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DUALITY MAPPINGS AND THE EXISTENCE OF PERIODIC SOLUTIONS FOR NON-AUTONOMOUS SECOND ORDER SYSTEMS

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1 – Introduction

The aim of this article is to describe a new method for proving the existence of periodic solutions for certain types of nonautonomous systems. This method is based on the properties of the so-called duality mappings defined on smooth Banach spaces. The method is illustrated by the case

$$\frac{d}{dt}\left(\|\dot u(t)\|^{p-2}\,\dot u(t)\right) \;=\; \|u(t)\|^{p-2}\,u(t) + F(t,u(t)) \tag{1.1}$$

$$u(0)-u(T) \;=\; \dot u(0)-\dot u(T) \;=\; 0\;, \tag{1.2}$$

where p is a real number so that $1<p<\infty$, $0<T<\infty$ is a constant and $F\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$, $(t,x)\to F(t,x)$ is a measurable function in t for each $x\in\mathbb{R}^N$ and continuous in x for a.e. $t\in[0,T]$. Clearly, the nonlinear operator $\frac{d}{dt}\left(\|\dot u(t)\|^{p-2}\,\dot u(t)\right)$ is a vector version of p-Laplacian operator.

In order to say what we understand by solution for the problem (1.1), (1.2) we remind some basic results concerning the $W^{1,p}_T$-spaces.

Let $\mathcal{C}^\infty_T$ be the space of indefinitely differentiable T-periodic functions from $\mathbb{R}$ to $\mathbb{R}^N$. We denote by $\langle\cdot,\cdot\rangle$ the inner product on $\mathbb{R}^N$ and by $\|\cdot\|$, the norm generated by this inner product (the same meaning is applied for the $\|\cdot\|$-norm involved in (1.1)).

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A function $v \in L^1(0,T;\mathbb{R}^N)$ is called a weak derivative of a function $u \in L^1(0,T;\mathbb{R}^N)$ if, for each $f \in \mathcal{C}_T^\infty$,

$$\int_0^T \langle u(t), f'(t) \rangle \, dt = - \int_0^T \langle v(t), f(t) \rangle \, dt \; . \tag{1.3}$$

The weak derivative of u will be denoted by $\dot{u}$ or $\dfrac{du}{dt}$. If $v = \dot{u}$ satisfies (1.3), then $\displaystyle\int_0^T v(s) \, ds = 0$ and there is $c \in \mathbb{R}^N$ such that $u(t) = \displaystyle\int_0^t v(s) \, ds + c$. The Sobolev space $W_T^{1,p}$, $1 < p < \infty$, is the space of functions $u \in L^p(0,T;\mathbb{R}^N)$ having the weak derivative $\dot{u} \in L^p(0,T;\mathbb{R}^N)$. According to the previous results, if $u \in W_T^{1,p}$ then

$$u(t) = \int_0^t \dot{u}(s) \, ds + c \quad \text{and} \quad u(0) = u(T) \; . \tag{1.4}$$

The norm over $W_T^{1,p}$ is defined by

$$\|u\|^p_{W_T^{1,p}} = \int_0^T \|u(t)\|^p \, dt + \int_0^T \|\dot{u}(t)\|^p \, dt \; . \tag{1.5}$$

It is a simple matter to verify that $W_T^{1,p}$ is a reflexive Banach space and $\mathcal{C}_T^\infty \subset W_T^{1,p}$. Let us also recall the following result (see Mawhin and Willem [9], propositions 1.1 and 1.2):

Theorem 1.1.

1) *There exists $c > 0$ such that*

$$\|u\|_\infty \le c \, \|u\|_{W_T^{1,p}}, \quad \textit{for all } \; u \in W_T^{1,p} \; . \tag{1.6}$$

Moreover, if $\int_0^T u(t) \, dt = 0$, then

$$\|u\|_\infty \le c \, \|\dot{u}\|_{L^p(0,T;\mathbb{R}^N)} \; . \tag{1.7}$$

2) *If the sequence (u_k) converges weakly to u in $W_T^{1,p}$, then (u_k) converges uniformly to u on $[0,T]$.* ∎

By Theorem 1.1, the injection of $W_T^{1,p}$ in $\mathcal{C}([0,T];\mathbb{R}^N)$ is compact. Consequently, the injection of $W_T^{1,p}$ in $L^p(0,T;\mathbb{R}^N)$ is also compact.

Now, suppose that $F\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies the following hypothesis:

($\mathcal{H}$) *for any* $u \in L^p(0,T;\mathbb{R}^N)$, *the map* $t \in [0,T] \to F(t,u(t))$ *belongs to* $L^{p'}(0,T;\mathbb{R}^N)$, $p' = \dfrac{p}{p-1}$.

Definition 1.1. By a solution of (1.1), (1.2) we mean a function $u \in W_T^{1,p}$, which satisfies (1.1) a.e. on [0,T] and $\dot u(0) = \dot u(T)$. □

Note that by (1.4) functions of $W_T^{1,p}$ are continuous and T-periodic, i.e. $u(0) = u(T)$. In addition, if (1.1) is satisfied a.e. then the condition $\dot u(0) = \dot u(T)$ is meaningful. Indeed, because (from the definition of $W_T^{1,p}$) $u \in L^p(0,T;\mathbb{R}^N)$ and $\dot u \in L^p(0,T;\mathbb{R}^N)$, we infer that the mappings $\|\dot u\|^{p-2}\,\dot u \in L^{p'}(0,T;\mathbb{R}^N)$, $\|u\|^{p-2}\,u + F(.,u(.)) \in L^{p'}(0,T;\mathbb{R}^N)$. Consequently, from (1.1) it follows that the map $t \in [0,T] \to \|\dot u(t)\|^{p-2}\,\dot u(t)$ belongs to $W_T^{1,p'}$ and is thus continuous on $[0,T]$ and satisfies

$$\|\dot u(0)\|^{p-2}\,\dot u(0) \;=\; \|\dot u(T)\|^{p-2}\,\dot u(T)\ ,$$

which implies $\dot u(0) = \dot u(T)$. Moreover, it follows from standard properties of the weak derivative that u is a solution of (1.1), (1.2), according to definition 1.1, if and only if $u \in W_T^{1,p}$ satisfies:

$$\int_0^T \Big\langle \|\dot u(t)\|^{p-2}\,\dot u(t),\, f'(t)\Big\rangle\,dt \;=\; -\int_0^T \Big\langle \|u(t)\|^{p-2}\,u(t) + F(t,u(t)),\, f(t)\Big\rangle\,dt \tag{1.8}$$
$$\text{for every}\quad f \in \mathcal{C}_T^\infty\ .$$

Thus in order to prove the existence of a solution for (1.1), (1.2) the adopted strategy will be to prove the existence of an element $u \in W_T^{1,p}$, which satisfies (1.8).

