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SINGULARITIES OF OPTIMAL AVERAGED PROFIT FOR STATIONARY STRATEGIES

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Abstract: For one-parametric families of polidynamical systems and of profit densities on the circle, we classify the generic singularities of the optimal averaged profit as a function of the parameter, when the averaged profit is provided by stationary strategies.

1 – Introduction

A *polidynamical system* on the circle S^1 (phase space) is a control system

$$\dot{x} = v(x,u) \ , \qquad x \in S^1, \quad u \in \{1,...,n\}, \ n \geq 2 \tag{1}$$

where the control u is a piecewise constant function defined on the time axis, assuming values on the finite set $\{1,...,n\}$.

When such a control is defined, every solution of the differential equation (1) is an *admissible motion*. Moreover, for every value i $(1 \leq i \leq n)$ of the control parameter, the vector field $v(\,\cdot\,,i)$ is an *admissible velocity* which we denote by v_i. Hence, the choice of a control parameter value corresponds to the choice of an admissible velocity.

A polidynamical system on the circle depending smoothly on a parameter value p of a one-dimensional manifold $\mathcal{P}$

$$\dot{x} = v(x,u,p) \ , \qquad x \in S^1, \quad u \in \{1,...,n\}, \ n \geq 2, \quad p \in \mathcal{P}$$

is a *one-parametric family of polidynamical systems on the circle*.

When additionally there is a profit density f (smooth function) on the circle depending smoothly on the parameter value p, that is, a *one-parametric family of*

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profit densities on the circle, then an admissible motion on the interval $[t_0, t_0+T]$, $T > 0$, provides the *profit*

$$P(T,p) = \int_{t_0}^{t_0+T} f\big(x(t),p\big)\, dt$$

and the *averaged profit*

$$A(T,p) = \frac{P(T,p)}{T} \ .$$

An important control problem concerns the choice, for each parameter value, of an admissible motion providing the maximum averaged profit on the infinite horizon, i.e., when $T \to \infty$. Such maximum is the *optimal averaged profit* and the strategy providing it is called *optimal*.

The optimal averaged profit is a function of the parameter and we will study its singularities. The problem of classifying these singularities was treated for the first time by V.I. Arnold in [1] in the case of a control system depending smoothly on a parameter p of a one-dimensional manifold $\mathcal{P}$

$$\dot{x} = v(x,u,p) \ , \qquad x \in S^1, \quad u \in S^1, \quad p \in \mathcal{P}$$

with the control u assuming values on the circle, and of a profit density that does not depend on the parameter.

If the control system has many equilibria then it is reasonable to consider the problem of maximizing the averaged profit in the infinite horizon among the stationary strategies, that is, strategies that consist in the permanent staying at a point. In this case, the typical singularities of the optimal averaged profit can be found in [1].

For polidynamical systems, if a parameter value is fixed then, in a generic case, the equilibrium points form a discrete set. However, there are open intervals with positive and negative admissible velocities, allowing admissible motions that in the infinite horizon provide the same averaged profit as the permanent staying at the points of such intervals.

For this reason we consider the choice of such motions as stationary strategies, enlarging this concept. In this context, we define a *stationary strategy point* as one where the zero velocity belongs to the convex hull of all of the admissible velocities at that point. The *stationary domain* is the set of all of these points.

In this paper, we consider the averaged profit restricted to stationary strategies and classify the generic singularities of the respective optimal profit as a function of the parameter. In our list, we obtain five singularities, three of which have been already presented by V.I. Arnold in [1] for stationary strategies at equilibria.

2 – Singularities of the stationary domain

We will consider the product space of the circle by the parameter space as a fibred space over the parameter, that is, with fibres $\mathcal{F}_p = S^1 \times \{p\}$, for every parameter value p.

Two objects of the same nature defined on a fibred space are $\mathcal{F}$-equivalent if one of them can be carried out to the other by a fibred diffeomorphism (diffeomorphism that sends fibres to fibres).

On the space of our objects (families of polidynamical systems or profit densities, pairs of them, etc.) we introduce the fine smooth Whitney topology. A property is *generic* (or *holds generically*) if the subset of elements satisfying it is open and dense.

