocycle, which is a linear cocycle over the Zorich renormalization operator. We are going to outline this connection, following Zorich [32] mostly, and then provide some motivation to the proof of the main result.

9 – Interval exchange maps

9.1. The geodesic flow in a given fixed direction may be analyzed through the return map of geodesics to convenient cross-sections. For typical directions this map is well-defined and an *interval exchange transformation*: there is a finite partition of the domain into subintervals restricted to which the return map is a translation; the transformation just reshuffles those subintervals. Interval exchange transformations are described by pairs (π, λ) where π determines the combinatorics of the subintervals before and after the transformation, and λ describes the lengths of the subintervals. See Marmi, Moussa, Yoccoz [22] and Figure 6.

The geodesic flow on the translation surface, in the chosen direction, may then be recovered as a suspension of the interval exchange transformation. This is illustrated in Figure 7: the surface is represented as a finite union of rectangles with appropriate identification of boundary segments (a *zippered rectangle* [28]), the interval exchange acts on the horizontal cross-section, and the geodesic flow is vertical on each of the rectangles. This construction involves additional parameters, especially a vector h that describes the heights of the rectangles (the suspension roof function).

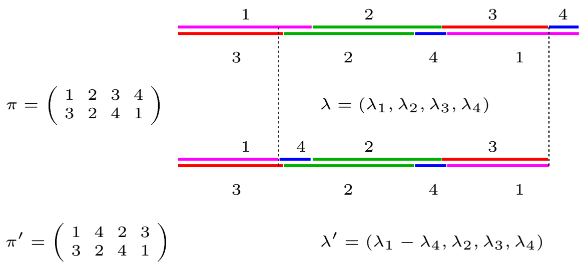


Figure 6

9.2. Then, to analyze the behavior of longer and longer geodesics, one considers return maps to shorter and shorter cross-sections. An efficient way to implement this idea is the *Rauzy–Veech induction operator* [25, 28]. At the level of interval exchange maps this corresponds to removing a convenient subinterval on the right of the domain and replacing the original transformation by its return map to the reduced domain. This is expressed by a map

$$\hat{R}(\pi, \lambda) = (\pi', \lambda') , \qquad \lambda' = A(\lambda)$$

where $A = A_{\pi,\lambda}$ is a linear map. An example is presented in Figure 6.

We also consider the *Rauzy–Veech renormalization operator* $R(\pi,\lambda)=(\pi',\lambda'')$, which is just the induction operator followed by rescaling of the reduced domain back to length 1. This is a Markov map and admits an invariant measure ν absolutely continuous with respect to Lebesgue measure in the λ-space and ergodic.

9.3. These operators also define a directed graph structure (*Rauzy diagram*) on the set of all admissible combinatorial data π: there is an arrow from π to π' if and only if $R(\pi,\lambda) = (\pi',\lambda'')$ for some λ and λ''. The *Rauzy classes* are the connected components of this graph. It is clear that the set of all pairs (π,λ) where π varies in a given Rauzy class and π varies on the whole simplex of length 1 positive vectors is invariant under the renormalization operator. In what follows we always consider our operators restricted to such an invariant set. Then the statements are meant for every choice of the corresponding Rauzy class.

10 – Lyapunov exponents

10.1. The induction and renormalization operators also act at the level of zippered rectangles: the action on the height vector h is described by $h' = B(h)$ where $B = B_{\pi,\lambda}$ is the linear map defined by

$$A^* \cdot B = \mathrm{id}\ .$$

An example is presented in Figure 7. These linear maps are particularly important for our purposes. Indeed, the point with taking successively smaller cross-sections is that this causes geodesic segments returning to the cross-section to become successively longer, and the map $h \mapsto h'$ describes exactly how this happens. Thus, the behavior of long geodesic segments corresponds to the asymptotic behavior of the *Rauzy–Veech cocycle*

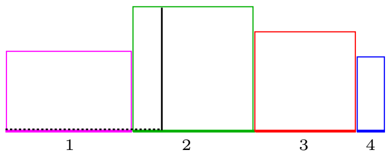
$$\mathcal{R}\big((\pi,\lambda),h\big) = \big(R(\pi,\lambda), B_{\pi,\lambda}(h)\big)\ ,$$

which is a linear cocycle over the Rauzy–Veech renormalization operator R.

