PORTUGALIAE MATHEMATICA
Vol. 63 Fasc. 1 – 2006
Nova Série

ON THE NON-DEFECTIVITY AND NON WEAK-DEFECTIVITY OF SEGRE-VERONESE EMBEDDINGS OF PRODUCTS OF PROJECTIVE SPACES

Edoardo Ballico *

Recommended by Arnaldo Garcia

Abstract: Fix integers $s \geq 2$ and $n \geq 1$. Set $\tilde{x_i} := n+i-1$ if $3 \leq i \leq s$ and $\tilde{x_2} := \max\{3, n+1\}$. Set $\tilde{x_1} := 9$ if $n = 1$ and $\tilde{x_1} = n!(n+1) - n$ if $n \geq 2$. Fix integers $x_i \geq \tilde{x_i}$, $1 \leq i \leq s$. Here we prove that the line bundle $\mathcal{O}_{\mathbf{P}^n \times (\mathbf{P}^1)^{s-1}}(x_1, \ldots, x_s)$ is not weakly defective, i.e. for every integer z such that $z(n+s)+1 \leq \binom{n+x_1}{n} \prod_{i=2}^{s}(x_i+1)$ the linear system $|\mathcal{I}_Z(x_1, \ldots, x_s)|$ has dimension $\binom{n+x_1}{n} \prod_{i=2}^{s}(x_i+1) - z(n+s) - 1$ and a general $T \in |\mathcal{I}_Z(x_1, \ldots, x_s)|$ has an ordinary double point at each point of Z_{red} as only singularities, where $Z \subset \mathbf{P}^n \times (\mathbf{P}^1)^{s-1}$ is a general union of z double points.

1 – Introduction

The main aim of this paper is to use the so-called Horace Method introduced by A. Hirschowitz to prove the non-defectivity and non-weak defectivity (in the sense of [10]) of "many" line bundles in $\mathbf{P}^n \times (\mathbf{P}^1)^{s-1}$. See [6], [7], [8] and [13] for several results on the defectivity or non-defectivity on certain multiprojective spaces and the linear algebra translation of any non-defectivity result for line bundles on arbitrary multiprojective spaces. First, we will prove the following result.

Received: November 22, 2004; *Revised*: May 11, 2005.
AMS Subject Classification: 14N05.
Keywords: Segre–Veronese variety; Segre–Veronese embedding; multiprojective space; products of projective spaces; weakly defective variety; zero-dimensional scheme; double point; fat point.
* The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

Theorem 1. *Fix integers $k > 0$, $s \geq 2$ and $n \geq 1$. Set $\tilde{x_i} := n + i - 1$ if $3 \leq i \leq s$ and $\tilde{x_2} := \max\{3, n+1\}$. Set $\tilde{x_1} := 9$ if $n = 1$ and $\tilde{x_1} = n!\,(n+1) - n$ if $n \geq 2$. Fix integers $x_i \geq \tilde{x_i}$, $1 \leq i \leq s$. Let $Z \subset \mathbf{P}^n \times (\mathbf{P}^1)^{s-1}$ be a general union of k double points. If $k(n+s) \leq \binom{n+x_1}{n} \prod_{i=2}^{s}(x_i+1)$, then $h^1(\mathbf{P}^n \times (\mathbf{P}^1)^{s-1}, \mathcal{I}_Z(x_1, \dots, x_s)) = 0$. If $k(n+s) \geq \binom{n+x_1}{n} \prod_{i=2}^{s}(x_i+1)$, then $h^0(\mathbf{P}^n \times (\mathbf{P}^1)^{s-1}, \mathcal{I}_Z(x_1, \dots, x_s)) = 0$.*

With the classical terminogy Theorem 1 says that for all $k > 0$ the line bundle $\mathcal{O}_{\mathbf{P}^n \times (\mathbf{P}^1)^{s-1}}(x_1, \dots, x_s)$ is not $(k-1)$-defective, i.e. that this line bundle is not defective. See Lemma 4 for a conditional inductive approach for an arbitrary multiprojective space. Theorem 1 was just the only case in which we were able to prove the initial step to carry over the inductive procedure.

Inspired from [15], Proof of Theorem 4.1, we will prove the following result.

Theorem 2. *Fix integers $s \geq 2$ and $n \geq 1$. Set $\tilde{x_i} := n + i - 1$ if $3 \leq i \leq s$ and $\tilde{x_2} := \max\{3, n+1\}$. Set $\tilde{x_1} := 9$ if $n = 1$ and $\tilde{x_1} = n!(n+1) - n$ if $n \geq 2$. Fix integers $x_i \geq \tilde{x_i}$, $1 \leq i \leq s$. Then the line bundle $\mathcal{O}_{\mathbf{P}^n \times (\mathbf{P}^1)^{s-1}}(x_1, \dots, x_s)$ is not weakly defective, i.e. for every integer z such that $z(n+s)+1 \leq \binom{n+x_1}{n} \prod_{i=2}^{s}(x_i+1)$ the linear system $|\mathcal{I}_Z(x_1, \dots, x_s)|$ has dimension $\binom{n+x_1}{n} \prod_{i=2}^{s}(x_i+1) - z(n+s) - 1$ and a general $T \in |\mathcal{I}_Z(x_1, \dots, x_s)|$ has an ordinary double point at each point of Z_{red} as only singularities, where $Z \subset \mathbf{P}^n \times (\mathbf{P}^1)^{s-1}$ is a general union of z double points.*

