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HOPF BIFURCATION WITH S_3-SYMMETRY

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Abstract: The aim of this paper is to study Hopf bifurcation with $\mathbf{S}_3$-symmetry assuming Birkhoff normal form. We consider the standard action of $\mathbf{S}_3$ on $\mathbb{R}^2$ obtained from the action of $\mathbf{S}_3$ on $\mathbb{R}^3$ by permutation of coordinates. This representation is absolutely irreducible and so the corresponding Hopf bifurcation occurs on $\mathbb{R}^2 \oplus \mathbb{R}^2$. Golubitsky, Stewart and Schaeffer (*Singularities and Groups in Bifurcation Theory: Vol. 2*. Applied Mathematical Sciences **69**, Springer-Verlag, New York 1988) and Wood (Hopf bifurcations in three coupled oscillators with internal $\mathbf{Z}_2$ symmetries, *Dynamics and Stability of Systems* **13**, 55–93, 1998) prove the generic existence of three branches of periodic solutions, up to conjugacy, in systems of ordinary differential equations with $\mathbf{S}_3$-symmetry, depending on one real parameter, that present Hopf bifurcation. These solutions are found by using the Equivariant Hopf Theorem. We describe the most general possible form of a $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant mapping (assuming Birkhoff normal form) for the standard $\mathbf{S}_3$-simple action on $\mathbb{R}^2 \oplus \mathbb{R}^2$. Moreover, we prove that generically these are the *only* branches of periodic solutions that bifurcate from the trivial solution.

1 – Introduction

The object of this paper is to study Hopf bifurcation with $\mathbf{S}_3$-symmetry assuming Birkhoff normal form. We consider the standard action of $\mathbf{S}_3$ on the two-dimensional irreducible space

$$U = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 + x_2 + x_3 = 0\Big\} \cong \mathbb{R}^2$$

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defined by

$$\sigma \cdot (x_1, x_2, x_3) = \big(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}\big) \qquad \big(\sigma \in \mathbf{S}_3,\ (x_1, x_2, x_3) \in U\big)\ .$$

Note that any $\mathbf{S}_3$-irreducible space is $\mathbf{S}_3$-isomorphic to U. Moreover the standard action of $\mathbf{D}_3$ on $\mathbb{C}$ is isomorphic to the above action of $\mathbf{S}_3$ on U.

Since U is $\mathbf{S}_3$-absolutely irreducible, the corresponding Hopf bifurcation occurs on

$$V = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 :\ z_1 + z_2 + z_3 = 0\Big\} \ \cong\ U \oplus U \ \cong\ \mathbb{R}^2 \oplus \mathbb{R}^2\ .$$

Suppose we have a system of ordinary differential equations (ODEs)

$$\dot{x} = f(x, \lambda) \tag{1.1}$$

where $x \in V$, $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $f\colon V \times \mathbb{R} \to V$ is smooth and commutes with $\mathbf{S}_3$:

$$f(\sigma \cdot x, \lambda) = \sigma \cdot f(x, \lambda) \qquad (\sigma \in \mathbf{S}_3,\ \ x \in V,\ \ \lambda \in \mathbb{R})\ .$$

With these conditions

$$f(0, \lambda) \equiv 0\ .$$

Assume that $(df)_{(0,0)}$ has an imaginary eigenvalue, say i, after rescaling time if necessary. Golubitsky *et al.* [3] and Wood [7] prove the generic existence of three branches of periodic solutions, up to conjugacy, of (1.1) bifurcating from the trivial solution. These solutions are found by using the Equivariant Hopf Theorem (Golubitsky *et al.* [3] Theorem XVI 4.1). They thus correspond to three (conjugacy classes of) isotropy subgroups of $\mathbf{S}_3 \times \mathbf{S}^1$ (acting on V), each having a two-dimensional fixed-point subspace. In this paper we prove in Theorem 5.2 that if we assume (1.1) satisfying the conditions of the Equivariant Hopf Theorem and f is in Birkhoff normal form then the *only* branches of small-amplitude periodic solutions of period near 2π of (1.1) that bifurcate from the trivial equilibrium are the branches of solutions guaranteed by the Equivariant Hopf Theorem.

This paper is organized in the following way. In Section 2 we start by reviewing a few concepts and results related with the general theory of Hopf bifurcation with symmetry — we follow the approach of Golubitsky *et al.* [3]. In Section 3 we recall the conjugacy classes of $\mathbf{S}_3 \times \mathbf{S}^1$ (with action on V) obtained by Golubitsky *et al.* [3] (see also Wood [7]). There are five conjugacy classes and three of them correspond to isotropy subgroups with two-dimensional subspaces.

The next step is to find the general form of the vector field f of (1.1). We assume that f is in Birkhoff normal form to all orders and so f commutes also with $\mathbf{S}^1$. Specifically, we choose coordinates such that

$$\theta \cdot z = e^{i\theta} z \qquad (\theta \in \mathbf{S}^1,\ \ z \in V) \ .$$

We show in Section 4.1 that the standard action of $\mathbf{D}_3 \times \mathbf{S}^1$ on $\mathbb{C}^2$ considered by Golubitsky *et al.* [3] is isomorphic to the action of $\mathbf{S}_3 \times \mathbf{S}^1$ on V (Lemma 4.1). In that way we can use an appropriate isomorphism between $\mathbb{C}^2$ and V and convert the invariant theory of $\mathbf{D}_3 \times \mathbf{S}^1$ on $\mathbb{C}^2$ (obtained by Golubitsky *et al.* [3]) into the invariant theory of $\mathbf{S}_3 \times \mathbf{S}^1$ on V (Proposition 4.2). We describe then in Theorem 4.4 and Corollary 4.6 the most general possible form of a $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant mapping f in (1.1): we obtain generators for the ring of the invariants and generators for the module of the equivariants over the ring of the invariants. Finally, in Theorem 5.2 of Section 5, we prove that generically the only branches of small-amplitude periodic solutions of (1.1) that bifurcate from the trivial equilibrium are those guaranteed by the Equivariant Hopf Theorem. The proof of this theorem relies mostly in the general form of f and the use of Morse Lemma.

We end this introduction by pointing out a few remarks. The main results of this paper are Theorem 4.4 and Theorem 5.2. The first one describes the $\mathbf{S}_3 \times \mathbf{S}^1$-invariant theory and relied upon the establishment of an appropriate isomorphism between $\mathbf{S}_3$ and $\mathbf{D}_3$-simple spaces. The second result proves the nonexistence of branches of periodic solutions of $\mathbf{S}_3$-bifurcation problems that are not guaranteed by the Equivariant Hopf Theorem. For $n > 3$, the groups $\mathbf{D}_n$ and $\mathbf{S}_n$ are not isomorphic. However, we hope that our approach for $\mathbf{S}_3$ will be useful when considering $\mathbf{S}_n$, for $n > 3$. In particular, we predict that the methods of the proof of Theorem 5.2 can be followed once the fifth order truncation of the Taylor series of a general $\mathbf{S}_n$-bifurcation problem in Birkhoff normal form is obtained. Finally, the proof of Theorem 5.2 relied upon Morse Lemma and the general form of the vector field. Both of these ingredients are available in the $\mathbf{D}_n$-case, for $n \geq 3$. Thus the method we followed should work for $n = 3$ using the appropriate coordinates for the $\mathbf{D}_3$-simple space, and we believe that can be adapted to the $\mathbf{D}_n$ case for general n.

2 – Background

We say that a system of ordinary differential equations (ODEs)

$$\dot{x} = f(x, \lambda) \ , \qquad f(0,0) = 0 \tag{2.1}$$

where $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ is the bifurcation parameter and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth function, undergoes a *Hopf bifurcation* at $\lambda = 0$ if $(df)_{0,0}$ has a pair of *simple* purely imaginary eigenvalues. Here $(df)_{0,0}$ denotes the $n \times n$ Jacobian matrix of derivatives of f with respect to the variables x_j, evaluated at $(x, \lambda) = (0, 0)$. Under additional hypotheses of nondegeneracy, the standard Hopf Theorem implies the occurrence of a branch of periodic solutions. See for example Golubitsky and Schaeffer [1] Theorem VIII 3.1. However the presence of symmetry in (2.1) imposes restrictions on the corresponding imaginary eigenspace that may complicate the analysis, and in general the standard Hopf Theorem does not apply directly. We outline the concepts and results involved in the study of (2.1) in presence of symmetry. We follow Golubitsky *et al.* [3] Chapter XVI. See also Golubitsky and Stewart [2] Chapter 4.

Let Γ be a compact Lie group with a linear action on $V = \mathbb{R}^n$ and suppose that f *commutes* with Γ (or it is Γ-*equivariant*):

$$f(\gamma \cdot x, \lambda) = \gamma \cdot f(x, \lambda) \qquad (\gamma \in \Gamma,\ \ x \in V,\ \ \lambda \in \mathbb{R}) \ .$$

We are interested in branches of periodic solutions of (2.1) where f commutes with a group Γ occurring by Hopf bifurcation from the trivial solution $(x, \lambda) = (0, 0)$.

Conditions for imaginary eigenvalues

Let W be a subspace of V. We say that W is Γ-*invariant* if $\gamma w \in W$ for all $\gamma \in \Gamma$ and for all $w \in W$. Moreover, if the only Γ-invariant subspaces of W are $\{0\}$ and W, then W is said to be Γ-*irreducible*. The space V is Γ-*absolutely irreducible* if the only linear mappings on V that commute with Γ are the scalar multiples of the identity. It is a well-known result that the absolute irreducibility of V implies the irreducibility of V ([3] Lema XXII 3.3).

Let V and W be real vector spaces of the same dimension, and Γ and Δ isomorphic Lie groups. Suppose we have an action denoted by $\cdot$ of Γ on V and an action of Δ on W denoted by $*$. We say that these actions are *isomorphic* if there exists a linear isomorphism $L \colon V \to W$ such that for all $\gamma \in \Gamma$ there exists a unique $\gamma' \in \Delta$ such that

$$L(\gamma \cdot x) = \gamma' * L(x) \tag{2.2}$$

for all $x \in V$.

We are interested in periodic solutions of (2.1) when $(df)_{(0,0)}$ has a pair of imaginary eigenvalues $\mp\omega i$. As we said before the symmetry Γ of f imposes restrictions on the corresponding imaginary eigenspace $E_{\omega i}$. Specifically, it must contain a Γ-*simple* subspace W of V ([3] Lemma XVI 1.2) that is either:

(**a**) $W \cong W_1 \oplus W_1$ where W_1 is absolutely irreducible for Γ; or

(**b**) W is irreducible but non-absolutely irreducible for Γ.

Moreover, generically the imaginary eigenspace itself is Γ-simple and coincides with the corresponding real generalized eigenspace of $(df)_{(0,0)}$. By rescaling time and choosing appropriate coordinates we may assume that $\omega = 1$ and

$$(df)_{0,0}\big|_{E_i} = \begin{pmatrix} 0 & -\mathrm{Id}_{m\times m} \\ \mathrm{Id}_{m\times m} & 0 \end{pmatrix} \equiv J$$

where $2m = \dim E_i$. See [3] Proposition XVI 1.4 and Lemma XVI 1.5.

Spatio-temporal symmetries

The method for finding periodic solutions to such a system rests on prescribing in advance the symmetry of the solutions we seek. Before we describe precisely what we mean by a symmetry of a periodic solution we recall a few definitions.