Theorem 1. *Consider the space of one-parametric families of polidynamical systems on the circle. Generically, the germ of the stationary domain of a family at any of its points is the germ at the origin of one of the five sets from the second column of Table 1 up to $\mathcal{F}$-equivalence, under the conditions and the number of admissible velocities in the third and fourth columns, respectively. Besides, in a generic case, the stationary domain of a family of vector fields is stable up to small perturbations of such families.*

Table 1

No.	**Singularities**	**Conditions**	n
1	$\mathbb{R}^2$	Interior point	≥ 2
2	$x \geq 0$	Nondegenerate equilibrium point of exactly one admissible velocity	≥ 2
3_+	$x^2 - p \leq 0$	Bifurcation point of exactly one admissible velocity	≥ 2
3_-	$-(x^2 - p) \leq 0$		
4_+	$x^2 - p^2 \leq 0$	Nondegenerate equilibrium point of exactly two admissible velocities	2
4_-	$-(x^2 - p^2) \leq 0$		
5_a	$x \geq -\lvert p \rvert$	Nondegenerate equilibrium point of exactly two admissible velocities	≥ 3
5_b	$p \leq \lvert x \rvert$		

The proof of this theorem is based in the following two lemmas:

Lemma 1. *Consider the space of one-parametric families of vector fields on the circle. Generically, the germ of the zero level of a family at any of its points is $\mathcal{F}$-equivalent to the germ at the origin of either $x = 0$ or $p = x^2$. Besides, in a generic case, all of the vector fields of the family change sign when crossing this level.* ∎

Lemma 2. *Consider the space of one-parametric families of polidynamical systems on the circle. Generically, the number of vanishing admissible velocities at a point is no more than 2. Besides, in a generic case, a point where two of the admissible velocities vanish is a nondegenerate equilibrium point of both of them and their zero levels are transversal at this point.*

Both lemmas are proved on the basis of Thom transversality theorem. Here we just present the proof of Lemma 2.

Proof of Lemma 2: Let C_n be the space of one-parametric families of polidynamical systems on the circle with n admissible velocities.

For $1 \leq i < j < k \leq n$, the set J_{ijk} of 0-jets of families in $C_n\,(n \geq 3)$ at common equilibrium points of the i-th, j-th and k-th admissible velocities is a codimension 3 closed submanifold of the space of 0-jets of such families. Due to Thom transversality theorem and to the fact that this codimension is greater than the two-dimensional product space of the circle by the parameter space, generically there are no points satisfying the above conditions.

Moreover, because there is only a finite number of possible choices of the triple (i, j, k), we conclude the first assertation of this lemma.

The other two statements are proved in an analogous manner. In fact, the set of 1-jets of families in $C_n\,(n \geq 2)$ at bifurcation points of the i-th admissible velocity where the j-th velocity vanishes is a codimension 3 closed submanifold of the space of 1-jets of such families. The same happens with the set of 1-jets of families in $C_n\,(n \geq 2)$ at common equilibrium points of the i-th and j-th velocities where the zero levels of these families are not transversal. ∎

Proof of Theorem 1: Obviously, the germ of the stationary domain at any of its interior points is $\mathcal{F}$-equivalent to the germ of the real plane at the origin.

It is easy to see that at a boundary point of the stationary domain at least one of the admissible velocities has to vanish and all the other that do not vanish have the same sign.

In fact, if at a point (x_0, p_0) there are two admissible velocities with opposite signs then, by continuity of these velocities, this also happens in a neighborhood of this point. Hence, (x_0, p_0) is an interior point of the stationary domain. For a similar reason, if at (x_0, p_0) all of the admissible velocities have the same nonvanishing sign then (x_0, p_0) is an exterior point of the stationary domain. Consequently, at a boundary point of the stationary domain none of the two previous situations can happen, concluding that at least one of the admissible velocities has to vanish and all the other that do not vanish have the same sign.

By Lemma 2, in a generic case, this point has to be an equilibrium point of exactly one or exactly two admissible velocities.

In the first case, the germ of the zero level of such admissible velocity at that point is $\mathcal{F}$-equivalent to the germ at the origin of one of the two smooth curves from Lemma 1. Hence, the stationary domain lies in one of the sides of such curves leading to the singularities 2 and $3_\pm$ of Table 1.

In the second case, by Lemma 2, the boundary point is a nondegenerate equilibrium point of both the vanishing admissible velocities and their zero levels are transversal at this point. Consequently, the germ of the union of those zero levels at such a point is $\mathcal{F}$-equivalent to the germ at the origin of the set $x^2 - p^2 = 0$. If there are only two admissible velocities we get the singularities $4_\pm$ of Table 1; otherwise, we get the singularities 5_a and 5_b of the same table.