2 – An operator equation associated to problem (1.1), (1.2)

Let $J_{p-1}\colon W_T^{1,p} \to (W_T^{1,p})^*$ defined as follows:

$$\big\langle J_{p-1}u, v\big\rangle_{W_T^{1,p},(W_T^{1,p})^*} = \int_0^T \Big\langle \|u(t)\|^{p-2}\,u(t),\, v(t)\Big\rangle\,dt + \int_0^T \Big\langle \|\dot u(t)\|^{p-2}\,\dot u(t),\, \dot v(t)\Big\rangle\,dt \tag{2.1}$$
$$\text{for all}\quad v \in W_T^{1,p}\ .$$

First, let us show that J_{p-1} is well-defined: for any $u \in W_T^{1,p}$, $J_{p-1}u$ defined by (2.1) is a linear and continuous functional on $W_T^{1,p}$. Because the linearity of $J_{p-1}u$ is obvious we shall prove the boundedness. Since $J_{p-1}0 = 0$ we may assume that $u \neq 0$. From (2.1) we deduce

$$
\begin{aligned}
&\left|\langle J_{p-1}u, v\rangle_{W_T^{1,p},\left(W_T^{1,p}\right)^*}\right| \leq \\
&\quad \leq \int_0^T \Big(\|u(t)\|^{p-1}\,\|v(t)\| + \|\dot u(t)\|^{p-1}\,\|\dot v(t)\|\Big)\,dt \\
&\quad \leq \left(\int_0^T \|u(t)\|^{(p-1)p'}\,dt\right)^{\frac{1}{p'}} \left(\int_0^T \|v(t)\|^{p}\,dt\right)^{\frac{1}{p}} \\
&\qquad + \left(\int_0^T \|\dot u(t)\|^{(p-1)p'}\,dt\right)^{\frac{1}{p'}} \left(\int_0^T \|\dot v(t)\|^{p}\,dt\right)^{\frac{1}{p}} \\
&\quad = \|u\|^{\frac{p}{p'}}_{L^p(0,T;\mathbb{R}^N)}\,\|v\|_{L^p(0,T;\mathbb{R}^N)} + \|\dot u\|^{\frac{p}{p'}}_{L^p(0,T;\mathbb{R}^N)}\,\|\dot v\|_{L^p(0,T;\mathbb{R}^N)} \\
&\quad \leq \frac{\|u\|^p_{L^p(0,T;\mathbb{R}^N)}}{p'} + \frac{\|v\|^p_{L^p(0,T;\mathbb{R}^N)}}{p} + \frac{\|\dot u\|^p_{L^p(0,T;\mathbb{R}^N)}}{p'} + \frac{\|\dot v\|^p_{L^p(0,T;\mathbb{R}^N)}}{p} \\
&\quad = \frac{\|u\|^p_{W_T^{1,p}}}{p'} + \frac{\|v\|^p_{W_T^{1,p}}}{p}\,.
\end{aligned}
$$

Thus we have

$$
\left|\langle J_{p-1}u, v\rangle_{W_T^{1,p},\left(W_T^{1,p}\right)^*}\right| \leq \frac{\|u\|^p_{W_T^{1,p}}}{p'} + \frac{\|v\|^p_{W_T^{1,p}}}{p}\,. \tag{2.2}
$$

For $v = z\|u\|_{W_T^{1,p}}$ and $\|z\| = 1$ we deduce from (2.2) that

$$
\left|\langle J_{p-1}u, z\rangle_{W_T^{1,p},\left(W_T^{1,p}\right)^*}\right| \leq \|u\|^{p-1}_{W_T^{1,p}}
$$

saying that $Ju \in \left(W_T^{1,p}\right)^*$ and $\|Ju\|_{\left(W_T^{1,p}\right)^*} \leq \|u\|^{p-1}_{W_T^{1,p}}$. On the other hand, from (2.1) one obtains

$$
\langle J_{p-1}u, u\rangle_{W_T^{1,p},\left(W_T^{1,p}\right)^*} = \|u\|^p_{W_T^{1,p}}
$$

which implies the contrary inequality $\|Ju\|_{\left(W_T^{1,p}\right)^*} \geq \|u\|^{p-1}_{W_T^{1,p}}$. We conclude that $J_{p-1}\colon W_T^{1,p} \to \left(W_T^{1,p}\right)^*$ defined by (2.1) has the following metric properties:

$$
\langle J_{p-1}u, u\rangle_{W_T^{1,p},\left(W_T^{1,p}\right)^*} = \|u\|^p_{W_T^{1,p}}\,, \qquad \|J_{p-1}u\|_{\left(W_T^{1,p}\right)^*} = \|u\|^{p-1}_{W_T^{1,p}}\,. \tag{2.3}
$$

Because, for any $u \in L^p(0,T;\mathbb{R}^N)$, the mapping $t \in [0,T] \to F(t,u(t))$ is supposed to belong to $L^{p'}(0,T;\mathbb{R}^N)$, we may consider the operator $A\colon L^p(0,T;\mathbb{R}^N) \to L^{p'}(0,T;\mathbb{R}^N)$ defined by

$$(Au)(t) = F(t,u(t)) \ \text{ a.e. on } [0,T], \qquad \text{for every } \ u \in L^p(0,T;\mathbb{R}^N)\ . \tag{2.4}$$

Let i be the compact injection of $W_T^{1,p}$ in $L^p(0,T;\mathbb{R}^N)$ and $i^*\colon L^{p'}(0,T;\mathbb{R}^N) \to \big(W_T^{1,p}\big)^*$ its adjoint:

$$i^* x^* = x^* \circ i\ , \qquad \forall\, x^* \in L^{p'}(0,T;\mathbb{R}^N)\ . \tag{2.5}$$

Clearly, (2.5) reads as follows: for every $v \in W_T^{1,p}$,

$$\langle i^* x^*, v\rangle_{W_T^{1,p},(W_T^{1,p})^*} = \langle x^*, i(v)\rangle_{L^p(0,T;\mathbb{R}^N),L^{p'}(0,T;\mathbb{R}^N)}\ . \tag{2.6}$$

Let $u \in W_T^{1,p}$ be a solution of equation

$$J_{p-1} u \;=\; -\,(i^* A i)\, u\ . \tag{2.7}$$

Then, for every $v \in W_T^{1,p}$, one has

$$\begin{aligned}
\big\langle J_{p-1}u, v\big\rangle_{W_T^{1,p},(W_T^{1,p})^*} &= -\big\langle (i^*Ai)\,u, v\big\rangle_{W_T^{1,p},(W_T^{1,p})^*}\\
&= -\big\langle A(i(u)), i(v)\big\rangle_{L^p(0,T;\mathbb{R}^N),L^{p'}(0,T;\mathbb{R}^N)}\\
&= -\big\langle A(u), v\big\rangle_{L^p(0,T;\mathbb{R}^N),L^{p'}(0,T;\mathbb{R}^N)}\\
&= -\int_0^T \big\langle (Au)(t), v(t)\big\rangle\, dt \;=\; -\int_0^T \big\langle F(t,u(t)), v(t)\big\rangle\, dt\ .
\end{aligned}$$

Taking into account (2.1), the equality

$$\big\langle J_{p-1}u, v\big\rangle_{W_T^{1,p},(W_T^{1,p})^*} \;=\; -\int_0^T \big\langle F(t,u(t)), v(t)\big\rangle\, dt$$

rewrites as

$$\int_0^T \Big\langle \|\dot u(t)\|^{p-2}\, \dot u(t),\ \dot v(t)\Big\rangle\, dt \;=\; -\int_0^T \Big\langle \|u(t)\|^{p-2}\, u(t) + F(t,u(t)),\ v(t)\Big\rangle\, dt$$
$$\text{for all} \quad v \in W_T^{1,p}\ . \tag{2.8}$$

In particular, (2.8) is satisfied for any $v = f \in \mathcal{C}_T^\infty \subset W_T^{1,p}$.

Consequently, if $u \in W_T^{1,p}$ is a solution of the operator equation (2.7), then u is a solution of problem (1.1), (1.2). Thus, in order to prove the existence of a solution for the problem (1.1), (1.2), it would be sufficient to prove the existence of a solution for the operator equation (2.7).

It is a simple matter to see that in the preceeding reasoning the operator A generated by the function $F(\cdot,\cdot)$ may be replaced by any operator $N: L^p(0,T;\mathbb{R}^N) \to L^{p'}(0,T;\mathbb{R}^N)$. Thus we obtain the following proposition:

Proposition 2.1. *Let $J_{p-1}\colon W^{1,p}_T \to \left(W^{1,p}_T\right)^*$, $1<p<\infty$ be defined by (2.1) and let $N\colon L^p(0,T;\mathbb{R}^N) \to L^{p'}(0,T;\mathbb{R}^N)$ be given. Let $i\colon W^{1,p}_T \to L^p(0,T;\mathbb{R}^N)$ be the compact injection of $W^{1,p}_T$ in $L^p(0,T;\mathbb{R}^N)$ and $i^*\colon L^{p'}(0,T;\mathbb{R}^N) \to \left(W^{1,p}_T\right)^*$ its adjoint.*

If $u \in W^{1,p}_T$ is a solution of the operator equation

$$J_{p-1}u \;=\; -\,(i^*Ni)\,u \tag{2.9}$$

then u is a solution for the problem

$$\frac{d}{dt}\left(\|\dot u(t)\|^{p-2}\,\dot u(t)\right) \;=\; \|u(t)\|^{p-2}\,u(t) + (Nu)(t)\;, \tag{2.10}$$

$$u(0)-u(T) \;=\; \dot u(0)-\dot u(T) \;=\; 0\;. \ \blacksquare \tag{2.11}$$

The operator J_{p-1} defined by (2.1) and occuring in (2.7), (2.9) is a particular example of the so-called “duality mapping”.