The *orbit* of the action of Γ on $x \in V$ is defined to be

$$\Gamma x = \Big\{ \gamma \cdot x : \ \gamma \in \Gamma \Big\}$$

and the *isotropy subgroup* of $x \in V$ is the subgroup Σ_x of Γ defined by

$$\Sigma_x = \Big\{ \gamma \in \Gamma : \ \gamma \cdot x = x \Big\} .$$

Points on the same group orbit have isotropy subgroups that are conjugate. Later we use this property to simplify the calculations of the isotropy lattice of (an action of) a group.

Note that if f as above is Γ-equivariant and if $x(t)$ is a solution of (2.1), then $\gamma \cdot x(t)$ is also a solution of (2.1). In particular, if f vanishes on $x \in V$, then it vanishes on the orbit Γx. Further, if the *fixed-point subspace* of $\Sigma \in \Gamma$ is

$$\mathrm{Fix}(\Sigma) = \Big\{ x \in V : \ \gamma \cdot x = x, \ \forall \gamma \in \Sigma \Big\} ,$$

then

$$f\big(\mathrm{Fix}(\Sigma)\big) \subseteq \mathrm{Fix}(\Sigma) .$$

To see this, note that if $x \in \mathrm{Fix}(\Sigma)$ and $\sigma \in \Sigma$ then $\sigma \cdot f(x) = f(\sigma \cdot x) = f(x)$ and so $f(x) \in \mathrm{Fix}(\Sigma)$. As a consequence if $x(t)$ is a solution of (2.1) then the isotropy subgroup of $x(t)$ is the isotropy subgroup of $x(0)$ for all $t \in \mathbb{R}$. In particular we can find an equilibrium solution with isotropy subgroup Σ by restricting the original vector field f to the subspace $\mathrm{Fix}(\Sigma)$.

We describe now what we mean by a symmetry of a periodic solution $x(t)$ of (2.1). Suppose that $x(t)$ is 2π-periodic in t (if not, we can rescale time to make the period 2π). Let $\gamma \in \Gamma$. Then $\gamma \cdot x(t)$ is another 2π-periodic solution of (2.1). If $\gamma \cdot x(t)$ and $x(t)$ intersect then the uniqueness of solutions implies that the trajectories must be identical. So either the two trajectories are identical or they do not intersect.

Suppose that the trajectories are identical. Then uniqueness of solutions implies that there exists $\theta \in \mathbf{S}^1$ (we identify the circle group $\mathbf{S}^1$ with $\mathbb{R}/2\pi\mathbf{Z}$) such that

$$\gamma \cdot x(t) = x(t-\theta)\ .$$

We call $(\gamma,\theta) \in \Gamma\times\mathbf{S}^1$ a *spatio-temporal symmetry* of the solution $x(t)$. Denote the space of 2π-periodic mappings by $\mathcal{C}_{2\pi}$. Note that $\mathbf{S}^1$ acts on $\mathcal{C}_{2\pi}$. This action of $\mathbf{S}^1$ is usually called the *phase-shift* action. The collection of all symmetries of $x(t)$ forms a subgroup

$$\Sigma_{x(t)} = \Big\{(\gamma,\theta) \in \Gamma\times\mathbf{S}^1:\ \gamma \cdot x(t) = x(t-\theta)\Big\}\ .$$

Moreover if we consider the natural action of $\Gamma\times\mathbf{S}^1$ on $\mathcal{C}_{2\pi}$ given by

$$(\gamma,\theta)\cdot x(t) = \gamma \cdot x(t-\theta)$$

where the Γ-action is induced from its *spatial* action on V and the $\mathbf{S}^1$ action is by phase shift, then $\Sigma_{x(t)}$ is the isotropy subgroup of $x(t)$ with respect to this action.

The Equivariant Hopf Theorem

We consider (2.1) where f commutes with a compact Lie group Γ and we assume the generic hypothesis that $L = (df)_{0,0}$ has only one pair of imaginary eigenvalues, say $\mp i$. Taking into account that we seek periodic solutions with period approximately 2π, we can apply a Liapunov–Schmidt reduction preserving symmetries that will induce a different action of $\mathbf{S}^1$ on a finite-dimensional space, which can be identified with the exponential of $L|_{E_i} = J$ acting on the imaginary eigenspace E_i of L. Moreover the reduced equation of f commutes with $\Gamma\times\mathbf{S}^1$. See [3] Lemma XXVI 3.2. The basic idea is that small-amplitude periodic solutions of (2.1) of period near 2π correspond to zeros of a reduced equation

$\phi(x,\lambda,\tau) = 0$ where τ is the period-perturbing parameter. To find periodic solutions of (2.1) with symmetries Σ is equivalent to find zeros of the reduced equation restricted to $\mathrm{Fix}(\Sigma)$. See [3] Chapter XVI Section 4.

Consider (2.1) where $f\colon \mathbb{R}^n{\times}\mathbb{R} \to \mathbb{R}^n$ is smooth and commutes with a compact Lie group Γ and make the generic hypothesis that $\mathbb{R}^n$ is Γ-simple. Choose coordinates so that

$$(df)_{(0,0)} = J$$

where $m = n/2$. The eigenvalues of $(df)_{0,\lambda}$ are $\sigma(\lambda) \mp i\rho(\lambda)$ where $\sigma(0) = 0$ and $\rho(0) = 1$ ([3] Lemma XVI 1.5). Suppose that

$$\sigma'(0) \neq 0 \ . \tag{2.3}$$

Consider the action of $\mathbf{S}^1$ on $\mathbb{R}^n$ defined by:

$$\theta \cdot x = e^{i\theta J} x \qquad (\theta \in \mathbf{S}^1,\ \ x \in \mathbb{R}^n) \ .$$

The following result states that for each isotropy subgroup of $\Gamma{\times}\mathbf{S}^1$ with two-dimensional fixed-point subspace there exists a unique branch of periodic solutions of (2.1) with that symmetry:

Theorem 2.1 (Equivariant Hopf Theorem)**.** *Let the system of ordinary differential equations (2.1) where $f\colon \mathbb{R}^n{\times}\mathbb{R} \to \mathbb{R}^n$ is smooth, commutes with a compact Lie group Γ and satisfies*

$$(df)_{0,0} = \begin{pmatrix} 0 & -\mathrm{Id}_{m\times m} \\ \mathrm{Id}_{m\times m} & 0 \end{pmatrix} \equiv J \tag{2.4}$$

and (2.3) where $\sigma(\lambda){\mp}i\rho(\lambda)$ are the eigenvalues of $(df)_{0,\lambda}$. Suppose that $\Sigma \subseteq \Gamma{\times}\mathbf{S}^1$ is an isotropy subgroup such that

$$\dim \mathrm{Fix}(\Sigma) = 2 \ .$$

Then there exists a unique branch of small-amplitude periodic solutions to (2.1) with period near 2π, having Σ as their group of symmetries.

Proof: See Golubitsky *et al.* [3] Theorem XVI 4.1. ∎

A tool for seeking periodic solutions that are not guaranteed by the Equivariant Hopf Theorem and also for calculating the stabilities of the periodic solutions is to use a *Birkhoff normal form* of f: by a suitable coordinate change, up to any given order, the vector field f can be made to commute with Γ and $\mathbf{S}^1$ (in the Hopf case). This result is the equivariant version of the Poincaré–Birkhoff Normal Form Theorem ([3] Theorem XVI 5.1).

Throughout this paper, we assume that the original vector field is in Birkhoff normal form (it commutes with $\Gamma\times\mathbf{S}^1$ where $\Gamma=\mathbf{S}_3$). Under this hypothesis is valid the following result:

Theorem 2.2. *Let the system of ordinary differential equations (2.1) where the vector field $f\colon \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ is smooth, commutes with a compact Lie group Γ and satisfies $(df)_{0,0}=J$ as in (2.4). Suppose that f in (2.1) is in Birkhoff normal. Then it is possible to perform a Liapunov–Schmidt reduction on* (2.1) *such that the reduced equation ϕ has the form*

$$\phi(v,\lambda,\tau) \;=\; f(v,\lambda)-(1+\tau)\,Jv$$

where τ is the period-scaling parameter.

Proof: See [3] Theorem XVI 10.1. ∎

Invariant theory

We finish this section by recalling a few results about invariant theory of compact groups. Let Γ be a compact Lie group and V a finite-dimensional (real) vector space. A function $f\colon V\to\mathbb{R}$ is said to be Γ-*invariant* if

$$f(\gamma\cdot x)=f(x) \qquad (\gamma\in\Gamma,\ \ x\in V)\ .$$

The Hilbert–Weyl Theorem ([3] Theorem XII 4.2) implies that there always exist finitely many Γ-invariant polynomials $u_1,...,u_s$ such that every Γ-invariant polynomial function f has the form

$$f(x)=p\big(u_1(x),...,u_s(x)\big) \qquad (x\in V)$$

for some polynomial function p. We denote by $\mathcal{P}(\Gamma)$ the set of all Γ-invariant polynomials from V to $\mathbb{R}$. This is a ring under the usual polynomial operations and the set $\{u_1,...,u_s\}$ is said to be a *Hilbert basis* of that ring. Schwarz [6] proves that if $\{u_1,...,u_s\}$ is a Hilbert basis for the ring $\mathcal{P}(\Gamma)$ and $f\colon V\to\mathbb{R}$ is a smooth Γ-invariant function then there exists a smooth function $h\colon\mathbb{R}^s\to\mathbb{R}$ such that

$$f(x)=h\big(u_1(x),...,u_s(x)\big) \qquad (x\in V)$$

(see [3] Theorem XII 4.3). The set of all Γ-equivariant polynomial mappings is a module over the ring $\mathcal{P}(\Gamma)$ and the Hilbert–Weyl Theorem also implies that there exists a finite-set of Γ-equivariant polynomial mappings $\mathbf{X}_1, ..., \mathbf{X}_t$ that *generate* the module over the ring $\mathcal{P}(\Gamma)$. That is, every Γ-equivariant polynomial mapping $g\colon V \to V$ has the form

$$g \;=\; f_1\mathbf{X}_1 + \cdots + f_t\mathbf{X}_t$$

where each polynomial function $f_j\colon V \to \mathbb{R}$ is Γ-invariant. See [3] Theorem XII 5.2. The generalization of this result to the module of the smooth Γ-equivariant mappings is due to Poénaru [4]. See [3] Theorem XII 5.3.

3 – The action of $\mathbf{S}_3 \times \mathbf{S}^1$

Let $\Gamma = \mathbf{S}_3$ be the group of bijections of the set $\{1, 2, 3\}$ under composition and let us consider the natural action of $\mathbf{S}_3$ on $\mathbb{C}^3$. That is,

$$\sigma \cdot (z_1, z_2, z_3) = \big(z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)}\big) \qquad \big(\sigma \in \mathbf{S}_3,\;\; (z_1, z_2, z_3) \in \mathbb{C}^3\big)\ . \tag{3.1}$$

The decomposition of $\mathbb{C}^3$ into irreducible subspaces for this action of $\mathbf{S}_3$ is

$$\mathbb{C}^3 \;\cong\; \mathbb{C}^3_0 \oplus V_1$$

where

$$\mathbb{C}^3_0 \;=\; \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3\colon\;\; z_1 + z_2 + z_3 = 0\Big\}$$

and

$$V_1 = \Big\{(z, ..., z)\colon\;\; z \in \mathbb{C}\Big\} \;\cong\; \mathbb{C}\ .$$

Note that the space $\mathbb{C}^3_0$ is $\mathbf{S}_3$-simple:

$$\mathbb{C}^3_0 \;\cong\; \mathbb{R}^3_0 \oplus \mathbb{R}^3_0$$

where

$$\mathbb{R}^3_0 \;\cong\; \mathbb{R}^2$$

is $\mathbf{S}_3$-absolutely irreducible and the action of $\mathbf{S}_3$ on V_1 is trivial.

