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AVERAGING TECHNIQUE AND OSCILLATION FOR EVEN ORDER DAMPED DELAY DIFFERENTIAL EQUATIONS

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Abstract: By using the averaging technique, some integral oscillation criteria are obtained for even order damped delay differential equations.

1 – Introduction

This paper deals with the oscillatory behavior of the even order damped delay differential equation

$$\begin{aligned}(1.1)\qquad &\big(\Phi(x^{n-1)}(t))\big)' + p(t)\,\Phi(x^{(n-1)}(t)) + f\Big(t,\, x[\,\tau_{01}(t)\,],\, ...,\, x[\,\tau_{0m}(t)\,],\\ &\qquad ...,\, x^{(n-1)}[\,\tau_{n-11}(t)\,],\, ...,\, x^{(n-1)}[\,\tau_{n-1m}(t)\,]\Big) = 0 \qquad \text{for} \quad t \ge t_0 > 0\ ,\end{aligned}$$

where $\Phi(s) = |s|^{\alpha-1}s$ with $\alpha > 0$ a fixed constant, and n is an even number.

Throughout this paper, we assume that

($\mathbf{A}_1$) $p \in C(I,\mathbb{R}_0)$ and $\lim\limits_{t\to\infty}\displaystyle\int_{\bar t}^{t}\left[\exp\left(-\int_{\bar t}^{s} p(\tau)\,d\tau\right)\right]^{1/\alpha} ds = \infty$ for every $\bar t \ge t_0$, where $I = [\,t_0,\infty\,)$ and $\mathbb{R}_0 = [0,\infty)$;

($\mathbf{A}_2$) $\tau_{ki} \in C(I,\mathbb{R})$ and $\lim_{t\to\infty}\tau_{ki}(t) = \infty$, $k = 0,1,...,n-1$, $i = 1,2,...,m$;

($\mathbf{A}_3$) $f \in C(I\times\mathbb{R}^{m\times n},\mathbb{R})$ satisfies the one-side estimate

$$f\Big(t, x_{01}, x_{02}, ..., x_{0m}, ..., x_{n-11}, ..., x_{n-1m}\Big)\,\mathrm{sign}\, x_{01} \;\ge\; q(t)\prod_{i=1}^{m} |x_{0i}|^{\alpha_i} \qquad \text{for} \quad x_{01}x_{0i} \ge 0 \quad (i = 1,2,...,m)\ ,$$

where $q \in C(I,\mathbb{R}_0)$ and $q(t)$ is not identically zero on any ray $[t_*,\infty)$, $\alpha_i \ge 0$ $(i = 1,2,...,m)$ are constants with $\sum_{i=1}^{m}\alpha_i = \alpha$.

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By a solution of Eq.(1.1) we mean a function $x(t) \in C^{n-1}([T_x, \infty), \mathbb{R})$ for some $T_x \geq t_0$ which has the property that $\Phi(x^{(n-1)}(t)) \in C^1(T_x, \mathbb{R})$ and satisfy Eq.(1.1) on $[T_x, \infty)$. A solution $x(t)$ of Eq.(1.1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

In the last decades, many results are obtained for the particular cases of Eq.(1.1) such as the even order nonlinear delay differential equation

$$\Big(|x^{(n-1)}(t)|^{\alpha-1}\, x^{(n-1)}(t)\Big)' + f\big(t, x[\tau(t)]\big) \;=\; 0 \;, \tag{1.2}$$

and the even order damped delay differential equation

$$\begin{aligned} x^{(n)}(t) \,+\, p(t)x^{(n-1)}(t) \,+\, f\Big(&t,\, x[\tau_{01}(t)],\, ...,\, x[\,\tau_{0m}(t)\,], \\ &...,\, x^{(n-1)}[\,\tau_{n-11}(t)\,],\, ...,\, x^{(n-1)}[\,\tau_{n-1m}(t)\,]\Big) \;=\; 0 \;. \end{aligned} \tag{1.3}$$

For this contributions we refer the reader to [1–3, 10–12] and the references cited therein. As far as we know that Eq.(1.1) in generalize form has never been the subject of systematic investigations.

The main objective of this paper is to establish some general oscillation criteria for Eq.(1.1) by introducing parameter functions $H(t,s)$, $\rho(s)$, $k(s)$ and using integral averaging techniques similar to that exploited by Kamenev [5] and Philos [8]. We also extend and improve the results in [1, 3, 10–12]. The relevance of our results is illustrated with two examples.

2 – Preliminaries

In order to discuss our main results, we first introduce the general mean similar to that exploited by Philos [8].

