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SOLUTIONS FOR SINGULAR CRITICAL GROWTH SCHRÖDINGER EQUATIONS WITH MAGNETIC FIELD

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Abstract: In this paper, we consider the semilinear stationary Schrödinger equation with a magnetic field: $-\Delta_A u - V(x)u = |u|^{2^*-2}u$ in $\mathbb{R}^N$, where A is the vector (or magnetic) potential and V is the scalar (or electric) potential. By means of variational method, we establish the existence of nontrivial solutions in the critical case.

1 – Introduction and main result

In this paper, we are concerned with the semilinear Schrödinger equation

$$-\Delta_A u - V(x)u = |u|^{2^*-2}u\ , \qquad x\in\mathbb{R}^N\ , \tag{1.1}$$

where $-\Delta_A = (-i\nabla+A)^2$, $u\colon\mathbb{R}^N\to\mathbb{C}$, $N\geq 3$, $2^*=\frac{2N}{N-2}$ denotes the critical Sobolev exponent, $A=(A_1,A_2,...,A_N)\colon\mathbb{R}^N\to\mathbb{R}^N$ is the vector (or magnetic) potential, the coefficient V is the scalar (or electric) potential and may be sign-changing.

The nonlinear Schrödinger equation arises in different physical theories (e.g., the description of Bose–Estein condensates and nonlinear optics), and has been widely considered in the literature, see [1, 6, 7, 8, 11, 13].

Throughout this paper, suppose $A\in L^2_{\rm loc}(\mathbb{R}^N,\mathbb{R}^N)$. Define

$$L^2(\mathbb{R}^N,V^-dx) := \left\{u\colon\mathbb{R}^N\to\mathbb{C}\ \Big|\ \int_{\mathbb{R}^N}|u|^2\,V^-\,dx<\infty\right\}$$

and

$$H^1_{A,V^-}(\mathbb{R}^N) := \left\{u\in L^2(\mathbb{R}^N,V^-dx)\ \Big|\ \nabla_A u\in L^2(\mathbb{R}^N)\right\}\ ,$$

where $\nabla_A=(\nabla+iA)$, $V^\pm=\max\{\pm V,0\}\not\equiv 0$.

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$H^1_{A,V^-}(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\int_{\mathbb{R}^N} \big(\nabla_A u \cdot \overline{\nabla_A v} + V^- u\overline{v}\big)\, dx\ ,$$

where the bar denotes complex conjugation.

It is known that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1_{A,V^-}(\mathbb{R}^N)$ (see [8]).

Definition 1.1. $u \in H^1_{A,V^-}(\mathbb{R}^N)$ is said to be a weak solution of problem (1.1) if

$$\int_{\mathbb{R}^N} \Big(\nabla_A u \cdot \overline{\nabla_A \varphi} - V(x)\, u\, \overline{\varphi} - |u|^{2^*-2}\, u\, \overline{\varphi}\Big)\, dx \;=\; 0 \qquad \forall\, \varphi \in H^1_{A,V^-}(\mathbb{R}^N)\ . \ \square \tag{1.2}$$

The corresponding energy functional of problem (1.1) is defined by

$$I_{A,V}(u) \;=\; \frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla_A u|^2 - V(x)|u|^2\big)\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*} dx,\ \ u \in H^1_{A,V^-}(\mathbb{R}^N)\,. \tag{1.3}$$

It is well known that the nontrivial solutions of problem (1.1) are equivalent to the nonzero critical points of $I_{A,V}$ in $H^1_{A,V^-}(\mathbb{R}^N)$.

Now we list some assumptions on the potential V:

($\mathbf{A}_1$) $0 \not\equiv V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and $V \in L^{\frac{N}{2}}(\mathbb{R}^N \backslash B_R(0))$ for any $R > 0$. Moreover, there exist $\delta > 0, \lambda > 0$ such that

$$|x|^2\, V(x) = \mu + \lambda |x|^\alpha\ , \qquad \forall\, x \in B_\delta(0)\ ,$$

where $0 < \mu < \overline{\mu} = (\frac{N-2}{2})^2$ and $0 < \alpha < \min\{2,\, 2\sqrt{\overline{\mu} - \mu}\}$.