This observation is crucial for understanding how the Zorich phenomenon can be handled with methods of dynamical systems and ergodic theory. However, there is still an important technical difficulty: the absolutely continuous invariant measure ν of the operator R is usually infinite. This was solved by Zorich [32], who constructed accelerated versions

$$Z(\pi,\lambda) = R^{n(\pi,\lambda)}(\pi,\lambda) \quad \text{and} \quad \mathcal{Z}\big((\pi,\lambda),h\big) = \mathcal{R}^{n(\pi,\lambda)}(\pi,\lambda,h)\ ,$$

of those objects, such that Z admits an invariant absolutely continuous *probability* μ and $\mathcal{Z}$ is an integrable linear cocycle over Z. In addition, μ is ergodic.



$$h_1' = h_1 \quad h_2' = h_2 \quad h_3' = h_3 \quad h_4' = h_1 + h_4$$

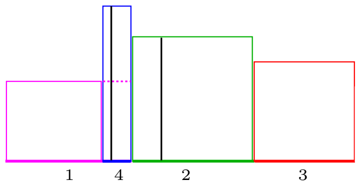


Figure 7

10.2. It follows, by Oseledets [24], that the Zorich cocycle $\mathcal{Z}$ has well-defined Lyapunov exponents. In addition, one can show that $\mathcal{Z}$ preserves a certain alternate 2-form α whose kernel $\{u\colon \alpha(u,\cdot) \equiv 0\}$ has dimension $\kappa - 1$ and such that all the Lyapunov exponents along the invariant subbundle defined by the cone vanish identically. This implies that the Lyapunov spectrum has the form

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0 = \cdots = 0 \geq -\lambda_g \geq \cdots \geq -\lambda_1 \ .$$

It is not difficult to show that $\lambda_1 > 0$. The interpretation of this cocycle outlined above gives that the λ_i are related to the exponents ν_i in the Zorich–Kontsevich conjecture through

$$\nu_i = \frac{\lambda_i}{\lambda_1} \quad \text{ for } \ i = 1, ..., g \ .$$

Moreover, if the λ_i are all distinct then the asymptotic flag is given by the Oseledets decomposition for the Zorich cocycle. Thus, the conjecture may be reformulated (see Conjecture 1 in [33] and Conditional Theorem 4 in [34]):

Conjecture 2. *The Lyapunov exponents of the Zorich cocycle satisfy*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_g > 0$$

on every Rauzy class.

This is the statement we actually prove in our forthcoming papers [3] and [4].

10.3. Typical translation surfaces in the same connected component of strata are special suspensions over interval exchange transformations whose combinatorics belong to the same Rauzy class. Moreover, the Zorich renormalization operator Z is closely related to the Poincaré map of the Teichmüller flow to a convenient cross-section. In fact, the Zorich cocycle $\mathcal{Z}$ is closely related to a Poincaré map of a continuous time linear cocycle over the Teichmüller flow itself, the Kontsevich–Zorich cocycle [19].

From these relations one deduces that the Lyapunov spectrum of $\mathcal{Z}$ determines the one of $\mathcal{T}^t$: the Lyapunov exponents of the Teichmüller flow are the numbers

$$\pm\left(1+\frac{\lambda}{\lambda_1}\right), \quad \text{where } \lambda \text{ is a Lyapunov exponent of } \mathcal{Z} .$$

Compare (4). This explains how the Lyapunov spectrum of the Teichmüller flow comes to be related with the exponents in the Zorich phenomenon.

10.4. Besides the Zorich phenomenon, the Lyapunov exponents of the Zorich cocycle are also linked to the behavior of ergodic averages of interval exchange transformations (a first result in this direction is given in [32]) and translation flows or area-preserving flows on surfaces [10]. Let us point out that it is now possible to treat the case of interval exchange transformations in a very elegant way, using the results of Marmi–Moussa–Yoccoz [22]. A result for translation flows can then be recovered by suspension, which also implies the result for area-preserving flows.

11 – Motivation of the proof

11.1. Our proof of the Zorich–Kontsevich conjecture has two distinct parts:

- A general criterion for the simplicity of the Lyapunov spectrum of locally constant cocycles.
- A combinatorial analysis of Rauzy diagrams to show that the criterion can be applied to the Zorich cocycle on any Rauzy class.