Set

$$D = \Big\{(t,s)\colon t \geq s \geq t_0\Big\} \qquad \text{and} \qquad D_0 = \Big\{(t,s)\colon t > s \geq t_0\Big\} \;.$$

We say a function $H \in C(D, \mathbb{R})$ belong to a class $\Im$, if

- **(H$_1$)** $H(t,t) = 0$ for $t \geq t_0$, and $H(t,s) > 0$ for $(t,s) \in D_0$;
- **(H$_2$)** H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable;

($\mathbf{H}_3$) There exist functions $h\in C(D_0,\mathbb{R})$, and $\rho, k \in C^1(I,\mathbb{R}_+)$ ($\mathbb{R}_+=(0,\infty)$) such that

$$\frac{\partial}{\partial s}\big(H(t,s)\,k(s)\big) + \left(\frac{\rho'(s)}{\rho(s)} - p(s)\right) H(t,s)\,k(s) \;=\; -h(t,s)\,\big(H(t,s)\,k(s)\big)^{\alpha/(\alpha+1)} \;.$$

The following three lemmas will be need in the proofs of our results. The first is the well-known Kiguardze's Lemma [7]. The second can be founded in [9]. The third is new and extend Lemma 5.1 in [6] for Eq.(1.1).

Lemma 2.1. *Let $u \in C^n(I,\mathbb{R}_+)$. If $u^{(n)}(t)$ is of constant sign and not identically zero on any interval of the form $[t^*,\infty)$, then there exist a $t_4 \ge t_0$ and integer l, $0 \le l \le n$, with $n+l$ even for $u^{(n)}(t) \ge 0$, or $n+l$ odd for $u^{(n)}(t) \le 0$ and such that*

$$l > 0 \quad \textit{implies that} \quad u^{(k)}(t) > 0 \quad \textit{for} \;\; t \ge t_4, \quad k = 0,1,...,l{-}1$$

and

$$l \le n{-}1 \quad \textit{implies that} \quad (-1)^{l+k}\,u^{(k)}(t) > 0 \quad \textit{for} \;\; t \ge t_4, \quad k = l, l{+}1, ..., n{-}1 \;. \blacksquare$$

Lemma 2.2. *If the function $u(t)$ is as in Lemma 2.1 and*

$$u^{(n-1)}(t)\,u^{(n)}(t) \le 0 \qquad \textit{for any} \;\; t \ge t_u \;,$$

then for every $\lambda \in (0,1)$, we have

$$u(\lambda t) \;\ge\; \frac{2^{1-n}}{(n-1)!}\left[\frac{1}{2} - \left|\lambda - \frac{1}{2}\right|\right]^{n-1} t^{n-1}\,|u^{(n-1)}(t)| \qquad \textit{for all large} \;\; t \;. \blacksquare$$

Lemma 2.3. *Let (A_1)–(A_3) hold. Then, if $x(t)$ is a nonoscillatory solution of Eq.(1.1), we have*

$$x(t)\,x^{(n-1)}(t) > 0, \quad x(t)\,x^{(n)}(t) \le 0 \quad \textit{and} \quad x(t)\,x'(t) > 0 \qquad \textit{for all large} \;\; t \;. \tag{2.1}$$

Proof: Without loss of generality, we may assume that $x(t) > 0$ on $[t_1,\infty)$ for some sufficiently large $t_1 \ge t_0$. As $\lim_{t\to\infty} \tau_{0i}(t) = \infty$, there exists $t_2 \ge t_1$ such that $\tau_{0i}(t) \ge t_1$ for $t \ge t_2$ $(i = 1,2,...,m)$. Hence $x(\tau_{0i}(t)) > 0$ for $t \ge t_2$ $(i = 1,2,...,m)$. By (A_3), we have

$$\big(\Phi(x^{(n-1)}(t))\big)' + p(t)\,\Phi(x^{(n-1)}(t)) \;\le\; 0 \qquad \text{for} \;\; t \ge t_2 \;, \tag{2.2}$$

that is

$$\left(\exp\left(\int_{t_2}^{t} p(s)\,ds\right)\Phi\big(x^{(n-1)}(t)\big)\right)' \le 0 ,$$

it follows that $\exp(\int_{t_2}^{t} p(s)\,ds)\,\Phi(x^{(n-1)}(t))$ is decreasing and $x^{(n-1)}(t)$ is eventually of one sign. If there exists $t_3 \ge t_2$ such that $x^{(n-1)}(t) < 0$ for $t \ge t_3$, we have

$$\begin{aligned}\exp\left(\int_{t_2}^{t} p(s)\,ds\right)\Phi\big(x^{(n-1)}(t)\big) &\le \exp\left(\int_{t_2}^{t_3} p(s)\,ds\right)\Phi\big(x^{(n-1)}(t_3)\big)\\ &=: -M^{\alpha}\exp\left(\int_{t_2}^{t_3} p(s)\,ds\right) , \quad (M>0) .\end{aligned}$$

So

$$\big(-x^{(n-1)}(t)\big)^{\alpha} \ge M^{\alpha}\exp\left(-\int_{t_3}^{t} p(s)\,ds\right) ,$$

that is

$$x^{(n-1)}(t) \le -M\left[\exp\left(-\int_{t_3}^{t} p(s)\,ds\right)\right]^{\frac{1}{\alpha}} \quad \text{for} \quad t \ge t_3 .$$

Integrating it from t_3 to t, we get

$$x^{(n-2)}(t) - x^{(n-2)}(t_3) \le -M\int_{t_3}^{t}\left[\exp\left(-\int_{t_2}^{s} p(\tau)\,d\tau\right)\right]^{\frac{1}{\alpha}} ds .$$

In view of (A_1), it follows that $\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$. Thus, we show that $x^{(n-2)}(t) < 0$ eventually. But, by Lemma 2.1, we find

$$x^{(n-1)}(t) < 0 \quad \text{implies that} \quad x^{(n-2)}(t) > 0 \ \text{ for sufficient large } \ t .$$

Hence, we get a desired contradiction. So we find that $x^{(n-1)}(t) > 0$ eventually.

