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NON COMPLETE INTEGRABILITY OF A SATELLITE IN CIRCULAR ORBIT

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Abstract: We consider the problem of a rigid body (for example a satellite) moving in a circular orbit around a fixed gravitational center whose inertia tensor's components A, B, C are positive real numbers satisfying $0 < A < B \le C = 1$. We prove the non complete meromorphic integrability of the satellite using a criterion based on a theorem of J.-J. Morales and J.-P. Ramis. This criterion relies on some local and global properties of a linear differential system, called normal variational system and depending rationally on A and $\sqrt{3\,(B-A)}$. Our proof uses tools from computer algebra and proceeds in two steps: first the satellite *with axial symmetry* (i.e. $0 < A < B = C = 1$) then the satellite *without axial symmetry* (i.e. $0 < A < B < C = 1$).

1 – Introduction

We consider the problem of a rigid body (the satellite) moving in a circular orbit around a fixed gravitational center ([15, 16, 17], [18]). This satellite is characterized by its inertia tensor which is related to the distribution of the mass throughout the body. Its components A, B and C are positive real numbers which can be assumed to satisfy the conditions $0 < A < B \le C = 1$. We want to know whether there exists values of these parameters such that the satellite may be completely integrable (with meromorphic first integrals). That means that there exists a sufficient number of meromorphic first integrals which are functionally independent and in involution to describe the behavior of the satellite (see [1], [9] and [21] for precise definitions of these notions).

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In the last twenty years, Ziglin (1982); Baider, Churchill, Rod and Singer (1996) and Morales and Ramis (1998) established new theories on complete integrability of Hamiltonian systems. They all found necessary conditions of complete (meromorphic) integrability based on the monodromy group ([25], [26]) or the differential Galois group ([10], [21]) of a first order linear differential system, the *normal variational system* (or *equation*) computed along a particular solution of the Hamiltonian system (see [1], [9]).

In [18], using Ziglin's theory, A. Maciejewski and K. Gozdziewski gave a *numerical* proof of the non complete integrability of the problem of the satellite *with axial symmetry* (when $0 < A < B = C = 1$). Furthermore, A. Maciejewski ([16, 17]) and M. Audin ([2]) also gave independent (formal) proofs of non complete integrability for the satellite with *axial symmetry* . They both applied J.-J. Morales and J.-P. Ramis' theorem to an order two variational equation using two different approaches.

In this paper we prove that for all initial values of A, B and C, the satellite is not completely integrable. We consider first the satellite *with axial symmetry* ($0 < A < B = C = 1$) and then the satellite *without axial symmetry* ($0 < A < B < C = 1$). Both parts of the proof ($B = 1$ and $B < 1$) rely on a criterion of non complete meromorphic integrability deduced from J.-J. Morales and J.-P. Ramis' theorem ([21]) and established in [7, 6]:

Criterion 1 ([7, 6])**.** *Let (S) be a Hamiltonian system and (NVS) be the normal variational system computed along a particular solution of (S). If*

- **(i)** *(NVS) is irreducible and has formal solutions with logarithmic terms at a singular point, or*
- **(ii)** *(NVS) has an irreducible factor which has formal solutions with logarithmic terms at a singular point,*

then (S) is not completely integrable (with meromorphic first integrals).

Unlike Kovacic algorithm, criterion 1 can be applied to normal variational systems ($Y'(z) = M(z)\,Y(z)$, $M(z) \in \mathcal{M}_{n,n}(\bar{\mathbb{Q}}(z))$) of size $n \times n$ where n can be greater than 2. For a recent overview of the existing algorithms, see chapter 4 of [22]. Here the normal variational system depends rationally on A and a new parameter w defined by $w^2 = 3\,(B - A)$ ($M(z) \in \mathcal{M}_{4,4}(\bar{\mathbb{Q}}(A, w)(z))$). So one needs to adapt the algorithms to a parameterized situation and in general it is not so easy because arithmetic constraints on the parameters may appear, which creates problems of undecidability (theorem 1 of [5]).

In section 2, we give the normal variational system computed by A. Maciejewski in [15, 16, 18].

In section 3 we consider the particular case when the satellite has an *axial symmetry* i.e. when $0 < A < B = C = 1$ or $0 < w^2 = 3\,(1 - A) < 3$. The normal variational system depends only on w. We prove that this system can be reduced to a linear differential equation of order 2 which is *irreducible* and has formal solutions with *logarithmic* terms when w is a non zero real number.

In section 4 we deal with the *general* case of the satellite, the satellite *without axial symmetry* i.e. when $0 < A < B < C = 1$ or $0 < w^2 < 3\,(1 - A) < 3$. We first prove that the 4×4 normal variational system has *two* linearly independent regular formal solutions with *logarithmic* terms and that it is *irreducible* under the conditions on the parameters. In particular, we used the initial polynomial inequalities and inequations on the parameters A and w to solve arithmetic problems which appeared in the study of the exponential solutions of the normal variational system.

In both sections 3 and 4, we conclude that the satellite is not meromorphically completely integrable.

2 – Normal variational system

In this section we recall the equations which define the problem of the satellite ([15, 16, 18]) and the computation of the normal variational system made by A. Maciejewski in [15].