($\mathbf{A}_2$) There is $\theta \in (0,1)$ such that

$$\int_{\mathbb{R}^N} V^+(x)\, |u|^2\, dx \;\le\; \theta \int_{\mathbb{R}^N}\big(|\nabla_A u|^2 + V^-|u|^2\big)\, dx \qquad \text{for any}\ \ u \in H^1_{A,V^-}(\mathbb{R}^N)\ .$$

Remark. There does exist such potential V satisfying assumptions (A_1), (A_2). For example, take $0 < \delta < 1$

$$V(x) \;=\; \begin{cases} \dfrac{\mu}{|x|^2} + \lambda |x|^{\alpha-2} & \text{if}\ \ |x| \le \delta\,, \\[2ex] -\dfrac{k}{|x|^\beta} & \text{if}\ \ |x| > \delta\ , \end{cases}$$

where $0 < \lambda < \overline{\mu} - \mu$, $\beta > 2$ and $k > 0$. $\square$

A direct computation shows that $V(x)$ satisfies assumptions $(\mathrm{A}_1), (\mathrm{A}_2)$.

Our main result is the following:

Theorem 1.1. *Assume that* $(A_1), (A_2)$ *hold, and* A *is continuous at* 0. *Then problem (1.1) admits at least one nontrivial solution.*

We prove Theorem 1.1 by critical point theory. However, since the functional $I_{A,V}$ does not satisfy the Palais–Smale condition due to the lack of compactness of the embedding: $H^1_{A,V^-}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, the standard variational argument is not applicable directly. In addition, from assumption (A_1), the potential V has a strong singularity at the origin, which also brings some difficulty in dealing with (1.1). Precisely, the embedding: $H^1_{A,V^-}(\Omega) \hookrightarrow L^2(\Omega, |x|^{-2}dx)$ is continuous but not compact, where $\Omega \ni 0$ is an arbitrary bounded set in $\mathbb{R}^N$. Nevertheless, we can prove that $I_{A,V}$ satisfies the $(P.S.)_c$ condition with c below some energy level. We need to construct a suitable $(P.S.)_c$ compact sequence, which is obtained by the mountain-pass theorem (see [3]).

Throughout this paper, we shall denote the norm of the space $H^1_{A,V^-}(\mathbb{R}^N)$ by $\|u\|_{H^1_{A,V^-}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V^-|u|^2)\,dx)^{\frac{1}{2}}$, and the positive constants (possibly different) by C, C_1, C_2, ...

2 – Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we introduce some notations and preliminary lemmas.

Set

$$S_\mu := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \mu\,\frac{u^2}{|x|^2}\right)dx}{\left(\displaystyle\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{\frac{2}{2^*}}}\,.$$

From [9, 10], S_μ is independent of any $\Omega \subset \mathbb{R}^N$ in the sense that if

$$S_\mu(\Omega) := \inf_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\Omega}\left(|\nabla u|^2 - \mu\,\frac{u^2}{|x|^2}\right)dx}{\left(\displaystyle\int_{\Omega}|u|^{2^*}dx\right)^{\frac{2}{2^*}}}\,,$$

then $S_\mu(\Omega) = S_\mu(\mathbb{R}^N) = S_\mu$.

Let $\bar{\mu} = (\frac{N-2}{2})^2$, $\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}$, $\gamma' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}$, F. Catrina and Z.Q. Wang [5], S. Terracini [14] proved that for $\epsilon > 0$

$$U_\epsilon(x) = \frac{\Big(4\,\epsilon^2 N(\bar{\mu} - \mu)/(N-2)\Big)^{\frac{N-2}{4}}}{\Big(\epsilon^2\,|x|^{\frac{\gamma'}{\sqrt{\bar{\mu}}}} + |x|^{\frac{\gamma}{\sqrt{\bar{\mu}}}}\Big)^{\sqrt{\bar{\mu}}}}$$

satisfies

$$\begin{cases} -\Delta u = |u|^{2^*-2}\,u + \mu\,\frac{u}{|x|^2} & \text{in } \ \mathbb{R}^N\backslash\{0\}\ , \\ u \to 0 & \text{as } \ |x| \to \infty\ . \end{cases}$$

Moreover, U_ϵ achieves S_μ.