On the other hand, by (A_1) and (2.2), we have

$$0 \ge \big(\Phi(x^{(n-1)}(t))\big)' = \alpha\left(x^{(n-1)}(t)\right)^{\alpha-1} x^{(n)}(t) ,$$

then $x^{(n)}(t) \le 0$ eventually. Further, when $x^{(n-1)}(t) > 0$ eventually then again from Lemma 2.1, we have $x'(t) > 0$ eventually. Thus, there exist a $t_4 > t_3$ such that

$$x'(t) > 0 , \quad x^{(n-1)}(t) > 0 \quad \text{and} \quad x^{(n)}(t) \le 0 \qquad \text{for all} \ \ t \ge t_4 .$$

This completes the proof. ∎

3 – Main results

For convenience of statement, we shall introduce the following notations without further mentioning. Put

$$M(n,\lambda)=\frac{\lambda\,2^{2-n}}{(n-2)!}\left[\frac{1}{2}-\left|\lambda-\frac{1}{2}\right|\right]^{n-2}, \qquad \beta=\frac{\alpha+1}{\alpha}\ ,$$

$$\theta=(\alpha+1)^{-(\alpha+1)}M^{-\alpha}(n,\lambda)\ , \qquad g(t)=\sigma'(t)\,\sigma^{n-2}(t)\,\rho^{-1/\alpha}(t)$$

and

$$A_T^H\big(\Theta(t,s),t\big) \;=\; \frac{1}{H(t,T)}\int_T^t \Theta(t,s)\,ds\ ,$$

where $\rho,\sigma\in C^1(I,\mathbb{R}^+)$, $H\in C(D,\mathbb{R})$, $\Theta\in C(D,\mathbb{R})$ and $\lambda\in(0,1)$.

In the sequel, we also assume that

($\mathbf{A}_4$) there exists a function $\sigma\in C^1(I,\mathbb{R}_+)$ such that

$$\sigma(t)\le\inf_{i\in J}\big\{t,\tau_{0i}(t)\big\}, \quad \lim_{t\to\infty}\sigma(t)=\infty \quad \text{and} \quad \sigma'(t)>0 \quad \text{for}\ \ t\ge t_0\ ,$$

where $J=\{i:\ \alpha_i>0,\ i=1,2,...,m\}$.

In this paper, we always assume that the conditions (A_1)–(A_4) hold.

Theorem 3.1. *Suppose that there exist functions $H\in C(D,\mathbb{R})$, $h\in C(D_0,\mathbb{R})$, $\rho,k\in C^1(I,\mathbb{R}_+)$ and a constant $\lambda\in(0,1)$ such that H belongs to the class $\Im$, and*

$$\limsup_{t\to\infty} A_{t_0}^H\Big(H(t,s)\,\rho(s)\,q(s)\,k(s)-\theta\,g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\ t\Big) \;=\; \infty\ . \tag{3.1}$$

Then Eq.(1.1) is oscillatory.

Proof: To obtain a contradiction, suppose that $x(t)$ is a nonoscillatory solution of Eq.(1.1). By Lemma 2.3, there exists a $T_0\ge t_4$ such that (2.1) holds. Without loss of generality, we may assume that

$$x(t)>0, \quad x'(t)>0, \quad x^{(n-1)}(t)>0, \quad \text{and} \quad x^{(n)}(t)\le 0 \quad \text{for}\ \ t\ge T_0\ . \tag{3.2}$$

It is easy to check that we can apply Lemma 2.2 for $u=x'$ and conclude that there exists a $T_1\ge T_0$ such that

$$\begin{aligned} x'[\lambda\,\sigma(t)] \;&\ge\; \frac{2^{2-n}}{(n-2)!}\left[\frac{1}{2}-\left|\lambda-\frac{1}{2}\right|\right]^{n-2}\sigma^{n-2}(t)\,x^{(n-1)}[\sigma(t)] \\ &\ge\; \frac{1}{\lambda}\,M(n,\lambda)\,\sigma^{n-2}(t)\,x^{(n-1)}(t) \qquad \text{for}\ \ t\ge T_1\ , \end{aligned} \tag{3.3}$$

since $x^{(n-1)}[\sigma(t)] \geq x^{(n-1)}(t)$ for $t \geq T_1$. Put

$$W(t) = \rho(t)\, \frac{\Phi\big(x^{(n-1)}(t)\big)}{x^{\alpha}[\lambda\, \sigma(t)]} \quad \text{for} \ \ t \geq T_1 \ . \tag{3.4}$$