The definition and some fundamental properties of duality mappings, as well as some existence results for operator equations involving duality mappings will be given in the next section. The existence of a solution for (2.9) will be obtained by particularizing some of these abstract existence results to one situation of duality mapping, which is J_{p-1} defined by (2.1).

3 – Duality mappings

Let X be a real Banach space and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a gauge function, such that φ is continuous, strictly increasing, $\varphi(0)=0$ and $\varphi(t)\to\infty$ as $t\to\infty$. The duality mapping corresponding to the gauge function φ is the set valued mapping $J_\varphi\colon X \to 2^{X^*}$, defined by

$$J_\varphi 0 = 0,\;\; J_\varphi x = \varphi(\|x\|_X)\left\{u^*\in X^* \mid\; \langle u^*,x\rangle = \|x\|_X,\;\; \|u^*\|_{X^*}=1\right\} \;\text{ if }\; x\neq 0\,, \tag{3.1}$$

or, with a definition that covers both cases,

$$J_\varphi x \;=\; \left\{x^*\in X^* \mid\; \langle x^*,x\rangle = \varphi(\|x\|_X)\,\|x\|_X,\;\; \|x^*\|_{X^*} = \varphi(\|x\|_X)\right\}\;. \tag{3.2}$$

According to the well known Hahn–Banach theorem, it follows that

$$D(J_\varphi) = \Big\{x \in X \mid J_\varphi x \neq \phi\Big\} = X .$$

From (3.1), it immediately follows that J_φ is single valued iff X is a smooth Banach space, that is, a Banach space having the property that: for any $x \neq 0$ there exists a unique $u^* \in X^*$ such that $\langle u^*, x\rangle = \|x\|_X$ and $\|u^*\|_{X^*} = 1$.

It is well known (see, for instance, Diestel [3], Zeidler [10]) that the smoothness of X is equivalent to the Gâteaux differentiability of the norm.

Consequently, if $(X, \| \ \|_X)$ is smooth, then for any $x \in X$, $x \neq 0$, the only element u^* with the properties $\langle u^*, x\rangle = \|x\|_X$ and $\|u^*\|_{X*} = 1$ is $u^* = \| \ \|'_X(x)$ (where, $\| \ \|'_X$ denotes the Gâteaux gradient of the norm).

Consequently, if X is smooth, then $J_\varphi \colon X \to X^*$ is defined by

$$J_\varphi 0 = 0 \ , \qquad J_\varphi x = \varphi(\|x\|_X) \, \| \ \|'_X(x), \quad x \neq 0 \ , \tag{3.3}$$

the following metric properties being consequent (see, for example [6]):

$$\|J_\varphi x\|_X = \varphi(\|x\|_X) \ , \qquad \langle J_\varphi x, x\rangle = \varphi(\|x\|_X) \, \|x\|_X \ . \tag{3.4}$$

In [4], [5], the following result has been proved (see, for instance, [5], Theorem 2):

Theorem 3.1. *Let X be a smooth reflexive real Banach space, compactly embedded in the real Banach space Z:*

$$X \xrightarrow{i} Z \ , \qquad \|i(x)\|_Z \le c_Z \, \|x\|_X, \quad \text{for all } x \in X \tag{3.5}$$

and i is compact.

We assume the following:

- **(i)** *the duality mapping $J_{p-1} \colon X \to X^*$ corresponding to the gauge function $\varphi(t) = t^{p-1}$, $1 < p < \infty$ satisfies the following condition:*

$$\Big[x_n \rightharpoonup x \ \text{ and } \ J_{p-1} x_n \to J_{p-1} x\Big] \implies x_n \to x \ ; \tag{$(S)_2$}$$

- **(ii)** *let $N : Z \to Z^*$ be a demicontinuous operator: $z_n \to z \Longrightarrow N z_n \rightharpoonup N z$ satisfying the growth condition*

$$\|Nz\|_{Z^*} \le c_1 \, \|z\|_Z^{p-1} + c_2 \qquad \text{for all } z \in Z \tag{3.6}$$

where $c_1 \in [0, \lambda_1)$, $\lambda_1 = \inf \left\{ \dfrac{\|x\|_X^p}{\|i(x)\|_Z^p} \mid x \neq 0 \right\}$, $c_2 \ge 0$.

Then the equation

$$J_{p-1}x = Nx \tag{3.7}$$

has a solution. ∎

In the statement of Theorem 3.1, as well as in all that follows, we denote by X^* (resp. Z^*), the dual space of X (resp. Z) and by "$\rightharpoonup$" (resp. "$\to$") the convergence in the weak (respectively strong) topology.

Let us also note that, because of the compact embedding of X into Z, λ_1 is attained and $\lambda_1^{-\frac{1}{p}}$ is the best constant c_z in the writing of the embedding of X into Z (inequality (3.5)).

In the statement of Theorem 3.1, the duality mapping is required to satisfy condition $(S)_2$. In the following we will state a sufficient condition for the duality mapping to satisfy condition $(S)_2$.

To do this, we mention that a Banach space has the Kadec–Klee property if it is strictly convex and $[x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|] \Longrightarrow x_n \to x$.

Proposition 3.1. *If X is a smooth Banach space having the Kadec–Klee property, then, any duality mapping is single-valued ($J_\varphi \colon X \to X^*$) and satisfies condition $(S)_2$.*

Proof: Indeed,

$$\begin{aligned}
\Big[x_n \rightharpoonup x \text{ and } J_\varphi x_n \to J_\varphi x\Big] &\Longrightarrow \Big[x_n \rightharpoonup x \text{ and } \|J_\varphi x_n\|_{X^*} \to \|J_\varphi x\|_{X^*}\Big] \\
&\Longrightarrow \Big[x_n \rightharpoonup x \text{ and } \varphi(\|x_n\|_X) \to \varphi(\|x\|_X)\Big] \\
&\Longrightarrow \Big[x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\|\Big] \\
&\Longrightarrow x_n \to x \,. \ \blacksquare
\end{aligned}$$

Corollary 3.1. *If X is a smooth and locally uniformly convex Banach space, then, any duality mapping $J_\varphi \colon X \to X^*$ satisfies condition $(S)_2$. Specifically, any duality mapping on a smooth and uniformly convex space satisfies condition $(S)_2$. More specifically, any duality mapping on a Hilbert space satisfies condition $(S)_2$.* ∎

Finally, let us remark that in relation with some other conditions invoked in the theory of non-linear operators, such as, e.g. $(S)_+$, (S), $(S)_0$ and $(S)_1$

(see Zeidler [10], p. 583), condition $(S)_2$ is placed as follows: $(S)_+ \Longrightarrow (S) \Longrightarrow (S)_0 \Longrightarrow (S)_1 \Longrightarrow (S)_2$.

We will put an end to the comments related to the Theorem 3.1.

With respect to the conclusion of Theorem 3.1, we stress that we understand by a solution of $J_{p-1}x = Nx$ an element $u \in X$ with satisfies $J_{p-1}u = Nu$ in the sense of X^*, that is

$$J_{p-1}u \;=\; (i^* N i)u \;, \tag{3.8}$$

where i is the compact injection on X in Z, and $i^*: Z^* \to X^*$, $i^* z^* = z^* \circ i$ for all $z^* \in Z^*$, is the adjoint of i: $\langle i^* z^*, x\rangle_{X^*, X} = \langle z^*, i(x)\rangle_{Z^*, Z}$ for all $x \in X$.