Throughout this paper let $V = \mathbb{C}^3_0$. Suppose we have a system of ODEs

$$\dot{x} = f(x, \lambda) \tag{3.2}$$

where $x \in V$, $\lambda \in \mathbb{R}$ is the bifurcation parameter and $f\colon V\times\mathbb{R} \to V$ is smooth and commutes with $\mathbf{S}_3$. Note that since $\mathrm{Fix}(\mathbf{S}_3) = \{0\}$ then

$$f(0,\lambda) \equiv 0 \ .$$

We suppose that $(df)_{0,0}$ has eigenvalues $\mp i$. Our aim is to study the generic existence of branches of periodic solutions of (3.2) near the bifurcation point $(x,\lambda) = (0,0)$. We assume that f is in Birkhoff normal form and so f also commutes with $\mathbf{S}^1$, where $\theta \in \mathbf{S}^1$ acts on V by

$$\theta \cdot z = e^{i\theta} z \qquad (\theta \in \mathbf{S}^1,\ \ z \in V) \ . \tag{3.3}$$

Remarks 3.1.

(i) Note that any (real) two-dimensional $\mathbf{S}_3$-irreducible space is isomorphic to $\mathbb{R}^3_0$.

(ii) We show in Section 4.1 that the action of $\mathbf{D}_3\times\mathbf{S}^1$ on $\mathbb{C}^2$ considered in [3] is isomorphic to the above action of $\mathbf{S}_3\times\mathbf{S}^1$ on $V = \mathbb{C}^3_0$ (see Lemma 4.1). Along this paper we often make reference to the results obtained by Golubitsky *et al.* [3] Chapter XVIII where they study Hopf bifurcation with $\mathbf{D}_n\times\mathbf{S}^1$ (the case we are interested is $n = 3$) and to the results obtained by Wood [7] related to Hopf bifurcation with $\mathbf{S}_3\times\mathbf{S}^1$. □

We continue by studying the (conjugacy classes of) isotropy subgroups for the above action of $\mathbf{S}_3\times\mathbf{S}^1$ on V.

The isotropy lattice

Consider the subgroups of $\mathbf{S}_3\times\mathbf{S}^1$ defined by

$$\widetilde{\mathbf{Z}}_3 = \big\langle\big((123), 2\pi i/3\big)\big\rangle\,, \quad \widetilde{\mathbf{Z}}_2 = \big\langle\big((12),\pi\big)\big\rangle\,, \quad \mathbf{S}_1\times\mathbf{S}_2 = \big\langle\big((23),0\big)\big\rangle \ . \tag{3.4}$$

In the next proposition we describe the isotropy subgroups of $\mathbf{S}_3\times\mathbf{S}^1$ and the respective fixed-point subspaces.

Proposition 3.2 ([3, 7]). *Let $V = \mathbb{C}^3_0$ and consider the action of $\mathbf{S}_3\times\mathbf{S}^1$ on V given by (3.1) and (3.3). Then there are five conjugacy classes of isotropy subgroups for the action of $\mathbf{S}_3\times\mathbf{S}^1$ on V. They are listed, together with their orbit representatives and fixed-point subspaces in Table 1.*

Proof: See Golubitsky *et al.* [3] (p. 368–370) (and recall Remark 3.1) or Wood [7] (Proposition 3.2.1, p. 19). ∎

Table 1 – Orbit representatives, isotropy subgroups and fixed-point subspaces of $\mathbf{S}_3 \times \mathbf{S}^1$ acting on V. The groups $\widetilde{\mathbf{Z}}_3$, $\widetilde{\mathbf{Z}}_2$ and $\mathbf{S}_1 \times \mathbf{S}_2$ are defined in (3.4).

Orbit representative	**Isotropy subgroup**	**Fixed-point subspace**
$(0,0,0)$	$\mathbf{S}_3 \times \mathbf{S}^1$	$\{(0,0,0)\}$
$(a, e^{i\frac{2\pi}{3}} a, e^{i\frac{4\pi}{3}} a),\ \ a>0$	$\widetilde{\mathbf{Z}}_3$	$\Big\{(w, e^{i\frac{2\pi}{3}} w, e^{i\frac{4\pi}{3}} w):\ w \in \mathbb{C}\Big\}$
$(a,-a,0),\ \ a>0$	$\widetilde{\mathbf{Z}}_2$	$\Big\{(w,-w,0):\ w \in \mathbb{C}\Big\}$
$(2a,-a,-a),\ \ a>0$	$\mathbf{S}_1 \times \mathbf{S}_2$	$\Big\{(2w,-w,-w):\ w \in \mathbb{C}\Big\}$
$\big(a,b,-(a+b)\big),\ \ a>b>0$	$\mathbf{1}$	$\Big\{\big(w_1,w_2,-(w_1+w_2)\big):\ w_1,w_2 \in \mathbb{C}\Big\}$

Up to conjugacy, we have three isotropy subgroups with two-dimensional fixed-point subspaces: $\widetilde{\mathbf{Z}}_3$, $\widetilde{\mathbf{Z}}_2$ and $\mathbf{S}_1 \times \mathbf{S}_2$. It follows from the Equivariant Hopf Theorem (Theorem 2.1), that there are (*at least*) three branches of periodic solutions occurring generically in Hopf bifurcation with $\mathbf{S}_3$-symmetry (or equivalently, with $\mathbf{D}_3$-symmetry). That is, to each isotropy subgroup Σ of $\mathbf{S}_3 \times \mathbf{S}^1$ with two-dimensional fixed-point subspace corresponds a unique branch of periodic solutions of (3.2) with period near 2π and with symmetry Σ, obtained by bifurcation from the trivial equilibrium (assuming that f satisfies the conditions of the cited theorem). Let us notice, however, that the periodic solutions whose existence is guaranteed by the Equivariant Hopf Theorem are not necessarily the only periodic solutions that bifurcate from $(0,0)$. In the Section 5 we prove in Theorem 5.2. that generically these are the only branches of periodic solutions of (3.2) assuming that f is in Birkhoff normal form.

4 – Invariant theory for $\mathbf{S}_3 \times \mathbf{S}^1$

In order to look for periodic solutions of (3.2) we calculate now the general form of a $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant bifurcation problem. In Theorem 4.4 we obtain a Hilbert basis for the ring of the invariant polynomials $V \to \mathbb{R}$ and a module basis for the equivariant mappings $V \to V$ with polynomial components for the action of the group $\mathbf{S}_3\times\mathbf{S}^1$ on V considered in Section 3. For that we show in Section 4.1 that the action of $\mathbf{D}_3\times\mathbf{S}^1$ on $\mathbb{C}^2$ considered in [3] is isomorphic to the action of $\mathbf{S}_3\times\mathbf{S}^1$ on V — Lemma 4.1. In particular we can use the isomorphism between $\mathbb{C}^2$ and V obtained in this lemma to convert the invariant theory of $\mathbf{D}_3\times\mathbf{S}^1$ on $\mathbb{C}^2$ into the invariant theory of $\mathbf{S}_3\times\mathbf{S}^1$ on V (Proposition 4.2). We then recall the invariant theory for $\mathbf{D}_3\times\mathbf{S}^1$ obtained in [3] and conclude with Theorem 4.4.

4.1. Isomorphic actions of $\mathbf{D}_3\times\mathbf{S}^1$ and $\mathbf{S}_3\times\mathbf{S}^1$

Consider the action of $\mathbf{D}_3\times\mathbf{S}^1$ on $\mathbb{C}^2$ defined by

$$\begin{aligned} \gamma\cdot(z_1,z_2) &= (e^{i\gamma}z_1, e^{-i\gamma}z_2) \qquad (\gamma\in\mathbf{Z}_3)\ , \\ k\cdot(z_1,z_2) &= (z_2,z_1)\ , \\ \theta\cdot(z_1,z_2) &= (e^{i\theta}z_1, e^{i\theta}z_2) \qquad (\theta\in\mathbf{S}^1) \end{aligned} \tag{4.1}$$

for $(z_1,z_2)\in\mathbb{C}^2$. Here $\mathbf{Z}_3=\langle\frac{2\pi}{3}\rangle$ and $\mathbf{D}_3=\langle\frac{2\pi}{3},k\rangle$.

The following results (Lemma 4.1 and Proposition 4.2) are presumably well known, but we provide a simple self-contained proof.

Lemma 4.1. *The action of $\mathbf{D}_3\times\mathbf{S}^1$ on $\mathbb{C}^2$ as in (4.1) is isomorphic to the action of $\mathbf{S}_3\times\mathbf{S}^1$ on $V=\mathbb{C}^3_0$ as defined in (3.1) and (3.3).*

Proof: Consider the following bases B_1 and B_2 of $\mathbb{C}^2$ and V, respectively, over the field $\mathbb{C}$:

$$\begin{aligned} B_1 &= \big((1,0),(0,1)\big)\ , \\ B_2 &= \Big(\big(e^{i\frac{2\pi}{3}},1,e^{i\frac{4\pi}{3}}\big),\big(1,e^{i\frac{2\pi}{3}},e^{i\frac{4\pi}{3}}\big)\Big) \end{aligned} \tag{4.2}$$

and define the $\mathbb{C}$-linear isomorphism $L\colon\mathbb{C}^2\to V$ by

$$\begin{aligned} L(1,0) &= \big(e^{i\frac{2\pi}{3}},1,e^{i\frac{4\pi}{3}}\big)\ , \\ L(0,1) &= \big(1,e^{i\frac{2\pi}{3}},e^{i\frac{4\pi}{3}}\big)\ . \end{aligned}$$

Let $z=(z_1,z_2)\in\mathbb{C}^2$ and let us denote the actions of $\mathbf{D}_3\times\mathbf{S}^1$ on $\mathbb{C}^2$ and $\mathbf{S}_3\times\mathbf{S}^1$ on V by $\cdot$ and $*$ respectively. Then for $\theta\in\mathbf{S}^1$ we have

$$\begin{aligned} L\big((\tfrac{2\pi}{3},\theta)\cdot(z_1,z_2)\big) &= \big((123),\theta\big)*L(z_1,z_2)\ , \\ L\big((k,\theta)\cdot(z_1,z_2)\big) &= \big((12),\theta\big)*L(z_1,z_2)\ . \end{aligned}$$

Therefore the actions of $\mathbf{D}_3\times\mathbf{S}^1$ on $\mathbb{C}^2$ and $\mathbf{S}_3\times\mathbf{S}^1$ on V are isomorphic (recall (2.2)). ∎

Let

$$B_3 = \big((1,0,-1),(0,1,-1)\big) \tag{4.3}$$

be another basis of V (over the complex field). Then the matrix of the $\mathbb{C}$-linear isomorphism $L\colon \mathbb{C}^2 \to V$ with respect to the bases B_1 and B_3 is

$$A = \begin{bmatrix} e^{i\frac{2\pi}{3}} & 1 \\ 1 & e^{i\frac{2\pi}{3}} \end{bmatrix} \tag{4.4}$$

and the matrix of L^{-1} with respect to the bases B_3 and B_1 is

$$A^{-1} = -\frac{\sqrt{3}}{3}\begin{bmatrix} e^{i\frac{\pi}{2}} & e^{i\frac{5\pi}{6}} \\ e^{i\frac{5\pi}{6}} & e^{i\frac{\pi}{2}} \end{bmatrix}. \tag{4.5}$$

Proposition 4.2. *Consider A and A^{-1} as in (4.4) and (4.5) and let us denote by $Z \equiv \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ and $z \equiv \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ the coordinates of $Z \in \mathbb{C}^2$ and $z \in V$ with respect to the bases B_1 and B_3 defined by (4.2) and (4.3), respectively. Then:*

(**i**) *A polynomial $P\colon \mathbb{C}^2 \to \mathbb{R}$ is $\mathbf{D}_3 \times \mathbf{S}^1$-invariant if and only if the polynomial $P'\colon V \to \mathbb{R}$ defined by*

$$P'(z) \equiv P(A^{-1}z) \tag{4.6}$$

is $\mathbf{S}_3 \times \mathbf{S}^1$-invariant.