The stability of the stationary domain is an immediate consequence of the transversality conditions. ∎

3 – Singularities of the optimal averaged profit for stationary strategies

A stationary strategy provides an averaged profit in the infinite horizon that equals the value of the profit densities at a stationary strategy point. In fact, as it was said before, a stationary strategy provides the same averaged profit in the infinite horizon as the permanent staying at a stationary strategy point. Moreover, for every point of the stationary domain, it is always possible to define a stationary strategy for which the averaged profit in the infinite horizon equals the value of the family of profit densities at the considered point.

So, we conclude that the *optimal averaged profit A_s for stationary strategies* is the solution of the extremal problem with constraints:

$$A_s(p) = \max_{x \in S(p)} f(x,p) \,, \tag{2}$$

where $S(p)$ is the set of all phase points x such that (x, p) belongs to the stationary domain. It is defined for all parameter values p such that $S(p)$ is not empty.

Denote by S^* the subset of the stationary domain whose points provide the optimal profit A_s.

Lemma 3. *Consider the space of one-parametric families of pairs of polidynamical systems and profit densities on the circle. Generically, if p is a parameter value then the set $\overline{S^*} \cap \mathcal{F}_p$ does not contain:*

- **a**) *two points providing the singularities 2-4 of Table 2;*
- **b**) *two points having the same level, where one of them provides one of the singularities 2-4 of Table 2;*
- **c**) *three points having the same level.*

Two points have the *same level* if the family of profit densities has the same value at these points.

Proof of Lemma 3: Because all of these three statements have a similar proof, we only present the proof of the first one.

The set of 2-multijets of one-parametric families of pairs of polidynamical systems and profit densities on the circle at points of a same fibre and providing the singularities 2-4 of Table 2 is a codimension 5 closed submanifold of the space of 2-multijets of such families (because these two singularities have codimension 2, and belonging to same fibre gives another condition).

Hence, because this codimension is greater than the dimension of the domain of 2-multijet extension of these families, generically there are no points satisfying the above conditions, due to multijet transversality theorem and to the fact that the circle is a compact manifold. ∎

Two germs of functions are Γ-*equivalent* if their graphs are $\mathcal{F}$-equivalent, considering the product space of the functions domain by the real axis as a fibred space over the domain. Such diffeomorphism can be written on the form $(p, a) \mapsto (\varphi(p), h(p, a))$.

R^+-*equivalence* is the particular case of Γ-equivalence when the second component h is of the form $a + c(p)$, for a smooth function c. It is clear that the germ of a smooth function at a point is R^+-equivalent to the germ of the zero function at the origin.

Theorem 2. *Consider the space of one-parametric families of pairs of polidynamical systems and profit densities on the circle. Generically,* $\#\left(\overline{S^*}\cap\mathcal{F}_p\right)\le 2$, *for any parameter value* p. *Besides, in a generic case, the germ of the optimal profit at* p *is the germ at the origin of one of*

- *the four functions from second column of Table 2 up to* R^+*-equivalence, with the singularity of stationary domain and under the conditions pointed out in the third and fourth columns, respectively, if* $\#\left(\overline{S^*}\cap\mathcal{F}_p\right)=1$, *and*
- *the two functions from second column of Table 3 up to the equivalence and under the conditions pointed out in the third and fourth columns, respectively, if* $\#\left(\overline{S^*}\cap\mathcal{F}_p\right)=2$.

Table 2

No.	**Singularities**	**Singularities of the stationary domain**	**Conditions**
1	0	$\mathbb{R}^2$	$f_x=0$
		$x\ge 0$	$f_x\ne 0$
2	$\sqrt{p},\ \ p\ge 0$	$p\ge x^2$	$f_x\ne 0$
3	$\lvert p\rvert$	$x^2-p^2\le 0$ or $x\ge -\lvert p\rvert$	$f_x\ne 0$
4	$p\lvert p\rvert$	$x\ge 0$	$f_{xp}\ne 0=f_x>f_{xx}$

Table 3

No.	**Singularities**	**Eq.**	**Conditions**
1	$\lvert p\rvert$	R^+	Competition of two stationary strategy points providing singularity 1 from Table 2
2	$\max\left\{0;\ 1+\sqrt{p},\ p\ge 0\right\}$	Γ	Competition of two stationary strategy points providing the singularities 1 and 2 from Table 2

Remark. Each one of these singularities is well-known in Singularity Theory and appears in various classifications [1], [2], [3]. In particular, all of them are presented in [1] as generic singularities of the optimal averaged profit but only three of them are presented for stationary strategies at equilibria. In this paper all of them appear as result of the classification of generic singularities of the optimal averaged profit only for stationary strategies. $\square$

Proof of Theorem 2: Due to Lemma 3, in order to prove Theorem 2 we just have to analyze two cases, namely, when the set $\overline{S^*} \cap \mathcal{F}_p$ consists of one or two points.