Lemma 2.1. *For any bounded set $\Omega \subset \mathbb{R}^N$, $0 \in \Omega$, the embedding: $H^1(\Omega) \hookrightarrow L^2(\Omega, |x|^l)$ is compact with $l > -2$.*

Proof: Let $\{u_m\} \subset H^1(\Omega)$ be a bounded sequence. Then, up to a subsequence, we may assume

$$u_m \rightharpoonup u \quad \text{weakly in } \ H^1(\Omega)\ ;$$

$$u_m \to u \quad \text{strongly in } \ L^p(\Omega) \ \text{ with } \ 1<p<2^*\ ;$$

$$u_m \to u \quad \text{a.e. in } \ \Omega\ .$$

Choose $\max\{2, \frac{2N}{N+l}\} < q < 2^*$. Then

$$\int_\Omega |x|^l\,|u_m - u|^2\,dx \ \le\ \Big(\int_\Omega |x|^{\frac{lq}{q-2}}\,dx\Big)^{\frac{q-2}{q}}\,\Big(\int_\Omega |u_m - u|^q\,dx\Big)^{\frac{2}{q}}.$$

By the choice of q, we easily have

$$\int_\Omega |x|^{\frac{lq}{q-2}}\,dx \ \le\ C \quad \text{and} \quad \lim_{m\to\infty}\int_\Omega |u_m - u|^q\,dx \ =\ 0\ .$$

Thus, $\lim\limits_{m\to\infty} \int_\Omega |x|^l |u_m - u|^2\,dx = 0$. ∎

Lemma 2.2. *The functional $I_{A,V}$ satisfies the $(P.S.)_c$ condition with $c < \frac{1}{N}\,S_\mu^{\frac{N}{2}}$.*

Proof: Assume that $\{u_m\} \subset H^1_{A,V^-}(\mathbb{R}^N)$ satisfies

$$I_{A,V}(u_m) \to c \quad \text{and} \quad dI_{A,V}(u_m) \to 0 \ \text{ as } \ m \to \infty \ .$$

Then, by assumption (A_2), we get

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} \Big(|\nabla_A u_m|^2 + V^-|u_m|^2\Big)\, dx \ = \\ = \ I_{A,V}(u_m) - \frac{1}{2^*}\, \langle dI_{A,V}(u_m), u_m\rangle + \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} V^+(x)\, |u_m|^2\, dx \\ \le \ c + \left(\frac{1}{2} - \frac{1}{2^*}\right) \theta \int_{\mathbb{R}^N} \Big(|\nabla_A u_m|^2 + V^-|u_m|^2\Big)\, dx \ + \ o(1) \ ,\end{aligned}$$

which implies $\displaystyle\int_{\mathbb{R}^N} \big(|\nabla_A u_m|^2 + V^-|u_m|^2\big)\, dx \le C$.

By choosing a subsequence if necessary, we may assume that

$$u_m \rightharpoonup u \quad \text{weakly in } \ H^1_{A,V^-}(\mathbb{R}^N) \qquad \text{and} \qquad u_m \to u \quad \text{a.e. on } \ \mathbb{R}^N \ .$$

It is easy to verify that u is a weak solution of problem (1.1). Set $u_m = v_m + u$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} \Big(|\nabla_A v_m|^2 - V|v_m|^2\Big)\, dx \ = \\ = \int_{\mathbb{R}^N} \Big(|\nabla_A u_m|^2 - V|u_m|^2\Big)\, dx \ - \int_{\mathbb{R}^N} \Big(|\nabla_A u|^2 - V|u|^2\Big)\, dx \ + \ o(1)\end{aligned}$$

and by Brezis–Lieb lemma (see [2])

$$\int_{\mathbb{R}^N} |v_m|^{2^*} dx \ = \int_{\mathbb{R}^N} |u_m|^{2^*} dx \ - \int_{\mathbb{R}^N} |u|^{2^*} dx \ + \ o(1) \ .$$