Because i is compact $i^*: Z^* \to X^*$ is also compact. As a consequence, because $N: Z^* \to Z$ is demicontinuous, the operator $(i^* N i): X \to X^*$ is compact.

Finally, (3.8) is equivalent with

$$\big\langle J_{p-1}u, v\big\rangle_{X^*, X} = \big\langle N(i(u)), i(v)\big\rangle_{Z^*, Z} \quad \text{for all } \; v \in X \;. \tag{3.9}$$

Remark 3.1. Notice that if X is an infinite dimensional reflexive and smooth real Banach space then a duality mapping on X is never compact (see Theorem 1 in [7]). □

Indeed, let $J_\varphi: X \to X^*$ be a duality mapping. Then J_φ is surjective.

The idea of the proof is the following: if X is reflexive and smooth, every duality mapping $J_\varphi: X \to X^*$ is demicontinuous ($x_n \to x \implies J_\varphi x_n \rightharpoonup J_\varphi x$). Because J_φ is also monotone,

$$\Big\langle J_\varphi x - J_\varphi y,\, x - y\Big\rangle \;\geq\; \Big(\varphi\big(\|x\|\big) - \varphi\big(\|y\|\big)\Big)\,\Big(\|x\| - \|y\|\Big) \;\geq\; 0$$

and coercive

$$\frac{\langle J_\varphi x, x\rangle}{\|x\|} = \varphi\big(\|x\|\big) \to \infty \quad \text{ as } \; \|x\| \to \infty \;,$$

the surjectivity of J_φ follows from the well known surjectivity result due to Browder (see, for example Browder [2]).

Now, the noncompactness of J_φ follows by Baire's cathegory theorem.

Indeed, suppose, by contradiction, that J_φ would be compact. Because $X = \bigcup_{n\geq 1} B(0,n)$ and J_φ is surjective we derive that

$$X^* \,=\, J_\varphi(X) \,= \bigcup_{n\geq 1} \overline{J_\varphi\big(B(0,n)\big)} \;.$$

By Baire's theorem $\overset{\circ}{\overline{J_\varphi\big(B(0,n)\big)}} \neq \varnothing$, for some n, which is impossible because $\overline{J_\varphi\big(B(0,n)\big)}$ is compact and X^* is infinite dimensional.

After obtaining the surjectivity of J_φ, another maner to conclude that J_φ cannot be compact is the following: by using the surjectivity of J_φ it can be easily shown that

$$J_\varphi(\partial B_{X,r}) \;=\; \partial B_{X^*,\varphi(r)} \;,$$

where $\partial B_{X,r} = \big\{x \in X \mid \|x\| = r\big\}$, $\partial B_{X^*,\varphi(r)} = \big\{x^* \in X^* \mid \|x^*\| = \varphi(r)\big\}$.

Because of $\dim X^* = \infty$, $\partial B_{X^*,\varphi(r)} = J_\varphi(\partial B_{X^*,\varphi(r)})$ is not compact. Consequently, J_φ is not compact.

In [7], some estimations for the Kuratowski measure of noncompactness of a duality mapping are given.

Remark 3.2. The statement of Theorem 3.1 is the same with that of Theorem 2 in [5]. But, it is evident that if N satisfies the hypotheses of Theorem 3.1 then $(-N)$ satisfies them too. Consequently, under the hypotheses of Theorem 3.1, equation

$$J_{p-1} u \;=\; -\,(i^* N i)\, u$$

has a solution. □

It is in this form that the conclution of the Theorem 3.1 will be used in the sequel.

4 – Existence results for equation $J_{p-1} u = -\,(i^* N i)\, u$

Let's go back to operator $J_{p-1}\colon W^{1,p}_T \to \big(W^{1,p}_T\big)^*$ defined by (2.1) and appearing in equation (2.9). Since J_{p-1} satisfies the metric relations (2.3) it follows that, for any $u \in W^{1,p}_T$, $J_{p-1} u \in J_\varphi u$, where J_φ designates the (eventually multivalued) duality mapping on $W^{1,p}_T$ corresponding to the gauge function $\varphi(t) = t^{p-1}$, $1<p<\infty$, $t \geq 0$ (to see that, write (3.2) for $X = W^{1,p}_T$ and $\varphi(t) = t^{p-1}$, $1<p<\infty$, $t \geq 0$). But, as we will see below, $W^{1,p}_T$ with $1 < p < \infty$ is a smooth Banach space. It will result that any duality mapping on $W^{1,p}_T$, $1 < p < \infty$ is single valued. Consequently, $J_{p-1}\colon W^{1,p}_T \to \big(W^{1,p}_T\big)^*$ defined by (2.1) is just the duality mapping corresponding to the gauge function $\varphi(t) = t^{p-1}$.

Theorem 4.1. *If $1 < p < \infty$, then:*

a) *the space $\left(W_T^{1,p}, \|\ \|_{W_T^{1,p}}\right)$, is uniformly convex and smooth;*

b) *the duality mapping on $W_T^{1,p}$ corresponding to the gauge function $\varphi(t) = t^{p-1}$, $t \geq 0$ is single valued, $\left(J_{p-1} : W_T^{1,p} \to \left(W_T^{1,p}\right)^*\right)$ satisfies condition $(S)_2$ and is defined as follows: if $u \in W_T^{1,p}$, then*

$$\left\langle J_{p-1}u, v\right\rangle_{W_T^{1,p},(W_T^{1,p})^*} = \int_0^T \|u(t)\|^{p-2} \left\langle u(t), v(t)\right\rangle dt + \int_0^T \|\dot{u}(t)\|^{p-2} \left\langle \dot{u}(t), \dot{v}(t)\right\rangle dt \tag{4.1}$$

for all $v \in W_T^{1,p}$.

Proof: For convenience of the reader we recall the details of the proof.

a) First, let us prove that for $1 < p < \infty$, $L^p(0,T;\mathbb{R}^N)$ is uniformly convex.

For $2 \leq p < \infty$, let $u, v \in L^p(0,T;\mathbb{R}^N)$ satisfy $\|u\|_{L^p(0,T;\mathbb{R}^N)} = \|v\|_{L^p(0,T;\mathbb{R}^N)} = 1$ and $\|u-v\|_{L^p(0,T;\mathbb{R}^N)} \geq \varepsilon > 0$. Then we have (see Adams [1], Lemma 2.27):

$$\left\|\frac{u(t)+v(t)}{2}\right\|^p + \left\|\frac{u(t)-v(t)}{2}\right\|^p \leq \frac{1}{2}\|u(t)\|^p + \frac{1}{2}\|v(t)\|^p \qquad \text{a.e. } t \in [0,T]\ .$$

By integrating from 0 to T one obtains

$$\left\|\frac{u+v}{2}\right\|^p_{L^p(0,T;\mathbb{R}^N)} + \left\|\frac{u-v}{2}\right\|^p_{L^p(0,T;\mathbb{R}^N)} \leq \frac{1}{2}\|u\|^p_{L^p(0,T;\mathbb{R}^N)} + \frac{1}{2}\|v\|^p_{L^p(0,T;\mathbb{R}^N)} = 1\ ,$$

which implies $\left\|\dfrac{u+v}{2}\right\|^p_{L^p(0,T;\mathbb{R}^N)} \leq 1 - \dfrac{\varepsilon^p}{2^p}$.

