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A CLASS OF KINETIC MODELS FOR CHEMOTAXIS WITH THRESHOLD TO PREVENT OVERCROWDING

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Abstract: We introduce three new examples of kinetic models for chemotaxis, where a kinetic equation for the phase-space density is coupled to a parabolic or elliptic equation for the chemo-attractant, in two or three dimensions. We prove that these models have global-in-time existence and rigorously converge, in the drift-diffusion limit to the Keller–Segel model. Furthermore, the cell density is uniformly-in-time bounded. This implies, in particular, that the limit model also has global existence of solutions.

1 – Introduction

The slime mold amoebae, *Dictyostelium Discoideum*, is an important biological example both experimentally and theoretically. From the modeling point of view, its study starts with the work of Patlak [32] and gained maturity with the Keller–Segel model [23, 24].

Keller and Segel modeled the initiation of the aggregation of the *D. Discoideum*, using a system of two parabolic partial differential equations, one for the cell density $\rho \geq 0$ and the second for the density of the cyclic adenosine monophosphate (cAMP) $S \geq 0$, the chemical substance that mediate aggregation. The cell movement induced by chemical substances is called chemotaxis, and, in this particular case, cells move toward higher concentrations of cAMP, produced by the cell themselves.

A general overview of chemotaxis and a large bibliography on the Keller–Segel model can be found in [18, 19].

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The blow up phenomena, i.e., the arbitrary increase of L^∞-norms of solutions ρ or S, is an important mathematical question that has some partial answers in [4, 12, 13, 14, 15, 22, 27, 28] and the surveys [18, 19] Recently, Dolbeault and Perthame [10], in the two dimensional case and with constant coefficients, were able to give an optimal critical mass for the blow-up/global existence problem. For higher dimensions, see [8].

The derivation of the Keller–Segel model in [23, 24] was originally made from the phenomenological point of view. In [34] this model was derived as limit dynamics of systems of moderately interacting stochastic many particle process. In [11] it was derived from a semi-linear differential hyperbolic system.

The Keller–Segel model was also derived from kinetic equations, for the first time in [3] for a prescribed chemical S (see also [1, 2, 29]). In [17, 30] it is formally shown that these models converge, in the macroscopic limit, to the Keller–Segel model. Rigorous derivations appeared in [7], where local-in-time convergence was proved for turning kernels depending only on S and ∇S and for a elliptic equation for S (i.e., the limit of high diffusion, when $D_0 \gg D$), in the 3-dimensional case. Furthermore, global-in-time existence was proved for turning kernels bounded by certain functionals of S. In [21] these results were generalized to the 2-dimensional case and the limit of high diffusion was proved not necessary (i.e., the equation for S was of parabolic or elliptic type). Global-in-time existence of solutions was proved under the same bound on the turning kernel. Finally, in [20], the previous results, concerning global-in-time existence, were extended for turning kernels with a more general dependence on S. It is important to stress that even for kinetic models with global existence the limit Keller–Segel model can present finite-time-blow-up. See [7].

Keller–Segel model with prevention of overcrowding (as in Reference [16]) is given by

$$\text{(1)} \qquad \begin{cases} \partial_t \rho \;=\; \nabla \cdot \big(D(S,\rho)\,\nabla\rho - \mathcal{V}(S,\rho)\,\nabla S \big) \;, \\ \partial_t S \;=\; D_0\,\Delta S + \varphi(S,\rho) \;, \end{cases}$$

where we consider that $D(S,\rho) = D_0$ is a constant, $\varphi(S,\rho) = g_1(S,\rho)\rho - g_2(S,\rho)S$, with $g_1 \geq 0$ and $g_2 \geq \delta_0 > 0$, $\mathcal{V}(S,\rho) = \chi(S)\,\beta(\rho)\rho$, where $\chi > 0$ and there is a $\bar\rho > 0$ such that $\beta(\rho) > 0$ for $\rho \in [0,\bar\rho)$ and $\beta(\rho) = 0$, $\rho \geq \bar\rho$. Initial conditions are supposed to be non-negative. Hillen and Painter were able to give sufficient conditions for global existence of solutions for this kind of model (see [16]), that we are able to obtain here also in a slightly different and more general framework.

Although mathematically interesting, the blow up phenomena representing cell overcrowding is considered by some authors as unrealistic from a biological context. According to [31] this is so because the finite size of individual cells and the behavior of cells at higher densities is ignored in the Keller–Segel model.

This work is concerned with kinetic models for chemotaxis with prevention of overcrowding and is structured as follows: in Section 2, we introduce kinetic models for chemotaxis and compute formally its macroscopic (drift-diffusion) limits. In Section 3 we show three new different kinetic models with a threshold that implies formal convergence to the Keller–Segel model (1), by extending examples in [7, 30]. Furthermore, the macroscopic density is uniformly-in-time bounded. Finally, in Section 4 we prove that these three examples rigorously converges to the Keller–Segel model, and conclude global existence of solutions to the limit model (1).