(**ii**) *A function $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ with polynomial components is $\mathbf{D}_3 \times \mathbf{S}^1$-equivariant if and only if $\widetilde{f}\colon V \to V$ defined by*

$$\widetilde{f}(z) \equiv Af(A^{-1}z) \tag{4.7}$$

is $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant.

Proof: If we take $Z \equiv \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$, the action of the elements $(\frac{2\pi}{3},\theta)$ and (k,θ) of $\mathbf{D}_3 \times \mathbf{S}^1$ on $\mathbb{C}^2$ is given by

$$\left(\frac{2\pi}{3},\theta\right)\cdot Z = M_1 Z\,, \qquad \text{where } M_1 = e^{i\theta}\begin{bmatrix} e^{i\frac{2\pi}{3}} & 0 \\ 0 & e^{i\frac{4\pi}{3}} \end{bmatrix}, \tag{4.8}$$

$$(k,\theta)\cdot Z = M_2 Z\,, \qquad \text{where } M_2 = e^{i\theta}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.9}$$

Similarly, if $z \equiv \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, the action of the elements $((123),\theta)$ and $((12),\theta)$ of $\mathbf{S}_3\times\mathbf{S}^1$ on V is defined by

$$(4.10) \qquad \big((123),\theta\big) * z \;=\; N_1 z, \qquad \text{where } N_1 = e^{i\theta}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},$$

$$(4.11) \qquad \big((12),\theta\big) * z \;=\; N_2 z, \qquad \text{where } N_2 = e^{i\theta}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

With this notation, by Lemma 4.1 the following equalities are valid:

$$(4.12) \qquad AM_1 = N_1 A \quad \text{and} \quad AM_2 = N_2 A\ .$$

Consequently

$$(4.13) \qquad M_1 A^{-1} = A^{-1} N_1 \quad \text{and} \quad M_2 A^{-1} = A^{-1} N_2\ .$$

Let us prove (i). Let $P\colon \mathbb{C}^2 \to \mathbb{R}$ be a $\mathbf{D}_3\times\mathbf{S}^1$-invariant polynomial and let us define $P'\colon V \to \mathbb{R}$ by $P'(z) \equiv P(A^{-1}z)$. Then for $i = 1,2$ we have

$$P'(N_i z) \;=\; P\big(A^{-1}(N_i z)\big) \;=\; P\big(M_i(A^{-1}z)\big) \;=\; P(A^{-1}z) \;=\; P'(z)$$

and so P' is $\mathbf{S}_3\times\mathbf{S}^1$-invariant. Suppose now that the polynomial $P\colon \mathbb{C}^2 \to \mathbb{R}$ is such that P' defined as in (4.6) is $\mathbf{S}_3\times\mathbf{S}^1$-invariant. As

$$P(Z) = P(A^{-1}AZ)\ ,$$

then for $i = 1,2$ it follows that

$$P(M_i Z) = P\big(A^{-1}A(M_i Z)\big) = P\big(A^{-1}(N_i AZ)\big) = P'\big(N_i(AZ)\big) = P'(AZ) = P(Z)$$

and P is $\mathbf{D}_3\times\mathbf{S}^1$-invariant.

The proof of (ii) is similar. ∎

4.2. Invariant theory for $\mathbf{D}_3\times\mathbf{S}^1$

Recall the action of $\mathbf{D}_3\times\mathbf{S}^1$ on $(z_1,z_2)\in\mathbb{C}^2$ defined by (4.1). In the next proposition we get a Hilbert basis for the ring of the $\mathbf{D}_3\times\mathbf{S}^1$-invariant polynomials and a module basis for the $\mathbf{D}_3\times\mathbf{S}^1$-equivariant smooth mappings (over the ring of the $\mathbf{D}_3\times\mathbf{S}^1$-invariant smooth functions):

Proposition 4.3 ([3]).

(**a**) *Every smooth* $\mathbf{D}_3 \times \mathbf{S}^1$*-invariant function* $f: \mathbb{C}^2 \to \mathbb{R}$ *has the form*

$$f(z_1, z_2) = h(P_1, P_2, P_3, P_4)$$

where

$$\begin{aligned} &P_1 = |z_1|^2 + |z_2|^2 , \qquad P_2 = |z_1|^2\,|z_2|^2 , \qquad P_3 = (z_1\overline{z}_2)^3 + (\overline{z}_1 z_2)^3 , \\ &P_4 = i\big(|z_1|^2 - |z_2|^2\big)\,\big((z_1\overline{z}_2)^3 - (\overline{z}_1 z_2)^3\big) \end{aligned} \tag{4.14}$$

and $h: \mathbb{R}^4 \to \mathbb{R}$ *is smooth.*

(**b**) *Every smooth* $\mathbf{D}_3 \times \mathbf{S}^1$*-equivariant function* $f: \mathbb{C}^2 \to \mathbb{C}^2$ *has the form*

$$f(z_1, z_2) \;=\; A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B\begin{bmatrix} z_1^2\overline{z}_1 \\ z_2^2\overline{z}_2 \end{bmatrix} + C\begin{bmatrix} \overline{z}_1^2 z_2^3 \\ \overline{z}_2^2 z_1^3 \end{bmatrix} + D\begin{bmatrix} z_1^4\overline{z}_2^3 \\ z_2^4\overline{z}_1^3 \end{bmatrix}$$

where A, B, C, D *are complex-valued* $\mathbf{D}_3 \times \mathbf{S}^1$*-invariant smooth functions.*

Proof: See Golubitsky *et al.* [3] Proposition XVIII 2.1 when $n = 3$. ∎

4.3. Invariant theory for $\mathbf{S}_3 \times \mathbf{S}^1$

We can use now Proposition 4.2 and Proposition 4.3 to describe the invariant theory for $\mathbf{S}_3 \times \mathbf{S}^1$:

Theorem 4.4. *Let* $z \equiv \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ *denote the coordinates of* $z \in V$ *with respect to the basis* B_3 *(recall (4.3)). Then:*

(**i**) *Every* $\mathbf{S}_3 \times \mathbf{S}^1$*-invariant polynomial* $f: V \to \mathbb{R}$ *has the form*

$$f(z) = h(N, P, S, T)$$

where

$$\begin{aligned} N \;&=\; 2|z_1|^2 + 2|z_2|^2 + z_1\overline{z}_2 + \overline{z}_1 z_2 , \\ P \;&=\; |z_1|^4 + |z_2|^4 + |z_1|^2\,|z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z}_2)\,\big(|z_1|^2 + |z_2|^2\big) + 2\,\mathrm{Re}(z_1^2\overline{z}_2^2) , \\ S \;&=\; 6\,\mathrm{Re}(z_1^2\overline{z}_2^2)\,\big(|z_1|^2 + |z_2|^2\big) + 4\,\mathrm{Re}(z_1^3\overline{z}_2^3) + 9|z_1|^4\,|z_2|^2 + 9|z_1|^2\,|z_2|^4 \\ &\quad - 2|z_1|^6 - 2|z_2|^6 + 6\,\mathrm{Re}(z_1\overline{z}_2)\,\big[6|z_1|^2\,|z_2|^2 - |z_1|^4 - |z_2|^4\big] , \\ T \;&=\; \mathrm{Im}(z_1\overline{z}_2)\,\big(|z_2|^2 - |z_1|^2\big)\,\Big[2\,\mathrm{Re}(z_1\overline{z}_2)\,\big(|z_1|^2 + |z_2|^2\big) + 2\,\mathrm{Re}(z_1\overline{z}_2)^2 + 3|z_1|^2\,|z_2|^2\Big] \end{aligned}$$

and $h: \mathbb{R}^4 \to \mathbb{R}$ *is polynomial.*

(**ii**) *Every* $\mathbf{S}_3{\times}\mathbf{S}^1$*-equivariant function with polynomial components* $g\colon V{\to}V$ *has the form*

$$g(z) \;=\; Ag_1(z) + Bg_2(z) + Cg_3(z) + Dg_4(z)$$

where

$$g_1(z) \;=\; \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad g_2(z) \;=\; \begin{bmatrix} |z_1|^2 z_1 + z_1^2\overline{z}_2 + 2z_1|z_2|^2 - \overline{z}_1 z_2^2 \\ |z_2|^2 z_2 + \overline{z}_1 z_2^2 + 2|z_1|^2 z_2 - z_1^2\overline{z}_2 \end{bmatrix},$$

$$g_3(z) \;=\; \begin{bmatrix} \overline{z}_1(\overline{z}_1 + 2\overline{z}_2)\,(z_2^3 - z_1^3) + 3z_1^2 z_2(\overline{z}_2^2 - \overline{z}_1^2) + 3z_1 z_2^2\overline{z}_2(2\overline{z}_1 + \overline{z}_2) \\ \overline{z}_2(2\overline{z}_1 + \overline{z}_2)\,(z_1^3 - z_2^3) + 3z_1 z_2^2(\overline{z}_1^2 - \overline{z}_2^2) + 3z_1^2\overline{z}_1 z_2(\overline{z}_1 + 2\overline{z}_2) \end{bmatrix},$$

$$g_4(z) \;=\; \begin{bmatrix} \widetilde{g}_4(z_1, z_2) \\ \widetilde{g}_4(z_2, z_1) \end{bmatrix},$$

$$\begin{aligned} \widetilde{g}_4(z_1, z_2) \;=\; & (\overline{z}_1^3 - \overline{z}_2^3)\,(6z_1^2 z_2^2 + 4z_1 z_2^3 - z_1^4) + 3\overline{z}_1\overline{z}_2\left(\overline{z}_2(z_1^4 - z_2^4) - \overline{z}_1 z_2^4\right) \\ & + 6|z_1|^2|z_2|^2\left(3|z_1|^2 z_2 - 2|z_2|^2 z_2 + 2z_1^2(\overline{z}_1 + \overline{z}_2)\right) \end{aligned}$$

and A, B, C, D *are* $\mathbf{S}_3{\times}\,\mathbf{S}^1$*-invariant polynomials from* V *to* $\mathbb{C}$.

Proof: We begin by proving (i). By Proposition 4.3 the polynomials P_1, P_2, P_3, P_4 as in (4.14) form a Hilbert basis for the ring of the $\mathbf{D}_3\times\mathbf{S}^1$-invariant polynomials. By Proposition 4.2 (i), the polynomials defined by

$$N = 3\,P_1(A^{-1}z)\;, \qquad P = 9\,P_2(A^{-1}z)\;,$$
$$S = 27\,P_3(A^{-1}z) \qquad \text{and} \qquad T = -\tfrac{9}{2}\,P_4(A^{-1}z)$$

are $\mathbf{S}_3{\times}\mathbf{S}^1$-invariants and form a Hilbert basis for the ring of the $\mathbf{S}_3{\times}\mathbf{S}^1$-invariant polynomials (for the action on V). Taking A^{-1} as in (4.5) we obtain the polynomials N, P, T, S as stated in the proposition.