For $1 < p < 2$, let $u, v \in L^p(0,T;\mathbb{R}^N)$ satisfy $\|u\|_{L^p(0,T;\mathbb{R}^N)} = \|v\|_{L^p(0,T;\mathbb{R}^N)} = 1$ and $\|u-v\|_{L^p(0,T;\mathbb{R}^N)} \geq \varepsilon > 0$. Then we have (see Adams [1], Lemma 2.27):

$$\left\|\frac{u(t)+v(t)}{2}\right\|^{p'} + \left\|\frac{u(t)-v(t)}{2}\right\|^{p'} \leq \left(\frac{1}{2}\|u(t)\|^p + \frac{1}{2}\|v(t)\|^p\right)^{\frac{1}{p-1}}\ .$$

Moreover, if $w \in L^p(0,T;\mathbb{R}^N)$ then $\|w(\cdot)\|^{p'} \in L^{p-1}(0,T;\mathbb{R}_+)$ and

$$\left\|\, \|w(\cdot)\|^{p'} \right\|_{L^{p-1}(0,T;\mathbb{R}_+)} = \|w\|^{p'}_{L^p(0,T;\mathbb{R}^N)}\ .$$

Finally, because of $0 < p-1 < 1$ we have (the reverse Minkowsky inequality):

$$\|w_1 + w_2\|_{L^{p-1}(0,T;\mathbb{R}_+)} \geq \|w_1\|_{L^{p-1}(0,T;\mathbb{R}_+)} + \|w_2\|_{L^{p-1}(0,T;\mathbb{R}_+)}\ ,$$

for any $w_1, w_2 \in L^{p-1}(0,T;\mathbb{R}_+)$.

By using these arguments we obtain:

$$\begin{aligned}
&\left\|\frac{u+v}{2}\right\|^{p'}_{L^p(0,T;\mathbb{R}^N)}+\left\|\frac{u-v}{2}\right\|^{p'}_{L^p(0,T;\mathbb{R}^N)}=\\
&\qquad=\left\|\left\|\frac{u(\cdot)+v(\cdot)}{2}\right\|^{p'}\right\|_{L^{p-1}(0,T;\mathbb{R}_+)}+\left\|\left\|\frac{u(\cdot)-v(\cdot)}{2}\right\|^{p'}\right\|_{L^{p-1}(0,T;\mathbb{R}_+)}\\
&\qquad\le\left\|\left\|\frac{u(\cdot)+v(\cdot)}{2}\right\|^{p'}+\left\|\frac{u(\cdot)-v(\cdot)}{2}\right\|^{p'}\right\|_{L^{p-1}(0,T;\mathbb{R}_+)}\\
&\qquad=\left(\int_0^T\left(\left\|\frac{u(t)+v(t)}{2}\right\|^{p'}+\left\|\frac{u(t)-v(t)}{2}\right\|^{p'}\right)^{p-1}dt\right)^{\frac{1}{p-1}}\\
&\qquad\le\left(\int_0^T\left(\frac{1}{2}\,\|u(t)\|^p+\frac{1}{2}\,\|v(t)\|^p\right)dt\right)^{\frac{1}{p-1}}\\
&\qquad=\left(\frac{1}{2}\,\|u\|^p_{L^p(0,T;\mathbb{R}^N)}+\frac{1}{2}\,\|v\|^p_{L^p(0,T;\mathbb{R}^N)}\right)^{\frac{1}{p-1}}\;=\;1\,.
\end{aligned}$$

Thus,

$$\left\|\frac{u+v}{2}\right\|^{p'}_{L^p(0,T;\mathbb{R}^N)}\le 1-\frac{\varepsilon^{p'}}{2^{p'}}\,.$$

In either case $(2\le p<\infty$ or $1<p<2)$ there exists $\delta=\delta(\varepsilon)>0$ such that $\left\|\frac{u+v}{2}\right\|^{p'}_{L^p(0,T;\mathbb{R}^N)}\le 1-\delta\,.$

Now, consider the Banach space $L^p(0,T;\mathbb{R}^N)\times L^p(0,T;\mathbb{R}^N)$, endowed with the norm

$$\|(u,v)\|\;=\;\Big(\|u\|^p_{L^p(0,T;\mathbb{R}^N)}+\|v\|^p_{L^p(0,T;\mathbb{R}^N)}\Big)^{\frac{1}{p}}\;.$$

By Theorem 1.22 in Adams [1] it follows that $L^p(0,T;\mathbb{R}^N)\times L^p(0,T;\mathbb{R}^N)$ is uniformly convex with respect to the $\|\ \|$-norm.

Let $P\colon\big(W^{1,p}_T,\|\ \|_{W^{1,p}_T}\big)\to\big(L^p(0,T;\mathbb{R}^N)\times L^p(0,T;\mathbb{R}^N),\,\|\ \|\big)$ defined as follows:

$$Pu=(u,\dot u)\qquad\text{for any}\;\;u\in W^{1,p}_T\,.$$

Since $\|Pu\|=\|u\|_{W^{1,p}_T}$, P is an isometric isomorphism of $\big(W^{1,p}_T,\|\ \|_{W^{1,p}_T}\big)$ onto a closed subspace of $\big(L^p(0,T;\mathbb{R}^N)\times L^p(0,T;\mathbb{R}^N),\,\|\ \|\big)$.

The later one being uniformly convex it follows that $\big(W^{1,p}_T,\|\ \|_{W^{1,p}_T}\big)$ is also uniformly convex.

The smoothness of $W_T^{1,p}$ is equivalent with the Gâteaux differentiability of the norm $\| \ \|_{W_T^{1,p}}$. But

$$\|u\|^p_{W_T^{1,p}} = \|u\|^p_{L^p(0,T;\mathbb{R}^N)} + \|\dot u\|^p_{L^p(0,T;\mathbb{R}^N)} \ , \tag{4.2}$$

and $\| \ \|_{L^p(0,T;\mathbb{R}^N)}$ is Gâteaux differentiable: for any $u \neq 0$,

$$\begin{aligned} &\| \ \|'_{L^p(0,T;\mathbb{R}^N)}(u) \ \in \ L^{p'}(0,T;\mathbb{R}^N) \ , \\ &\Big[\| \ \|'_{L^p(0,T;\mathbb{R}^N)}(u)\Big](t) \ = \ \frac{\|u(t)\|^{p-2}\,u(t)}{\|u\|^{p-1}_{L^p(0,T;\mathbb{R}^N)}} \qquad \text{a.e.} \ \ t\in[0,T] \ . \end{aligned} \tag{4.3}$$

From (4.3) and (4.2) we deduce the Gâteaux differentiability of the norm $\| \ \|_{W_T^{1,p}}$. ∎

Corollary 4.1.

a) *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a gauge function. The duality mapping on $W_T^{1,p}$, $1<p<\infty$ corresponding to φ is single valued $\Big(J_\varphi\colon W_T^{1,p} \to \big(W_T^{1,p}\big)^*\Big)$ and is defined as follows:*

$$J_\varphi 0 \ = \ 0 \ ,$$

$$\begin{aligned} &\big\langle J_\varphi u, v\big\rangle_{W_T^{1,p},(W_T^{1,p})^*} \ = \\ &\quad = \frac{\varphi\big(\|u\|_{W_T^{1,p}}\big)}{\|u\|^{p-1}_{W_T^{1,p}}} \left[\int_0^T \Big\langle \|u(t)\|^{p-2}\,u(t),\, v(t)\Big\rangle\,dt + \int_0^T \Big\langle \|\dot u(t)\|^{p-2}\,\dot u(t),\, \dot v(t)\Big\rangle\,dt\right] \\ &\qquad\qquad \textit{for all} \ \ v\in W_T^{1,p}, \ \ \textit{if} \ \ u\neq 0 \ . \end{aligned} \tag{4.4}$$

b) *The gradient (in the Gâteau sense) of $\| \ \|_{W_T^{1,p}}$ -norm is defined as follows:*

$$\big(\nabla\| \ \|_{W_T^{1,p}}\big)\colon \ W_T^{1,p}\setminus\{0\} \to \big(W_T^{1,p}\big)^* \ ,$$

$$\begin{aligned} &\Big\langle \big(\nabla\| \ \|_{W_T^{1,p}}\big)(u),\, v\Big\rangle_{W_T^{1,p},(W_T^{1,p})^*} \ = \\ &\quad = \ \frac{1}{\|u\|^{p-1}_{W_T^{1,p}}} \left[\int_0^T \Big\langle \|u(t)\|^{p-2}\,u(t),\, v(t)\Big\rangle\,dt + \int_0^T \Big\langle \|\dot u(t)\|^{p-2}\,\dot u(t),\, \dot v(t)\Big\rangle\,dt\right] \\ &\qquad\qquad \textit{for all} \ \ v\in W_T^{1,p} \ . \end{aligned} \tag{4.5}$$