Although it may be mathematically not unexpected, we believe these models with threshold we consider in this work provide a first applied example where a nonlinear factor in kinetic models may yield in the limit to existence of both convection/diffusion and pure diffusion (when the threshold is attained) regimes. The numerical simulation of pattern formation in the limit model of [16] raises interesting open questions at the kinetic level.

2 – Models and formal asymptotic expansions

We consider a kinetic model for chemotaxis as presented in [7], i.e., we consider the cell density $f_\varepsilon(x,v,t) \geq 0$ and the chemo-attractant density $S_\varepsilon(x,t) \geq 0$ in a point $(x,v,t) \in \mathbb{R}^n \times V \times \mathbb{R}_+$ and $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, respectively, where V is the compact and rotationally invariant set of all possible velocities, $V \subset B_{v_{\max}} \subset \mathbb{R}^n$, where B_r is the ball with center in 0 and radius r. We also consider $T_\varepsilon[S,\rho](x,v,v',t)$, the turning rate from velocity v' to v in a space-time point (x,t) where $(x,v,v',t) \in \mathbb{R}^n \times V \times V \times \mathbb{R}_+$ in the presence of cells and chemo-attractants with densities ρ and S, respectively. Above, $\varepsilon > 0$ is the ratio between the microscopic variables and macroscopic variables and the limit $\varepsilon \to 0$ corresponds to the drift-diffusion limit of the model.

We now obtain, formally, the system satisfied by the macroscopic densities $\rho_0 = \lim_{\varepsilon\to 0} \int_V f_\varepsilon \, dv$ and $S_0 = \lim_{\varepsilon \to 0} S_\varepsilon$, from the one obeyed by the microscopic densities f_ε and S_ε.

We introduce the following notation

$$
\begin{aligned}
f_\varepsilon &= f_\varepsilon(x,v,t)\ ,\\
f'_\varepsilon &= f_\varepsilon(x,v',t)\ ,\\
T_\varepsilon[S,\rho] &= T_\varepsilon[S,\rho](x,v,v',t)\ ,\\
T^*_\varepsilon[S,\rho] &= T_\varepsilon[S,\rho](x,v',v,t)\ .
\end{aligned}
$$

We consider the kinetic model in $(\mathbb{R}^n\times V\times\mathbb{R}_+)$, with $n=2$ or 3.

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v\cdot\nabla f_\varepsilon = -\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon[S_\varepsilon,\rho_\varepsilon](f_\varepsilon)\ , \tag{2}$$

$$\mathcal{T}_\varepsilon[S,\rho](f) := \int_V \left(T^*_\varepsilon[S,\rho]f - T_\varepsilon[S,\rho]f'\right) dv'\ , \tag{3}$$

$$\rho_\varepsilon := \int_V f_\varepsilon\, dv\ , \tag{4}$$

$$\delta\,\partial_t S_\varepsilon = \Delta S_\varepsilon + \rho_\varepsilon - \delta\,\gamma\, S_\varepsilon\ , \tag{5}$$

with initial conditions given by

$$f_\varepsilon(x,v,0) = f^{\mathrm{I}}(x,v) \ \geq\ 0\ , \tag{6}$$

$$S_\varepsilon(x,0) = S^{\mathrm{I}}(x) \ \geq\ 0\ . \tag{7}$$

See [7] for the derivation of the system (2–7). Equation (4) defines the macroscopic (physical space) density ρ_ε as a function of the microscopic (phase space) density f_ε, when integrated over all possible velocities. We assume $\delta,\gamma\geq 0$ and that the ε-independent initial conditions are in suitable spaces. For simplicity, we also assume $S^{\mathrm{I}}\equiv 0$ in most part of this work and in Remark 6 we extend our results to the more general case given by Equation (7). Of course, if $\delta=0$ in (5), the condition (7) is unnecessary.

Remark 1. If the initial condition f^{I} is compactly supported, then f_ε is compactly supported for every t. More precisely, if $f^{\mathrm{I}}\subset B_r$, then

$$\operatorname{supp} f_\varepsilon \subset B_{r+v_{\max}t/\varepsilon}\ .\ \square$$

The formal asymptotic is obtained in the same way as in [7]. Namely, we impose the expansion

$$\begin{aligned} f_\varepsilon &= f_0 + \varepsilon f_1 + \cdots \ , \\ \rho_k &:= \int_v f_k \, dv \ , \\ S_\varepsilon &= S_0 + \varepsilon S_1 + \cdots \ , \\ T_\varepsilon &= T_0 + \varepsilon T_1 + \cdots \ . \end{aligned}$$

We assume the kernel $T_0[S,\rho](x,v,v',t) = \lambda[S,\rho](x,t)F(v)$, such that

(A1) $F = F(|v|) > 0$,

(A2) $T_0[S,\rho]F' = T_0^*[S,\rho]F$,

(A3) $\int_V F\, dv = 1$,

(A4) $\int_V vF\, dv = 0$.