Therefore, we get

$$\big\langle dI_{A,V}(v_m), v_m\big\rangle \ = \ \big\langle dI_{A,V}(u_m), u_m\big\rangle - \big\langle dI_{A,V}(u), u\big\rangle + o(1) \ = \ o(1) \ .$$

Thus,

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \Big(|\nabla_A v_m|^2 - V|v_m|^2\Big)\, dx \ = \lim_{m\to\infty} \int_{\mathbb{R}^N} |v_m|^{2^*} dx \ = \ a \ , \tag{2.1}$$

where a is a nonnegative number.

If $a = 0$, then we infer

$$\begin{aligned}\lim_{m\to\infty} \int_{\mathbb{R}^N} \Big(|\nabla_A v_m|^2 + V^-|v_m|^2\Big)\, dx \ &= \ \lim_{m\to\infty} \int_{\mathbb{R}^N} V^+|v_m|^2\, dx \\ &\le \ \theta \lim_{m\to\infty} \int_{\mathbb{R}^N} \Big(|\nabla_A v_m|^2 + V^-|v_m|^2\Big)\, dx \ ,\end{aligned}$$

which implies

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}\Big(|\nabla_A v_m|^2+V^-|v_m|^2\Big)\,dx \;=\; 0\;. \tag{2.2}$$

If $a>0$ then, by Sobolev inequality, we obtain

$$\begin{aligned}\Big(\int_{\mathbb{R}^N}|v_m|^{2^*}dx\Big)^{\frac{2}{2^*}} &\le\; S_\mu^{-1}\int_{\mathbb{R}^N}\Big(\big|\nabla|v_m|\big|^2-\frac{\mu|v_m|^2}{|x|^2}\Big)\,dx\\ &\le\; S_\mu^{-1}\int_{\mathbb{R}^N}\Big(|\nabla_A v_m|^2-\frac{\mu|v_m|^2}{|x|^2}\Big)\,dx\\ &\le\; S_\mu^{-1}\Big(\int_{\mathbb{R}^N}|\nabla_A v_m|^2\,dx\;-\int_{B_\delta(0)}\frac{\mu|v_m|^2}{|x|^2}\,dx\Big)\\ &\le\; S_\mu^{-1}\int_{\mathbb{R}^N}\Big(|\nabla_A v_m|^2-V(x)\,|v_m|^2\Big)\,dx\\ &\quad+\lambda\,S_\mu^{-1}\int_{B_\delta(0)}|x|^{\alpha-2}\,|v_m|^2\,dx\;+\;S_\mu^{-1}\int_{\mathbb{R}^N\setminus B_\delta(0)}V(x)\,|v_m|^2\,dx\;,\end{aligned} \tag{2.3}$$

where we use the diamagnetic inequality in the above argument (see [12]):

$$\big|\nabla|u|\big|\le|\nabla_A u|\qquad\text{a.e. in }\;\mathbb{R}^N\;.$$

Hence, by assumption (A_1), Lemma 2.1 and (2.1), (2.3), we derive $a^{\frac{2}{2^*}}\le S_\mu^{-1}a$, and then $a\ge S_\mu^{\frac{N}{2}}$.