Then, differentiating (3.4), using (1.1), (3.3) and observing $x[\lambda\, \sigma(t)] \leq x[\tau_{0i}(t)]$, we obtain

$$\begin{aligned} W'(t) \ &\leq \ -\rho(t)\, q(t)\, \frac{\prod_{i=1}^{m} x^{\alpha_i}[\,\tau_{0i}(t)\,]}{x^{\alpha}[\,\lambda\sigma(t)\,]} + \left[\frac{\rho'(t)}{\rho(t)} - p(t)\right] W(t) \\ &\quad - \frac{\alpha\, \lambda\, \rho(t)\, \sigma'(t) [\, x^{(n-1)}(t)\,]^{\alpha}}{x^{\alpha+1}[\,\lambda\sigma(t)\,]}\; x'[\,\lambda\, \sigma(t)\,] \\ &\leq \ -\rho(t)\, q(t) + \left[\frac{\rho'(t)}{\rho(t)} - p(t)\right] W(t) - \alpha\, M(n,\lambda)\, g(t)\, W^{\beta}(t) \ , \end{aligned}$$

that is, for $t \geq T_1$,

$$\rho(t)\, q(t) \ \leq \ -W'(t) + \left(\frac{\rho'(t)}{\rho(t)} - p(t)\right) W(t) - \alpha\, M(n,\lambda)\, g(t)\, W^{\beta}(t) \ . \tag{3.5}$$

Multiplying inequality (3.5) by $H(t,s)\, k(s)$ and integrating from T to t, which in view of (H_3) leads to

$$\begin{aligned} &\int_T^t H(t,s)\, \rho(s)\, q(s)\, k(s)\ ds \\ &\qquad \leq \ H(t,T)\, k(T)\, W(T) \ + \int_T^t |h(t,s)|\, [H(t,s)\, k(s)]^{1/\beta}\, W(s)\ ds \\ &\qquad\quad - \alpha\, M(n,\lambda) \int_T^t H(t,s)\, k(s)\, g(t)\, W^{\beta}(s)\ ds \ . \end{aligned} \tag{3.6}$$

By the Young inequality

$$\begin{aligned} &|h(t,s)|\, [H(t,s)\, k(s)]^{1/\beta}\, W(s) \\ &\qquad \leq \ \theta\, g^{-\alpha}(s)\, |h(t,s)|^{\alpha+1} + \alpha\, M(n,\lambda)\, H(t,s)\, k(s)\, g(s)\, W^{\beta}(s) \ . \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain, for $t > T \geq T_1$,

$$\begin{aligned} &\int_T^t H(t,s)\, \rho(s)\, q(s)\, k(s)\ ds \\ &\qquad \leq \ H(t,T)\, k(T)\, W(T) + \theta \int_T^t g^{-\alpha}(s)\, |h(t,s)|^{\alpha+1}\, ds \ . \end{aligned} \tag{3.8}$$

Then, for $t \geq t_0$,

$$
\begin{aligned}
H(t,t_0)\; A^H_{t_0}\Big(H(t,s)\,\rho(s)\,q(s)\,k(s) - \theta\, g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\ t\Big)& \\
= \ & H(T_1,t_0)\, A^H_{t_0}\Big(H(t,s)\,\rho(s)\,q(s)\,k(s) - \theta\, g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\ T_1\Big) \\
& + H(t,T_1)\, A^H_{T_1}\Big(H(t,s)\,\rho(s)\,q(s)\,k(s) - \theta\, g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\ t\Big) \\
\leq \ & H(t,t_0)\, \left\{ \int_{t_0}^{T_1} \rho(s)\,q(s)\,k(s)\ ds \ +\ k(T_1)\,W(T_1) \right\}\ .
\end{aligned}
$$

Divide the above inequality by $H(t,t_0)$ and take the upper limit as $t \to \infty$. Using (3.1), we obtain a contradiction. This completes the proof. ∎

Remark 3.1. Taking $H(t,s) = t - s$ and $k(s) = 1$, then Theorem 3.1 improves Theorem 2.1 in [1] for Eq.(1.2), and taking $H(t,s) = (t-s)^{v-1}$, $k(s) = 1$ and $\rho(s) = s^l$, for some $v > 2$ and some constant l in case of Eq.(1.3), Theorem 3.1 reduces to the oscillation criteria in [3]. □

Remark 3.2. For Eq.(1.2), Theorem 3.1 improves Theorem 2.1 in [12] by dropping the restriction "$\rho'(t) \geq 0$". For Eq.(1.3), we obtain Theorem 2.1 (X) in [10] and Theorem 2.1 in [11] from Theorem 3.1. □

It may be happen that condition (3.1) in Theorem 3.1 fails to hold. Consequently, Theorem 3.1 does not apply. In the remainder of this paper we treat this cases and give new oscillation theorems for Eq.(1.1).