The proof of (ii) is analogous. Again, we use Proposition 4.2 (ii) and Proposition 4.3. ∎

Remark 4.5. A function $f = (f_1, f_2, f_3)$ from V to V that commutes with $\mathbf{S}_3{\times}\,\mathbf{S}^1$ has the form

$$f(z_1, z_2, z_3) \;=\; \begin{bmatrix} f_1(z_1, z_2, z_3) \\ f_1(z_2, z_1, z_3) \\ f_1(z_3, z_2, z_1) \end{bmatrix}.$$

Note that from $f((1i)\cdot(z_1,z_2,z_3)) = (1i)\cdot f(z_1,z_2,z_3)$ for $i=2,3$ and for all $(z_1,z_2,z_3)\in V$, it follows that $f_2(z_1,z_2,z_3) = f_1(z_2,z_1,z_3)$ and $f_3(z_1,z_2,z_3) = f_1(z_3,z_2,z_1)$. $\square$

Corollary 4.6. *Let $z=(z_1,z_2,z_3)\in V$ and so $z_3=-z_1-z_2$, and let*

$$u_1 = z_1\overline{z}_1\ , \qquad u_2 = z_2\overline{z}_2 \qquad \text{and} \qquad u_3 = z_3\overline{z}_3\ .$$

Then:

(i) *Every smooth function $\widetilde{f}\colon V\to\mathbb{R}$ invariant under $\mathbf{S}_3\times\mathbf{S}^1$ has the form*

$$\widetilde{f}(z_1,z_2,z_3) \;=\; \widetilde{h}(\widetilde{N},\widetilde{P},\widetilde{S},\widetilde{T})$$

where

$$\begin{aligned} \widetilde{N} &= u_1+u_2+u_3\ ,\\ \widetilde{P} &= u_1^2+u_2^2+u_3^2\ ,\\ \widetilde{S} &= u_1^3+u_2^3+u_3^3+6u_1u_2u_3\ ,\\ \widetilde{T} &= \mathrm{Im}(z_1\overline{z}_2)\left[u_1u_2(u_2-u_1)+u_2u_3(u_3-u_2)+u_1u_3(u_1-u_3)\right] \end{aligned} \tag{4.15}$$

and $\widetilde{h}\colon\mathbb{R}^4\to\mathbb{R}$ is smooth.

(ii) *Every $\mathbf{S}_3\times\mathbf{S}^1$-equivariant and smooth function $\widetilde{g}\colon V\to V$ can be written as*

$$\widetilde{g}(z) \;=\; A\mathbf{X}_1+B\mathbf{X}_2+C\mathbf{X}_3+D\mathbf{X}_4$$

where

$$\mathbf{X}_1=\begin{bmatrix} z_1\\ z_2\\ z_3\end{bmatrix},\quad \mathbf{X}_2=\begin{bmatrix} 2z_1u_1-(z_2u_2+z_3u_3)\\ 2z_2u_2-(z_1u_1+z_3u_3)\\ 2z_3u_3-(z_2u_2+z_1u_1)\end{bmatrix},\quad \mathbf{X}_3=\begin{bmatrix} 2z_1u_1^2-(z_2u_2^2+z_3u_3^2)\\ 2z_2u_2^2-(z_1u_1^2+z_3u_3^2)\\ 2z_3u_3^2-(z_2u_2^2+z_1u_1^2)\end{bmatrix},$$

(4.16)

$$\mathbf{X}_4=\begin{bmatrix} (\overline{z}_1^3-\overline{z}_2^3)(6z_1^2z_2^2+4z_1z_2^3-z_1^4)+6u_1u_2(3u_1z_2-2z_1^2\overline{z}_3-2u_2z_2)+3\overline{z}_1\overline{z}_2(z_1^4\overline{z}_2+z_2^4\overline{z}_3)\\ (\overline{z}_2^3-\overline{z}_1^3)(6z_1^2z_2^2+4z_1^3z_2-z_2^4)+6u_1u_2(3u_2z_1-2z_2^2\overline{z}_3-2u_1z_1)+3\overline{z}_1\overline{z}_2(\overline{z}_1z_2^4+z_1^4\overline{z}_3)\\ (\overline{z}_3^3-\overline{z}_2^3)(6z_2^2z_3^2+4z_2^3z_3-z_3^4)+6u_2u_3(3u_3z_2-2z_3^2\overline{z}_1-2u_2z_2)+3\overline{z}_2\overline{z}_3(\overline{z}_2z_3^4+z_2^4\overline{z}_1)\end{bmatrix}$$

and A,B,C,D are $\mathbf{S}_3\times\mathbf{S}^1$-invariant and smooth functions from V to $\mathbb{C}$.

Proof: By Schwarz and Poénaru Theorems (see Schwarz [6] or [3] Theorem XII 4.3 and Poénaru [4] or [3] Theorem XII 5.3) we may suppose that $\widetilde{f}$ is polynomial and $\widetilde{g}$ has polynomial components. As $|z_3|^2 = |z_1+z_2|^2 = |z_1|^2+|z_2|^2+z_1\overline{z}_2+\overline{z}_1 z_2$ then

$$\begin{aligned}
2\,\mathrm{Re}(z_1\overline{z}_2) \;&=\; z_1\overline{z}_2+\overline{z}_1 z_2 \;=\; |z_3|^2-|z_1|^2-|z_2|^2 \;=\; u_3-u_1-u_2\ ,\\
2\,\mathrm{Re}(z_1^2\overline{z}_2^2) \;&=\; z_1^2\overline{z}_2^2+\overline{z}_1^2 z_2^2 \;=\; u_1^2+u_2^2+u_3^2-2u_1u_3-2u_2u_3\ ,\\
2\,\mathrm{Re}(z_1^3\overline{z}_2^3) \;&=\; z_1^3\overline{z}_2^3+\overline{z}_1^3 z_2^3 \;=\; u_3^3-u_1^3-u_2^3-3u_1u_3^2-3u_2u_3^2\\
&\qquad\qquad\qquad\qquad\; +3u_1^2u_3+3u_2^2u_3+3u_1u_2u_3\ .
\end{aligned} \tag{4.17}$$

Consider the polynomials N, P, S and T as defined in Theorem 4.4. Using the equalities (4.17) we obtain

$$\begin{aligned}
N \;&=\; u_1+u_2+u_3\ ,\\
P \;&=\; u_1^2+u_2^2+u_3^2-u_1u_2-u_1u_3-u_2u_3\ ,\\
S \;&=\; 2u_1^3+2u_2^3+2u_3^3-3u_1u_2(u_1+u_2)-3u_2u_3(u_2+u_3)\\
&\quad -3u_1u_3(u_1+u_3)+12u_1u_2u_3\ ,\\
T \;&=\; \mathrm{Im}(z_1\overline{z}_2)\,\big[u_1u_2(u_2-u_1)+u_2u_3(u_3-u_2)+u_1u_3(u_1-u_3)\big]\ .
\end{aligned}$$

Let $\widetilde{N}$, $\widetilde{P}$, $\widetilde{S}$, $\widetilde{T}$ be the $\mathbf{S}_3\times\mathbf{S}^1$-invariant polynomials defined in (4.15). Then

$$N=\widetilde{N}\ ,\qquad P=\frac{3}{2}\widetilde{P}-\frac{1}{2}\widetilde{N}^2\ ,\qquad S=3\widetilde{S}-\widetilde{N}^3\ ,\qquad T=\widetilde{T}\ .$$

By Theorem 4.4 the polynomials N, P, S, T form a Hilbert basis for the ring of the $\mathbf{S}_3\times\mathbf{S}^1$-invariant polynomials. Therefore $\widetilde{N}$, $\widetilde{P}$, $\widetilde{S}$, $\widetilde{T}$ also form a Hilbert basis for this ring.

We prove now (ii). Let g_1, g_2, g_3, g_4 be as in Theorem 4.4. Replacing $-z_1-z_2$ by z_3 in each one of the g_i we obtain through routine calculations

$$g_1=\begin{bmatrix} z_1\\ z_2\end{bmatrix},\qquad g_2=\begin{bmatrix} 2z_1u_2-z_1^2\overline{z}_3-\overline{z}_1z_2^2\\ 2u_1z_2-z_2^2\overline{z}_3-z_1^2\overline{z}_2\end{bmatrix},$$

$$g_3=\begin{bmatrix} (\overline{z}_3^2-\overline{z}_2^2)(z_2^3-z_1^3)+3z_2z_1^2(\overline{z}_2^2-\overline{z}_1^2)+3z_1z_2^2(\overline{z}_3^2-\overline{z}_1^2)\\ (\overline{z}_3^2-\overline{z}_1^2)(z_1^3-z_2^3)+3z_1z_2^2(\overline{z}_1^2-\overline{z}_2^2)+3z_2z_1^2(\overline{z}_3^2-\overline{z}_2^2)\end{bmatrix}$$

and

$$g_4=\begin{bmatrix} (\overline{z}_1^3-\overline{z}_2^3)(6z_1^2z_2^2+4z_1z_2^3-z_1^4)+6u_1u_2(3u_1z_2-2z_1^2\overline{z}_3-2u_2z_2)+3\overline{z}_1\overline{z}_2(z_1^4\overline{z}_2+z_2^4\overline{z}_3)\\ (\overline{z}_2^3-\overline{z}_1^3)(6z_1^2z_2^2+4z_2z_1^3-z_2^4)+6u_1u_2(3u_2z_1-2z_2^2\overline{z}_3-2u_1z_1)+3\overline{z}_1\overline{z}_2(z_2^4\overline{z}_1+z_1^4\overline{z}_3)\end{bmatrix}.$$

We obtain the $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant functions $\widetilde{X}_i$ from g_i (for $i = 1, 2, 3, 4$) keeping the components of g_i and considering the third component as described in Remark 4.5:

$$\widetilde{\mathbf{X}}_1 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \qquad \widetilde{\mathbf{X}}_2 = \begin{bmatrix} 2z_1u_2 - z_1^2\overline{z}_3 - \overline{z}_1 z_2^2 \\ 2z_2u_1 - z_2^2\overline{z}_3 - \overline{z}_2 z_1^2 \\ 2z_3u_2 - z_3^2\overline{z}_1 - \overline{z}_3 z_2^2 \end{bmatrix},$$

$$\widetilde{\mathbf{X}}_3 = \begin{bmatrix} (\overline{z}_3^2 - \overline{z}_2^2)(z_2^3 - z_1^3) + 3z_2z_1^2(\overline{z}_2^2 - \overline{z}_1^2) + 3z_2^2z_1(\overline{z}_3^2 - \overline{z}_1^2) \\ (\overline{z}_3^2 - \overline{z}_1^2)(z_1^3 - z_2^3) + 3z_1z_2^2(\overline{z}_1^2 - \overline{z}_2^2) + 3z_1^2z_2(\overline{z}_3^2 - \overline{z}_2^2) \\ (\overline{z}_1^2 - \overline{z}_2^2)(z_2^3 - z_3^3) + 3z_2z_3^2(\overline{z}_2^2 - \overline{z}_3^2) + 3z_2^2z_3(\overline{z}_1^2 - \overline{z}_3^2) \end{bmatrix}$$

and

$$\widetilde{\mathbf{X}}_4 = \begin{bmatrix} (\overline{z}_1^3 - \overline{z}_2^3)(6z_1^2z_2^2 + 4z_1z_2^3 - z_1^4) + 6u_1u_2(3u_1z_2 - 2z_1^2\overline{z}_3 - 2u_2z_2) + 3\overline{z}_1\overline{z}_2(z_1^4\overline{z}_2 + z_2^4\overline{z}_3) \\ (\overline{z}_2^3 - \overline{z}_1^3)(6z_1^2z_2^2 + 4z_1^3z_2 - z_2^4) + 6u_1u_2(3u_2z_1 - 2z_2^2\overline{z}_3 - 2u_1z_1) + 3\overline{z}_1\overline{z}_2(\overline{z}_1z_2^4 + z_1^4\overline{z}_3) \\ (\overline{z}_3^3 - \overline{z}_2^3)(6z_2^2z_3^2 + 4z_2^3z_3 - z_3^4) + 6u_2u_3(3u_3z_2 - 2\overline{z}_1z_3^2 - 2u_2z_2) + 3\overline{z}_2\overline{z}_3(\overline{z}_2z_3^4 + \overline{z}_1z_2^4) \end{bmatrix}.$$