The equations of the rotational motion of the body can be written with a 9×9 Lie-Poisson system:

$$\mathsf{x}'(t) = \mathcal{J}(\mathsf{x}(t))\,\nabla H(\mathsf{x}(t))$$

where $\mathsf{x} = {}^{\mathsf{t}}(\mathsf{m}, \gamma, \mathsf{n})$,

$$H = \frac{1}{2}\,\langle \mathsf{m}, I^{-1}\mathsf{m}\rangle - \langle \mathsf{m}, \mathsf{n}\rangle + \frac{3}{2}\langle \gamma, I\gamma\rangle \ ,$$

$\mathcal{J}(\mathsf{x})$ is the 9×9 matrix

$$\mathcal{J}(\mathsf{x}) = \begin{pmatrix} J(\mathsf{m}) & J(\gamma) & J(\mathsf{n}) \\ J(\gamma) & 0 & 0 \\ J(\mathsf{n}) & 0 & 0 \end{pmatrix}$$

and for $x \in \{\mathsf{m}, \gamma, \mathsf{n}\}$, $J(x)$ is the 3×3 matrix

$$J(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} .$$

The vector n is the unit vector normal to the plane of the orbit; the vector γ is the unit vector in the direction of the radius vector of the center of mass of the satellite and the vector m is the angular momentum ($\mathsf{m} = I\Omega$ where Ω is the angular velocity).

The matrix I is the inertia tensor, it is related to the distribution of the mass throughout the body:

$$I = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

where the parameters A, B and C are positive real numbers. Without loss of generality, one can assume that they satisfy

$$0 < A < B \leq C = 1 \ . \tag{1}$$

In [15], the author reduces this 9×9 differential system to a Hamiltonian one on a symplectic manifold.

He considers new variables $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ defined by:

$$\mathsf{m} = K^{-1}\, p \ ,$$

$$K = \begin{pmatrix} -s_2 & s_3\, c_2 & c_3\, c_2 \\ 0 & c_3 & -s_3 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathsf{n} = \begin{pmatrix} -s_2 \\ s_3\, c_2 \\ c_3\, c_2 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} c_2\, c_1 \\ s_3\, s_2\, c_1 - c_3\, s_1 \\ c_3\, s_2\, c_1 + s_3\, s_1 \end{pmatrix},$$

$$\forall\, i \in \{1, 2, 3\} \ , \qquad c_i = \cos(q_i), \quad s_i = \sin(q_i) \ .$$

He defines an heteroclinic solution on the invariant manifold

$$\Big\{(q, p) \in \mathbb{R}^6, \ q_2 = q_3 = p_2 = p_3 = 0\Big\}$$

by:

$$\cos(q_1) = \frac{1-z^2}{1+z^2}$$

$$\sin(q_1) = \frac{-2z}{1+z^2}$$

$$p_1 = 1 - w\,\frac{1-z^2}{1+z^2}$$

$$\frac{dz}{dt} = \frac{w}{2}\,(1-z^2)$$

where w is a new parameter defined by

$$w^2 = 3\,(B-A)\ . \tag{2}$$

He gets a normal variational system with coefficients in $\mathbb{Q}(w, A)(z)$.

Remark 1. To compute the normal part of the variational system one can compute the variational system first and make a symplectic transformation on it. We do not show this computation which leads to the same normal variational system as in [15]. □

The normal variational system we work with is then

$$Y'(z) = M(z)\,Y(z) \tag{3}$$

where

$$M(z) = \begin{pmatrix} M_1 & M_2 \\ M_3 & -{}^tM_1 \end{pmatrix}$$

with

$$M_1 = \begin{pmatrix} 0 & \frac{2(w^2-3+3\,A)}{w\,(3\,A+w^2)}\,\frac{(1+w)\,z^2+1-w}{(z-1)\,(z+1)\,(z^2+1)} \\ \frac{-2}{w}\,\frac{(1+w)\,z^2+1-w}{(z+1)\,(z-1)\,(z^2+1)} & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \frac{-6}{w\,(3\,A+w^2)}\,\frac{1}{(z+1)\,(z-1)} & 0 \\ 0 & \frac{-2}{A\,w}\,\frac{1}{(z+1)\,(z-1)} \end{pmatrix}$$

$M_3 =$

$$\begin{pmatrix} \frac{2}{w}\frac{(w^2+2\,w+4-3\,A)\,z^4+(6\,A-4-2\,w^2)\,z^2w^2-2\,w+4-3\,A}{(z-1)\,(z+1)\,(z^2+1)^2} & \frac{-4\,(w^2-3+3\,A)}{w}\,\frac{z}{(z^2+1)^2} \\ \frac{-4\,(w^2-3+3\,A)}{w}\,\frac{z}{(z^2+1)^2} & \frac{-2\,(w^2-3+3\,A)}{w\,(3\,A+w^2)}\,\frac{(1+w)^2\,z^4+(12\,A+2+2\,w^2)\,z^2+(w-1)^2}{(z-1)\,(z+1)\,(z^2+1)^2} \end{pmatrix}.$$

The conditions (1) and (2) become

$$\boxed{0 < w^2 \leq 3\,(1-A) < 3} \quad .$$

In the following section, we consider the special case when the satellite has an axial symmetry.