In addition,

$$I_{A,V}(u)\;=\;I_{A,V}(u)-\frac{1}{2}\big\langle dI_{A,V}(u),u\big\rangle\;=\;\frac{1}{N}\int_{\mathbb{R}^N}|u|^{2^*}dx\;\ge\;0\;.$$

Therefore,

$$\begin{aligned}c\;&=\;I_{A,V}(u_m)+o(1)\\ &=\;I_{A,V}(v_m)+I_{A,V}(u)+o(1)\\ &\ge\;\frac{1}{2}\int_{\mathbb{R}^N}\Big(|\nabla_A v_m|^2-V(x)\,|v_m|^2\Big)\,dx\;-\;\frac{1}{2^*}\int_{\mathbb{R}^N}|v_m|^{2^*}dx\;+\;o(1)\\ &=\;\Big(\frac{1}{2}-\frac{1}{2^*}\Big)\,a\\ &\ge\;\frac{1}{N}\,S_\mu^{\frac{N}{2}}\;,\end{aligned}$$

which contradicts $c<\frac{1}{N}\,S_\mu^{\frac{N}{2}}$. ∎

Define

$$c_A \;=\; \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{A,V}(\gamma(t)) \;,$$

where $\Gamma = \bigl\{\gamma \in C\bigl([0,1], H^1_{A,V^-}(\mathbb{R}^N)\bigr) \bigm| \gamma(0)=0,\; I_{A,V}(\gamma(1))<0\bigr\}$.

Lemma 2.3. *Let the assumptions of Theorem 1.1 hold. Then* $c_A < \frac{1}{N}\, S_\mu^{\frac{N}{2}}$.

Proof: Since $A(x)$ is continuous at 0, we infer $|A(x)| \le c_0$ for all $|x| \le \eta(\le \delta)$. Set $u_\epsilon(x) = \psi(x)\, U_\epsilon(x)$, where ψ is a cut off function satisfying $\psi(x) \equiv 1$ if $|x| \le \frac{\eta}{2}$, $\psi(x) \equiv 0$ if $|x| \ge \eta$ and $0 \le \psi(x) \le 1$.

Following [3] and after a detailed calculation, we have the following estimates:

$$\int_{\mathbb{R}^N} \left(|\nabla(\psi U_\epsilon)|^2 - \mu\frac{|\psi U_\epsilon|^2}{|x|^2}\right) dx \;=\; S_\mu^{\frac{N}{2}} + O(\epsilon^{N-2}) \;, \tag{2.4}$$

$$\int_{\mathbb{R}^N} |\psi U_\epsilon|^{2^*}\, dx \;=\; S_\mu^{\frac{N}{2}} + O(\epsilon^{N}) \;, \tag{2.5}$$

$$\int_{\mathbb{R}^N} |\psi U_\epsilon|^2\, dx \;\approx\; \beta(\epsilon) \;=\; \begin{cases} \epsilon^{\frac{N-2}{\sqrt{\overline{\mu}-\mu}}}, & \text{if } \; 0<\mu<\overline{\mu}-1\,, \\ \epsilon^{N-2}\,|\log\epsilon| & \text{if } \; \mu=\overline{\mu}-1\,, \\ \epsilon^{N-2} & \text{if } \; \overline{\mu}-1<\mu<\overline{\mu}\;, \end{cases} \tag{2.6}$$

$$\int_{\mathbb{R}^N} |x|^{\alpha-2}\, |\psi U_\epsilon|^2\, dx \;\approx\; \epsilon^{\frac{\alpha\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}} \;, \tag{2.7}$$

where $A_\epsilon \approx B_\epsilon$ means $C_1 B_\epsilon \le A_\epsilon \le C_2 B_\epsilon$.

Observe that

$$\begin{aligned} \int_{\mathbb{R}^N} \Bigl(|\nabla_A u_\epsilon|^2 - V(x)\,|u_\epsilon|^2\Bigr)\, dx \;&= \\ &= \int_{\mathbb{R}^N} \Bigl(|\nabla(\psi U_\epsilon)|^2 + |A|^2\,|\psi U_\epsilon|^2 - V(x)\,|\psi U_\epsilon|^2\Bigr)\, dx \\ &= \int_{\mathbb{R}^N} \left(|\nabla(\psi U_\epsilon)|^2 - \mu\,\frac{|\psi U_\epsilon|^2}{|x|^2}\right) dx \\ &\quad + \int_{\mathbb{R}^N} |A|^2\,|\psi U_\epsilon|^2\, dx \;-\; \lambda \int_{\mathbb{R}^N} |x|^{\alpha-2}\,|\psi U_\epsilon|^2\, dx \\ &\le\; S_\mu^{\frac{N}{2}} - C_1\,\epsilon^{\frac{\alpha\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}} + C_2\,\beta(\epsilon) + O(\epsilon^{N-2}) \;, \end{aligned} \tag{2.8}$$

where $\beta(\epsilon)$ is given by (2.6).