Theorem 2 will be an easy corollary of Theorems 1 and 3. To state Theorem 3 we need to introduce the following notation. Fix integers $s \geq 1$, $n_1 \geq \cdots \geq n_s > 0$ and $t_i \geq 0$, $1 \leq i \leq s$. In some inductive step we will allow the case $n_s = 0$, just taking a point as $\mathbf{P}^{n_s}$. Even if $n_i = 0$ for some i define the integers $a_{(n_1,\dots,n_s;t_1,\dots,t_s)}$, $b_{(n_1,\dots,n_s;t_1,\dots,t_s)}$, $c_{(n_1,\dots,n_s;t_1,\dots,t_s)}$ and $d_{(n_1,\dots,n_s;t_1,\dots,t_s)}$ by the following relations:

$$(1) \qquad \left(1 + \sum_{i=1}^{s} n_i\right) a_{(n_1,\dots,n_s;t_1,\dots,t_s)} + b_{(n_1,\dots,n_s;t_1,\dots,t_s)} = \prod_{i=1}^{s} \binom{n_i + t_i}{n_i},$$

$$(2) \qquad 0 \leq b_{(n_1,\dots,n_s;t_1,\dots,t_s)} \leq \sum_{i=1}^{s} n_i,$$

$$(3) \qquad \left(1 + \sum_{i=1}^{s} n_i\right) c_{(n_1,\dots,n_s;t_1,\dots,t_s)} + d_{(n_1,\dots,n_s;t_1,\dots,t_s)} + 1 = \prod_{i=1}^{s} \binom{n_i + t_i}{n_i},$$

$$(4) \qquad 0 \leq d_{(n_1,\dots,n_s;t_1,\dots,t_s)} \leq \sum_{i=1}^{s} n_i.$$

Notice that

$$a_{(n_1,\dots,n_s;t_1,\dots,t_s)} = c_{(n_1,\dots,n_s;t_1,\dots,t_s)}$$

and

$$d_{(n_1,\dots,n_s;t_1,\dots,t_s)} = b_{(n_1,\dots,n_s;t_1,\dots,t_s)} - 1$$

if $b_{(n_1,\dots,n_s;t_1,\dots,t_s)} > 0$, while

$$a_{(n_1,\dots,n_s;t_1,\dots,t_s)} = c_{(n_1,\dots,n_s;t_1,\dots,t_s)} + 1$$

and

$$d_{(n_1,\dots,n_s;t_1,\dots,t_s)} = \sum_{i=1}^{s} n_i$$

if $b_{(n_1,\dots,n_s;t_1,\dots,t_s)} = 0$.

Notice that

$$\begin{aligned} a_{(n_1,\dots,n_{s-1},0;t_1,\dots,t_s)} &= a_{(n_1,\dots,n_{s-1};t_1,\dots,t_{s-1})}\ , \\ b_{(n_1,\dots,n_{s-1},0;t_1,\dots,t_s)} &= b_{(n_1,\dots,n_{s-1};t_1,\dots,t_{s-1})}\ , \\ c_{(n_1,\dots,n_{s-1},0;t_1,\dots,t_s)} &= d_{(n_1,\dots,n_{s-1};t_1,\dots,t_{s-1})}\ , \\ d_{(n_1,\dots,n_{s-1},0;t_1,\dots,t_s)} &= d_{(n_1,\dots,n_{s-1};t_1,\dots,t_{s-1})}\ . \end{aligned}$$

Theorem 3. *Fix integers $k > 0$, $s \geq 2$, $n_1 \geq \cdots \geq n_s > 0$, $x_i \geq 3$, $1 \leq i \leq s$, such that $k(n_1 + \cdots + n_s + 1) \geq \prod_{i=1}^{s} \binom{n_i+x_i}{n_s}$. Fix a hyperplane H of $\mathbf{P}^{n_j}$ and set $M := \prod_{i=1}^{s} \mathbf{P}^{n_i}$, $E := \prod_{i=1}^{j-1} \mathbf{P}^{n_i} \times H \times \prod_{i=j+1}^{s} \mathbf{P}^{n_i}$. Assume the existence of an integer j such that $1 \leq j \leq s$ and the following properties hold:*

(**a**) *The line bundles $\mathcal{O}_M(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_s)$, $\mathcal{O}_M(x_1, \dots, x_{j-1}, x_j - 1, x_{j+1}, \dots, x_s)$ and $\mathcal{O}_M(x_1, \dots, x_{j-1}, x_j - 2, x_{j+1}, \dots, x_s)$ are not defective.*

(**b**) *For every integer $z > 0$ such that*

$$z(n_1 + \cdots + n_s + 1) + a_{(n_1,\dots,n_{j-1},n_j-1,n_{j+1},\dots,n_s;x_1,\dots,x_s)} \leq \tag{5}$$

$$\leq \prod_{i=1}^{j-1} \binom{n_i + x_i}{n_i} \cdot \binom{n_j + x_j - 1}{n_j} \cdot \prod_{i=j+1}^{s} \binom{n_i + x_i}{n_i} \tag{6}$$

and any general union $W \subset M$ of z double points of M a general hypersurface of multidegree $(x_1, \dots, x_{j-1}, x_j - 1, x_{j+1}, \dots, x_s)$ of $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ singular at each point of Z_{red} has an isolated singularity at at least one point of W_{red}.

Let $Z \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ be a general union of k double points. Then a general hypersurface of multidegree $(x_1, \ldots, x_s)$ of $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ singular at each point of Z_{red} has an ordinary node at each point of Z_{red} and no other singularity.

With the terminology of [10], Theorem 2 means that the Segre–Veronese embedding of $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ with multidegree $(x_1, \ldots, x_s)$ is not weakly $(k-1)$-defective.