3 – Satellite *with axial symmetry*

We assume that two of the components of the inertia tensor are equal: $B = C = 1$ or

$$\boxed{0 < w^2 = 3\,(1-A) < 3} \quad .$$

If we replace A with $A = \frac{3-w^2}{3}$, the normal variational system (3) becomes:
$Y'(z) =$

$$\begin{pmatrix} 0 & 0 & \frac{-2}{w}\,\frac{1}{(z-1)\,(z+1)} & 0 \\ \frac{-2}{w}\,\frac{(w+1)\,z^2+1-w}{(z+1)\,(z-1)\,(z^2+1)} & 0 & 0 & \frac{6}{w\,(w^2-3)}\,\frac{1}{(z-1)\,(z+1)} \\ \frac{2}{w}\,\frac{(2w^2+2w+1)\,z^4+(2-4\,w^2)\,z^2+2w^2-2w+1}{(z+1)\,(z-1)\,(z^2+1)^2} & 0 & 0 & \frac{2}{w}\,\frac{(w+1)z^2+1-w}{(z+1)\,(z-1)\,(z^2+1)} \\ 0 & 0 & 0 & 0 \end{pmatrix} Y(z)$$

where the parameter w is such that $0 < w^2 < 3$.

This system has the particular solution ${}^t(0,1,0,0)$. We construct a symplectic matrix P (such that ${}^tP\,J\,P - J = 0$ where $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$) with the vector

${}^t(0,1,0,0)$ on its first row:

$$P = \begin{pmatrix} \boxed{\begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \end{pmatrix} .$$

We make the Gauge transformation $Y = P\,Z$ and get the equivalent linear differential system satisfied by Z whose first column and third row are zero by construction:

$$Z'(z) = \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \alpha(z) \\ 0 & 0 & 0 & 0 \\ 0 & \beta(z) & 0 & 0 \end{pmatrix} Z(z)$$

where

$$\alpha(z) = \frac{-2}{w\,(z^2-1)}$$

$$\beta(z) = 2\,\frac{(2w^2+2w+1)\,z^4 + (2-4w^2)\,z^2 + 1 - 2w + 2w^2}{w(z^2+1)^2\,(z^2-1)} .$$

We can remove the first and third columns and rows to get the following 2×2 reduced system:

$$\nu'(z) = \begin{pmatrix} 0 & \alpha(z) \\ \beta(z) & 0 \end{pmatrix} \nu(z) .$$

According to [21] (or proposition 4.2 p. 76 of [20]), this symplectic transformation enables to apply our criterion directly on this reduced 2×2 linear differential system instead of applying it on the initial 4×4 system.

We first transform it into a linear differential equation of order 2 namely the one satisfied by the first component y of ν:

$$y''(z) - \frac{\alpha'(z)}{\alpha(z)}\,y'(z) - \alpha(z)\,\beta(z)\,y(z) \;=\; 0$$

i.e.

$$y''(z) + \frac{2\,z}{(z-1)\,(z+1)}\,y'(z) - \frac{4\Big((2\,w^2+2\,w+1)\,z^4 + (2-4\,w^2)\,z^2 + 2\,w^2 - 2\,w + 1\Big)}{w^2\,(z+1)^2\,(z-1)^2\,(z^2+1)^2}\,y(z) \;=\; 0 . \tag{4}$$

Proposition 1. *For each non zero real number w, the equation (4) has formal solutions with logarithmic terms at the point i and is irreducible.*

Proof: The singularities i, $-i$, 1 and -1 are regular singular points and their exponents are:

$$\begin{array}{ll} e_i, e_{-i} \in \{-1, 2\} & \text{at } \; i \; \text{ and } \; -i\,; \\ e_1, e_{-1} \in \{\frac{i}{w}, \frac{-i}{w}\} & \text{at } \; 1 \; \text{ and } \; -1\,. \end{array}$$

The point infinity is an ordinary point.

At the points i and $-i$, the exponents differ from an integer so there may be a formal solution with logarithmic terms. Using the criterion of detection of logarithmic terms (16.3, [14]), we prove that under the condition $w \neq 0$, there is a formal solution with a logarithm at the point i. One can also use the procedure `formal_sol` of the package `DEtools` in Maple which returns the following two formal solutions:

$$f_1 = h_1 \;\text{ and }\; f_2 = ln(X)\, h_1 + h_2$$

where

$$\begin{aligned} h_1 &= \frac{i}{9w}\,(X^2 + \frac{i\,(2\,w-1)}{2\,w}\,X^3 - \frac{13\,w^2 - 15\,w + 4}{20\,w^2}\,X^4) + h.o.t. \\ h_2 &= -\frac{1}{3}\,\frac{1}{X} + \frac{i\,(\,w-2)}{6\,w} - \frac{1}{12}\,\frac{\left(w^2-2\right)}{w^2}\,X - \frac{i\left(27\,w^3 + 99 + 133\,w^2 - 279\,w\right)}{1080\,w^3}\,X^2 \\ &+ \frac{\left(387\,w - 189 + 9\,w^4 - 661\,w^2 + 164\,w^3\right)}{2160\,w^4}\,X^3 \\ &- \frac{i\left(9\,w^5 - 576 - 5290\,w^2 + 5274\,w^3 - 631\,w^4 + 3591\,w\right)}{21600\,w^5}\,X^4 + h.o.t. \end{aligned}$$

with $X = z - i$.

