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# REAL GELFAND–MAZUR ALGEBRAS

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*Recommended by A.F. dos Santos*

**Abstract:** Several classes of real Gelfand–Mazur algebras are described. Conditions, when the trace $M \cap B$ of a closed maximal left (right) ideal $M$ of a real topological algebra $A$ would be a maximal ideal in a subalgebra $B$ of the center of $A$ are given.

## 1 – Introduction

**1.** Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, $A$ a *topological algebra* over $\mathbb{K}$ with associative separately continuous multiplication (in short, topological algebra) and $m(A)$ the set of all closed regular (or modular) two-sided ideals of $A$, which are maximal as left or right ideals. If the quotient algebra $A/M$ (in the quotient topology) is topologically isomorphic to $\mathbb{K}$ for each $M \in m(A)$, then $A$ is called a *Gelfand–Mazur algebra* (see [1], [2], [3] or [4]). Herewith, $A$ is a *real Gelfand–Mazur algebra* if $\mathbb{K} = \mathbb{R}$ and is a *complex Gelfand–Mazur algebra* if $\mathbb{K} = \mathbb{C}$.

Moreover, a unital topological algebra $A$ is a *Q-algebra* if the set $\mathrm{Inv}A$ of all invertible elements of $A$ is open in $A$; is a *Waelbroeck algebra* or a *topological algebra with continuous inverse* if $A$ is a $Q$-algebra in which the inversion $a \to a^{-1}$ is continuous in $\mathrm{Inv}A$; is a *Fréchet algebra* if the underlying linear topological space

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of $A$ is complete and metrizable; is a *topological algebra with bounded elements* if every $a \in A$ is bounded in $A$ that is, there is a nonzero complex number $\lambda_a$ such that the set

$$\left\{\left(\frac{a}{\lambda_a}\right)^n : \ n \in \mathbb{N}\right\}$$

is bounded in $A$; is an *exponentially galbed algebra* if its underlying topological vector space is an *exponentially galbed space* that is, for each neighbourhood $O$ of zero in $A$ there exists another neighbourhood $U$ of zero in $A$ such that

$$\left\{\sum_{k=0}^{n} \frac{a_k}{2^k} : \ a_0, ..., a_n \in U\right\} \subset O$$

for each $n \in \mathbb{N}$ and is a *locally pseudoconvex algebra* if $A$ has a base $\mathcal{B} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of neighbourhoods of zero consisting of *balanced* (i.e. $\lambda U_\alpha \in U_\alpha$, whenever $|\lambda| \leqslant 1$) and *pseudoconvex* (i.e. $U_\alpha + U_\alpha \subset \mu U_\alpha$ for some $\mu \geqslant 2$) sets. Moreover, a locally pseudoconvex algebra $A$ is *locally $A$-pseudoconvex* if for each $U_\alpha \in \mathcal{B}$ and $a \in A$ there is a number $\mu_a > 0$ such that $aU_\alpha$, $U_\alpha a \subset \mu_a U_\alpha$ and is *locally $m$-pseudoconvex* if $U_\alpha^2 \subset U_\alpha$ for each $U_\alpha \in \mathcal{B}$.

**2.** Let now $A$ be a real topological algebra,

$$Z(A) = \left\{z \in A: \ za = az \ \text{ for each } \ a \in A\right\}$$

the center of $A$ and $B$ a closed subalgebra of $Z(A)$ in the subset topology. An ideal $M \in m(B)$ is called *extendible to $A$* if

$$I(M) = \mathrm{cl}_A\left\{\sum_{k=1}^{n} a_k m_k : \ n \in \mathbb{N}, \ a_1, ..., a_n \in A; \ m_1, ..., m_n \in M\right\} \neq A,$$

where $\mathrm{cl}_A(M)$ denotes the closure of $M$ in the topology of $A$. We denote by $m_e(B)$ the set of all ideals $M \in m(B)$, which are extendible to $A$.

**3.** Let $A$ be a (real or complex) topological algebra, $M$ a maximal regular left (right) ideal of $A$ and $P_M = \{a \in A : aA \subset M\}$ ($P_M = \{a \in A : Aa \subset M\}$, respectively) the *primitive ideal* of $A$ defined by $M$. If $\{\theta_A\}$ is a primitive ideal of $A$, then $A$ is called a *primitive algebra* and if there is a closed maximal regular left (right) ideal $M$ of $A$ such that $P_M = \{\theta_A\}$, then $A$ is called a *topologically primitive algebra* (see [2], p. 21).