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. Our proof of Theorem 1 will be characteristic free, while our proofs of Theorems 2 and 3 depend heavily from the characteristic zero assumption: a key tool will be [10], Th. 1.4. To prove Theorems 2 and 3 we will use an idea of Mella ([15], proof of Th. 4.1). To start the induction we will also use a theorem of weak non-defectivity for $\mathbf{P}^{n_1}$ ([15], Cor. 4.5).See [1], [2], [3], [4] or [9] for Alexander–Hirschowitz theorem on non-defectivity of line bundles on $\mathbf{P}^n$. For several results on non-defectivity for Segre–Veronese embeddings of multiprojective spaces (many of them with low x_1 not covered by Theorem 1), see [6] (which also contain a linear algebra interpretation of Theorem 1), [7], [8]. For related results for $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ and a similar inductive proof, see [13]. See [12], [9], Remark 6.2, (which quotes [16]) and [15], Remark 4.4, for several examples of weak defective line bundles on projective spaces.

2 – The proofs

For any scheme A and any $P \in A_{reg}$ let $2P$ (or $2\{P, A\}$ if there is any danger of misunderstandings) denote the first infinitesimal neighborhood of P in A, i.e. the closed zero-dimensional subscheme of A with $(\mathcal{I}_P)^2$ as its ideal sheaf. We have $\mathrm{length}(2P) = \dim_P(A) + 1$. We will say that $2P$ is the double point of A with P as its support. For any finite subset $S \subset A_{reg}$ set $2\{S, A\} := \cup_{P \in S} 2\{P, A\}$ and write $2S$ instead of $2\{S, A\}$ if there is no danger of misunderstandings. Let $D \subset A$ be an effective Cartier divisor of A and $Z \subset A$ any closed subscheme of A. Let $\mathrm{Res}_D(Z)$ denote the residual subscheme of Z with respect to D, i.e. the closed subscheme of A with $\mathcal{I}_{Z,A} : \mathcal{O}_A(-D)$ as its ideal sheaf. For instance, $\mathrm{Res}_D(2P) = \{P\}$ if $P \in D_{reg}$ and $\mathrm{Res}_D(2P) = 2P$ if $P \notin D_{red}$. By the very definition of residual scheme for any $L \in \mathrm{Pic}(A)$ we have the following exact sequence:

$$(7) \qquad 0 \to \mathcal{I}_{\mathrm{Res}_D(Z),A} \otimes L \to \mathcal{I}_{Z,A} \otimes L \to \mathcal{I}_{Z \cap D, D} \otimes L_{|D} \to 0 .$$

From the cohomology exact sequence of the exact sequence (7) we get at once the following lemma which is a very elementary version of the so-called Horace Lemma and that we will always call "the Horace Lemma".

Lemma 1. *Let A be a projective scheme, D an effective Cartier divisor of A, Z a closed subscheme of A and $L \in Pic(A)$. Then:*

(i) $h^0(A, \mathcal{I}_{Z,A} \otimes L) \leq h^0(A, \mathcal{I}_{\mathrm{Res}_D(Z),A} \otimes L) + h^0(D, \mathcal{I}_{Z \cap D, D} \otimes L_{|D})$;

(ii) $h^1(A, \mathcal{I}_{Z,A} \otimes L) \leq h^1(A, \mathcal{I}_{\mathrm{Res}_D(Z),A} \otimes L) + h^1(D, \mathcal{I}_{Z \cap D, D} \otimes L_{|D})$. ∎

The following result is a very particular case of [5], Lemma 2.3 (see in particular Fig. 1 at p. 308).

Lemma 2. *Let A be an integral projective variety, $L \in Pic(A)$, D an integral effective Cartier divisor of A, $Z \subset A$ a closed subscheme of A not containing D and s a positive integer. Let U be the union of Z and s general double points of A. Let S be the union of s general points of D. Let $E \subset D$ be the union of s general double points of D (not double points of A, i.e. each of them has length $\dim(A)$). To prove $h^1(A, \mathcal{I}_{U,A} \otimes L) = 0$ (resp. $h^0(A, \mathcal{I}_{U,A} \otimes L) = 0$) it is sufficient to prove $h^1(D, \mathcal{I}_{(Z \cap D) \cup S} \otimes (L_{|D})) = h^1(A, \mathcal{I}_{\mathrm{Res}_D(Z) \cup E, A} \otimes L(-D)) = 0$ (resp. $h^0(D, \mathcal{I}_{(Z \cap D) \cup S} \otimes (L_{|D})) = h^0(A, \mathcal{I}_{\mathrm{Res}_D(Z) \cup E, A} \otimes L(-D)) = 0$).* ∎

Remark 1. Here we assume $s = 2$ and $n_2 = 1$. The following inequality

$$\binom{n_1 + x_1}{n_1} \geq (n_1 + 2)^2 \tag{8}$$

is satisfied if and only if either $n_1 = 1$ and $x_1 \geq 8$ or $n_1 \geq 2$ and $x_1 \geq 3$. □

Remark 2. Fix integers $s \geq 2$, $n_1 \geq \cdots \geq n_s > 0$ and $x_i > 0$, $1 \leq i \leq s$. Here we will discuss when the inequality

$$\binom{n_s + x_s - 1}{n_s} \prod_{j=1}^{s-1} \binom{n_j + x_j}{n_j} \geq \left(1 + \sum_{i=1}^{s} n_i\right) \left(\sum_{i=1}^{s} n_i\right) \tag{9}$$

holds. However, since in all applications of this inequality we will need to use an induction on s starting from the case $s = 2, n_s = 1$, we will need to assume also that the inequality (8) is satified, i.e. we need to assume also either $n_1 = 1$ and $x_1 \geq 8$ or $n_1 \geq 2$ and $x_1 \geq 3$. Under these assumptions the inequality (9) is always satisfied. □

The following lemma will be used implicitly several times in the proofs of Theorems 1, 2 and 3 to avoid that a certain set has negative cardinality.