(A5) The turning rate $T_0[S,\rho]$ is bounded, and there exists a constant $\lambda_{\min} > 0$ such that $T_0[S,\rho]/F \geq \lambda_{\min}$, $\forall\, (v,v') \in V \times V$, $x \in \mathbb{R}^n$, $t \geq 0$.

From Assumption (A2) we see that $\mathcal{T}_0[S,\rho](F) = 0$, i.e., F is the non-perturbed equilibrium distribution. This assumption is called “detailed balance”. Assumption (A3) is a unimportant normalization while (A4) means that the equilibrium distribution does not cause drift. The others one are technical assumptions.

We put the expansions in the System (2–5) and match terms of the same order in ε. Arguing as in [7], to order 0, we find that

$$\mathcal{T}_0[S_0,\rho_0](f_0) = 0 \ ,$$

and then $f_0 = \rho_0 F$. We also find that

$$v \cdot \nabla f_0 = -\mathcal{T}_0[S_0,\rho_0](f_1) - \mathcal{T}_1[S_0,\rho_0](f_0) \ .$$

This implies that

$$f_1(x,v,t) = -\kappa(x,v,t) \cdot \nabla \rho_0(x,t) - \Theta(x,v,t)\, \rho_0(x,t) + \rho_1(x,t)\, F(v) \ ,$$

where

$$\begin{aligned} \mathcal{T}_0[S_0,\rho_0](\kappa) &= vF \ , \\ \mathcal{T}_0[S_0,\rho_0](\Theta) &= \mathcal{T}_1[S_0,\rho_0](F) \ . \end{aligned}$$

We integrate Equation (2) over V and finally find that the macroscopic system is given by

$$\partial_t\rho_0 \;=\; \nabla\cdot\Big(D[S_0,\rho_0]\,\nabla\rho_0 - \Gamma[S_0,\rho_0]\,\rho_0\Big)\;, \tag{8}$$

$$\delta\,\partial_t S_0 \;=\; \Delta S_0 + \rho_0 - \delta\,\gamma\,S_0\;, \tag{9}$$

where

$$D[S_0,\rho_0] \;=\; \int_V v\otimes\kappa[S_0,\rho_0](x,v,t)\,dv\;, \tag{10}$$

$$\Gamma[S_0,\rho_0] \;=\; -\int_V v\,\Theta[S_0,\rho_0](x,v,t)\,dv\;. \tag{11}$$

For simplicity we consider $\gamma=0$, which means that we do not consider the chemical decay of the chemo-attractant, and we normalize $\delta=1$ (except in Remark 5, where $\delta=0$). Furthermore, the matrix $D[S_0,\rho_0]$ is symmetric and positive definite (see Remark 2 in [7]), and $\Gamma[S_0,\rho_0]$ is the convection term.

Assumptions (A1–A5) imply that Equations (10) and (11) can be written simply as

$$D[S_0,\rho_0] \;=\; \frac{1}{n\,\lambda[S_0,\rho_0]}\int_V |v|^2\,F(|v|)\,dv\,I\;, \tag{12}$$

$$\Gamma[S_0,\rho_0] \;=\; -\frac{1}{\lambda[S_0,\rho_0]}\int_V v\,\mathcal{T}_1[S_0,\rho_0](F)\,dv\;, \tag{13}$$

where I is the $n\times n$ identity matrix.

Let us introduce three different models and obtain, formally, their drift-diffusion limit:

(M1) In the first model we have $T_\varepsilon = T_0+\varepsilon T_1$, where $T_0[S,\rho]=\lambda[S,\rho]F$ is a non-oriented turning kernel and the chemotactical perturbation is given by

$$T_1[S,\rho](x,v,v',t) \;=\; F(v)\Big(a\big(S(x,t),\rho(x,t)\big)v - b\big(S(x,t),\rho(x,t)\big)v'\Big)\cdot\nabla S(x,t)\;,$$

where a and b are real continuous functions defined in $[0,\infty)\times[0,\infty)$, such that $0<a(S,\rho)<\bar a(S)$, $0<b(S,\rho)<\bar b(S)$ for $\rho\in[0,\bar\rho)$ and $a(S,\rho)=b(S,\rho)=0$, for $\rho\geq\bar\rho$. We immediately see that if v points in the direction of ∇S (or, v' points in the opposite direction) the turning rates increases. Then, intuitively, the overall effect is to make the cell walk upward the gradient. Similar kinds of models appear in [7, 17].