**4.** Properties of real Banach algebras have been studied in several books and articles (see, for example, [7], [8], [9] and [10]); of real $k$-normed and real $k$-Banach algebras in [6]; of real Waelbroeck algebras in [6] and in [12]; of real locally $m$-convex algebras in [11] and of real locally pseudoconvex division algebras in [5]. Properties of several classes of real Gelfand–Mazur algebras are studied and conditions for a real topological algebra $A$ that the trace $M \cap B$ of a closed maximal left (right) ideal $M$ of $A$ in a subalgebra $B$ of the center $Z(A)$ to be a closed maximal ideal in $B$ are given in the present paper.

## 2 – Properties of the center and of the quotient algebra

Let $(A, \tau)$ be a real topological algebra, $I$ a closed two-sided ideal of $A$ and $\pi_I$ the canonical homomorphism of $A$ onto $A/I$. By $\tau_{A/I}$ we denote the quotient topology on $A/I$, defined by $\tau$ and $\pi_I$, and by $\tau_I$ the subset topology on $Z(A/I)$ defined by $\tau_{A/I}$. Similarily as in the complex case (see [2], pp. 26–28) we have the following result.

**Proposition 1.** *Let $A$ be a real topological algebra and $I$ a closed two-sided ideal of $A$. If there exists a topology $\tau$ on $A$ such that*

- **a)** *$(A, \tau)$ is locally pseudoconvex, then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real locally pseudoconvex algebras;*
- **b)** *$(A, \tau)$ is locally $A$-pseudoconvex (in particular, locally $m$-pseudoconvex), then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real locally $A$-pseudoconvex (respectively, locally $m$-pseudoconvex) algebras;*
- **c)** *$(A, \tau)$ is an exponentially galbed algebra with bounded elements, then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real exponentially galbed algebras with bounded elements;*
- **d)** *$(A, \tau)$ is a locally pseudoconvex Fréchet algebra, then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real locally pseudoconvex Fréchet algebras;*
- **e)** *$(A, \tau)$ is a real topological algebra with jointly continuous multiplication, then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real topological algebras with jointly continuous multiplication.*

*Moreover, if $I$ is a regular ideal, $u$ a right modular unit for $I$ and for each $a \in A$ there is a $\lambda \in \mathbb{R}$ such that $a - \lambda u \in I$, then $\mathrm{sp}_{A/I}(x)$ is not empty for each $x \in A/I$ and $\mathrm{sp}_{Z(A/I)}(y) = \mathrm{sp}_{A/I}(y)$ for each $y \in Z(A/I)$.* ∎

## 3 – Complexification of real algebras

**1.** Let $A$ be a (not necessarily topological) real algebra and let $\tilde{A} = A + iA$ be the comlexification of $A$. Then every element $\tilde{a}$ of $\tilde{A}$ is representable in the form $\tilde{a} = a + ib$, where $a, b \in A$ and $i^2 = -1$. If we define the addition in $\tilde{A}$, the multiplication over $\mathbb{C}$ and the multiplication in $\tilde{A}$ by

$$(a + ib) + (c + id) = (a + c) + i(b + d),$$

$$(\alpha + i\beta)(a + ib) = (\alpha a - \beta b) + i(\alpha b + \beta a),$$

and

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

for all $a, b, c, d \in A$ and $\alpha, \beta \in \mathbb{R}$, then $\tilde{A}$ is a complex algebra with zero element $\theta_{\tilde{A}} = \theta_A + i\theta_A$ (here and later on $\theta_A$ denotes the zero element of $A$). If $A$ is an algebra with unit element $e_A$, then $e_{\tilde{A}} = e_A + i\theta_A$ is the unit element of $\tilde{A}$. Herewith, $\tilde{A}$ is an associative (commutative) algebra if $A$ is an associative (respectively, commutative) algebra. We can consider $A$ as a real subalgebra of $\tilde{A}$ under the imbedding $\nu$ from $A$ into $\tilde{A}$ defined by $\nu(a) = a + i\theta_A$ for each $a \in A$.

**2.** Let $A$ be an algebra over $\mathbb{K}$ with unit $e_A$ and

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{K} : a - \lambda e_A \notin \mathrm{Inv} A\}$$

for each $a \in A$. Then $\mathrm{sp}_A(a)$ is the *spectrum* of $a$. Herewith, elements of $\mathrm{sp}_A(a)$ are complex numbers if $A$ is a complex algebra and real numbers if $A$ is a real algebra.