Theorem 3.2. *Let the functions H, h, ρ, k, and constant λ be as in Theorem 3.1. Further, assume that*

$$
0 < \inf_{s \geq t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,T_0)} \right\} \leq \infty\ , \tag{3.9}
$$

and

$$
\limsup_{t\to\infty}\ A^H_{t_0}\Big(g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\ t\Big)\ <\ \infty\ . \tag{3.10}
$$

If there exists a function $\varphi \in C(I,\mathbb{R})$ such that for $t \geq t_0$, $T \geq t_0$,

$$
\int^{\infty} g(s)\,k^{-1/\alpha}(s)\,[\varphi_+(s)]^{\beta}\ ds\ =\ \infty \tag{3.11}
$$

and

$$\limsup_{t\to\infty} A_T^H\Big(H(t,s)\,\rho(s)\,q(s)\,k(s) - \theta\,g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\ t\Big) \ \ge\ \varphi(T)\ , \tag{3.12}$$

where $\varphi_+ = \max\{\varphi, 0\}$. *Then Eq.(1.1) is oscillatory.*

Proof: Proceeding as in proof of Theorem 3.1 we get (3.6) and (3.8) hold, and return to inequality (3.8). Therefore, for $t > T \ge T_1$,

$$\limsup_{t\to\infty} A_T^H\Big(H(t,s)\,\rho(s)\,q(s)\,k(s) - \theta\,g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\ t\Big) \ \le\ W(T)\,k(T)\ .$$

By (3.12), we have

$$k(T)\,W(t) \ge \varphi(T) \qquad \text{for } \ T \ge T_1\ , \tag{3.13}$$

and

$$\limsup_{t\to\infty} A_{T_1}^H\Big(H(t,s)\,\rho(s)\,q(s)\,k(s),\ t\Big) \ \ge\ \varphi(T_1)\ . \tag{3.14}$$

By (3.6) and (3.14), we see that

$$\begin{aligned} &\liminf_{t\to\infty}\Big\{L A_{T_1}^H\Big(H(t,s)\,k(s)\,g(s)\,W^\beta(s),\ t\Big) - A_{T_1}^H\Big(|h(t,s)|\,\big[H(t,s)\,k(s)\big]^{1/\beta} W(s),\ t\Big)\Big\} \\ &\qquad \le\ k(T_1)\,W(T_1) - \limsup_{t\to\infty} A_{T_1}^H\Big(H(t,s)\,\rho(s)\,q(s)\,k(s),\ t\Big) \\ &\qquad \le\ k(T_1)\,W(T_1) - \varphi(T_1)\ <\ \infty\ , \end{aligned} \tag{3.15}$$

where $L = \alpha\,M(n,\lambda)$.

Now, we claim that

$$\int_{T_1}^{\infty} k(s)\,g(s)\,W^\beta(s)\ ds\ <\ \infty\ . \tag{3.16}$$

Suppose to the contrary that

$$\int_{T_1}^{\infty} k(s)\,g(s)\,W^\beta(s)\ ds\ =\ \infty\ . \tag{3.17}$$

By (3.9), there exists a positive constant $\eta > 0$ satisfying

$$\inf_{s\ge t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\} >\ \eta\ >\ 0\ . \tag{3.18}$$

It follows from (3.17) that for any arbitrary positive number ν there exists a $T_2 \geq T_1$ such that

$$\int_{T_1}^{\infty} k(s)\, g(s)\, W^{\beta}(s)\ ds \ \geq\ \frac{\nu}{\eta} \qquad \text{for all } \ t \geq T_2\ .$$

Therefore

$$\begin{aligned} A^H_{T_1}&\Big(H(t,s)\, k(s)\, g(s)\, W^{\beta}(s),\ t\Big) \\ &= \frac{1}{H(t,T_1)} \int_{T_1}^{t} H(t,s)\ d\bigg(\int_{T_1}^{s} k(\tau)\, g(\tau)\, W^{\beta}(\tau)\ d\tau\bigg) \\ &\geq \frac{1}{H(t,T_1)} \int_{T_2}^{t} \bigg(\int_{T_1}^{s} k(\tau)\, g(\tau)\, W^{\beta}(\tau)\ d\tau\bigg) \Big(-\frac{\partial}{\partial s} H(t,s)\Big)\ ds \\ &\geq \frac{\nu}{\eta}\, \frac{1}{H(t,T_1)} \int_{T_2}^{t} \Big(-\frac{\partial}{\partial s} H(t,s)\Big)\, ds \ = \ \frac{\nu}{\eta}\, \frac{H(t,T_2)}{H(t,T_1)}\ . \end{aligned}$$

By (3.18), there exists a $T_3 \geq T_2$ such that $H(t,T_2)/H(t,T_1) \geq \eta$ for all $t \geq T_3$, which implies

$$A^H_{T_1}\Big(H(t,s)\, k(s)\, g(s)\, W^{\beta}(s),\ t\Big) \ \geq\ \nu \qquad \text{for all } \ t \geq T_3\ .$$