Consider $\widetilde{N}$, $\widetilde{P}$, $\widetilde{S}$, $\widetilde{T}$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$ as in (4.15) and (4.16). Note that $\widetilde{\mathbf{X}}_1 = \mathbf{X}_1$ and $\widetilde{\mathbf{X}}_4 = \mathbf{X}_4$. Routine calculations show that

$$\widetilde{\mathbf{X}}_2 = 2\widetilde{N}\mathbf{X}_1 - \mathbf{X}_2 \,, \qquad \widetilde{\mathbf{X}}_3 = (2\widetilde{N}^2 - 3\widetilde{P})\mathbf{X}_1 - 2\widetilde{N}\mathbf{X}_2 + 3\mathbf{X}_3 \,.$$

By Theorem 4.4 the $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant functions $\mathbf{X}_j : V \to V$ and $i\mathbf{X}_j : V \to V$ for $j = 1, ..., 4$, generate the module of the $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant functions over the ring of the $\mathbf{S}_3 \times \mathbf{S}^1$-invariants. ∎

5 – Hopf bifurcation with $\mathbf{S}_3$-symmetry

In Section 3 we determined the conjugacy classes of isotropy subgroups for the action of $\mathbf{S}_3 \times \mathbf{S}^1$ on $V = \mathbb{C}_0^3$ (Proposition 3.2). Up to conjugacy, we have three isotropy subgroups with two-dimensional fixed-point subspaces: $\widetilde{\mathbf{Z}}_3$, $\widetilde{\mathbf{Z}}_2$ and $\mathbf{S}_1 \times \mathbf{S}_2$. It follows from the Equivariant Hopf Theorem, that there are (*at least*) three branches of periodic solutions corresponding to each one of these isotropy subgroups of $\mathbf{S}_3 \times \mathbf{S}^1$ occurring generically in Hopf bifurcation with $\mathbf{S}_3$-symmetry. We prove in Theorem 5.2 that generically these are the only branches of periodic solutions obtained through bifurcation from the trivial equilibrium in bifurcation problems with $\mathbf{S}_3$-symmetry and assuming Birkhoff normal form.

Suppose that the function $f: V{\times}\mathbb{R} \to V$ is $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant and smooth, and satisfies the conditions of the Equivariant Hopf Theorem. Thus we assume that

$$(df)_{0,\lambda}(z) = \mu(\lambda)z \tag{5.1}$$

where μ is a smooth function from $\mathbb{R}$ to $\mathbb{C}$ such that

$$\mu(0) = i \ \wedge \ \mathrm{Re}\big(\mu'(0)\big) \neq 0 \ . \tag{5.2}$$

From Theorem 2.2 the small-amplitude periodic solutions of the equation

$$\dot{z} = f(z,\lambda) \tag{5.3}$$

of period near 2π are in one to one correspondence with the zeros of the equation

$$g(z,\lambda,\tau) = 0 \tag{5.4}$$

where $g = f - (1+\tau)iz$ and τ is the period-scaling parameter. From Corollary 4.6 and Remark 4.5 the general form of $f = (f_1, f_2, f_3)$ is

$$\begin{aligned} f_1(z_1,z_2,z_3,\lambda) &= \mu(\lambda)z_1 + Az_1 + B\mathbf{X}_{2,1} + C\mathbf{X}_{3,1} + D\mathbf{X}_{4,1} \ , \\ f_2(z_1,z_2,z_3,\lambda) &= f_1(z_2,z_1,z_3,\lambda) \ , \\ f_3(z_1,z_2,z_3,\lambda) &= f_1(z_3,z_2,z_1,\lambda) \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} \mathbf{X}_{2,1} &= 2z_1u_1 - (z_2u_2 + z_3u_3) \ , \\ \mathbf{X}_{3,1} &= 2z_1u_1^2 - (z_2u_2^2 + z_3u_3^2) \ , \\ \mathbf{X}_{4,1} &= (\overline{z}_1^3 - \overline{z}_2^3)\,(6z_1^2z_2^2 + 4z_1z_2^3 - z_1^4) + 6u_1u_2(3u_1z_2 - 2z_1^2\overline{z}_3 - 2u_2z_2) \\ &\quad + 3\overline{z}_1\overline{z}_2(z_1^4\overline{z}_2 + z_2^4\overline{z}_3) \end{aligned}$$

and A, B, C and D are smooth $\mathbf{S}_3{\times}\,\mathbf{S}^1$-invariant functions from $V{\times}\mathbb{R}$ to $\mathbb{C}$ (thus they may depend on λ). Since we are assuming (5.1) it follows that $A(0,\lambda) \equiv 0$. Recall that $u_j = z_j\overline{z}_j$ for $j = 1,2,3$.

Lemma 5.1. *Consider f as in (5.5). Let $z_3 = -z_1 - z_2$ where $(z_1,z_2) = (r_1e^{i\phi_1}, r_2e^{i\phi_2})$ with $r_1,r_2 \in \mathbb{R}$ and $\phi = \phi_2 - \phi_1$. Then we can write the first two components of f as*

$$\begin{bmatrix} r_1e^{i\phi_1}h(r_1,r_2,\phi,\lambda) \\ r_2e^{i\phi_2}h(r_2,r_1,-\phi,\lambda) \end{bmatrix}$$

where h is a smooth function from $\mathbb{R}^4$ to $\mathbb{C}$.

Proof: Let $\widetilde{N}$, $\widetilde{P}$, $\widetilde{S}$, $\widetilde{T}$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$ be as in the Corollary 4.6. Taking $z_3 = -z_1 - z_2$, $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ and $\phi = \phi_2 - \phi_1$ we can write each of the invariant polynomials in the form

$$\begin{aligned}
\widetilde{N} &= 2r_1^2 + 2r_2^2 + 2r_1 r_2 \cos\phi \ , \\
\widetilde{P} &= r_1^4 + r_2^4 + (r_1^2 + r_2^2 + 2r_1 r_2 \cos\phi)^2 \ , \\
\widetilde{S} &= r_1^6 + r_2^6 + (r_1^2 + r_2^2 + 2r_1 r_2 \cos\phi)^3 + 6r_1^2 r_2^2 (r_1^2 + r_2^2 + 2r_1 r_2 \cos\phi) \ , \\
\widetilde{T} &= r_1 r_2^2 \sin\phi \Big(2r_1^4 r_2 (1{+}2\cos^2\phi) - r_1^2 (r_1 r_2 + r_2^2)(r_1 + 2r_2\cos\phi)^2 \\
&\quad + \big(2r_1^5 + 4r_1^3 r_2^2 - 2r_1^4 r_2^2 (r_1 + 2r_2\cos\phi)^3\big) \cos\phi \Big)
\end{aligned}$$

and the first two components of $\mathbf{X}_j$ for $j = 2,3,4$ as

$$\begin{bmatrix} \mathbf{X}_{i,1} \\ \mathbf{X}_{i,2} \end{bmatrix} = \begin{bmatrix} r_1 e^{i\phi_1} h_i(r_1, r_2, \phi) \\ r_2 e^{i\phi_2} h_i(r_2, r_1, -\phi) \end{bmatrix} \tag{5.6}$$

where

$$\begin{aligned}
h_2(r_1, r_2, \phi) &= 3r_1^2 + \big(i\sin(2\phi) + 2 + \cos(2\phi)\big) r_2^2 + (3\cos\phi + i\sin\phi) r_1 r_2 \ , \\
h_3(r_1, r_2, \phi) &= \big(9\cos\phi + 3i\sin\phi + \cos(3\phi) + i\sin(3\phi)\big)\, r_1 r_2^3 + 3r_1^4 \\
&\quad + (5\cos\phi + i\sin\phi) r_1^3 r_2 + \big(6 + 4\cos(2\phi) + 2i\sin(2\phi)\big)\, r_1^2 r_2^2 \\
&\quad + \big(3 + 2\cos(2\phi) + 2i\sin(2\phi)\big)\, r_2^4 \ , \\
h_4(r_1, r_2, \lambda) &= \Big(30\cos\phi + i\big(3\sin(3\phi) + 6\sin\phi\big) + 5\cos(3\phi)\Big)\, r_1^3 r_2^3 - r_1^6 \\
&\quad - (21\cos\phi + 9i\sin\phi) r_1 r_2^5 - \big(3i\sin(2\phi) + 4 + 3\cos(2\phi)\big)\, r_2^6 \\
&\quad + \big(9\cos(2\phi) + 12 + 3i\sin(2\phi)\big)\, r_1^4 r_2^2 \ .
\end{aligned} \tag{5.7}$$

It follows the result if we consider (5.5). ∎

Theorem 5.2. *Consider the system (5.3) with f as in (5.5) where $A(0,\lambda) \equiv 0$ and $\mu\colon \mathbb{R} \to \mathbb{C}$ is smooth and satisfies (5.2). Generically (5.3) admits only branches of periodic solutions that bifurcate from $(0,0)$ with symmetry (conjugate to) $\mathbf{S}_1{\times}\mathbf{S}_2$, $\widetilde{\mathbf{Z}}_2$, $\widetilde{\mathbf{Z}}_3$.*

Proof: Consider the Taylor expansion of f as in (5.5) around $z = 0$ and recall Corollary 4.6. Then we can write f in the form

$$
\begin{aligned}
f_1(z_1,z_2,z_3,\lambda) &= \Big[\mu(\lambda) + a\big(|z_1|^2+|z_2|^2+|z_3|^2\big)\Big]\, z_1 \\
&\quad + b\big(2|z_1|^2 z_1 - |z_2|^2 z_2 - |z_3|^2 z_3\big) + \text{terms of degree} \geq 5 \;, \\
f_2(z_1,z_2,z_3,\lambda) &= f_1(z_2,z_1,z_3,\lambda) \;, \\
f_3(z_1,z_2,z_3,\lambda) &= f_1(z_3,z_2,z_1,\lambda) \;,
\end{aligned} \tag{5.8}
$$

where $\mu(0)=i$, $\mathrm{Re}(\mu'(0))\neq 0$ and a, b are smooth complex-valued functions of λ.