A real (not necessarily topological) algebra $A$ is *formally real* if from $a, b \in A$ and $a^2 + b^2 = \theta_A$ follows that $a = b = \theta_A$ and is *strictly real* if $\mathrm{sp}_{\tilde{A}}(a + i\theta_A) \subset \mathbb{R}$. It is known (see, for example, [6], Proposition 1.9.14) that every formally real division algebra is strictly real and every commutative strictly real division algebra is formally real. Moreover, the complexification $\tilde{A}$ of a commutative real division algebra $A$ is division algebra if and only if $A$ is formally normal (see [6], Proposition 1.6.20).

**Lemma 1.** *Let $A$ be a real algebra and $I$ a two-sided ideal of $A$. Then the quotient algebra $A/I$ is formally real if and only if $I$ satisfies the condition*

($\alpha$) *from $a, b \in A$ and $a^2 + b^2 \in I$ follows that $a, b \in I$.*

**Proof:** Let $A$ be a real algebra, $I$ a two-sided ideal in $A$, $\pi_I$ the quotient map of $A$ onto $A/I$ and let $a,\ b \in A$ be such that $a^2 + b^2 \in I$. Then

$$\pi_I(a)^2 + \pi_I(b)^2 = \pi_I(a^2 + b^2) = \theta_{A/I}.$$

If $A/I$ is formally real, then $\pi_I(a) = \pi_I(b) = \theta_{A/I}$ or $a \in I$ and $b \in I$.

Let now a two-sided ideal $I$ satisfy the condition $(\alpha)$ and $x,\ y \in A/I$ be such that $x^2 + y^2 = \theta_{A/I}$. Then there are $a,\ b \in A$ such that $x = \pi(a)$, $y = \pi(b)$ and

$$\pi_I(a^2 + b^2) = x^2 + y^2 = \theta_{A/I}.$$

Hence, from $a^2 + b^2 \in I$ follows that $x = y = \theta_{A/I}$ by the condition $(\alpha)$. ∎

**3.** Let now $(A, \tau)$ be a real topological algebra and $\{U_\alpha : \alpha \in \mathcal{A}\}$ a base of neigbourhoods of zero of $(A, \tau)$. As usual (see [6] or [12] ), we endow $\tilde{A}$ with the topology $\tilde{\tau}$ in which $\{U_\alpha + i\, U_\alpha : \alpha \in \mathcal{A}\}$ is a base of neighbourhoods of zero. It is known that $(\tilde{A}, \tilde{\tau})$ is a complex topological algebra and the multiplication in $(\tilde{A}, \tilde{\tau})$ is jointly continuous if the multiplication in $(A, \tau)$ is jointly continuous (see [6], Proposition 2.2.10). Moreover, the underlying topological space of $(\tilde{A}, \tilde{\tau})$ is a Hausdorff space if $(A, \tau)$ is a Hausdorff algebra.

Let $M$ be a maximal regular left (right or two-sided) ideal of $A$. Then (see [6], Proposition 1.6.12, p. 46) $\widetilde{M} = M + iM$ is a maximal regular left (right or two-sided) ideal in $\widetilde{A}$.

**Proposition 2.** *Let $A$ be a real topological algebra, $M$ a closed maximal regular left (right) ideal of $A$ and $P_M$ the primitive ideal of $A$ defined by $M$. Then*

**a)** *the primitive ideal $\widetilde{P}_{\widetilde{M}}$ of $\widetilde{A}$ defined by $\widetilde{M}$ is representable in the form $\widetilde{P}_{\widetilde{M}} = P_M + iP_M$;*

**b)** $\widetilde{A}/\widetilde{P}_{\widetilde{M}} = A/P_M + iA/P_M$;

**c)** $Z(\widetilde{A}) = Z(A) + iZ(A)$.

**Proof: a)** Let $A$ be a real topological algebra, $a,\ b \in P_M$ and $v + iw \in \widetilde{A}$. Since

$$(a + ib)(v + iw) = av - bw + i(aw + bv) \in \widetilde{M},$$

then $P_M + iP_M \subset \widetilde{P}_{\widetilde{M}}$. Let now $a + ib \in \widetilde{P}_{\widetilde{M}}$ and $v + i\theta_A \in \widetilde{A}$. Then

$$(a + ib)(v + i\theta_A) = av + ibv \in \widetilde{M}$$

if and only if $av,\ bv \in M$ or $a,\ b \in P_M$. Thus $\widetilde{P}_{\widetilde{M}} \subset P_M + iP_M$.