We will see in section 4.1 a like-Ince criterion to detect logarithmic terms in regular formal solutions of first order homogeneous linear differential systems without computing the terms of these solutions.

Now let us prove that the equation is irreducible. As it is of order 2, it suffices to prove that it has no right factor of degree one $(\frac{d}{dz} - r, r \in \overline{\mathbb{Q}}(w)(z))$, i.e. no exponential solution (y such that $\frac{y'(z)}{y(z)} = r \in \overline{\mathbb{Q}}(w)(z)$).

But an exponential solution can be written

$$(z-i)^{e_i}\,(z+i)^{e_{-i}}\,(z-1)^{e_1}\,(z+1)^{e_{-1}}\,q(z)$$

where q is a polynomial and for x_0 in $\{1,-1,i,-i\}$, e_{x_0} is an exponent at the (regular singular) point x_0. As the point infinity is an ordinary point for the equation, the degree of q is $-(e_1+e_{-1}+e_i+e_{-i})$ which is at most $2-e_1-e_{-1}$ where $e_1^2\,w^2+1=e_{-1}^2\,w^2+1=0$. So we must have $2-e_1-e_{-1}\in\mathbb{N}$. As the parameter w is real, the only possibility for (e_1,e_{-1}) is: $(e_1,e_{-1})\in\{(\frac{i}{w},\frac{-i}{w}),(\frac{-i}{w},\frac{i}{w})\}$ and then the degree of the polynomial q does not depend on the parameters.

So if there is an exponential solution then it can be written

$$\left(\frac{z+1}{z-1}\right)^e\frac{q(z)}{(z^2+1)}$$

where $q=q_0+q_1\,z+q_2\,z^2$ and $e^2\,w^2+1=0$.

The coefficients q_0,q_1,q_2 satisfy a linear homogeneous system whose coefficients depend on e and w and whose size is numerically fixed. More precisely this system is $\mathcal{A}\,^t(q_0,q_1,q_2)=0$ where $\mathcal{A}$ is a 5×3 matrix. A Groebner Basis ([13]) for the polynomial system made of the 3×3 sub-determinants of the matrix $\mathcal{A}$ and of the polynomial $e^2\,w^2+1$ is reduced to [1]. So there exists no polynomial q, no non zero exponential solution and the order two linear differential equation (4) is irreducible. ∎

Remark 2. One can notice that if an exponential solution exists, then its valuation at the point i (resp. $-i$) will be equal to 2. Indeed the valuation of the only formal solution at the point i (resp. $-i$) without logarithmic term, f_1, is equal to 2. So an exponential solution would be $\left(\frac{z+1}{z-1}\right)^e q(z)\,(z^2+1)^2$ and the degree of the polynomial $q(z)$ would be -4, which is impossible. We thank Jacques-Arthur Weil for this remark. □

From criterion 1 and proposition 1, we can now state:

Proposition 2. *The satellite with axial symmetry is not completely integrable.* ∎

We now extend the proof to the case of the satellite *without axial symmetry*.

4 – General case: satellite *without axial symmetry*

Now we assume that there is no axial symmetry i.e. $B\neq C=1$. We get the conditions

$$(\mathcal{C}):\boxed{0<w^2<3\,(1-A)<3}\ .$$

4.1. Formal solutions with logarithmic terms

We count the number of formal solutions with logarithmic terms at the point i ([4], [23]) without computing any term of the solutions thanks to the following lemma:

Lemma 1. *Let $z\,Y'(z) = \mathbf{A}(z)\,Y(z)$ be a first order linear differential system of size $n \times n$ ($\mathbf{A} \in \mathcal{M}_{n,n}(C(z))$, C field of characteristic 0) with 0 a regular singular point:*

$$\mathbf{A}(z) = A_0 + A_1\,z + \cdots + A_k\,z^k + \cdots \ , \qquad A_0 \neq 0 \ .$$

Assume that the eigenvalues of the matrix A_0 (i.e. the exponents at 0), $\rho_0 < \cdots < \rho_{n-1}$, differ each other from non zero integers.

Let m be the biggest difference between two exponents: $m = \rho_{n-1} - \rho_0$ ($\in \mathbb{N}$) and let $\mathcal{M}$ be the $m \times m$ triangular block-matrix (i.e. $m\,n \times m\,n$ matrix):
$\mathcal{M} =$

$$\begin{pmatrix}
\rho_0\,I_n - A_0 & 0 & 0 & 0 & 0 & 0 \\
-A_1 & (\rho_0+1)\,I_n - A_0 & 0 & 0 & 0 & 0 \\
-A_2 & -A_1 & (\rho_0+2)\,I_n - A_0 & 0 & 0 & 0 \\
 & -A_2 & & & & \\
\vdots & & & & \ddots & \\
 & & & & & \\
-A_{m-1} & -A_{m-2} & & & -A_1 & (\rho_0+m-1)\,I_n - A_0
\end{pmatrix} .$$

The number of linearly independent formal solutions with logarithmic terms at the point 0 is equal to $n - N$ where N is the dimension of the kernel of the matrix $\mathcal{M}$.