They correspond, roughly, to our two manuscripts [3] and [4].

11.2. The basic idea of the criterion is that it suffices to find orbits of the base dynamics over which the cocycle exhibits certain forms of behavior, that we call twisting and pinching. Roughly speaking, *twisting* means that certain families of subspaces are put in general position, and *pinching* means that a large part of the Grassmannian (consisting of subspaces in general position) is concentrated in a small region, by the action of the cocycle. We will be a bit more precise in a while. We call the cocycle *simple* if it meets both requirements (notice that "simple" really means that the cocycle's behavior is quite rich). Then, according to our criterion, the Lyapunov spectrum is simple.

It should be noted that the orbits on which these types of behavior are observed are very particular and, a priori, correspond to zero measure subsets. Nevertheless, they are able to "persuade" almost every orbit to have a simple Lyapunov spectrum. For this, one assumes that the base dynamics is rather chaotic (which is the case for the Teichmüller flow). In a purely random situation, this persuasion mechanism has been understood for quite some time, through the works of Furstenberg, Kesten [11, 12], Guivarc'h, Raugi [14], and Gol'dsheid, Margulis [13]. That this happens also in chaotic, but not random, situations was unveiled by the works of Ledrappier [20] and Bonatti, Gomez-Mont, Viana [5, 6, 30]. Our own papers [3, 4] extend those conclusions to the specific situation needed in the present context.

11.3. The strategy to prove that the Zorich cocycle is "rich", in the sense described above, is to use induction on the complexity. The geometric motivation is more transparent when one thinks in terms of the Kontsevich–Zorich cocycle. As explained, we want to find orbits of the Teichmüller flow inside any connected component of a stratum $\mathcal{S} = \mathcal{A}_g(m_1, ..., m_\kappa)$ with some given behavior. To this end, we look at orbits that spend a long time near the boundary of $\mathcal{S}$. While there, these orbits pick up the behavior of the boundary dynamics of the Teichmüller flow, which contains the dynamics of the Teichmüller flow restricted to connected components of strata $\mathcal{S}'$ with simpler combinatorics (corresponding to certain ways to degenerate $\mathcal{S}$).

This is easy to make sense of when the stratum $\mathcal{S}$ is not closed in the moduli space, since the whole Teichmüller flow provides a broader ambient dynamics where everything takes place. It is less clear how to formalize the idea when $\mathcal{S}$ is closed (this is the case when $\mathcal{S} = \mathcal{A}_g(2g-2)$). In this case, the boundary dynamics corresponds to the Teichmüller flow acting on surfaces of smaller genuses and a geometric interpretation of the degeneration is more subtle. Kontsevich, Zorich [19] considered the inverse of such a degeneration process, that they called "bubbling a handle".

On the other hand, our degeneration process is very simple when viewed in terms of interval exchange transformations: we just make one interval very small. This small interval remains untouched by the renormalization process for a very long time, while the other intervals are acted upon by a degenerate renormalization process. This is what allows us to put in place an inductive argument. To really control the effect, we must choose our small interval very carefully. It is also sometimes useful to choose particular permutations in the Rauzy class we are analyzing. A particularly sophisticated choice is needed when we must change the genus of the underlying translation surface.

11.4. Let us say a few more words on how the Zorich cocycle is shown to be simple. The fact that the cocycle is symplectic is important for the arguments. By induction, we show that it acts minimally on the space of Lagrangian flags. This is used to derive that the cocycle is twisting. In the induction, there must be some gain of information at each step, when we must change genus: in this case, this gain regards the action of the cocycle on lines, and it comes from the rather easy fact that this action is minimal.

Also by induction, we show that certain orbits of the cocycle are pinching. Here the gain of information when we must change genus has to come from the action on Lagrangian spaces, and it is far from obvious. One can use Forni's theorem [10] and, indeed, we did so in a previous version of the arguments. However, the proof is now independent of his result, so that our work gives a new proof of Forni's main theorem. Indeed, in our (combinatorial) argument the pinching of Lagrangian subspaces comes from orbits that have a pair of zero Lyapunov exponents, but present some parabolic behavior in the central subspace.

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