Therefore, from (2.4)–(2.8), we conclude

$$
\begin{aligned}
c_A &\le \max_{t\ge 0} I_{A,V}(tu_\epsilon) \\
&= \frac{1}{N}\left(\frac{\int_{\mathbb{R}^N}\left(|\nabla_A u_\epsilon|^2 - V(x)\,|u_\epsilon|^2\right)dx}{\left(\int_{\mathbb{R}^N}|u_\epsilon|^{2^*}dx\right)^{\frac{2}{2^*}}}\right)^{\frac{N}{2}} \\
&\le \frac{1}{N}\left(\frac{S_\mu^{\frac{N}{2}} - C_1\,\epsilon^{\frac{\alpha\sqrt{\bar\mu}}{\sqrt{\bar\mu-\mu}}} + C_2\,\beta(\epsilon) + O(\epsilon^{N-2})}{S_\mu^{\frac{N-2}{2}} + O(\epsilon^{N-2})}\right)^{\frac{N}{2}} \\
&< \frac{1}{N}\,S_\mu^{\frac{N}{2}} \quad \left(\text{by the choice of } \alpha:\ 0<\alpha<\min\left\{2,\,2\sqrt{\bar\mu-\mu}\right\}\right). \ \blacksquare
\end{aligned}
$$

Proof of Theorem 1.1: By assumption (A_2), for any $u \in H^1_{A,V^-}(\mathbb{R}^N)$, we have

$$
\begin{aligned}
I_{A,V}(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\Big(|\nabla_A u|^2 - V(x)\,|u|^2\Big)\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx \\
&\ge \frac{1-\theta}{2}\int_{\mathbb{R}^N}\Big(|\nabla_A u|^2 + V^-|u|^2\Big)\,dx - C\bigg(\int_{\mathbb{R}^N}\Big(|\nabla_A u|^2 + V^-|u|^2\Big)\,dx\bigg)^{\frac{2^*}{2}}.
\end{aligned}
$$

Thus, there exists a sufficiently small constant $\rho > 0$ such that

$$
b(u) := \inf_{\|u\|_{H^1_{A,V^-}(\mathbb{R}^N)}=\rho} I_{A,V}(u) > 0 = I_{A,V}(0)\ .
$$

In addition, for any $v \in H^1_{A,V^-}(\mathbb{R}^N)\backslash\{0\}$, $I_{A,V}(tv) \to -\infty$ as $t \to \infty$. Hence, there is a $t_0 > 0$ such that $\|t_0 v\| > \rho$ and $I_{A,V}(t_0 v) < 0$. By using a variant of the mountain pass theorem (see [3]), there exists a sequence $\{u_m\} \subset H^1_{A,V^-}(\mathbb{R}^N)$ such that as $m \to \infty$

$$
I_{A,V}(u_m) \to c_A\ , \qquad dI_{A,V}(u_m) \to 0\ .
$$

By Lemmas 2.2, 2.3, the sequence $\{u_m\}$ is relatively compact in $H^1_{A,V^-}(\mathbb{R}^N)$. So there exist a subsequence, still denoted by $\{u_m\}$, and a function $u \in H^1_{A,V^-}(\mathbb{R}^N)$ such that

$$
u_m \to u \quad \text{strongly in } H^1_{A,V^-}(\mathbb{R}^N)\ .
$$

Thus c_A is a critical value of $I_{A,V}$, and u is a corresponding critical point of $I_{A,V}$ in $H^1_{A,V^-}(\mathbb{R}^N)$. $\blacksquare$

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