Proof: Let $Y(z)$ be a formal solution without logarithmic term:

$$Y(z) = z^{\rho_0}\,(Y_0 + Y_1\,z + Y_2\,z^2 + Y_3\,z^3 + \cdots) \ .$$

The coefficients Y_k satisfy the following recurrence relation:

$$\big((k+\rho_0)\,I_n - A_0\big)\,Y_k = \sum_{j=1}^{k} A_j\,Y_{k-j} \ , \qquad k \in \mathbb{N} \ .$$

The coefficient Y_k is uniquely determined when $k + \rho_0 > \rho_{n-1}$. The m first equations can be written

$$\mathcal{M}\,\mathcal{Y} = 0$$

where $\mathcal{Y}$ is the vector ${}^t({}^tY_0,\ldots,{}^tY_{m-1})$.

So the number of linearly independent formal solutions without logarithmic term is equal to the dimension N of the kernel of $\mathcal{M}$, which gives the result. ∎

We notice here that there may be formal solutions with logarithmic terms even if the matrix A_0 is diagonalizable. To detect them we need further terms in the development at the point 0 of the matrix $\mathbf{A}$ as it is the case for the detection of logarithms in the formal solutions of linear differential equations (16.3, [14]).

From this lemma we deduce the following proposition:

Proposition 3. *Under the conditions $(\mathcal{C})$ on the parameters, the normal variational system (3) has two linearly independent formal solutions with logarithmic terms at the regular singular point i.*

Proof: Using a Moser reduction at the point i ([23]) and moving the singularity i to the point 0, one can compute an equivalent linear differential system:

$$z\,Y'(z) = (A_0 + A_1\,z + \cdots + A_k\,z^k + \cdots)\,Y(z)\ .$$

As the matrix A_0 is non zero, the point i is a regular singularity of the system (3) and its exponents are the eigenvalues of the matrix A_0: $\rho_0 = -2$, $\rho_1 = -1$, $\rho_2 = 0$, $\rho_3 = 1$. They differ each other from non zero integers.

Let us consider the 16×16 matrix

$$\mathcal{M} = \begin{pmatrix} -2I_4 - A_0 & 0 & 0 & 0 \\ -A_1 & -I_4 - A_0 & 0 & 0 \\ -A_2 & -A_1 & -A_0 & 0 \\ -A_3 & -A_2 & -A_1 & I_4 - A_0 \end{pmatrix}.$$

Every sub-determinant of order 15 is zero. Among the sub-determinants of order 14, there is at least one which is non zero under the inequations given by $(\mathcal{C})$. For example the determinant of the sub-matrix obtained after removing the rows 2 and 6 and the columns 12 and 16 from the matrix $\mathcal{M}$ is:

$$\frac{2592\,(w^2+3)^2\,(w^2+3\,A-3)^3\,(A-1)}{w^6\,(3\,A+w^2)^4}\ .$$

So the matrix $\mathcal{M}$ is of rank 14 under the inequations given by $(\mathcal{C})$ and according to lemma 1, there are two linearly independent formal solutions with logarithmic terms. ∎

Remark 3. Adapting the function `formal_sols` of the package `Isolde` ([23]) in Maple to the parameterized situation we deal with, we are able to compute "*generically*" formal solutions of parameterized linear differential systems at any point whose exponents do not depend on the parameters. Generically we get two formal solutions with logarithmic terms at the point i which are defined for all values satisfying the conditions $(\mathcal{C})$ except for (A, w) such that $3 - w^2 - 6A = 0$. This exceptional value of (A, w) does not annihilate the 14×14 sub-determinant in the proof of the proposition 3 and we have checked that for $A = \frac{3-w^2}{6}$, we get again two linearly independent formal solutions with logarithmic terms at the point i. □

Remark 4. Another technique to detect logarithms in formal solutions of linear differential systems will be seen in a forthcoming paper. □

4.2. Irreducibility

We prove the following proposition.

Proposition 4. *Under the condition* $(\mathcal{C})$, *the normal variational system (3) is irreducible.*

Proof: We prove that there is no factor of degree one, no factor of degree three and no factor of degree two in the three following subsections. ∎

4.2.1. Factor of degree one

A factor of degree one corresponds to an exponential solution. Deciding whether a first order linear differential system has an exponential solution requires looking for polynomial solutions of an auxiliary linear differential system ([3],[23]). But the degree of the searched polynomial solution may depend on the parameters. Here we meet two situations:

1. When the degree of the searched polynomial depends on the parameters A and w, then we get an arithmetic condition on A and w and we prove that it cannot be satisfied under the inequalities and inequations of $(\mathcal{C})$ (lemma 2). Such a proof uses tools from real algebraic geometry.
2. When the degree of the polynomial does not depend on the parameters, then we get only algebraic conditions on the parameters and we prove that they are not satisfied under the inequations given by $(\mathcal{C})$ (lemma 3).