In this case we have

$$\Gamma[S,\rho] \;=\; \frac{1}{n\,\lambda[S,\rho]}\,\big[a(S,\rho)+b(S,\rho)\big]\int_V v^2\,F(|v|)\,dv\,\nabla S\;;$$

(M2) Let us define, following [17], the "non-local gradient":

$$\overset{\circ}{\underset{R}{\nabla}} S(x,t;R) \;=\; \frac{n}{R w_{n-1}} \int_{S^{n-1}} \nu\, S(x+R\nu,t)\, d\nu \ ,$$

where w_{n-1} is the area of the $n-1$-dimensional sphere. The turning kernel is defined by

$$T_\varepsilon[S,\rho] \;=\; \lambda[S,\rho]\, F(v) + F(v)\Big[\overset{\circ}{a}(S,\rho)\, v \;-\overset{\circ}{b}(S,\rho)\, v'\Big] \cdot \overset{\circ}{\underset{R}{\nabla}} S(x,t;\varepsilon R) \ ,$$

where $\overset{\circ}{a}$ and $\overset{\circ}{b}$ are real continuous functions defined in $[0,\infty)\times[0,\infty)$, such that $0<\overset{\circ}{a}(S,\rho)<\bar{\overset{\circ}{a}}(S)$, $0<\overset{\circ}{b}(S,\rho)<\bar{\overset{\circ}{b}}(S)$ for $\rho\in[0,\bar\rho)$ and $\overset{\circ}{a}(S,\rho)=\overset{\circ}{b}(S,\rho)=0$, for $\rho\ge\bar\rho$. From the fact that, at least formally,

$$\lim_{R\to 0} \overset{\circ}{\underset{R}{\nabla}} S(x,t;\varepsilon R) \;=\; \varepsilon\, \nabla S(x,t) \ ,$$

we see that the "non-local gradient" is an approximation of the gradient ∇S (for small R) and thus the interpretation is similar to the case (M1). Formally, T_0 and T_1 are the same as in model (M1) (with a and b replaced by $\overset{\circ}{a}$ and $\overset{\circ}{b}$), and so is $\Gamma[S,\rho]$; and

(M3) We define a third kernel given by

$$\begin{aligned} T_\varepsilon[S,\rho](x,v,v',t) \;=\;& c_+\, \psi\big(S(x,t), S(x+\varepsilon\mu_+(\rho)v,t)\big) F(v) \\ &+ c_-\, \psi\big(S(x,t), S(x-\varepsilon\mu_-(\rho)v',t)\big) F(v) \ , \end{aligned}$$

where $\psi : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable, non-decreasing (in the second variable) function. We interpret $\varepsilon\mu_\pm(\rho)v_{\max}$ as the effective radius of the cell, with the sign $+$ indicating its ability to access future directions and $-$ its memory of past directions. These functions $\mu_\pm$ are real continuous functions defined in $[0,\infty)$ such that $0<\mu_\pm(\rho)<\mu_{\max}$ for $\rho\in[0,\bar\rho)$ and $\mu_\pm(\rho)=0$, for $\rho\ge\bar\rho$, i.e., if concentration is higher that a certain threshold the cell becomes "blind". We write the expansion $T_\varepsilon = T_0+\varepsilon T_1 + O(\varepsilon^2)$, where

$$\begin{aligned} T_0[S,\rho] \;&=\; (c_+ + c_-)\, \psi(S,S)\, F(v) \ , \\ T_1[S,\rho] \;&=\; \partial_2\psi(S,S)\, F(v)\, \big(c_+\mu_+(\rho)v - c_-\mu_-(\rho)v'\big)\cdot\nabla S \ , \end{aligned}$$

where $\partial_2\psi$ means differentiation with respect to the second variable. Finally,

$$\Gamma[S,\rho] \;=\; \frac{\partial_2\psi(S,S)}{n(c_+ + c_-)\, \psi(S,S)} \big(c_+\mu_+(\rho) + c_-\mu_-(\rho)\big) \int_V v^2\, F(|v|)\, dv\, \nabla S \ .$$

So the three models converge formally to Keller–Segel equation (1) with diffusion coefficient given by Equation (12) and chemotactical sensitivity $\chi(S)$ given by

$$\chi(S)\,\beta(\rho)\,\nabla S \;=\; \Gamma[S,\rho]\ .$$

For given functions D, χ, β it is necessary to find new functions λ, a and b, or $\mathring{a}$ and $\mathring{b}$ or ψ, μ_+ and μ_- which obey the above equation and Equation (12).