Lemma 3. *Fix integers $s \ge 2$, $n_1 \ge \cdots \ge n_s > 0$ and $x_i > 0$, $1 \le i \le s$, such that either $n_1 = 1$ and $x_1 \ge 8$ or $n_1 \ge 2$ and $x_1 \ge 3$. Then $a_{(n_1,\dots,n_s-1;x_1,\dots,x_s)} \ge n_1 + \cdots + n_s$.*

Proof: Since $a_{(n_1,\dots,n_s-1;x_1,\dots,x_s)} - n_1 - \cdots - n_s$ is a non-decreasing function of $x_2, \dots, x_s$, we may assume $x_2 = \cdots = x_s = 1$. By the definition (1) of $a_{(n_1,\dots,n_s-1;x_1,1,\dots,x_s)}$ is sufficient to show $\tau(n_1, x_1, n_2, \dots, n_s) := \binom{n_1+x_1}{n_1} \prod_{j=1}^{s-1}(n_j+1)n_s - (n_1+\cdots+n_s)^2 \ge 0$. It is easy to check that function τ is a non-decreasing function of $n_2, \dots, n_s$. One easily verifies that $\tau(1, 8, 1, \dots, 1) \ge 0$ and $\tau(2, 3, 1, \dots, 1) \ge 0$, concluding the proof. ∎

Lemma 4. *Let X be an integral m-dimensional projective variety and L, R very ample line bundles on X such that $h^i(X, L) = h^i(X, L \otimes R) = h^i(X, L \otimes R^{\otimes 2}) = 0$ for all $i > 0$. Fix an integral $D \in |R|$. For all integers $i \ge 0$ set $a_{L\otimes R^{\otimes i}} := \lfloor h^0(X, L \otimes R^{\otimes i})/(m+1)\rfloor$, $b_{L\otimes R^{\otimes i}} := h^0(X, L \otimes R^{\otimes i}) - (m+1)a_{L\otimes R^{\otimes i}}$, $\alpha := \lfloor (h^0(X, L \otimes R^{\otimes 2}) - h^0(X, L \otimes R)/m\rfloor$ and $\beta := h^0(X, L \otimes R^{\otimes 2}) - h^0(X, L \otimes R) - m\alpha$. Assume:*

- **(i)** *$h^1(X, \mathcal{I}_{2A} \otimes L \otimes R) = h^1(D, \mathcal{I}_{2\{B,D\},D} \otimes (L \otimes R^{\otimes 2})_{|D}) = 0$ for general $A \subset X$, $B \subset D$ such that $\sharp(A) = a_{L\otimes R^{\otimes 2}} - \alpha$ and $\sharp(B) = \alpha$.*
- **(ii)** *$h^0(X, \mathcal{I}_{2S} \otimes L) \le h^0(X, L \otimes R) - (m+1)a_{L\otimes R^{\otimes 2}} + \beta$ for a general $S \subset X$ such that $\sharp(S) = a_{L\otimes R^{\otimes 2}} - \alpha - \beta$.*

Then L is not defective, i.e. for every integer $k > 0$ we have $h^0(X, \mathcal{I}_Z \otimes L \otimes R^{\otimes 2}) = \max\{0, h^0(X, L \otimes R^{\otimes 2}) - k(m+1)\}$ (or, equivalently, $h^1(X, \mathcal{I}_Z \otimes L \otimes R^{\otimes 2}) = \max\{0, k(m+1) - h^0(X, L \otimes R^{\otimes 2})\}$) for a general union of k double points of X.

Proof: We will only check that $h^1(X, \mathcal{I}_Z \otimes L \otimes R^{\otimes 2}) = 0$ for a general union Z of $a_{L\otimes R^{\otimes 2}}$ double points of X, because the proof that $h^0(X, \mathcal{I}_W \otimes L \otimes R^{\otimes 2}) = 0$ for a general union W of $a_{L\otimes R^{\otimes 2}} + 1$ double points of X is similar and all cases in which $k \le a_{L\otimes R^{\otimes 2}}$ (the surjectivity range of the restriction map) follow from the case $k = a_{L\otimes R^{\otimes 2}}$, while all cases with $k \ge a_{L\otimes R^{\otimes 2}} + 1$ (the injectivity range for the restriction map) follow from the case $k = a_{L\otimes R^{\otimes 2}} + 1$. Since $h^1(X, L \otimes R) = 0$, we have $h^0(D, (L \otimes R^{\otimes 2})_{|D}) = h^0(X, L \otimes R^{\otimes 2}) - h^0(X, L \otimes R)$. By assumption $h^1(D, \mathcal{I}_{2\{B,D\},D} \otimes (L \otimes R^{\otimes 2})_{|D}) = 0$ (i.e. $h^0(D, \mathcal{I}_{2\{B,D\},D} \otimes (L \otimes R^{\otimes 2})_{|D}) = \beta$)for a general $B \subset E$ such that $\sharp(B) = \alpha$. Hence $h^1(D, \mathcal{I}_{F\cup 2\{B,D\},D} \otimes (L \otimes R^{\otimes 2})_{|D}) =$