An exponential solution will be of the form ([23])

$$(z-1)^{e_1}\,(z+1)^{e_{-1}}\,(z-i)^{e_i}\,(z+i)^{e_{-i}}\,P(z)$$

where $P(z)$ is a polynomial vector and for x_0 in $\{1,-1,i,-i\}$, x_0 is a regular singular point whose exponent, e_{x_0}, is a root of the polynomial E_{x_0}:

$$E_i(n)=E_{-i}(n)=(n+2)\,(n+1)\,n\,(n-1)$$

$$E_1(n)=E_{-1}(n)=n^4+a\,n^2+b \tag{5}$$

with

$$a=\frac{(w^2+A+1)\,(3-3\,A-w^2)+3\,A}{w^2\,A\,(3\,A+w^2)}\quad\text{and}\quad b=\frac{4\,(1-A)\,(3-3\,A-w^2)}{A\,w^4\,(3\,A+w^2)}\ . \tag{6}$$

As the exponents at the points i and $-i$ are $-2,-1,0$ and 1, we can look for exponential solutions of the form

$$(z-1)^{e_1}\,(z+1)^{e_{-1}}\,(z^2+1)^{-2}\,P(z)$$

and $P(z)$ satisfies now the linear differential system

$$Y'(z)=\left(M(z)-\left(\frac{e_1}{z-1}+\frac{e_{-1}}{z+1}-\frac{4z}{z^2+1}\right)I_4\right)Y(z)\ .$$

As this system is simple at infinity (see [3]) and of indicial polynomial $(n-(4-e_{-1}-e_1))^4$, the polynomial vector $P(z)$ is of degree at most $4-e_{-1}-e_1$.

We will distinguish two cases: either $e_{-1}+e_1$ is non zero then with the conditions $(\mathcal{C})$ on w and A the quantity $4-e_{-1}-e_1$ cannot be an integer (lemma 2); either $e_{-1}+e_1$ is equal to zero then there is no value of A and w satisfying $(\mathcal{C})$ and leading to a polynomial solution of degree 4 (lemma 3).

Lemma 2. *Under the conditions $(\mathcal{C})$, the normal variational system (3) has no non zero exponential solution*

$$(z-1)^{e_1}\,(z+1)^{e_{-1}}\,(z^2+1)^{-2}\,P(z)$$

where $P(z)$ is a non zero polynomial vector and where e_1+e_{-1} is non zero.

Proof: If such an exponential solution exists then $4-e_{-1}-e_1$ (i.e. the degree of P) is in $\mathbb{N}$. So a necessary condition is that there exists e_1, e_{-1} such that

$$E_1(e_1) = E_{-1}(e_{-1}) = 0 \quad \text{and} \quad e_1 + e_{-1} \in \mathbb{Z} \; .$$

The polynomial whose zeroes are the sums of the roots of E_1 and E_{-1} is $Res_x(E_1(u), Res_v(E_{-1}(v), x-(u+v)))$ where $Res_x(f(x), g(x))$ is the resultant in x of $f(x)$ and $g(x) \in C[x]$. Taking its squarefree part, we get the necessary condition:

$$\exists\, \alpha \in \mathbb{Z} \; , \qquad E(\alpha) = \alpha\,(\alpha^4 + 4\,a\,\alpha^2 + 16\,b)\,(\alpha^4 + 2\,a\,\alpha^2 + a^2 - 4\,b) = 0$$

(in fact a stronger condition is $4-\alpha \in \mathbb{N}$).

We prove that under the initial constraints $(\mathcal{C})$, the polynomial E has no non zero *real* root. It suffices to notice that a, b and $a^2 - 4\,b$ are positive.

The positivity of a and b is directly deduced from the conditions $1 - A > 0$, $A > 0$, $w^2 > 0$ and $3 - 3\,A - w^2 > 0$.

Now the expression of $a^2 - 4\,b$ is given below:

$$\begin{aligned} &w^4\,A^2\,(w^2+3\,A)^2\,(a^2-4b) \;= \\ &\quad w^8 + 4\,(2\,A-1)\,w^6 + 2\,(3\,A^2-3\,A-1)\,w^4 - 12\,(2\,A-1)\,(3\,A^2-3\,A+1)\,w^2 \\ &\quad - 9\,(5\,A^2-5\,A-1)\,(3\,A^2-3\,A+1)\; . \end{aligned}$$

Using the algorithm of Cylindric Algebraic Decomposition ([12], [24]) one can prove that the polynomial

$$\begin{aligned} &W^4 + 4\,(2\,A-1)\,W^3 + 2\,(3\,A^2-3\,A-1)\,W^2 - 12\,(2\,A-1)\,(3\,A^2-3\,A+1)\,W - \\ &\qquad\qquad 9\,(5\,A^2-5\,A-1)\,(3\,A^2-3\,A+1) \end{aligned}$$

is always positive when

$$A > 0, \quad W > 0, \quad 1 - A > 0, \quad 3 - 3\,A - W > 0 \; .$$

This is a simple computation for the specialists, which was made in Magma by M. Safey ([24]) who also noticed that the condition $0 < A < 1$ suffices to prove the positivity of this polynomial. You can also check it using the function `Experimental‘ImpliesRealQ` in Mathematica (version 4.0.1). We conclude that $a^2 - 4\,b$ is positive.

So under the conditions $(\mathcal{C})$, the polynomial E has no non zero real root and in particular it has no non zero integer root. ∎

Now if there is an exponential solution then $e_1 + e_{-1} = 0$ is the only possibility.