**b**) Let $a,\ b \in A$. Then

$$a + P_M + i(b + P_M) = (a + ib) + (P_M + iP_M) = (a + ib) + \widetilde{P}_{\widetilde{M}} \in \widetilde{A}/\widetilde{P}_{\widetilde{M}}.$$

Hence, $A/P_M + iA/P_M \subset \widetilde{A}/\widetilde{P}_{\widetilde{M}}$ and similarily $\widetilde{A}/\widetilde{P}_{\widetilde{M}} \subset A/P_M + iA/P_M$.

**c**) It is clear that $Z(A) + iZ(A) \subset Z(\widetilde{A})$. Let now $a_0 + ib_0 \in Z(\widetilde{A})$. Since

$$aa_0 + iab_0 = (a + i\theta_A)(a_0 + ib_0) = (a_0 + ib_0)(a + i\theta_A) = a_0a + ib_0a$$

for each $a \in A$, then $a_0, b_0 \in Z(A)$. ∎

**Corollary 1.** *If $A$ is a real topologically primitive topological algebra, then the complexification of $A$ is a complex topologically primitive topological algebra.*

**Proof:** Let $A$ be a real topologically primitive topological algebra. Then there exists a closed maximal regular left (right) ideal $M$ in $A$ such that $P_M = \{\theta_A\}$. Since

$$\widetilde{P}_{\widetilde{M}} = P_M + iP_M = \{\theta_A + i\theta_A\} = \{\theta_{\tilde{A}}\}$$

and $\widetilde{M}$ is a closed maximal regular left (right) ideal of $\widetilde{A}$, then $\widetilde{A}$ is a complex topologically primitive topological algebra. ∎

## 4 – Commutative real Gelfand–Mazur algebras

To describe real Gelfand–Mazur algebras, we need the following result proved in [5], Corollary 5.5:

**Proposition 3.** *Let $A$ be a commutative strictly real division algebra. If $A$ has a topology*(1) *$\tau$ such that $(A, \tau)$ is*

- **a**) *a locally pseudoconvex Hausdorff algebra with continuous inversion;*
- **b**) *a locally $A$-pseudoconvex (in particular, locally $m$-pseudoconvex) Hausdorff algebra;*
- **c**) *a locally pseudoconvex Fréchet algebra;*
- **d**) *an exponentially galbed Hausdorff algebra with jointly continuous multiplication and bounded elements;*

(1) Which can be different from the preliminary topology of $A$.

**e**) *a topological Hausdorff algebra for which the spectrum* $\mathrm{sp}_A(a)$ *is not empty for each* $a \in A$,

*then A is a commutative real Gelfand–Mazur division algebra.* ∎

Now we prove

**Theorem 1.** *Let A be a commutative real topological algebra. If A has a topology*(2) $\tau$ *such that* $(A, \tau)$ *satisfies the condition* $(\alpha)$ *for each* $I \in m((A, \tau))$ *and belongs in one of the following classes of topological algebras:*

**a**) *locally pseudoconvex Waelbroeck algebras;*

**b**) *locally A-pseudoconvex (in particular, locally m-pseudoconvex) algebras;*

**c**) *locally pseudoconvex Fréchet algebras;*

**d**) *exponentially galbed algebras with jointly continuous multiplication and bounded elements;*

**e**) *topological algebras in which for any element* $a \in A$ *and* $M \in m((A, \tau))$ *there is a* $\lambda \in \mathbb{R}$ *such that* $a - \lambda u \in M$ *(here u is a modular unit for M),*

*then A is a commutative real Gelfand–Mazur algebra.*

**Proof:** Let $(A, \tau)$ be a commutative real topological algebra which satisfies the condition $(\alpha)$ for each $I \in m((A, \tau))$ and $M$ a fixed element of $m((A, \tau))$. Then $(A/M, \tau_{A/M})$ is a commutative strictly real topological division Hausdorff algebra by Lemma 1. If now $(A, \tau)$ satisfies

**1**) the condition a), then $(A/M, \tau_{A/M})$ is a commutative strictly real locally pseudoconvex Waelbroeck division algebra by the statement a) of Proposition 1 and Corollary 3.6.27 from [6];

**2**) the condition b), then $(A/M, \tau_{A/M})$ is a commutative strictly real locally $A$-pseudoconvex (in particular, $m$-pseudoconvex) Hausdorff division algebra by the statement b) of Proposition 1;

**3**) the condition c), then $(A/M, \tau_{A/M})$ is a commutative strictly real locally pseudoconvex Fréchet division algebra by the statement d) of Proposition 1;

**4**) the condition d), then $(A/M, \tau_{A/M})$ is a commutative strictly real exponentially galbed Hausdorff division algebra with jointly continuous multiplication and bounded elements by the statements c) and f) of Proposition 1;

(2) See the footnote 1.