$h^0(D, \mathcal{I}_{F\cup 2\{B,D\},D} \otimes (L \otimes R^{\otimes 2})_{|D}) = 0$ for a general $F \subset E$ such that $\sharp(F) = \beta$. Fix a general $S \subset X$ such that $\sharp(S) = a_{L\otimes R^{\otimes 2}} - \alpha - \beta$. To check the vanishing of $h^1(X, \mathcal{I}_Z \otimes L \otimes R^{\otimes 2})$ it is sufficient to prove $h^1(X, \mathcal{I}_{2G\cup 2S\cup 2B} \otimes L \otimes R^{\otimes 2})$, where $G \subset X$ is a general subset such that $\sharp(G) = \beta$. We have $\mathrm{Res}_D(2B) = B$ and $2B \cap D = 2\{B, D\}$. By Lemma 2 it is sufficient to prove $h^1(X, \mathcal{I}_{B\cup 2S\cup 2\{G,D\}} \otimes L \otimes R) = 0$. First, we will check that $h^1(X, \mathcal{I}_{2S\cup 2\{G,D\}} \otimes L \otimes R) = 0$. Since $2\{G, D\} \subset 2G$, it is sufficient to prove $h^1(X, \mathcal{I}_{2S\cup 2G} \otimes L \otimes R) = 0$. By assumption we have $h^1(X, \mathcal{I}_{2S} \otimes L \otimes R) = 0$. Even more is true. Indeed, by assumption we have $h^1(X, \mathcal{I}_{2S\cup 2J} \otimes L \otimes R) = 0$ for a general $J \subset X$ such that $\sharp(J) = \beta$; more precisely, it is sufficient to assume that $S \cup J$ is general in X. By semicontinuity we may assume that our vanishing is true not only for D, but for a general $D' \in |D|$. Since R is very ample, there is an integral $D' \in |D|$ passing through m general points of X. Since $\beta \le m$ and we may choose S after choosing G, the condition $G \subset D$ is not restrictive, i.e. we may take G as J. Hence $h^1(X, \mathcal{I}_{2S\cup 2G} \otimes L \otimes R) = 0$ and thus $h^1(X, \mathcal{I}_{2S\cup 2\{G,D\}} \otimes L \otimes R) = 0$. Since $\mathrm{Res}_D(2S \cup B \cup 2\{G, D\}) = 2S$, $h^1(X, \mathcal{I}_{2S\cup 2\{G,D\}} \otimes L \otimes R) = 0$ and B is general in D, we have $h^1(X, \mathcal{I}_{B\cup 2S\cup 2\{G,D\}} \otimes L \otimes R) = 0$ if and only if $h^0(X, \mathcal{I}_{2S\cup 2\{G,D\}} \otimes L \otimes R) - h^0(X, \mathcal{I}_{2S} \otimes L) \ge \sharp(B)$ ([9], Lemma 3). i.e. if and only if $h^0(X, L\otimes R) - (m+1)a_{L\otimes R^{\otimes 2}} + (m+1)\alpha + (m+1)\beta - m\beta - h^0(X, \mathcal{I}_{2S} \otimes L) \ge \alpha$, i.e. if and only if $h^0(X, \mathcal{I}_{2S} \otimes L) \le h^0(X, L\otimes R) - (m+1)a_{L\otimes R^{\otimes 2}} + \beta$ for a general $S \subset X$ such that $\sharp(S) = a_{L\otimes R^{\otimes 2}} - \alpha - \beta$, which is true by our last assumption. ∎

Proof of Theorem 1: Set $M := \mathbf{P}^n \times (\mathbf{P}^1)^{s-1}$. Fix $P \in \mathbf{P}^1$ and set $E := \mathbf{P}^n \times (\mathbf{P}^1)^{s-2} \times \{P\}$ (seen as a hypersurface of multidegree $(0, \dots, 0, 1)$ of M). We divide the proof into 5 steps.

(**a**) Here we assume $s = 2$, $n \ge 2$, $n_2 = 1$, $x_1 = n!(n+1) - n$ and $x_2 = n+1$. Set $\alpha := \binom{n+x_1}{n}/(n+1) = \binom{n!(n+1)}{n}/(n+1)$. Notice that $\alpha \in \mathbb{Z}$ and that $\binom{n+x_1}{n}(x_2+1)/(n+2) = \binom{n+x_1}{n}(n+2)/(n+2) = (n_1+1)\alpha$. Fix a general union $S \subset E$ of α points of E. Notice that $\mathcal{O}_E(x, t) \cong \mathcal{O}_{\mathbf{P}^n}(x)$ for all x, t. Take $n+1$ distinct points $Q_1, \dots, Q_{n+1} \in \mathbf{P}^1$ and set $E_i := \mathbf{P}^n \times \{Q_i\} \cong E \subset M$. Let $S_i \subset E_i$ be a general union of α points of E_i. Hence $2S_i \cap E_i = 2\{S_i, E_i\}$ and $\mathrm{Res}_{E_i}(2S_i) = S_i$. Set $Z_1 := Z := \cup_{i=1}^{n_1+1} 2S_i$. To prove Theorem 1 it is sufficient to prove $h^1(M, \mathcal{I}_Z(x_1, n+1)) = 0$ (or, equivalently, $h^0(M, \mathcal{I}_Z(x_1, n+1)) = 0$). For $2 \le i \le n+1$ set $Z_i := \bigcup_{x=i}^{n+1} 2S_x \cup \bigcup_{y=1}^{i-1} S_y$. Hence $\mathrm{Res}_{E_i}(Z_i) = Z_{i+1}$ for all $1 \le i \le n$. By Lemma 1 to prove $h^1(M, \mathcal{I}_{Z_i}(x_1, n+2-i)) = 0$ it is sufficient to prove $h^1(M, \mathcal{I}_{Z_{i+1}}(x_1, n+1-i)) = 0$. Hence after $n+1$ steps we reduce to check that $h^1(M, \mathcal{I}_{\cup_{i=1}^{n+1} S_i}(x_1, 0)) = 0$. Let S be the union of the projections on E of all sets S_i, $1 \le i \le n_1 + 1$. By the generality of each S_i the set S is a general union of $(n+1)\alpha$ points of E and hence $h^i(E, \mathcal{I}_S(x_1, 0)) = 0$ for $i = 0, 1$, concluding the proof in this case.