Since ν is arbitrary, we conclude that

$$\lim_{t\to\infty} A^H_{T_1}\Big(H(t,s)\, k(s)\, g(s)\, W^{\beta}(s),\ t\Big) \ =\ \infty\ . \tag{3.19}$$

Next, let us consider a sequence $\{t_j\}_1^\infty$ in $[t_0,\infty)$ with $\lim_{j\to\infty} t_j = \infty$ satisfying

$$\begin{aligned} &\lim_{j\to\infty}\Big\{L A^H_{T_1}\Big(H(t_j,s)\, k(s)\, g(s)\, W^{\beta}(s), t_j\Big) - A^H_{T_1}\Big(|h(t_j,s)|\big[H(t_j,s)k(s)\big]^{1/\beta} W(s), t_j\Big)\Big\} \\ &= \liminf_{t\to\infty}\Big\{L A^H_{T_1}\Big(H(t,s)\, k(s)\, g(s)\, W^{\beta}(s), t\Big) - A^H_{T_1}\Big(|h(t,s)|\big[H(t,s)k(s)\big]^{1/\beta} W(s), t\Big)\Big\}. \end{aligned}$$

In view of (3.15), there exists a constant M_0 such that

$$\begin{aligned} L\, A^H_{T_1}&\Big(H(t_j,s)\, k(s)\, g(s)\, W^{\beta}(s),\ t_j\Big) \\ &- A^H_{T_1}\Big(|h(t_j,s)|\ \big[H(t_j,s)\, k(s)\big]^{1/\beta}\, W(s),\ t_j\Big) \ \leq\ M_0\ , \end{aligned} \tag{3.20}$$

for all sufficient large j. It follows from (3.19) that

$$\lim_{j\to\infty} A^H_{T_1}\Big(L\, H(t_j,s)\, k(s)\, g(s)\, W^{\beta}(s),\ t_j\Big) \ =\ \infty\ . \tag{3.21}$$

This and (3.20) give

$$\lim_{j\to\infty} A^H_{T_1}\Big(|h(t_j,s)|\,\big[H(t_j,s)\,k(s)\big]^{1/\beta}\,W(s),\,t_j\Big) \;=\; \infty\;. \tag{3.22}$$

Thus, by (3.20) and (3.21), for all enough large j,

$$\frac{A^H_{T_1}\Big(|h(t_j,s)|\,\big[H(t_j,s)\,k(s)\big]^{1/\beta}\,W(s),\,t_j\Big)}{L\,A^H_{T_1}\Big(H(t_j,s)\,k(s)\,g(s)\,W^{\beta}(s),\,t_j\Big)} - 1 \;\geq\; -\frac{1}{2}\;.$$

That is

$$\frac{A^H_{T_1}\Big(|h(t_j,s)|\,\big[H(t_j,s)\,k(s)\big]^{1/\beta}\,W(s),\,t_j\Big)}{A^H_{T_1}\Big(H(t_j,s)\,k(s)\,g(s)\,W^{\beta}(s),\,t_j\Big)} \;\geq\; \frac{1}{2}\,L \qquad \text{for all large enough } \; j\;.$$

This and (3.22) imply

$$\frac{\Big[A^H_{T_1}\Big(|h(t_j,s)|\,\big[H(t_j,s)\,k(s)\big]^{1/\beta}\,W(s),\;t_j\Big)\Big]^{\alpha+1}}{\Big[A^H_{T_1}\Big(H(t_j,s)\,k(s)\,g(s)\,W^{\beta}(s),\;t_j\Big)\Big]^{\alpha}} \;=\; \infty\;. \tag{3.23}$$

On the other hand, by Hölder's inequality, we have

$$\begin{aligned}&\Big[A^H_{T_1}\Big(|h(t_j,s)|\,\big[H(t_j,s)\,k(s)\big]^{1/\beta}\,W(s),\;t_j\Big)\Big]^{\alpha+1}\\ &\qquad\leq\;\Big[A^H_{T_1}\Big(H(t_j,s)\,k(s)\,g(s)\,W^{\beta}(s),\;t_j\Big)\Big]^{\alpha}\,\Big[A^H_{T_1}\Big(g^{-\alpha}(s)\,|h(t_j,s)|^{\alpha+1},\;t_j\Big)\Big].\end{aligned}$$

It follows that, for all large enough j,

$$\frac{\Big[A^H_{T_1}\Big(|h(t_j,s)|\,\big[H(t_j,s)\,k(s)\big]^{1/\beta}\,W(s),\;t_j\Big)\Big]^{\alpha+1}}{\Big[A^H_{T_1}\Big(H(t_j,s)\,k(s)\,g(s)\,W^{\beta}(s),\;t_j\Big)\Big]^{\alpha}} \;\leq\; A^H_{T_1}\Big(g^{-\alpha}(s)\,|h(t_j,s)|^{\alpha+1},\,t_j\Big)\;.$$

By (3.23), we find

$$\lim_{j\to\infty} A^H_{T_1}\Big(g^{-\alpha}(s)\,|h(t_j,s)|^{\alpha+1},\,t_j\Big) \;=\; \infty\;,$$

which contradicts to (3.10). Hence, (3.16) holds. Finally, by (3.13), we obtain

$$\int_{t_0}^{\infty} g(s)\,k^{-1/\alpha}(s)\,[\varphi_+(s)]^{\beta}\;ds \;\leq\; \int_{t_0}^{\infty} k(s)\,g(s)\,W^{\beta}(s)\;ds \;<\;\infty\;,$$

which contradicts (3.11). This completes the proof. ∎

Following the procedure of the proof of Theorem 3.2, we can also prove the following two theorems.