Lemma 3. *Under the conditions $(\mathcal{C})$, the normal variational system (3) has no non zero exponential solution*

$$(z+1)^e\,(z-1)^{-e}\,(z^2+1)^{-2}\,P(z)$$

where $P(z)$ is a non zero polynomial vector.

Proof: If it exists, the polynomial P will be of degree at most 4 (independent of the parameters) and will be a solution of the system:

$$z\,Y'(z) = N(z)\,Y(z)$$

where $N(z) = z\,(M(z) + (\frac{2e}{z^2-1} + \frac{4z}{z^2+1})\,I_4)$. This new system can also be written:

$$z\,Y'(z) = (N_0 + N_1 z^{-1} + N_2\,z^{-2} + N_3\,z^{-3} + N_4\,z^{-4} + \cdots)\,Y(z)\ .$$

The matrix N_0 is the scalar matrix $4\,I_4$.

Let us denote $P = P_4\,z^4 + P_3\,z^3 + P_2\,z^2 + P_1\,z + P_0$. The coefficients P_0, P_1, P_2, P_3 and P_4 satisfy the following relations:

$$\begin{cases} (4\,I_4 - N_0)\,P_4 &= 0 \\ (3\,I_4 - N_0)\,P_3 &= N_1\,P_4 \\ (2\,I_4 - N_0)\,P_2 &= N_1\,P_3 + N_2\,P_4 \\ (I_4 - N_0)\,P_1 &= N_1\,P_2 + N_2\,P_3 + N_3\,P_4 \\ -N_0\,P_0 &= N_1\,P_1 + N_2\,P_2 + N_3\,P_3 + N_4\,P_4 \\ \vdots & \vdots \end{cases}$$

As $N_0 = 4\,I_4$, the first equation gives no condition on P_4 and the vectors P_3, P_2, P_1, P_0 are uniquely determined in function of $P_4 =^t (z_1, z_2, z_3, z_4)$ by the four other equations. Then the equation $z\,P'(z) - N(z)\,P(z) = 0$ gives five linear homogeneous systems $\mathcal{M}_k\,{}^t(z_1, z_2, z_3, z_4) = 0$, $k = 1 \ldots 5$.

The next step consists in proving that the determinants of the matrices $\mathcal{M}_k$ cannot simultaneously cancel under the conditions $(\mathcal{C})$. The following computations were made in Magma. Let us denote d_k the determinant of the matrix $\mathcal{M}_k$.

$$\begin{cases} d_1 = (3\,A + w^2)^9\,A^4\,\tilde{d}_1 \\ d_2 = (3\,A + w^2)\,w^4\,A^5\,\tilde{d}_2 \\ d_3 = (3\,A + w^2)^8\,A^4\,\tilde{d}_3 \\ d_4 = 0 \\ d_5 = 0 \end{cases}$$

where $\tilde{d}_1$, $\tilde{d}_2$ and $\tilde{d}_3$ are polynomials of respective total degree 55, 46 and 57 in the variables A, w and e.

For each k in $\{1,2,3\}$ we compute the resultants $r_k = Res_e(\tilde{d}_k, numer(e^4 + a\,e^2 + b))$ where a and b are given by (6):

$$\begin{cases} r_1 = w^{100}\,A^{10}\,(A-1)^2\,(w^2+3\,A)^{10}\,(w^2+3\,A-3)^2\,f_1^2 \\ r_2 = w^{76}\,A^{8}\,(A-1)^2\,(w^2+3\,A)^{10}\,(w^2+3\,A-3)^4\,f_2^2 \\ r_3 = w^{96}\,A^{12}\,(A-1)^2\,(w^2+3\,A)^{12}\,(w^2+3\,A-3)^2\,f_3^2\ . \end{cases}$$

The polynomials f_1, f_2 and f_3 are of respective degrees 26, 20 and 26 in the variable A.

Now we compute $s_1 = Res_A(f_1, f_2)$ and $s_2 = Res_A(f_2, f_3)$:

$$\begin{cases} s_1 = w^{25}\,(w+1)^2\,(w^2-3)^{154}\,(w^2+3)^{88}\,t_1\,t_2 \\ s_2 = w^{33}\,(w+1)\,(w^2-3)^{143}\,(w^2+3)^{88}\,t_3\,t_4\ . \end{cases}$$

Lastly the greatest common divisor of the polynomials $t_1\,t_2$ and $t_3\,t_4$ is equal to 1. So with the inequations given by $(\mathcal{C})$, the determinants d_k cannot be simultaneously zero and there is no non zero polynomial $P(z)$ satisfying $z\,P'(z) = N(z)\,P(z)$. ∎

4.2.2. Factor of degree three

There is a factor of degree three to the normal variational system if and only if there is a factor of degree one to the adjoint of the normal variational system

$$Y' = -{}^t M\,Y\ .$$

But here the matrix M has a special structure due to the Hamiltonian system, indeed,

$$M\,J + J\,{}^t M = 0 \quad \text{with} \;\; J^2 = -I\ .$$

So, the adjoint system can be written

$$J\,Y' = J\,(-{}^t M\,Y) = M\,(J\,Y) = (J\,Y)'$$

and each exponential solution Y to the adjoint system corresponds to an exponential solution $J\,Y$ to the initial system.