Remark 2. In the Keller–Segel model (1), we have that $D[S_0,\rho_0] = D_0$ is a constant. Then, we find that λ is a constant (see Equation (12)), and then $T_0[S,\rho] = \lambda F$. In this work, we will consider however the more general dependence $T_0[S,\rho](x,v,v',t) = \lambda(t)F(v)$, where $\lambda(t) \in [\lambda_{\min},\lambda_{\max}]$, $\lambda_{\min},\lambda_{\max} \in (0,\infty)$, $\forall t \in \mathbb{R}_+$, is a continuous function. □

Remark 3. The value $\bar\rho$ is called *saturation value*. For space-time points (x,t) such that $\rho(x,t) \geq \bar\rho$ the movement is purely random, without any chemotactical effect. We will prove in the following sections that this (with some other assumptions) prevents blow-up. In fact a stronger conclusion holds, that is, the cell concentration in each point never increases beyond that value, or beyond the initial condition.

These three models, however, are different in its chemotactical part, i.e., wherever $\rho(x,t) < \bar\rho$. In the first model cells are directly able to measure gradients of the concentration. It is not clear that they really can do so, see [30]. In the second case cells measure only concentration on its surface (for all practical purposes, we consider cells as spheres centered in x and with radius εR) and integrate over all directions. Finally, in (M3), all they need is to access the concentration value in some effective radius, but no "integration ability" is required. □

3 – Global existence of kinetic solutions

In this section we introduce further restrictions in order to prove the global existence of solutions for the kinetic models. These assumptions will not be necessary to prove convergence to the drift-diffusion limit in Section 4.

For kinetic models, local-in-time existence and uniqueness of solutions are guaranteed, see [5] or [33]. The positivity (≥ 0) of solutions is a simple consequence of the positivity of the turning rate $T_\varepsilon[S,\rho]$ and of the initial conditions.

We prove global existence in the kinetic level for the models (M1), (M2) and (M3) subject to Remark 2 and with some other assumptions to be soon introduced. For simplicity, we omit $\varepsilon > 0$ wherever its omission causes no confusion. In particular, we write $f := f_\varepsilon$, $\rho := \rho_\varepsilon$ and $S := S_\varepsilon$.

We introduce the following assumptions in models (M1), (M2) and (M3) respectively:

(B1) We assume that $b \equiv 0$, that $a(S,\rho)/(\bar\rho - \rho)$ is a non-increasing function of ρ and

$$\sup_{S\ge 0,\rho\ge 0} \frac{a(S,\rho)}{\bar\rho-\rho} \le \frac{a_{\max}}{\bar\rho} ,$$

where

$$a_{\max} := \sup_{S\ge 0} a(S,0) .$$

(B2) We assume that $\mathring{b} \equiv 0$, that $\mathring{a}(S,\rho)/(\bar\rho - \rho)$ is a non-increasing function of ρ and

$$\sup_{S\ge 0,\rho\ge 0} \frac{\mathring{a}(S,\rho)}{\bar\rho-\rho} \le \frac{\mathring{a}_{\max}}{\bar\rho} ,$$

where

$$\underset{\max}{\mathring{a}} := \sup_{S\ge 0} \mathring{a}(S,0) .$$

(B3) We impose $c_- = 0$, $c_+ = 1$ and $\mu := \mu_+$. We also impose

$$\sup_{\rho\ge 0} \frac{\mu(\rho)}{\bar\rho-\rho} \le \frac{\mu_{\max}}{\bar\rho} ,$$

with

$$\mu_{\max} := \sup_{\rho\ge 0} \mu(\rho) .$$

From Remark 2, we have that $\psi(S,S) = \lambda \ge \lambda_{\min} > 0$ and we impose that

$$\sup_{S,S'\ge 0} \frac{\partial\psi(S,S')}{\partial S'} = \psi_1 \in (0,\infty) .$$

We define

$$\Lambda_0 := \frac{1}{n}\left[\frac{2^{n-2}(n-1)}{\pi^{1/2}\,\Gamma(n)\max\left\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)},\bar\rho\right\}}\right]^{(n-1)/n}\left[\frac{2^{n+1}\,\pi^{n/2}}{\sqrt{2}\,e^{-1/2}\,\|\rho^{\mathrm{I}}\|_{L^1(\mathbb{R}^n)}}\right]^{1/n},$$

and our main result reads:

Theorem 1. *Let $i=1,2,3$. Assume $\varepsilon<\varepsilon_i$, where*

$$\varepsilon_1 := \frac{\lambda_{\min}}{a_{\max}\,v_{\max}}\,\Lambda_0\ ,$$

$$\varepsilon_2 := \frac{\lambda_{\min}}{n\,\overset{\circ}{a}_{\max}\,v_{\max}}\,\Lambda_0\ ,$$

$$\varepsilon_3 := \frac{\lambda_{\min}}{2\,\psi_1\,\mu_{\max}\,v_{\max}}\,\Lambda_0\ .$$