Theorem 3.3. *Let the functions H, h and ρ, k and constant λ be as in Theorem 3.1, and assume that (3.9) holds. Suppose that there exists a function $\varphi \in C(I, \mathbb{R})$ such that (3.11) and the condition*

$$\liminf_{t\to\infty} A_{t_0}^H\Big(g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\, t\Big) \;<\; \infty \tag{3.24}$$

and

$$\liminf_{t\to\infty} A_T^H\Big(H(t,s)\,\rho(s)\,q(s)\,k(s) - \theta\, g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\, t\Big) \;\geq\; \varphi(T) \tag{3.25}$$

hold for all $T \geq t_0$. Then Eq.(1.1) is oscillatory. ∎

Theorem 3.4. *Let the functions H, h, ρ, k and constant λ as in Theorem 3.1, and assume that (3.9) holds. Suppose that there exists a function $\varphi \in C(I, \mathbb{R})$ such that (3.11), (3.25) and the condition*

$$\liminf_{t\to\infty} A_{t_0}^H\Big(H(t,s)\,\rho(s)\,q(s)\,k(s),\, t\Big) \;<\; \infty \tag{3.26}$$

hold for all $T \geq t_0$. Then Eq.(1.1) is oscillatory. ∎

Remark 3.3. The results obtained here are presented in a form which is essentially new. Since the functions $\tau_{ki}(t)$ $(k = 0, 2, ..., n-1,\; i = 1, 2, ..., m)$ have not to assume any particular form, Eq.(1.1) can be any ordinary, retarded, advanced or mixed type equations. Hence Theorems 3.1–3.4 hold for all that kind of equations. □

Remark 3.4. The above Theorems 3.2–3.4 extend and improve Theorems 2.2–2.4 in [12]. □

For illustration, we consider the following two examples.

Example 3.1. Consider the following delay differential equation

$$\Big(|x^{(n-1)}(t)|\, x^{(n-1)}(t)\Big)' + p(t)\,|x^{(n-1)}(t)|\, x^{(n-1)}(t) + q(t)\, x(t-\tau)\, x(t-\sigma) = 0\,, \tag{3.27}$$

for $t \geq t_0 = \{1, 1+\max\{\tau, \sigma\}\}$, where $\alpha = 2$, n is a even number, and τ, σ are constants, $p, q \in C(I, \mathbb{R}_0)$, $0 \leq p(t) \leq c_1 t^{-1}$, $0 \leq c_1 \leq 1$.

Let $\rho(t)=\exp(\int_{t_0}^{t}p(u)\,du\,)$, then $\rho(t)\le t_0^{-c_1}t^{c_1}$, and

$$\frac{\rho'(t)}{\rho(t)}-p(t)=0\ ,\qquad \exp\Big(-\int_{t_0}^{t}p(u)\,du\Big)\ge\frac{t_0^{c_1}}{t}\ .$$

So, conditions (A_1) and (A_2) are satisfied. Here, we define

$$\sigma(t)\ =\ \begin{cases} t, & \text{if}\ \ \tau,\sigma\le 0\,,\\ t-\tau, & \text{if}\ \ \tau\ge 0,\ \ \tau\ge\sigma\,,\\ t-\sigma, & \text{if}\ \ \sigma\ge 0,\ \ \sigma\ge\tau\ ,\end{cases}$$

then $\sigma(t)\ge 1$, $\sigma'(t)=1$ and $g(t)\ge t_0^{c_1/2}t^{-c_1/2}$ for all $t\ge t_0$.

Choosing $q(t)$ such that $\rho(t)\,q(t)\ge c_2\,t^{-1}$, $(c_2>0)$, and taking $H(t,s)=(t-s)^\delta$, $k(s)\equiv 1$, $\delta<3$ is integer, then $h(t,s)=\delta(t-s)^{\delta/3-1}$. Thus

$$g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1}\ \le\ \delta^3\,t_0^{-c_1}(t-s)^{(\delta-3)}\,s\ .$$

It follows from [4] that

$$(t-s)^\delta\ge t^\delta-\delta st^{\delta-1}\qquad\text{for}\ \ t\ge s\ge 1\ .$$