Proof:

a) If $(X, \|\ \|)$ is a real normed space and φ and ψ are two gauge functions then it is easy to check that

$$J_\varphi u = \frac{\varphi(\|u\|)}{\psi(\|u\|)}\, J_\psi u, \qquad \text{for all} \quad u \in X \setminus \{0\}\ . \tag{4.6}$$

Consequently, if J_φ is the duality mapping on $W_T^{1,p}$ corresponding to the gauge function φ, then one has

$$J_\varphi u = \frac{\varphi\big(\|u\|_{W_T^{1,p}}\big)}{\|u\|^{p-1}_{W_T^{1,p}}}\, J_{p-1} u, \qquad \text{for all} \quad u \in X \setminus \{0\}\ . \tag{4.7}$$

Formula (4.4) follows by comparing (4.7) and (2.1).

b) Because $W_T^{1,p}$, $1 < p < \infty$ is smooth, the duality mapping on $W_T^{1,p}$ corresponding to the gauge function φ is given by

$$J_\varphi u = \varphi\big(\|u\|_{W_T^{1,p}}\big) \big(\nabla \|\ \|_{W_T^{1,p}}\big)(u), \qquad \text{for all} \quad u \in X \setminus \{0\}\ . \tag{4.8}$$

In order to obtain (4.5) it suffices to compare (4.8) and (4.4). ∎

Theorem 4.2. *Let i be the compact injection of $W_T^{1,p}$, $1 < p < \infty$ in $L^p(0,T;\mathbb{R}^N)$ and i^* its dual. Let J_{p-1} (given by (2.1)) be the duality mapping on $W_T^{1,p}$ corresponding to the gauge function $\varphi(t) = t^{p-1}$, $t \geq 0$.*

Suppose that $N\colon L^p(0,T;\mathbb{R}^N) \to L^{p'}(0,T;\mathbb{R}^N)$ is a demicontinuous operator which satisfies the growth condition

$$\|Nu\|_{L^{p'}(0,T;\mathbb{R}^N)} \leq c_1 \|u\|^{p-1}_{L^p(0,T;\mathbb{R}^N)} + c_2\,, \qquad \textit{for all} \quad u \in L^p(0,T;\mathbb{R}^N) \tag{4.9}$$

where

$$c_1 \in [0, \lambda_1)\ , \qquad \lambda_1 = \inf \left\{ \frac{\|u\|^p_{W_T^{1,p}}}{\|i(u)\|^p_{L^p(0,T;\mathbb{R}^N)}} \ \middle|\ u \neq 0 \right\}, \qquad c_2 \geq 0\ .$$

Then, the equation

$$J_{p-1} u = -\,(i^* N i)\, u \tag{4.10}$$

has a solution in $W_T^{1,p}$.

Consequently, the problem

$$\frac{d}{dt}\left(\|\dot u(t)\|^{p-2}\,\dot u(t)\right) \;=\; \|u(t)\|^{p-1}\,u(t) + (Nu)(t)\ , \tag{4.11}$$

$$u(0)-u(T) \;=\; u'(0)-u'(T) \;=\; 0 \tag{4.12}$$

has a solution in $W_T^{1,p}$.

Proof: The hypotheses of the Theorem 4.2 entail the satisfaction of the hypotheses of Theorem 3.1 under the following choice:

- $X = W_T^{1,p},\ 1<p<\infty,\ Z = L^p(0,T;\mathbb{R}^N)$;
- $J_{p-1} = J_{p-1,W_T^{1,p}}\colon\, W_T^{1,p} \to \left(W_T^{1,p}\right)^*$ the duality mapping on $W_T^{1,p}$ corresponding to the gauge function $\varphi(t) = t^{p-1},\ t\ge 0$;
- $N\colon\, Z = L^p(0,T;\mathbb{R}^N) \to Z^* = L^{p'}(0,T;\mathbb{R}^N)$ satisfying (4.9).

Indeed,

- $X = W_T^{1,p},\ 1<p<\infty,$ is reflexive, smooth (Theorem 4.1) and compactly embedded in $Z = L^p(0,T;\mathbb{R}^N)$;
- $X = W_T^{1,p},\ 1<p<\infty$ is uniformly convex (Theorem 4.1). Consequently, $J_{p-1,W_T^{1,p}}$ satisfies condition (S_2) (Corrolary 3.1);
- $N\colon\, Z = L^p(0,T;\mathbb{R}^N) \to Z^* = L^{p'}(0,T;\mathbb{R}^N)$ is supposed to be demicontinous and satisfying the growth condition (4.9).

Consequently, according to the Theorem 3.1, there is a solution in $W_T^{1,p}$ of equation (4.10). According to proposition (2.1), every solution in $W_T^{1,p}$ of equation (4.10) satisfies (4.11) a.e. in $[0,T]$ and the boundary conditions (4.12). ∎

As application of Theorem 4.2 we give an existence result for problem (1.1), (1.2) which we repeat here for the convenience of the reader:

$$\frac{d}{dt}\left(\|\dot u(t)\|^{p-2}\,\dot u(t)\right) \;=\; \|u(t)\|^{p-2}\,u(t) + F(t,u(t))\ , \tag{4.13}$$

$$u(0)-u(T) \;=\; \dot u(0)-\dot u(T) \;=\; 0, \quad \text{with}\ \ 1<p<\infty\ . \tag{4.14}$$

This result is contained in the following theorem.

Theorem 4.3. *Let* $F : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N,\ (t,x) \mapsto F(t,x)$ *be a function measurable in* t *for each* $x\in\mathbb{R}^N$ *and continuous in* x *for a.e.* $t\in[0,T]$.

Assume that:

$$\|F(t,x)\| \;\le\; \gamma(t)\,\big(\|x\|^r+1\big)\;, \tag{4.15}$$

where $\gamma \in L^{2p'}(0,T;\mathbb{R}_+)$, $\dfrac{1}{p}+\dfrac{1}{p'}=1$, $r=\dfrac{p}{2p'}$, $1<p<\infty$.

Then, problem (4.13), (4.14) has a solution in $W_T^{1,p}$.

To prove this, let us start by the following lemma:

Lemma 4.1. *Denote by* i *the compact injection of* $W_T^{1,p}$ *into* $L^p(0,T;\mathbb{R}^N)$ *and let*

$$\lambda_1 \;=\; \inf\left\{ \frac{\|u\|^p_{W_T^{1,p}}}{\|i(u)\|^p_{L^p(0,T;\mathbb{R}^N)}} \;\middle|\; u\neq 0 \right\}\;, \tag{4.16}$$

In the hypotheses of Theorem 4.3, the operator $N\colon L^p(0,T;\mathbb{R}^N)\to L^{p'}(0,T;\mathbb{R}^N)$ *defined by*

$$(Nu)(t)=F(t,u(t)) \qquad \text{for every } \; u\in L^p(0,T;\mathbb{R}^N) \tag{4.17}$$

is continuous. Moreover, for any $c_1\in\Big(0,2^{\left(\frac{1}{p'}-\frac{1}{p}\right)}\lambda_1\Big)$, N *satisfies the growth condition*

$$\|Nu\|_{L^{p'}(0,T;\mathbb{R}^N)} \;\le\; 2^{\left(\frac{1}{p}-\frac{1}{p'}\right)}c_1\|u\|^{p-1}_{L^p(0,T;\mathbb{R}^N)}+c_2\;, \tag{4.18}$$

where $\gamma(t)=c_1\beta(t)$ *and*

$$c_2^{p'} \;=\; 2^{p'-2}\,c_1^{p'}\int_0^T\Big[\beta(t)^{2p'}+2\,(\beta(t))^{p'}\Big]\,dt\;. \tag{4.19}$$