Consider $g = f-(1+\tau)iz$ as in (5.4) and so the first two coordinates of g are:

$$
\begin{aligned}
g_1(z,\lambda,\tau) &= \Big[\nu + a\big(|z_1|^2+|z_2|^2+|z_3|^2\big)\Big]\, z_1 \\
&\quad + b\big(2|z_1|^2 z_1 - |z_2|^2 z_2 - |z_3|^2 z_3\big) + \text{terms of degree} \geq 5 \;, \\
g_2(z,\lambda,\tau) &= \Big[\nu + a\big(|z_1|^2+|z_2|^2+|z_3|^2\big)\Big]\, z_2 \\
&\quad + b\big(2|z_2|^2 z_2 - |z_1|^2 z_1 - |z_3|^2 z_3\big) + \text{terms of degree} \geq 5 \;,
\end{aligned} \tag{5.9}
$$

where $\nu = \mu(\lambda) - (1+\tau)i$.

We have that $\mathrm{Fix}_V(\mathbf{S}_3) = \{0\}$, consequently $f(0,\lambda) \equiv 0$. Therefore $(0,\lambda)$ is an equilibrium point of (5.3) for all values of λ. Since we are assuming that $(df)_{0,\lambda}(z) = \mu(\lambda)z$, where $\mu(0) = i$ and $\mathrm{Re}(\mu'(0)) \neq 0$, the stability of this equilibrium varies when λ crosses zero.

The space V is $\mathbf{S}_3$-simple and we are assuming (5.1) and (5.2) and so the conditions of the Equivariant Hopf Theorem are satisfied. Since the isotropy subgroups $\mathbf{S}_1{\times}\mathbf{S}_2$, $\widetilde{\mathbf{Z}}_2$ and $\widetilde{\mathbf{Z}}_3$ have two-dimensional fixed-point subspaces, by the Equivariant Hopf Theorem the system (5.3) admits branches of periodic solutions with symmetry $\mathbf{S}_1{\times}\mathbf{S}_2$, $\widetilde{\mathbf{Z}}_2$, $\widetilde{\mathbf{Z}}_3$ and conjugate to these groups by bifurcation from $(z,\lambda) = (0,0)$. Moreover, these correspond to zeros of (5.4) with the corresponding symmetry. We study now the existence of branches of periodic solutions of (5.3) with trivial symmetry that bifurcate from $(0,0)$. For that we look for branches of zeros (z_1,z_2) of (5.9) with $z_1 z_2 \neq 0$. These satisfy

$$
\begin{aligned}
\frac{g_1(z,\lambda,\tau)}{z_1} &= 0 \;, \\
\frac{g_2(z,\lambda,\tau)}{z_2} &= 0 \;.
\end{aligned} \tag{5.10}
$$

Taking $z_3 = -z_1 - z_2$, $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ and $\phi = \phi_2 - \phi_1$, by Lemma 5.1 we can write the first two components of f in the form

$$\begin{bmatrix} r_1 e^{i\phi_1} h(r_1, r_2, \phi, \lambda) \\ r_2 e^{i\phi_2} h(r_2, r_1, -\phi, \lambda \end{bmatrix}$$

and so (5.10) can be written as

$$(5.11)\qquad \begin{cases} \nu + \big((2a+3b)\cos\phi + ib\sin\phi\big)r_1 r_2 + (2a+3b)r_1^2 \\ \quad + \big(2a + 2b + b\cos(2\phi) + ib\sin(2\phi)\big)r_2^2 + P_1(r_1, r_2, \phi, \lambda) \;=\; 0 \\ \nu + \big((2a+3b)\cos\phi - ib\sin\phi\big)r_1 r_2 + (2a+3b)r_2^2 \\ \quad + \big(2a + 2b - ib\sin(2\phi) + b\cos(2\phi)\big)r_1^2 + P_1(r_2, r_1, -\phi, \lambda) \;=\; 0\ , \end{cases}$$

where P_1 is smooth (whose Taylor expansion around $(r_1, r_2) = (0, 0)$ has terms (in r_1 and r_2) of degree greater or equal to 4). Recall (5.9).

Taking the difference of the equations of (5.11) we obtain

$$(5.12)\qquad \begin{aligned} b\Big[\big(2\sin^2\phi + i\sin(2\phi)\big)r_1^2 + \big(i\sin(2\phi) - 2\sin^2\phi\big)r_2^2 + 2i\sin\phi\, r_1 r_2\Big] + \\ + P_1(r_1, r_2, \phi, \lambda) - P_1(r_2, r_1, -\phi, \lambda) \;=\; 0\,. \end{aligned}$$

Consider the generic hypothesis

$$b(0) \neq 0$$

and let

$$P_2(r_1, r_2, \phi, \lambda) \;=\; \frac{P_1(r_1, r_2, \phi, \lambda) - P_1(r_2, r_1, -\phi, \lambda)}{2b}\ .$$

Then equation (5.12) is equivalent to

$$(5.13)\qquad \sin^2\phi(r_1^2 - r_2^2) + i\sin\phi\big(\cos\phi(r_1^2 + r_2^2) + r_1 r_2\big) + P_2(r_1, r_2, \phi, \lambda) \;=\; 0$$

and so the real and imaginary parts of (5.13) should verify:

$$(5.14)\qquad \begin{cases} \sin^2\phi(r_1^2 - r_2^2) + \mathrm{Re}\big(P_2(r_1, r_2, \phi, \lambda)\big) \;=\; 0 \\ \sin\phi(\cos\phi\, r_1^2 + \cos\phi\, r_2^2 + r_1 r_2) + \mathrm{Im}\big(P_2(r_1, r_2, \phi, \lambda)\big) \;=\; 0\ . \end{cases}$$

The degree two truncation of the system (5.14) is equivalent to

$$(5.15)\qquad [\, r_1,\ r_2 \,]\, A \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0\ , \qquad [\, r_1,\ r_2 \,]\, B \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0$$

where

$$A = \begin{bmatrix} \sin^2\phi & 0 \\ 0 & -\sin^2\phi \end{bmatrix}, \qquad B = \begin{bmatrix} \sin\phi\cos\phi & \dfrac{\sin\phi}{2} \\ \dfrac{\sin\phi}{2} & \sin\phi\cos\phi \end{bmatrix}.$$

Note that

$$\det A = -\sin^4\phi\ , \qquad \det B = \sin^2\phi\left(\cos^2\phi - \frac{1}{4}\right).$$

We study now the existence of smooth branches of zeros of the system (5.14) by bifurcation from $(0,0)$. We use (5.15) in some cases.

(**i**) We begin with the cases where $\det(A) < 0$ and $\det(B) < 0$. Let ϕ be such that $\sin\phi \neq 0$ and $\cos^2\phi < \frac{1}{4}$. The system (5.15) is equivalent to

$$\text{(5.16)} \qquad \begin{cases} (r_1 - r_2)\,(r_1 + r_2) \;=\; 0 \\ \cos\phi\, r_1^2 + r_1 r_2 + \cos\phi\, r_2^2 \;=\; 0\ , \end{cases}$$

which admits only the solution $(r_1, r_2) = (0,0)$. Moreover the system

$$\text{(5.17)} \qquad \begin{cases} (r_1 - r_2)\,(r_1 + r_2) + \dfrac{\mathrm{Re}\big(P_2(r_1,r_2,\phi,\lambda)\big)}{\sin^2\phi} \;=\; 0 \\ \cos\phi\, r_1^2 + r_1 r_2 + \cos\phi\, r_2^2 + \dfrac{\mathrm{Im}\big(P_2(r_1,r_2,\phi,\lambda)\big)}{\sin\phi} \;=\; 0\ , \end{cases}$$

is equivalent to (5.14). The solutions in $\mathbb{R}^2$ of the first equation of (5.16) correspond to the points of the lines $r_1 = r_2$ and $r_1 = -r_2$. Denote those lines by l_1 and l_2. The solutions of the second equation of (5.16) correspond to the points of two lines l_3, l_4, whose slopes are distinct from the slopes of the lines l_1 and l_2. So $(0,0)$ is the only solution of (5.16). Moreover $(r_1, r_2) = (0,0)$ is a critical nondegenerate point of each one of the functions

$$\begin{aligned} h_1(r_1,r_2) \;&=\; (r_1 - r_2)(r_1 + r_2)\ , \\ h_2(r_1,r_2) \;&=\; \cos\phi\, r_1^2 + r_1 r_2 + \cos\phi\, r_2^2\ , \\ h_3(r_1,r_2) \;&=\; h_1(r_1,r_2) + \frac{\mathrm{Re}\big(P_2(r_1,r_2,\phi,\lambda)\big)}{\sin^2\phi}\ , \\ h_4(r_1,r_2) \;&=\; h_2(r_1,r_2) + \frac{\mathrm{Im}\big(P_2(r_1,r_2,\phi,\lambda)\big)}{\sin\phi}\ . \end{aligned}$$

By Morse Lemma (see for example Poston and Stewart [5] Theorem 4.2) the solutions of each one of the equations of (5.17) correspond to smooth curves, say c_1, c_2 and c_3, c_4, tangent in $(0,0)$ to each one of the lines l_1, l_2 and l_3, l_4. Therefore, in a sufficiently small neighborhood of the origin, the system (5.17) admits only the trivial solution $(r_1, r_2) = (0,0)$.

(ii) We consider now the case where $\det(A) < 0$ and $\det(B) = 0$. Let ϕ be such that $\sin\phi \neq 0$ and $\cos^2\phi = \frac{1}{4}$. If $\phi = \frac{2}{3}\pi$ the system (5.15) is equivalent to

$$\begin{cases} (r_1 - r_2)\,(r_1 + r_2) \;=\; 0 \\ (r_1 - r_2)^2 \;=\; 0\;. \end{cases}$$

We obtain the solutions such that $r_1 = r_2$ (and $\phi = \frac{2\pi}{3}$). These solutions correspond to the periodic solutions with symmetry $\widetilde{\mathbf{Z}}_3$ of (5.3) whose existence is guaranteed by the Equivariant Hopf Theorem (when f is truncated to the third order). Consider now the system

$$\begin{cases} (r_1 - r_2)\,(r_1 + r_2) + \dfrac{4}{3}\,\mathrm{Re}\Big(P_2\Big(r_1, r_2, \dfrac{2}{3}\pi, \lambda\Big)\Big) \;=\; 0 \\ (r_1 - r_2)^2 - \dfrac{4\sqrt{3}}{3}\,\mathrm{Im}\Big(P_2\Big(r_1, r_2, \dfrac{2}{3}\pi, \lambda\Big)\Big) \;=\; 0\;. \end{cases} \tag{5.18}$$

As the Equivariant Hopf Theorem guarantees that if $\phi = \frac{2}{3}\pi$ the system (5.18) still admits the solution $r_1 = r_2$ then there are smooth functions $\widetilde{P}_i(r_1, r_2, \lambda)$ for $i = 1, 2$ (whose Taylor expansion around $(r_1, r_2) = (0,0)$ has terms in r_1, r_2 of degree greater or equal to three) such that the system (5.18) is equivalent to

$$\begin{cases} (r_1 - r_2)\,\big(r_1 + r_2 + \widetilde{P}_1(r_1, r_2, \lambda)\big) \;=\; 0 \\ (r_1 - r_2)\,\big(r_1 - r_2 + \widetilde{P}_2(r_1, r_2, \lambda)\big) \;=\; 0\;, \end{cases} \tag{5.19}$$

and so for (r_1, r_2) sufficiently close to $(0,0)$ this system admits only the solutions with $r_2 = r_1$. These correspond to the branch of periodic solutions of (5.3) with symmetry $\widetilde{\mathbf{Z}}_3$ guaranteed by the Equivariant Hopf Theorem. When $\phi = \frac{4}{3}\pi$ the situation is similar.