(**b**) Here we assume $s = 2$, $n \ge 2$, $x_1 = n!(n+1) - n$ and $x_2 = n+2$. Take a general $S \subset E$ such that $\sharp(S) = \alpha$ and general $A, B \subset M$ such that $\sharp(A) = \lfloor (n+1)\alpha(n+3)/(n+2) \rfloor - \alpha$ and $\sharp(B) = \lceil (n+1)\alpha(n+3)/(n+2) \rceil - \alpha$. To prove Theorem 1 in this case it is sufficient to prove $h^1(M, \mathcal{I}_{2S \cup 2A}(x_1, n+2)) = h^0(M, \mathcal{I}_{2S \cup 2B}(x_1, n+2)) = 0$. By the definition of α and Horace Lemma 1 it is sufficient to prove $h^1(M, \mathcal{I}_{S \cup 2A}(x_1, n+1)) = h^0(M, \mathcal{I}_{S \cup 2B}(x_1, n+1)) = 0$. We will only check $h^1(M, \mathcal{I}_{S \cup 2A}(x_1, n+1)) = 0$, the other vanishing being similar. By the generality of S in E it is sufficient to prove $h^1(M, \mathcal{I}_{2A}(x_1, n+1)) = 0$ and $h^0(M, \mathcal{I}_{2A}(x_1, n+1)) - h^0(M, \mathcal{I}_{2A}(x_1, n)) \ge \sharp(S) = \alpha$ (see e.g. [9], Lemma 3). Since $\lfloor (n+1)\alpha(n+3)/(n+2) \rfloor - \alpha \le (n+1)\alpha$ and A is general in M, we have $h^1(M, \mathcal{I}_{2A}(x_1, n+1)) = 0$ by part (a) and hence $h^0(M, \mathcal{I}_{2A}(x_1, n+1)) = (n+2)(n+1)\alpha - (n+2)\lfloor (n+1)\alpha(n+3)/(n+2) \rfloor - \alpha$. Hence it is sufficient to prove $h^1(M, \mathcal{I}_{2A}(x_1, n)) \le \alpha$. Let $J \subset M$ be a general union of $n\alpha$ points. We repeat the proof of part (a) taking only n hypersurfaces E_j, $1 \le j \le n$, and obtain $h^1(M, \mathcal{I}_{2J}(x_1, n)) = 0$. Since $\lfloor (n+1)\alpha(n+3)/(n+2) \rfloor \ge n\alpha$, we have $h^0(M, \mathcal{I}_{2A}(x_1, n)) \le h^0(M, \mathcal{I}_{2A}(x_1, n))$, concluding this case.

(**c**) Here we assume $s = 2$, $n \ge 2$, $x_1 = n!(n+1) - n$ and $x_2 \ge n+1$. By parts (a) and (b) and induction on the integer x_2 we may assume $x_2 \ge n+2$ and that the result is true for all x_2' such that $n+1 \le x_2' \le x_2 - 1$ and in particular for $x_2' = x_2 - 1$ and $x_2' = x_2 - 2$. We may repeat the proof of part (b); actually, now this case is easier because we may assume that the lemma is true for the integer $x_2 - 2$ and hence $h^1(M, \mathcal{I}_{2A}(x_1, x_2 - 2)) = 0$ and hence $h^0(M, \mathcal{I}_{2A}(x_1, x_2 - 1)) - h^1(M, \mathcal{I}_{2A}(x_1, x_2 - 2)) = (n+1)\alpha$.

(**d**) Here we assume $s = 2$, $n \ge 2$, $x_1 \ge n!(n+1) - n$ and $x_2 \ge n+1$. By parts (a), (b) and (c) and induction on the integer x_1 we may assume that the result is true for the integers $x_2 - 1$ and $x_2 - 2$. Hence we may repeat (with heavy simplifications) the proof of part (b).

(**e**) Now assume $n = 1$. By Remarks 1 and 2 the same proof work taking $\tilde{x}_1 = 9$ as starting point, because the integer $h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(9)) = 10$ is even, i.e. it is divible by $n+1$. ∎

The proof of the following lemma was suggested from the proofs in [15], §3 and §4.

Lemma 5. *Let X be an integral m-dimensional projective variety and L, R very ample line bundles on X such that $h^i(X, L) = h^i(X, L \otimes R) = h^i(X, L \otimes R^{\otimes 2}) = 0$ for all $i > 0$. Fix an integral $D \in |R|$. For all integers $i \geq 0$ set $a_{L\otimes R^{\otimes i}} := \lfloor h^0(X, L \otimes R^{\otimes i})/(m+1) \rfloor$, $b_{L\otimes R^{\otimes i}} := h^0(X, L \otimes R^{\otimes i}) - (m+1)a_{L\otimes R^{\otimes i}}$, $\alpha := \lfloor (h^0(X, L \otimes R^{\otimes 2}) - h^0(X, L \otimes R)/m \rfloor$ and $\beta := h^0(X, L \otimes R^{\otimes 2}) - h^0(X, L \otimes R) - m\alpha$. Set $c_{L\otimes R^{\otimes 2}} := \lfloor (h^0(X, L \otimes R^{\otimes 2}) - 1)/(m+1) \rfloor$. Assume:*