Proof: For any $u\in L^p(0,T;\mathbb{R}^N)$, we have (see (4.15) and (4.17)):

$$\begin{aligned}
\|(Nu)(t)\|^{p'} \;&=\; \|F(t,u(t))\|^{p'}\\
&\le\; c_1^{p'}\beta(t)^{p'}\big(\|u(t)\|^r+1\big)^{p'}\\
&\ge\; c_1^{p'}\beta(t)^{p'}2^{p'-1}\big(\|u(t)\|^{rp'}+1\big)\\
&=\; c_1^{p'}\beta(t)^{p'}2^{p'-1}\big(\|u(t)\|^{\frac{p}{2}}+1\big)\\
&=\; c_1^{p'}\beta(t)^{p'}2^{p'-1}\|u(t)\|^{\frac{p}{2}}+2^{p'-1}c_1^{p'}\beta(t)^{p'}\\
&\le\; 2^{p'-1}\,c_1^{p'}\left[\frac{\beta(t)^{2p'}}{2}+\frac{\|u(t)\|^p}{2}\right]+2^{p'-1}\,c_1^{p'}\beta(t)^{p'}\\
&=\; 2^{p'-2}\,c_1^{p'}\,\|u(t)\|^p+2^{p'-2}\,c_1^{p'}\,\Big[\beta(t)^{2p'}+2\,\beta(t)^{p'}\Big]\;.
\end{aligned}$$

Consequently,

$$\|(Nu)(t)\|^{p'} \le\; 2^{p'-2}\, c_1^{p'}\, \|u(t)\|^p + 2^{p'-2}\, c_1^{p'} \Big[\beta(t)^{2p'} + 2\,\beta(t)^{p'}\Big] \;, \tag{4.20}$$

where from, by means of integration from 0 to T,

$$\|Nu\|^{p'}_{L^{p'}(0,T;\mathbb{R}^N)} \le\; 2^{p'-2}\, c_1^{p'}\, \|u\|^p_{L^p(0,T;\mathbb{R}^N)} + c_2^{p'} \;, \tag{4.21}$$

which implies (4.18).

To prove N's continuity we will use a technique inspired by that employed in order to prove the continuity of Nemytskii type operators.

Consider $(u_n) \subset L^p(0,T;\mathbb{R}^N)$ and $u \subset L^p(0,T;\mathbb{R}^N)$ such that $\|u_n - u\|_{L^p(0,T;\mathbb{R}^N)} \to 0$. Let us prove that $\|Nu_n - Nu\|_{L^{p'}(0,T;\mathbb{R}^N)} \to 0,\; n \to \infty$.

Assume the contrary: $\|Nu_n - Nu\|_{L^{p'}(0,T;\mathbb{R}^N)} \not\to 0$. Then there is $\varepsilon_0 > 0$ and a subsequence $(u'_n) \subset (u_n)$ such that

$$\|Nu'_n - Nu\|_{L^{p'}(0,T;\mathbb{R}^N)} \;\ge\; \varepsilon_0 \;. \tag{4.22}$$

We define $f'_n \colon [0,T] \to \mathbb{R}_+,\; f'_n(t) = \|u'_n(t) - u(t)\|$ a.e. $t \in [0,T]$. It is obvious that

$$f'_n \in L^p(0,T;\mathbb{R}_+) \;, \qquad \|f'_n\|_{L^p(0,T;\mathbb{R}_+)} = \|u'_n - u\|_{L^p(0,T;\mathbb{R}^N)} \to 0, \quad n \to \infty \;.$$

According to a classical result, there exists a subsequence $(f''_n) \subset (f'_n)$ and a function $g \in L^p(0,T;\mathbb{R}_+)$ such that

$$f''_n(t) \to 0 \qquad \text{a.e. } \; t \in [0,T] \;,$$
$$|f''_n(t)| = f''_n(t) \le g(t) \qquad \text{a.e. } \; t \in [0,T] \;,$$

that is, there is $(u''_n) \subset (u'_n)$ such that

$$\|u''_n(t) - u(t)\| \to 0 \;, \qquad n \to \infty, \quad \text{a.e. } \; t \in [0,T] \;, \tag{4.23}$$

$$\|u''_n(t) - u(t)\| \le g(t), \qquad \text{a.e. } \; t \in [0,T] \;. \tag{4.24}$$

From (4.23), we derive,

$$(Nu''_n)(t) = F(t, u''_n(t)) \;\to\; F(t, u(t)) = (Nu)(t) \qquad \text{a.e. } \; t \in [0,T] \;,$$

that is

$$\big\|(Nu''_n)(t) - (Nu)(t)\big\| \to 0 \;, \qquad n \to \infty, \quad \text{a.e. } \; t \in [0,T] \;. \tag{4.25}$$

On the other hand, from the definition of N and the growth condition (4.15) we derive:

$$\begin{aligned}\big\|(Nu_n'')(t)-(Nu)(t)\big\| &\le c_1\,\beta(t)\,\big[\|u_n''(t)\|^r+1\big]+\|(Nu)(t)\|\\ &\le \frac{c_1}{2}\Big[\beta^2(t)+\big(\|u_n''(t)\|^r+1\big)^2\Big]+\|(Nu)(t)\|\\ &\le \frac{c_1}{2}\Big[\beta^2(t)+2\left(\|u_n''(t)\|^{2r}+1\right)\Big]+\|(Nu)(t)\|\\ &\le \frac{c_1}{2}\Big[\beta^2(t)+2\Big(\big(g(t)+\|u(t)\|\big)^{2r}+1\Big)\Big]+\|(Nu)(t)\|\\ &= \frac{c_1}{2}\Big[\beta^2(t)+2\Big(\big(g(t)+\|u(t)\|\big)^{\frac{p}{p'}}+1\Big)\Big]+\|(Nu)(t)\|\ .\end{aligned}$$

Consequently,

$$\big\|(Nu_n'')(t)-(Nu)(t)\big\| \le \frac{c_1}{2}\left[2+\beta^2(t)+2\Big(g(t)+\|u(t)\|\Big)^{\frac{p}{p'}}\right]+\|(Nu)(t)\|$$

$$\text{a.e. } t\in[0,T]\ . \tag{4.26}$$

Since

$$\big\|(Nu_n'')(\cdot)-(Nu)(\cdot)\big\| \in L^{p'}(0,T;\mathbb{R}_+)\ ,$$

$$\frac{c_1}{2}\left[2+\beta^2(\cdot)+2\Big(g(\cdot)+\|u(\cdot)\|\Big)^{\frac{p}{p'}}\right]+\|(Nu)(\cdot)\| \in L^{p'}(0,T;\mathbb{R}_+)\ ,$$

from (4.25) and (4.26), by means of Lebesque dominated convergence theorem, it follows that $\|(Nu_n'')(\cdot)-(Nu)(\cdot)\|\to 0$ in $L^{p'}(0,T;\mathbb{R}_+)$ that is $\|Nu_n''-Nu\|_{L^{p'}(0,T;\mathbb{R}^N)}\to 0,\ n\to\infty$ which, taking into account $(u_n'')\subset(u_n')$, contradicts (4.22).