For the cases $\phi = \frac{\pi}{3}$ and $\phi = \frac{5}{3}\pi$ we observe the following. We have that

$$\begin{aligned} (z_1, z_2) \;&=\; \big(r_1 e^{i\phi_1}, r_2 e^{i(\phi_1 + \frac{2\pi}{3})}\big) \;=\; \big(r_1 e^{i\phi_1}, -r_2 e^{i(\phi_1 + \frac{5\pi}{3})}\big)\;, \\ (z_1, z_2) \;&=\; \big(r_1 e^{i\phi_1}, r_2 e^{i(\phi_1 + \frac{4\pi}{3})}\big) \;=\; \big(r_1 e^{i\phi_1}, -r_2 e^{i(\phi_1 + \frac{\pi}{3})}\big)\;. \end{aligned}$$

Therefore the solutions (r_1, r_2) of the system (5.14) with $\phi = \frac{2\pi}{3}$ correspond to the solutions $(r_1, -r_2)$ of the system (5.14) with $\phi = \frac{5\pi}{3}$. Similarly, the solutions (r_1, r_2) of (5.14) with $\phi = \frac{4\pi}{3}$ correspond to the solutions $(r_1, -r_2)$ of the system (5.14) with $\phi = \frac{\pi}{3}$. Therefore from the cases $\phi = \frac{\pi}{3}$ and $\phi = \frac{5\pi}{3}$ we do not obtain new solutions (besides the solutions with symmetry conjugate to $\widetilde{\mathbf{Z}}_3$).

(**iii**) We study now the cases where $\det A < 0$ and $\det B > 0$. That is, we consider values of ϕ such that $\sin\phi \neq 0$ and $\cos^2\phi > \frac{1}{4}$. Again we consider the system (5.17). The point $(0,0)$ is a nondegenerate critical point of the function defined by $h(r_1, r_2, \phi) = \cos\phi \; r_1^2 + r_1 r_2 + \cos\phi \; r_2^2 + \frac{\operatorname{Im}(P_2(r_1,r_2,\phi))}{\sin\phi}$. In these conditions, Morse Lemma guarantees that the solutions of the second equation of the system (5.17) in a sufficiently small neighborhood of $(0,0)$ are in one to one correspondence with the solutions of the equation $\cos\phi\, r_1^2 + r_1 r_2 + \cos\phi\, r_2^2 = 0$. As $\det B > 0$ we conclude that (5.17) in a sufficiently small neighborhood of the origin admits only the solution $(r_1, r_2) = (0,0)$.

(**iv**) We consider now the cases where $\det(A) = \det(B) = 0$. That is, $\phi = 0$ or $\phi = \pi$. Let f be as in (5.5) and $g = f - (1+\tau)iz$. By Lemma 5.1, if we take $z_3 = -z_1 - z_2$, $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ and $\phi = \phi_2 - \phi_1$ we obtain a function $\widetilde{g} = (\widetilde{g_1}, \widetilde{g_2}, \widetilde{g_3})$ such that

$$
\begin{aligned}
\widetilde{g_1}(r_1, r_2, \phi, \phi_1, \lambda, \tau) &= (\nu + A) r_1 e^{i\phi_1} + B\mathbf{X}_{2,1} + C\mathbf{X}_{3,1} + D\mathbf{X}_{4,1} \;, \\
\widetilde{g_2}(r_1, r_2, \phi, \phi_2, \lambda, \tau) &= (\nu + A) r_2 e^{i\phi_2} + B\mathbf{X}_{2,2} + C\mathbf{X}_{3,2} + D\mathbf{X}_{4,2}
\end{aligned}
\tag{5.20}
$$

where $\nu = \mu(\lambda) - (1+\tau)i$ and A, B, C, D are written in the new coordinates. Taking $\phi = 0$ in (5.6) and (5.7) we obtain

$$
\begin{aligned}
\left(\frac{\mathbf{X}_{2,1}}{r_1 e^{i\phi_1}}\right)_{\phi=0} &= \left(\frac{\mathbf{X}_{2,2}}{r_2 e^{i\phi_2}}\right)_{\phi=0} = 3\,(r_1^2 + r_2^2 + r_1 r_2) \;, \\
\left(\frac{\mathbf{X}_{3,1}}{r_1 e^{i\phi_1}}\right)_{\phi=0} &= h_3(r_1, r_2, 0) \;=\; 3\, r_1^4 + 5\, r_2^4 + 5\, r_1^3 r_2 + 10\, r_1 r_2^3 + 10\, r_1^2 r_2^2 \;, \\
\left(\frac{\mathbf{X}_{3,2}}{r_2 e^{i\phi_2}}\right)_{\phi=0} &= h_3(r_2, r_1, 0) \;, \\
\left(\frac{\mathbf{X}_{4,1}}{r_1 e^{i\phi_1}}\right)_{\phi=0} &= h_4(r_1, r_2, 0) \;=\; 21\, r_1^4 r_2^2 + 35\, r_1^3 r_2^3 - r_1^6 - 21\, r_1 r_2^5 - 7\, r_2^6 \;, \\
\left(\frac{\mathbf{X}_{4,2}}{r_2 e^{i\phi_2}}\right)_{\phi=0} &= h_4(r_2, r_1, 0)
\end{aligned}
$$

and so

$$\begin{aligned}\left(\frac{\widetilde{g_1}(r_1,r_2,\phi,\phi_1,\lambda,\tau)}{r_1e^{i\phi_1}}\right)_{\phi=0} &= \nu + A_{\phi=0} + 3B_{\phi=0}\big(r_1^2+r_2^2+r_1r_2\big)\\ &\quad + C_{\phi=0}\big(3r_1^4+5r_2^4+5r_1^3r_2+10r_1r_2^3+10r_1^2r_2^2\big)\\ &\quad + D_{\phi=0}\big(21r_1^4r_2^2+35r_1^3r_2^3-r_1^6-21r_1r_2^5-7r_2^6\big)\ ,\\ \left(\frac{\widetilde{g_2}(r_1,r_2,\phi,\phi_2,\lambda,\tau)}{r_2e^{i\phi_2}}\right)_{\phi=0} &= \nu + A_{\phi=0} + 3B_{\phi=0}\big(r_1^2+r_2^2+r_1r_2\big)\\ &\quad + C_{\phi=0}\big(5r_1^4+3r_2^4+10r_1^2r_2^2+10r_1^3r_2+5r_1r_2^3\big)\\ &\quad + D_{\phi=0}\big(21r_1^2r_2^4+35r_1^3r_2^3-r_2^6-21r_1^5r_2-7r_1^6\big)\ .\end{aligned}$$

Then the equation

$$\left(\frac{\widetilde{g_1}(r_1,r_2,\phi,\phi_1,\lambda,\tau)}{r_1e^{i\phi_1}}\right)_{\phi=0} - \left(\frac{\widetilde{g_2}(r_1,r_2,\phi,\phi_2,\lambda,\tau)}{r_2e^{i\phi_2}}\right)_{\phi=0} = 0 \tag{5.21}$$

can be written as

$$(r_2-r_1)\,(r_1+2r_2)\,(2r_1+r_2)\,(r_1+r_2)\,\big(C+3(r_1^2+r_2^2+r_1r_2)D\big)_{\phi=0} = 0 \tag{5.22}$$

where C, D are smooth $\mathbf{S}_3\times\mathbf{S}^1$-invariant functions. Assuming the generic hypothesis

$$C(0)\neq 0$$

from (5.22) we obtain only branches of solutions of (5.4) corresponding to the branches of periodic solutions of (5.3) with symmetry (conjugate to) $\widetilde{\mathbf{Z}}_2$ and $\mathbf{S}_1\times\mathbf{S}_2$. We recall that

$$\mathrm{Fix}(\widetilde{\mathbf{Z}}_2) = \Big\{(w,-w,0):\ w\in\mathbb{C}\Big\}\ .$$

So, periodic solutions of (5.3) with symmetry $\widetilde{\mathbf{Z}}_2$ correspond to zeros of (5.4) where

$$(r_1=r_2 \ \text{ and }\ \phi=\pi) \quad \text{or} \quad (r_1=-r_2 \ \text{ and }\ \phi=0)\ .$$

From there the factor r_1+r_2 in the equation (5.22). In the case of $\mathbf{S}_1\times\mathbf{S}_2$, we have that

$$\mathrm{Fix}(\mathbf{S}_1\times\mathbf{S}_2) = \Big\{(2w,-w,-w):\ w\in\mathbb{C}\Big\}\ .$$

Periodic solutions of (5.3) with symmetry $\mathbf{S}_1\times\mathbf{S}_2$ or conjugate to $\mathbf{S}_1\times\mathbf{S}_2$ correspond to zeros of (5.4) where

$$\begin{aligned}&(r_1=2r_2 \ \text{ and }\ \phi=\pi) \quad \text{or} \quad (r_1=-2r_2 \ \text{ and }\ \phi=0)\ ,\\ &(r_2=2r_1 \ \text{ and }\ \phi=\pi) \quad \text{or} \quad (r_2=-2r_1 \ \text{ and }\ \phi=0)\ ,\\ &(r_1=-r_2 \ \text{ and }\ \phi=\pi) \quad \text{or} \quad (r_1=r_2 \ \text{ and }\ \phi=0)\ .\end{aligned}$$

So, we have the factors r_1+2r_2, r_2+2r_1 and r_1-r_2 in the equation (5.22). The case $\phi=\pi$ is similar.

(**v**) Finally, we study the cases where $z_1=0$ and $z_2\neq 0$. Let $\widetilde{N}$, $\widetilde{P}$, $\widetilde{S}$, $\widetilde{T}$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$ be as in Corollary 4.6. In that case $\widetilde{N}=2|z_2|^2$, $\widetilde{P}=|z_2|^4$, $\widetilde{S}=-2|z_2|^6$, $\widetilde{T}=0$ and

$$\mathbf{X}_1=\begin{bmatrix}0\\ z_2\\ -z_2\end{bmatrix},\quad \mathbf{X}_2=\begin{bmatrix}0\\ z_2|z_2|^2\\ -z_2|z_2|^2\end{bmatrix},\quad \mathbf{X}_3=\begin{bmatrix}0\\ -z_2|z_2|^4\\ z_2|z_2|^4\end{bmatrix},\quad \mathbf{X}_4=\begin{bmatrix}0\\ -z_2|z_2|^6\\ z_2|z_2|^6\end{bmatrix}.$$

Replacing in the system (5.4) where $g=f-(1+\tau)iz$ and f appears in (5.5) we obtain

$$\begin{aligned} g_1(z,\lambda,\tau) &= 0\,,\\ g_2(z,\lambda,\tau) &= z_2\big(\nu+h(z_2,\lambda)\big) \end{aligned}$$

where h is smooth and $\nu=\mu(\lambda)-(1+\tau)i$. In this case we obtain zeros corresponding to a branch of periodic solutions with symmetry conjugate to $\widetilde{\mathbf{Z}}_2$.

If $z_2=0$ and $z_1\neq 0$ the situation is similar to the previous one.

From the study (i)–(v) we conclude that the system (5.3) generically only admits branches of periodic solutions guaranteed by the Equivariant Hopf Theorem. ∎

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