Suppose that $\overline{S^*} \cap \mathcal{F}_{p_0} = \{(x_0, p_0)\}$ and assume that locally near this point the stationary domain is already in one of the normal forms of Table 1.

Generically, at least one of the derivatives f_x, f_{xx} and f_{xxx} does not vanish, due to Thom transversality theorem. Consequently, if the point (x_0, p_0) is in the interior of the stationary domain, then we must have $f_x(x_0, p_0) = 0$ and $f_{xx}(x_0, p_0) < 0$. That implies that the profit A_s is a smooth function near the point p_0, and so it has the first singularity of Table 2.

If at the point (x_0, p_0) the stationary domain has the singularity 2 of Table 1, then in a generic case, at this point either $f_x \neq 0$ or $f_x = 0$ and $f_{xx} \neq 0 \neq f_{xp}$, due to Thom transversality theorem. Besides, $f_{xx}(x_0, p_0) < 0$ due to optimality. Simple calculations imply that these two situations lead to the singularities 1 and 4 of Table 2, respectively.

If at (x_0, p_0) the stationary domain has a singularity of codimension 2 (which has to be one of the singularities $3_\pm$, $4_\pm$, 5_a or 5_b of Table 1), then generically $f_x(x_0, p_0) \neq 0$, due to Thom transversality theorem. So if at that point the stationary domain has one of the singularities 3_+, 4_+ or 5_a then the profit A_s has at the point p_0 the singularity 2 of Table 2 in the first case, and the singularity 3 of the same table in the last two.

Due to optimality, the derivative f_x at (x_0, p_0) has to vanish if x_0 belongs to the interior of $S(p_0)$. Therefore, in a generic case, at this point the stationary domain cannot have the singularities 3_-, 4_- and 5_b of Table 1.

Suppose now that the set $\overline{S^*} \cap \mathcal{F}_{p_0}$ consists of two distinct points (x_1, p_0) and (x_2, p_0). Due to Lemma 3, in a generic case, either both these points provide the singularity 1 from Table 2 or one of them provides the singularity 1 and the other gives one of the singularities 2-4 of the same table.

In the first case the points must have the same level. All the subcases that have to be considered are similar and we just analyze the situation when the points (x_1, p_0) and (x_2, p_0) are a boundary and an interior point of the stationary domain, respectively.

Hence, three equalities have to be satisfied: $f_x(x_2, p_0) = 0$, $f(x_1, p_0) = f(x_2, p_0)$ and $v(x_1, p_0) = 0$, for some admissible velocity v. These conditions form in the space of 2-multi 1-jets of one-parametric families of pairs of polidynamical systems and profit densities a codimension 3 submanifold.

Besides, in a generic case, the image of the multijet extension of a family of such pairs has no intersection with this submanifold in some neighborhood of the diagonal. That, together with multijet transversality theorem, implies that generically this extension is transversal to the submanifold, and so the matrix

$$\begin{pmatrix} v_x(x_1, p_0) & 0 & v_p(x_1, p_0) \\ 0 & f_{xx}(x_2, p_0) & f_{x,p}(x_2, p_0) \\ f_x(x_1, p_0) & 0 & f_p(x_1, p_0) - f_p(x_2, p_0) \end{pmatrix}$$

has maximum rank. This rank does not depend from the local coordinates near the points (x_1, p_0) and (x_2, p_0).

In local coordinates near these points in which the equations $v = 0$ and $f_x = 0$ are $x = x_1$ and $x = x_2$, respectively, that implies $f_p(x_1, p_0) - f_p(x_2, p_0) \neq 0$ because $v_p(x_1, p_0) = 0$. But in these coordinates $A_s = \max\{f(x_1, p);\, f(x_2, p)\}$ near the point p_0. So the germ (A_s, p_0) is R^+-equivalent to the germ of the function $|p|$ at the origin.

In the second case, due to statement (b) of Lemma 3, the points (x_1, p_0) and (x_2, p_0) must have different levels. Due to the presence of competition, one of them provides the singularity 1 of Table 2 and the other gives the singularity 2 of the same table and has the higher level. Consequently, the germ (A_s, p_0) is Γ-equivalent to the germ of the function $\max\{0;\, 1+\sqrt{p},\, p \geq 0\}$ at the origin. ∎

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