By using this inequality, we obtain that

$$\begin{aligned}\limsup_{t\to\infty}A_{t_0}^{H}&\Big(H(t,s)\,\rho(s)\,q(s)\,k(s)-\theta\,g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\ t\Big)\\ &\ge\ \limsup_{t\to\infty}\frac{1}{(t-t_0)^\delta}\int_{t_0}^{t}\Big[\frac{c_2(t^\delta-\delta st^{\delta-1})}{s}-\theta\,\delta^3\,t_{t_0}^{c_1}(t-s)^{\delta-3}\,s\Big]\,ds\\ &=\ \limsup_{t\to\infty}\big\{c_2(\,\ln t-\delta\,)\big\}\ =\ \infty\ .\end{aligned}$$

Thus, all conditions of Theorem 3.1 are satisfied and Eq.(3.27) is oscillatory. ▫

Example 3.2. Consider even order nonlinear delay equation

$$\begin{aligned}\Big(|x^{(n-1)}(t)|^{\alpha-1}\,x^{(n-1)}(t)\Big)'&+\,c_1\,t^{-1}\,|x^{(n-1)}(t)|^{\alpha-1}\,x^{(n-1)}(t)\\ &+\,q(t)\,\Big|x\Big(\frac{t}{2}\Big)\Big|^{\alpha-1}x\Big(\frac{3t}{4}\Big)\ =\ 0\ ,\end{aligned}\tag{3.28}$$

for $t\ge 1$, where n is even number, $q\in C(I,\mathbb{R}_0)$, $0\le c_1\le 1$, $2>\alpha>0$ with $n\alpha\ge 2(\alpha+1)$.

Choosing $\rho(t) = \exp(\int_1^t p(u)\,du\,) = t^{c_1}$ such that $\rho(t)\,q(t) \geq c_2/t^2$. Note that

$$\sigma(t) = \frac{t}{2}, \qquad \text{and} \qquad g(t) = \frac{1}{2^{n-1}}\, t^{(n-2)-c_1/\alpha} \ .$$

Taking $H(t,s) = (t-s)^2$, $k(s) \equiv 1$ for $t \geq s \geq 1$, then $|h(t,s)|^{\alpha+1} = 2^{\alpha+1}(t-s)^{1-\alpha}$. Since, $n\alpha \geq 2\alpha + 1$, we have

$$\begin{aligned} A_1^H\Big(g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\, t\Big) \;&=\; \frac{2^{\alpha n+1}}{(t-1)^2}\int_1^t s^{c_1+(2-n)\alpha}\,(t-s)^{1-\alpha}\;ds \\ &\leq\; \frac{2^{\alpha n+1}}{(t-1)^2}\int_1^t s^{1+(2-n)\alpha}\,(t-s)^{1-\alpha}\;ds \\ &\leq\; \frac{2^{\alpha n+1}}{(t-1)^2}\,\frac{(t-1)^{2-\alpha}}{2-\alpha}\ . \end{aligned}$$

So, Condition (3.10) is satisfied. On the other hand, for $t \geq T$,

$$\begin{aligned} &\limsup_{t\to\infty} A_T^H\Big(H(t,s)\,\rho(s)\,q(s)\,k(s) - \theta\, g^{-\alpha}(s)\,|h(t,s)|^{\alpha+1},\, t\Big) \\ &\qquad \geq\; \limsup_{t\to\infty}\left\{\frac{1}{(t-T)^2}\int_T^t (t-s)^2\,\frac{c_2}{s^2}\;ds \;-\; \frac{\theta\, 2^{\alpha n+1}}{2-\alpha}\,\frac{1}{(t-T)^\alpha}\Big(1-\frac{T}{t}\Big)^{2-\alpha}\right\} \\ &\qquad \geq\; \frac{c_2}{T}\ . \end{aligned}$$

Set $\varphi(T) = c_2/T$. It is clear that

$$\begin{aligned} \int^\infty g(s)\,k^{-1/\alpha}(s)\,[\varphi(s)]_+^\beta\;ds \;&\geq\; \frac{c_2^\beta}{2^{n-1}}\int^\infty s^{(n-3)-2/\alpha}\,ds \\ &\geq\; \frac{c_2^\beta}{2^{n-1}}\int^\infty s^{-1}\,ds \;=\; \infty\ . \end{aligned}$$

Thus, all hypotheses of Theorem 3.2 are satisfied, and Eq.(3.28) is oscillatory. □

Remark 3.5. The results in this paper are presented in the form of a high degree of generality. New oscillatory criteria can be obtained with the appropriate choices of the functions H and k. For instance, one can apply Theorems 3.1–3.4 with

$$H(t,s) = \left(\int_s^t \frac{d\tau}{\xi(\tau)}\right)^{\delta-1}, \qquad (t,s)\in D, \quad k(s) = s^l\ .$$

where $\delta > \alpha$ and l are constants, $\xi \in C(I,\mathbb{R}_+)$ with $\int_{t_0}^\infty 1/\xi(\tau)\,d\tau = \infty$. For example, an important particular case is $\xi(\tau)=\tau^\gamma$, $\gamma\leq 1$ is real number. □

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