Let us consider the model (Mi), subject to Assumptions (A1–A5), (Bi) and Remark 2 with initial conditions given by $f^{\mathrm{I}}(x,v)=\rho^{\mathrm{I}}(x)F(v)$, $\rho^{\mathrm{I}}\in L^1_+\cap L^\infty(\mathbb{R}^n)$, $S^{\mathrm{I}}=0$. Then the solution (f,S) of the nonlinear system (2–7) with $\delta=1$ and $\gamma=0$ exists globally: $f\in L^\infty(0,\infty;L^1_+\cap L^\infty(\mathbb{R}^n\times V))$, $S\in L^\infty(0,t;L^p(\mathbb{R}^n))$, $p\in(n/2,\infty]$, $\forall t\in(0,\infty)$. Furthermore,

$$\|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \left\|\frac{f(\cdot,\cdot,t)}{F}\right\|_{L^\infty(\mathbb{R}^n\times V)} \le \max\left\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)},\bar\rho\right\}\ ,\qquad \forall\, t\in\mathbb{R}_+\ .$$

The proof will involve several lemmas. We prove each lemma for $i=1$ and then extend it for $i=2$ and 3. Let us first explain the general idea in the proof. We first start with Lemma 1 where we show that $\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ is bounded by both $\|\rho(\cdot,s)\|_{L^1(\mathbb{R}^n)}$ and $\|\rho(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}$, $s\in[0,t]$. This is identically valid regardless of the case i. Then, we show that if and while the turning kernel is positive, then $\|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ is uniformly-in-time bounded (Lemma 2). Putting together this two lemmas, we prove that $\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ is uniformly-in-time bounded (Lemma 3). This allows the extension of the turning kernel (Mi) to all times $t\in\mathbb{R}_+$ (Lemma 4), and applying Lemmas 2 and 3 once more we finish the proof.

Lemma 1. *Let S be the solution of (5) with $\delta=1$ and $\gamma=0$, $q\in(n,\infty]$, $S^{\mathrm{I}}=0$, and let $t_0>0$ be fixed. Then, there are constants $c_0=c_0(q,n)$ and $c_1=c_1(n)$ such that*

$$\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c_0\int_0^t (t-s)^{-\frac{n}{2q}-\frac{1}{2}}\,\|\rho(\cdot,s)\|_{L^q(\mathbb{R}^n)}\,ds\ , \tag{14}$$

$$(15)\qquad \|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;\le\; \frac{2\,q}{q-n}\; c_0\, t^{(q-n)/(2q)} \sup_{s\in[0,t]} \|\rho(\cdot,s)\|_{L^q(\mathbb{R}^n)} \;,$$

for $t>0$ and

$$(16)\qquad \|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;\le$$
$$\le\; \frac{2\,q}{q-n}\, c_0 \sup_{s\in[0,t_0]} \|\rho(\cdot,t-s)\|_{L^q(\mathbb{R}^n)}\, t_0^{(q-n)/(2q)} + \frac{c_1\|\rho^{\mathrm{I}}\|_{L^1(\mathbb{R}^n)}}{t_0^{(n-1)/2}} \;,$$

for $t>t_0$. (In the above, if $q=\infty$, then $(q-n)/q=1$.)

Proof: We write $S=\Upsilon*\rho$, where

$$\Upsilon(x,t) \;=\; \frac{1}{(4\pi t)^{n/2}}\; e^{-x^2/(4t)} \;,$$

and $*$ denotes space and time convolution. Then $\nabla S=\nabla\Upsilon*\rho$, where

$$\nabla\Upsilon(x,t) \;=\; -\frac{x\,e^{-x^2/(4t)}}{2(4\pi)^{n/2}\,t^{n/2+1}} \;.$$

We use the bound $|x|e^{-x^2/(4t)} \le \sqrt{2t}\,e^{-1/2}$ and prove that

$$\|\nabla\Upsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;\le\; \frac{1}{2(4\pi)^{n/2}\,t^{(n+2)/2}} \sup_{x\in\mathbb{R}^n}\Big\{|x|\,e^{-x^2/(4t)}\Big\} \;\le\; \frac{\sqrt{2}\,e^{-1/2}}{2(4\pi)^{n/2}}\,\frac{1}{t^{(n+1)/2}} \;.$$