**5**) the condition e), then $(A/M, \tau_{A/M})$ is a commutative strictly real topological Hausdorff algebra for which the spectrum $\mathrm{sp}_{A/M}(x)$ is not empty for each $x \in A/M$ by Proposition 1.

Hence, in all these cases $A/M$ (in the quotient topology defined by the preliminary topology of $A$) is topologically isomorphic to $\mathbb{R}$ for each $M \in m(A)$ by Proposition 3. Therefore $A$ is a commutative real Gelfand–Mazur algebra. ∎

## 5 – Maximality of traces of ideals

Let $A$ be a unital real topological algebra, $B$ a subalgebra of $Z(A)$ and $M$ a closed maximal left (right) ideal of $A$. It is easy to see that the trace $M \cap B$ of $M$ is a closed ideal in $B$. To find the conditions for $A$ that the trace $M \cap B$ of $M$ to be maximal in $B$, we need

**Proposition 4.** *Let $A$ be a real locally $A$-pseudoconvex algebra* (*or a real locally pseudoconvex Fréchet algebra*) *with a unit element $e_A$, $M$ a closed maximal left* (*right*) *ideal of $A$ and $P_M$ a primitive ideal of $A$ defined by $M$. If $P_M$ satisfies the condition*

($\beta$) *from $a, b \in A$ and $a^2 + b^2 \in P_M$ follows that $a, b \in P_M$,*

*then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$.*

**Proof:** Let $(A, \tau)$ be a unital real locally $A$-pseudoconvex (locally pseudoconvex Fréchet) algebra, $M$ a closed maximal regular left (right) ideal of $A$, $P_M$ a primitive ideal in $A$ defined by $M$, $\pi_M$ the canonical homomorphism of $A$ onto $A/P_M$ and $\tau_M$ the quotient topology on $A/P_M$ defined by $\tau$ and $\pi_M$. Then $(A/P_M, \tau_M)$ is a unital real locally $A$-pseudoconvex Hausdorff (respectively, locally pseudoconvex Fréchet) algebra by Proposition 1. Since the complexification of $A/P_M$ is $\widetilde{A}/\widetilde{P}_{\widetilde{M}}$, where $\widetilde{P}_{\widetilde{M}}$ is a closed primitive ideal in $\widetilde{A}$ by Proposition 2, then $(\widetilde{A}/\widetilde{P}_{\widetilde{M}}, \widetilde{\tau}_M)$ is a unital complex locally $A$-pseudoconvex Hausdorff (respectively, locally pseudoconvex Fréchet) algebra by Theorem 3.3 and Corollary 3.2 from [5]. Hence, $Z(\widetilde{A}/\widetilde{P}_{\widetilde{M}})$ is topologically isomorphic to $\mathbb{C}$ by Theorem 1 from [1] or by Theorem 2.17 from [2]. Therefore, $Z(\widetilde{A}/\widetilde{P}_{\widetilde{M}})$ is a complex division algebra. As $Z(\widetilde{A}/\widetilde{P}_{\widetilde{M}}) = Z(A/P_M) + iZ(A/P_M)$ by Proposition 2, then $Z(A/P_M)$ is formally real by Proposition 1.6.20 from [6] (by condition ($\beta$) the quotient algebra $A/P_M$ is formally real by Lemma 1, hence $Z(A/P_M)$ is formally real too).

Now, every element $x \in Z(A/P_M)$ is representable in the form $x = \lambda_x e_A$ for some $\lambda_x \in \mathbb{R}$. Therefore, $Z(A/P_M)$ is isomorphic to $\mathbb{R}$. In the same way as in the complex case (see, e.g. [2], p. 47) it is easy to show that this isomorphism is a topological isomorphism because $Z(A/P_M)$ is a Hausdorff space in the subset topology. ∎

**Corollary 2.** *Let $A$ be a real locally $m$-pseudoconvex topological algebra with unit, $P_M$ a primitive ideal of $A$ defined by a closed maximal regular left (right) ideal $M$ of $A$. If $P_M$ satisfies the condition $(\beta)$, then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$.*