- **(i)** *$h^1(X, \mathcal{I}_{2A} \otimes L \otimes R) = h^1(D, \mathcal{I}_{2\{B,D\},D} \otimes (L \otimes R^{\otimes 2})_{|D}) = 0$ for general $A \subset X$, $B \subset D$ such that $\sharp(A) = c_{L\otimes R^{\otimes 2}} - \alpha$ and $\sharp(B) = \alpha$.*
- **(ii)** *$h^0(X, \mathcal{I}_{2S} \otimes L) \leq h^0(X, L \otimes R) - (m+1)c_{L\otimes R^{\otimes 2}} + \beta$ for a general $S \subset X$ such that $\sharp(S) = c_{L\otimes R^{\otimes 2}} - \alpha - \beta$.*
- **(iii)** *$L \otimes R$ is not $(c_{L\otimes R^{\otimes 2}} - \alpha - \beta - 1)$ weakly defective, i.e. for a general $U \subset X$ such that $\sharp(U) = c_{L\otimes R^{\otimes 2}} - \alpha - \beta$ a general element of $|\mathcal{I}_{2U}(L \otimes R)|$ has an isolated singular point (which is an ordinary double point) at each point of U and no other singularity contained in X_{reg}.*

Then L is not weakly defective, i.e. it is not defective and for every integer $z > 0$ such that $(m+1)z + 1 \leq h^0(X, L \otimes R^{\otimes 2})$ and any general $U \subset X$ such that $\sharp(U) = z$ a general member of $|\mathcal{I}_{2U} \otimes L \otimes R^{\otimes 2}|$ has an isolated singular point at each point of U and no other singularity contained in X_{reg}.

Proof: Notice that $c_{L\otimes R^{\otimes 2}} = a_{L\otimes R^{\otimes 2}}$ if $b_{L\otimes R^{\otimes 2}} \neq 0$ and $c_{L\otimes R^{\otimes 2}} = a_{L\otimes R^{\otimes 2}} - 1$ if $b_{L\otimes R^{\otimes 2}} = 0$. Hence the non defectivity of $L \otimes R^{\otimes 2}$ follows from Lemma 4. To check its non weak defectivity it is sufficient to check the case of $c_{L\otimes R^{\otimes 2}}$ singular points. More precisely, by semicontinuity and [10], Th. 1.4, it is sufficient to prove the existence of $W \subset X_{reg}$ such that $\sharp(W) = c_{L\otimes R^{\otimes 2}}$, $h^1(X, \mathcal{I}_{2W} \otimes L \otimes R^{\otimes 2}) = 0$ and a general $\Gamma \in |\mathcal{I}_{2W} \otimes L \otimes R^{\otimes 2}|$ has an isolated singularity at one point of W. We will copy the proof of Lemma 4 using the integer $c_{L\otimes R^{\otimes 2}}$ instead of the integer $a_{L\otimes R^{\otimes 2}}$ and use the notation of that proof. By assumption (iii) a general $Y \in |\mathcal{I}_{2S\cup 2G} \otimes L \otimes R|$ has an isolated singular point at each point of S for a general $S \cup G \subset X$ such that $\sharp(S \cup G) = c_{L\otimes R^{\otimes 2}} - \alpha$. Set $\tilde{Y} := Y \cup D \in |L \otimes R^{\otimes 2}|$. The proof of Lemma 4 gives that $\mathrm{Sing}(\tilde{Y})$ contains a finite set W containing S and such that $h^1(X, \mathcal{I}_{2W} \otimes L \otimes R^{\otimes 2}) = 0$. Since $D \cap S = \emptyset$, $\tilde{Y}$ has an isolated singular point at each point of S, concluding the proof. ∎

Proof of Theorem 3: It is sufficient to prove Theorem 3 for the integer $k = c_{(n_1,\dots,n_s;x_1,\dots,x_s)}$. Set $M := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$. By assumption $x_i \geq 3$ for all i and there is an integer j such that $1 \leq j \leq s$ and the line bundles $\mathcal{O}_M(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_s)$, $\mathcal{O}_M(x_1, \dots, x_{j-1}, x_j - 1, x_{j+1}, \dots, x_s)$ and

$\mathcal{O}_M(x_1, \dots, x_{j-1}, x_j - 2, x_{j+1}, \dots, x_s)$ are not defective. Notice that E is a hypersurface of M with multidegree $(0, \dots, 0, 1, 0, \dots, 0)$. We want to apply Lemma 4 taking $X := M$, $D := E$, $L := \mathcal{O}_M(x_1, \dots, x_{j-1}, x_j - 2, x_{j+1}, \dots, x_s)$ and $R := \mathcal{O}_M(0, \dots, 0, 1, 0, \dots 0)$. Since $L \times R^{\otimes i}$ is not defective for $i = 0, 1, 2$, assumptions (i) and (ii) of Lemma 4 are satisfied by our assumptions. Since L is not defective, the assumption (iii) of Lemma 4 is true by Remarks 1 and 2. ∎

Proof of Theorem 2: Set $M := \mathbf{P}^n \times (\mathbf{P}^1)^{s-1}$. Fix $P \in \mathbf{P}^1$ and set $E := \mathbf{P}^n \times (\mathbf{P}^1)^{s-2} \times \{P\}$ (seen as a hypersurface of multidegree $(0, \dots, 0, 1)$ of M). Set $\tilde{x_i} := n + i - 1$ if $2 \leq i \leq s$. Set $\tilde{x_1} := 9$ if $n = 1$ and $\tilde{x_1} = n!(n+1) - n$ if $n \geq 2$. Set $\alpha := \binom{n+\tilde{x_1}}{n}/(n+1)$. Notice that $\alpha \in \mathbb{Z}$.