We also show that

$$\begin{aligned}
\|\nabla\Upsilon(\cdot,t)\|^p_{L^p(\mathbb{R}^n)} \;&=\; \int_{\mathbb{R}^n} \frac{x^p\,e^{-px^2/(4t)}}{2^p(4\pi)^{np/2}\,t^{p(n+2)/2}}\;dx\\
&=\; \frac{\omega_{n-1}}{2^p(4\pi)^{np/2}\,t^{p(n+2)/2}} \int_0^\infty x^{p+n-1}\,e^{-px^2/(4t)}\,dx\\
&=\; \frac{2^{n-1}\,\omega_{n-1}}{(4\pi)^{np/2}\,p^{(p+n)/2}}\;\Gamma\Big(\frac{p+n}{2}\Big)\;t^{-(n(p-1)+p)/2} \;,
\end{aligned}$$

where $\omega_{n-1}=|S^{n-1}|=2\pi^{n/2}/\Gamma(\frac{n}{2})$. Finally, we have

$$\|\nabla\Upsilon(\cdot,t)\|_{L^p(\mathbb{R}^n)} \;=\; c_0(q,n)\,t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}} \;,$$

with

$$c_0(q,n) = \left[\frac{\Gamma\big(\frac{n}{2}\big)\;\Gamma\big(\frac{p+n}{2}\big)}{\pi^{n(p-1)/2}\,p^{(p+n)/2}}\right]^{1/p}, \qquad \frac{1}{q}+\frac{1}{p}=1 \;.$$

From the properties of the Gamma functions, we have

$$c_0(\infty, n) = \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right) = \frac{\pi^{n/2}\,\Gamma(n)}{2^{n-1}} .$$

We use Young's inequality to prove that

$$\begin{aligned}
\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\leq \int_0^t \|\nabla \Upsilon(\cdot, t-s)\|_{L^p(\mathbb{R}^n)} \|\rho(\cdot,s)\|_{L^q(\mathbb{R}^n)}\, ds \\
&\leq c_0 \sup_{s\in[0,t]} \|\rho(\cdot,s)\|_{L^q(\mathbb{R}^n)} \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2q}}\, ds \\
&= \frac{2\,q\,c_0}{q-n} \sup_{s\in[0,t]} \|\rho(\cdot,s)\|_{L^q(\mathbb{R}^n)}\, t^{\frac{q-n}{2q}} ,
\end{aligned}$$

which proves Equations (14) and (15).

Now, we fix a certain time $t_0 > 0$ and write for $t > t_0$

$$\begin{aligned}
&\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \\
&\leq \int_0^{t_0} \|\nabla \Upsilon(\cdot, s)\|_{L^p(\mathbb{R}^n)}\ \|\rho(\cdot,t-s)\|_{L^q(\mathbb{R}^n)}\, ds \\
&\quad + \int_{t_0}^t \|\nabla \Upsilon(\cdot, s)\|_{L^\infty(\mathbb{R}^n)}\ \|\rho(\cdot,t-s)\|_{L^1(\mathbb{R}^n)}\, ds \\
&\leq c_0 \sup_{s\in[0,t_0]} \|\rho(\cdot,t-s)\|_{L^q(\mathbb{R}^n)} \int_0^{t_0} s^{-\frac{n}{2q}-\frac{1}{2}}\, ds + \frac{\sqrt{2}\,e^{-1/2}}{2(4\pi)^{n/2}} \|\rho^{\mathrm{I}}\|_{L^1(\mathbb{R}^n)} \int_{t_0}^t \frac{ds}{s^{(n+1)/2}} \\
&\leq \frac{2\,q}{q-n}\, c_0 \sup_{s\in[0,t_0]} \|\rho(\cdot,t-s)\|_{L^q(\mathbb{R}^n)}\, t_0^{(q-n)/(2q)} + c_1 \|\rho^{\mathrm{I}}\|_{L^1(\mathbb{R}^n)} \left[\frac{1}{t_0^{(n-1)/2}} - \frac{1}{t^{(n-1)/2}}\right] \\
&\leq \frac{2\,q}{q-n}\, c_0 \sup_{s\in[0,t_0]} \|\rho(\cdot,t-s)\|_{L^q(\mathbb{R}^n)}\, t_0^{(q-n)/(2q)} + c_1 \|\rho^{\mathrm{I}}\|_{L^1(\mathbb{R}^n)} \frac{1}{t_0^{(n-1)/2}} ,
\end{aligned}$$

with

$$c_1(n) = \frac{\sqrt{2}\,e^{-1/2}}{2^{n+1}\,\pi^{n/2}} . \ \blacksquare$$

Remark 4. The central idea in Lemma 1 is to use the estimate

$$\sup_{s\in[0,t]} \|\nabla S(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} \leq c \left(\sup_{s\in[0,t]} \|\rho(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} + \sup_{s\in[0,t]} \|\rho(\cdot,s)\|_{L^1(\mathbb{R}^n)} \right) ,$$

for a certain constant c, which is valid in general when

$$\partial_t S - \Delta S = \rho$$

for $\rho \in L^1_+ \cap L^\infty(\mathbb{R}^n)$. This estimation, however, is unable to provide an explicit value for ε_i, $i=1,2,3$ as in Theorem 1. □

Lemma 2. *Consider a time $t_* > 0$ such that $T_\varepsilon[S,\rho] \geq 0$ for all $(x,v,v',t) \in (\mathbb{R}^n \times V \times V \times [0,t_*])$ and consider the assumptions as in Theorem 1. Then*

$$\sup_{t\in[0,t_*]} \|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \max\left\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)}, \bar\rho\right\} . \tag{17}$$

Proof: Initially, we prove for $i = 1$.