Since the operator $N\colon L^p(0,T;\mathbb{R}^N)\to L^{p'}(0,T;\mathbb{R}^N)$ generated by the function $F(\cdot,\cdot)$ is continuous and satisfies the growth condition (4.18), with $0<2^{\left(\frac{1}{p}-\frac{1}{p'}\right)}c_1<\lambda_1$, the existence of a solution in $W_T^{1,p}$ for problem (4.13), (4.14) is a direct consequence of Theorem 4.2. ∎

5 – An alternative existence proof via fixed point techniques

This technique is inspired by Manasevich and Mawhin [8]. As shown in the previous section, under the hypotheses of Theorem 4.2, in order to prove the existence of a solution for problem (4.13), (4.14) it is sufficient to prove the existence of a solution for

$$J_{p-1,W_T^{1,p}}\,u = -(i^*Ni)\,u \tag{5.1}$$

where, under the hypotheses of Theorem 4.3, N is defined by (4.17) and $(i^*Ni)\colon W_T^{1,p} \to \big(W_T^{1,p}\big)^* = W_T^{1,p'}$ is compact.

To prove the compactness of (i^*Ni) it is enough to observe that

$i\colon W_T^{1,p} \to L^p(0,T;\mathbb{R}^N)$ *is compact,*
$N\colon L^p(0,T;\mathbb{R}^N) \to L^{p'}(0,T;\mathbb{R}^N)$ *is continuous, and*
$i^*\colon L^{p'}(0,T;\mathbb{R}^N) \to \big(W_T^{1,p}\big)^*$ *is compact.*

On the other hand, the following theorem holds:

Theorem 5.1. *Let X be a smooth, reflexive Banach space having the Kadeč–Klee property. Then, for any gauge function φ, the duality mapping J_φ is a bijection of X on X^* having a continuous inverse.* ∎

The partial results employed in the proof may be found, for instance, in Zeidler [10].

Below, we describe the main steps of the proof.

1) Since X is smooth, J_φ is single-valued: $J_\varphi\colon X \to X^*$.
2) Since X is reflexive, J_φ is demicontinuous.
3) J_φ is monotone:

$$\forall\, x,y \in X, \quad \big\langle J_\varphi x - J_\varphi y,\, x-y\big\rangle \;\geq\; \big(\varphi(\|x\|) - \varphi(\|y\|)\big)\,\big(\|x\| - \|y\|\big) \;\geq\; 0\,.$$

4) J_φ is coercive

$$\frac{\langle J_\varphi x, x\rangle}{\|x\|} = \varphi(\|x\|) \to \infty \qquad \text{as} \quad \|x\| \to \infty\ .$$

5) Out of 2), 3), 4) it follows, via well know surjectivity theorem of Browder [2], that J_φ is a surjection of X to X^*.
6) Since X is strictly convex, J_φ is strictly monotone (injective, in particular).
7) From 5) and 6), it follows that J_φ is a bijection of X onto X^*.
8) $J_\varphi^{-1} = \chi^{-1} J^*_{\varphi^{-1}}$ where $J^*_{\varphi^{-1}}\colon X^* \to X^{**}$ is the duality mapping on X^* corresponding to the gauge function φ^{-1}, and χ^{-1} is the inverse of $\chi\colon X \to X^{**}$, $\langle \chi(x), x^*\rangle = \langle x^*, x\rangle$, for all $x \in X$ and $x^* \in X^*$. Note that because of X being reflexive and strictly convex it follows that X^{**} is strictly convex.

   Consequently, X^* is smooth, thus $J^*_{\varphi^{-1}}$ is single valued.

9) $J^*_{\varphi^{-1}}$ is demicontinuous (like any single-valued duality mapping on a reflexive space (see 2) above)).

10) From 8) and 9) it follows J_φ^{-1} is demicontinuous:

$$x_n^* \to x^* \text{ (in } X^*) \implies J_\varphi^{-1} x_n^* = \chi^{-1} J^*_{\varphi^{-1}} x_n^* \rightharpoonup J_\varphi^{-1} x^* = \chi^{-1} J^*_{\varphi^{-1}} x^* \text{ (in } X).$$

11) Moreover, $\|J_\varphi^{-1} x_n^*\| = \|J^*_{\varphi^{-1}} x_n^*\| = \varphi^{-1}(\|x_n^*\|) \to \varphi^{-1}(\|x^*\|) = \|J_\varphi^{-1} x^*\|$.

12) Since X has the Kadec–Klee property, from 10) and 11), we get $J_\varphi^{-1} x_n^* \to J_\varphi^{-1} x^*$.

Now, since $W_T^{1,p}$ has all properties imposed to the space X in the Theorem 5.1, it follows that $J_{p-1,W_T^{1,p}}$ is a bijection of $W_T^{1,p}$ on $\left(W_T^{1,p}\right)^*$ with a continuous inverse.

Consequently, showing that there exists $u \in W_T^{1,p}$ which satisfies (5.1) is equivalent with demonstrating that exists $u \in W_T^{1,p}$ such that

$$u = Tu \,, \tag{5.2}$$

with $T = \left(J_{p-1,W_T^{1,p}}\right)^{-1} [-i^* N i] \colon W_T^{1,p} \to W_T^{1,p}$ compact.

In order to prove that T has a fixed point, we will use a priori estimate technique: there is $r > 0$ such that

$$\left\{u \in W_T^{1,p} \mid \exists\, t \in (0,1] \text{ such that } u = t\,Tu\right\} \subset B(0,r)\,.$$

Indeed, consider $t \in (0,1]$ and $u_t \in W_T^{1,p}$ such that $u_t = t\,Tu_t$. Then

$$J_{p-1,W_T^{1,p}}\left(\frac{u_t}{t}\right) = (i^* N i)\, u_t\,. \tag{5.3}$$

Because of

$$\|i(u)\|_{L^p(0,T;\mathbb{R}^N)} \le \lambda_1^{-\frac{1}{p}} \|u\|_{W_T^{1,p}}\,, \qquad \text{for all } u \in W_T^{1,p}\,,$$

it follows that $\|i\| = \|i^*\| \le \lambda_1^{-\frac{1}{p}}$.

By using this fact and notating, for simplicity, $c_1' = 2^{\left(\frac{1}{p} - \frac{1}{p'}\right)} c_1$ in the growth condition (4.18) we derive:

$$\begin{aligned} \|u_t\|^{p-1}_{W_T^{1,p}} &\le \left\|\frac{u_t}{t}\right\|^{p-1}_{W_T^{1,p}} \\ &= \left\|J_{p-1,W_T^{1,p}}\left(\frac{u_t}{t}\right)\right\|_{W_T^{1,p'}} \\ &= \left\|i^*(N(i(u_t)))\right\|_{W_T^{1,p'}} \le \end{aligned}$$

$$\begin{aligned}
&\le \lambda_1^{-\frac{1}{p}} \left\|N(i(u_t))\right\|_{L^{p'}(0,T;\mathbb{R}^N)} \\
&\le \lambda_1^{-\frac{1}{p}} \left[c_1' \|i(u_t)\|^{p-1}_{L^p(0,T;\mathbb{R}^N)} + c_2\right] \\
&\le \lambda_1^{-\frac{1}{p}} \left[c_1' \, \lambda_1^{-\frac{p-1}{p}} \|u_t\|^{p-1}_{W_T^{1,p}} + c_2\right] \\
&= c_1' \, \lambda_1^{-1} \|u_t\|^{p-1}_{W_T^{1,p}} + \lambda_1^{-\frac{1}{p}} c_2 \ .
\end{aligned}$$

Thus, we have

$$(1 - c_1' \lambda_1^{-1}) \, \|u_t\|^{p-1}_{W_T^{1,p}} - \lambda_1^{-\frac{1}{p}} c_2 \le 0 \ . \tag{5.4}$$

Since $0 \le c_1' < \lambda_1$ we deduce that $(1 - c_1'\lambda_1^{-1}) \, \xi^{p-1} - \lambda_1^{-\frac{1}{p}} c_2 \to \infty$ as $\xi \to \infty$. Therefore, there exists $r > 0$ such that $(1 - c_1'\lambda_1^{-1}) \, \xi^{p-1} - \lambda_1^{-\frac{1}{p}} c_2 > 0$ for any $\xi \ge r$.

Consequently, (5.4) holds only for u_t satisfying $\|u_t\|_{W_T^{1,p}} < r$. ∎

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