As the normal variational system has no exponential solution, we conclude that there is no factor of degree three.

4.2.3. Factor of degree two

The factors of degree 2 of (3) correspond to the exponential solutions of its 6×6 second exterior system which are pure tensors i.e. which can be written as the exterior product of two solutions of the normal variational system (proposition 4.14 of [22]). The vector $W =^t (0,1,0,0,1,0)$ is an exponential solution of the second exterior system of the normal variational system (3) which comes from the symplectic structure of (3). Furthermore W is not a pure tensor, otherwise there would exist v_1, v_2 such that $W = v_1 \wedge v_2$ so there would exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ not all zero such that $(\lambda_1\, e_1 + \lambda_2\, e_2 + \lambda_3\, e_3 + \lambda_4\, e_4) \wedge (e_1 \wedge e_3 + e_2 \wedge e_4) = 0$ where (e_1, e_2, e_3, e_4) is the canonical basis, which is impossible.

So a necessary condition to have a factor of degree two for the initial system is that the second exterior system has another exponential solution.

Again the point infinity is an ordinary point and the points i, $-i$, 1 and -1 are the singular points of the second exterior system. The exponents at the singularity x_0 are the roots of the polynomial $F_{x_0}(n)$:

$$F_i(n) = F_{-i}(n) = n^2\,(n+1)\,(n-1)\,(n+2)\,(n+3) = 0$$

$$F_1(n) = F_{-1}(n) = n^2\,(n^4 + 2an^2 + a^2 - 4b) = 0$$

If there is an exponential solution it is:

$$(z-1)^{f_1}\,(z+1)^{f_{-1}}\,(z^2+1)^{-3}\,P(z)$$

where P is polynomial of degree at most $6-f_1-f_{-1}$ (it depends on the parameters!).

A necessary condition is that there exists f_1 and f_{-1} such that

$$F_1(f_1) = F_{-1}(f_{-1}) = 0 \qquad \text{and} \qquad \beta = f_1 + f_{-1} \in \mathbb{Z}$$

(a stronger condition is $6 - \beta \in \mathbb{N}$).

So a necessary condition is that there exists $\beta \in \mathbb{Z}$ such that

$$F(\beta) = \beta\,(\beta^4 + 4a\,\beta^2 + 16\,b)\,(\beta^4 + 2\,a\,\beta^2 + a^2 - 4\,b)\,(\beta^4 + 8\,a\,\beta^2 + 16\,(a^2 - 4\,b))$$

But according to lemma 2, under the conditions $(\mathcal{C})$, a, b and $a^2 - 4\,b$ are positive so the polynomial F has no non zero real solution. So the only possibility is $\beta = 0$ and one looks for exponential solutions of the form:

$$\frac{1}{(z^2+1)^3}\left(\frac{z+1}{z-1}\right)^f P(z)$$

where $f\,(f^4+2\,a\,f^2+a^2-4\,b)=0$ and P is a polynomial of degree 6 (it does not depend on the parameters!).

If $f=0$ we find the only solution ${}^t(0,1,0,0,1,0)$.

If $f\neq 0$ we get a system of polynomial equations that we solve using resultants under the conditions $(\mathcal{C})$:

$$\begin{cases} d_1=(e^6+31\,e^4+94\,e^2-36)\tilde{d}_1\\ d_2=(e^4+25\,e^2+64)\,\tilde{d}_2\\ d_3=(e^6+40\,e^4+229\,e^2+90)\,\tilde{d}_3\end{cases}$$

For each k, we compute $r_k=Res_e(\tilde{d}_k,f^4+2\,a\,f^2+a^2-4\,b)$ where a and b are given in (6).

$$\begin{cases} r_1=w^{140}\,A^{56}\,(3\,A+w^2)\,r_{1,1}\,r_{1,2}\\ r_2=w^{120}\,A^{50}\,(3\,A+w^2)\,r_{2,1}\,r_{2,2}\\ r_3=w^{140}\,A^{60}\,(3\,A+w^2)\,r_{3,1}\,r_{3,2}\end{cases}$$

where $r_{1,2}$, $r_{1,2}$, $r_{2,1}$, $r_{2,2}$, $r_{3,1}$ and $r_{3,2}$ are of respective degree in A: 12, 66, 8, 57, 12 and 64.

Using the same technique as in the two previous sub-sections, we prove that each of the following eight systems of three equations

$$r_{1,i_1}=r_{2,i_2}=r_{3,i_3}=0$$

has no solution under the conditions $(\mathcal{C})$.

We then conclude that the second exterior system has no exponential solution besides the solution $W={}^t\,(0,1,0,0,1,0)$ and that there is no factor of degree two under the conditions $(\mathcal{C})$.

4.3. Conclusion

From the propositions 4 and 3 and the criterion 1, one can state

Proposition 5. *The satellite without axial symmetry is not completely integrable.* ∎

5 – Conclusion

We gave a proof of the non complete meromorphic integrability of the satellite (a rigid body moving in a circular orbit around a fixed gravitational center) *with* and *without axial symmetry.* This result (propositions 2 and 5) completes the results of [16], [1] and [18] on the non complete integrability of the satellite *with axial symmetry.* A further study of the satellite in a magnetic field will be done in a forthcoming paper.

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