**Proof:** Since every locally $m$-pseudoconvex algebra is locally $A$-pseudoconvex, then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$ by Proposition 4. ∎

**Corollary 3.** *Let $A$ be a unital strictly real topologically primitive locally $A$-pseudoconvex Hausdorff algebra or a unital real topologically primitive locally pseudoconvex Fréchet algebra. Then $Z(A)$ is topologically isomorphic to $\mathbb{R}$.* ∎

**Theorem 2.** *Let*(3) *$A$ be a real locally $A$-pseudoconvex (in particular, a locally $m$-pseudoconvex) algebra with unit $e_A$ or a real locally pseudoconvex Fréchet algebra with unit $e_A$, $M$ a closed maximal left (right or two-sided) ideal of $A$, $P_M$ the primitive ideal in $A$ defined by $M$ and $B$ a closed subalgebra of $Z(A)$, containing $e_A$. If $P_M$ satisfies the condition $(\beta)$, then*

**1**) *every $b \in B$ defines a number $\lambda \in \mathbb{R}$ such that $b - \lambda e_A \in M$;*

**2**) *$M \cap B \in m_e(B)$.*

**Proof:** Similarily as in [1], the proof of Corollary 1, or in [2], the proof of Proposition 3.1, it is easy to show that Theorem 2 holds by Proposition 4 and Corollary 2. ∎

**Corollary 4.** *Let $A$ be a real locally $A$-pseudoconvex (in particular, a locally $m$-pseudoconvex) algebra with unit $e_A$ or a real locally pseudoconvex Fréchet algebra with unit $e_A$, $M$ a closed maximal left (right or two-sided) ideal of $A$ and $P_M$ the primitive ideal in $A$ defined by $M$. If $P_M$ satisfies the condition $(\beta)$, then*

**1**) *every $z \in Z(A)$ defines a number $\lambda \in \mathbb{R}$ such that $z - \lambda e_A \in M$;*

**2**) *$M \cap Z(A) \in m_e(Z(A))$.* ∎

(3) For complex locally $A$-pseudoconvex (in particular, locally $m$-pseudoconvex) algebras with unit and for complex locally pseudoconvex Fréchet algebras with unit similar result has been published in [1], Corollary 1, and in [2], Proposition 3.1.

## REFERENCES

[**1**] Abel, Mart – Description of closed maximal ideals in Gelfand–Mazur algebras, General Topological Algebras (Tartu, 1999), 7–13, *Math. Stud.* (Tartu), 1, *Est. Math. Soc.*, Tartu, 2001.

[**2**] Abel, Mart – Structure of Gelfand–Mazur algebras, *Dissertationes Mathematicae Universitatis Tartuensis,* 31, Tartu Univ. Press, Tartu 2003.

[**3**] Abel, Mati – *Gelfand–Mazur algebras*, in "Topological Vector Spaces, Algebras and Related Areas" (Hamilton, ON, 1994), 116–129, Pitman Res. Notes Math. Ser., 316, Longman Sci. Tech., Harlow, 1994.

[**4**] Abel, Mati – *Survey of results on Gelfand–Mazur algebras*, in "Non-Normed Topological Algebras" (Rabat, 2000), 14–25, Ecole Normale Supérieure Takaddoum, Rabat, 2004.

[**5**] Abel, Mati and Panova, O. – Real Gelfand–Mazur division algebras, *Int. J. Math. Math. Sci.,* (2003), 1–12.

[**6**] Balachandran, V.K. – *Topological Algebras*, North-Holland Math. Studies, 185, Elsevier, Amsterdam, 2000.

[**7**] Bonsall, F.F. and Duncan, J. – *Complete Normed Algebras*, Springer-Verlag, Berlin, 1973.

[**8**] Ingelstam, L. – Real Banach algebras, *Ark. Math.,* 5 (1964), 239–270.

[**9**] Li, B. and Tam, P. – Real Banach $*$-algebras, *Acta Math. Sin. (Engl. Ser.),* 16(3) (2000), 469–486.

[**10**] Rickart, C.E. – *General Theory of Banach Algebras*, D. van Nostrand Co, Inc., Princeton, 1960.

[**11**] Żelazko, W. – On generalized topological divisors of zero in real $m$-convex algebras, *Studia Math.,* 28 (1966/1967), 241–244.

[**12**] Żelazko, W. – On $m$-convexity of commutative real Waelbroeck algebras, *Comment. Math. Prace Mat.,* 40 (2000), 235–240.

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