(**a**) Assume $s = 2$, $n \geq 2$, $x_1 = \tilde{x_1}$ and fix a general $S \subset E \cong \mathbf{P}^n$ such that $\sharp(S) = \alpha - 1$. By [15], Cor. 4.5, the linear system $|\mathcal{I}_{2\{S,E\},E}(x_1, 0)|$ on E has the expected dimension at its general member has isolated singularities at each point of S. We immediately get that the linear system $|I_{2S}(x_1, 1)|$ on M has the expected dimension and that it contains hypersurfaces whose singular locus is $S \times \mathbf{P}^1$, i.e. hypersurfaces whose singular set has finitely many points as projection in the first factor $\mathbf{P}^n$ of M. Counting dimension we get that a general $Y \in |\mathcal{I}_{2\{S,E\},E}(x_1, 0)|$ has not this property and hence that it has an isolated singularity at at least one point of S. By [10], Th. 1.4, the line bundle $\mathcal{O}_M(x_1, 1)$ is not weakly $(\alpha - 2)$-defective. Then we continue as in part (b) of the proof of Theorem 1, but using Lemma 5 instead of Lemma 4, obtaining that for every integer t such that $1 \leq t \leq \tilde{x_2}$ the line bundle $\mathcal{O}_M(\tilde{x_1}, t)$ is not weakly $(t\alpha - 2)$-defective.

(**b**) Assume $s = 2$, $n \geq 2$, $x_1 = \tilde{x_1}$ and $x_2 \geq \tilde{x_2}$. We use part (a), Lemma 5 and induction on the integer x_2 to obtain the theorem in this case.

(**c**) Assume $s = 2$, $n \geq 2$, $x_1 \geq \tilde{x_1}$ and $x_2 \geq \tilde{x_2}$. Use induction on x_2 and Lemma 5 to check this case.

(**d**) Assume $s = 3$ and $n \geq 2$. Use the inductive proof of parts (a), (b) and (c). The starting point of the induction is the line bundle $\mathcal{O}_M(\tilde{x}_1, \dots, \tilde{x}_{s-1}, 0)$ on E (whose non weak defectivity when $s = 2$ was checked at the end of part (a)) instead of [15], Cor. 4.5.

(**e**) Assume $n = 1$. The same inductive proof works, since our bounds in the case $s = 2$ are very far from being sharp: for instance, the conditions $x_1 \geq 3$ and $x_2 \geq 3$ are sufficient for the non-defectivity of the line bundle $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(x_1, x_2)$ ([14])). ∎

REFERENCES

[1] Alexander, J. – Singularités imposables en position générale aux hypersurfaces de $\mathbb{P}^n$, *Compositio Math.,* 68 (1988), 305–354.

[2] Alexander, J. and Hirschowitz, A. – Un lemme d'Horace différentiel: application aux singularité hyperquartiques de $\mathbb{P}^5$, *J. Algebraic Geom.,* 1 (1992), 411–426.

[3] Alexander, J. and Hirschowitz, A. – La méthode d'Horace éclaté: application à l'interpolation en degré quatre, *Invent. Math.,* 107 (1992), 585–602.

[4] Alexander, J. and Hirschowitz, A. – Polynomial interpolation in several variables, *J. Algebraic Geom.,* 4 (1995), 201–222.

[5] Alexander, J. and Hirschowitz, A. – An asymptotic vanishing theorem for generic unions of multiple points, *Invent. Math.,* 140 (2000), 303–325.

[6] Catalisano, M.V.; Geramita, A.V. and Gimigliano, A. – Rank of tensors, secant varieties of Segre varieties and fat points, *Linear Algebra Appl.,* 355 (2002), 263–285.

[7] Catalisano, M.V.; Geramita, A.V. and Gimigliano, A. – Higher secant varieties of the Segre varieties, *J. Pure Appl. Algebra* (to appear).

[8] Catalisano, M.V.; Geramita, A.V. and Gimigliano, A. – *Secant defectivity for Segre–Veronese embeddings of* $\mathbb{P}^1 \times \mathbb{P}^1$ *and* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, preprint, 2003.

[9] Chandler, K. – A brief proof of a maximal rank theorem for generic double points in projective space, *Trans. Amer. Math. Soc.,* 353(5) (2000), 1907–1920.

[10] Chiantini, L. and Ciliberto, C. – Weakly defective varieties, *Trans. Amer. Math. Soc.,* 454(1) (2002), 151–178.

[11] Ciliberto, C. – *Geometric aspects of polynomial interpolation in more variables and of Waring's problem*, European Congress of Mathematics (Barcelona, 2000), 289–316, Progress in Math. 201, Birkhäuser, Basel, 2001.

[12] Ciliberto, C. and Hirschowitz, A. – Hypercubique de $\mathbb{P}^4$ avec sept points singulieres génériques, *C. R. Acad. Sci. Paris,* 313(I) (1991), 135–137.

[13] Fontanari, C. – *On Waring's problem for partially symmetric tensors*, e-print arXiv math. AG/040784.

[14] Laface, A. – On linear systems of curves on rational scrolls, *Geom. Dedicata,* 90 (2002), 127–144.

[15] Mella, M. – *Singularities of linear systems and the Waring problem*, e-print arXiv math. AG/0406288.

[16] Terracini, A. – Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari, *Ann. Mat. Pura e Appl.,* 24 (1915), 91–100.

Edoardo Ballico,
Dept. of Mathematics, University of Trento,
38050 Povo (TN) – ITALY
E-mail: `ballico@science.unitn.it`