Consider first that $\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)} \leq \bar\rho$. Then, we define

$$\begin{aligned}
\tilde f &= \bar\rho F - f , \\
\tilde\rho &= \int_V \tilde f = \bar\rho - \rho , \\
\tilde S &= \bar\rho t - S , \\
\tilde a(\tilde S, \tilde\rho) &= \frac{a(S,\rho)\rho}{\bar\rho - \rho} .
\end{aligned}$$

First we prove that

$$\tilde a(\tilde S,\tilde\rho) \leq \frac{a_{\max}\bar\rho}{\bar\rho} \leq a_{\max} ,$$

and conclude that

$$\tilde T_\varepsilon[\tilde S,\tilde\rho] := \lambda F + \tilde a(\tilde S,\tilde\rho)\, F v\cdot\nabla\tilde S \geq 0 , \quad \forall (x,v,v',t) \in (\mathbb{R}^n\times V\times V\times[0,t_*]) . \tag{18}$$

We easily see that

$$\nabla\tilde S = -\nabla S .$$

(f,S) is solution of

$$\begin{aligned}
\varepsilon^2\,\partial_t f + \varepsilon\, v\cdot\nabla f + \lambda f &= \lambda F\rho + \varepsilon\, F a(S,\rho)\, v\cdot\nabla S\rho , \\
\partial_t S - \Delta S &= \rho := \int_V f\, dv ,
\end{aligned}$$

with $f^{\mathrm{I}}=\rho^{\mathrm{I}}F$, $S^{\mathrm{I}}=0$, while $(\tilde f,\tilde S)$ satisfies the system

$$\begin{aligned}\varepsilon^2\,\partial_t\tilde f+\varepsilon\,v\cdot\nabla\tilde f+\lambda\,\tilde f\;&=\;\lambda\,F\tilde\rho-\varepsilon\,Fa(S,\rho)\,v\cdot\nabla S\rho\;=\;\lambda\,F\tilde\rho+\varepsilon\,F\tilde a(\tilde\rho,\tilde S)\,v\cdot\nabla\tilde S\tilde\rho\;,\\ \partial_t\tilde S-\Delta\tilde S\;&=\;\tilde\rho\;:=\int_V\tilde f\,dv\;,\end{aligned}$$

with initial conditions given by $\tilde f^{\mathrm{I}}=\tilde\rho^{\mathrm{I}}F=(\bar\rho-\rho^{\mathrm{I}})F>0$ and $\tilde S^{\mathrm{I}}=0$. Using the positivity of the turning kernel, Equation (18), we conclude the positivity of the solution $\tilde f$, i.e.,

$$0\;\le\;\bar\rho\,F-f \tag{19}$$

and then

$$\rho\;=\int_V f\,dv\;\le\;\bar\rho\;.$$

Now, let us suppose that $\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)}>\bar\rho$. Let $x\in\mathbb{R}^n$ be such that there is a neighborhood U of x such that $\rho^{\mathrm{I}}(x)>\bar\rho$ for $x\in U$, and a time $t_{\max}$ such that the ball with center in x and radius $v_{\max}t_{\max}$ is included in U. Then, in $U\times V\times[0,t_{\max}]$, we write:

$$\varepsilon^2\,\partial_t f+\varepsilon\,\lambda f+v\cdot\nabla f\;=\;\lambda\,F\rho\;,$$

or, equivalently,

$$\begin{aligned}e^{\frac{1}{\varepsilon^2}\int_0^t\lambda(\tau)d\tau}f(x,v,t)\;&=\;f\Big(x-\frac{v}{\varepsilon}t,v,0\Big)\\ &\quad+\int_0^t e^{\frac{1}{\varepsilon^2}\int_0^s\lambda(\tau)d\tau}\,\frac{\lambda(s)}{\varepsilon^2}\,F(v)\,\rho\Big(x-\frac{v(t-s)}{\varepsilon},s\Big)\,ds\;.\end{aligned} \tag{20}$$

We integrate over V and find that

$$e^{\frac{1}{\varepsilon^2}\int_0^t\lambda(\tau)d\tau}\rho(x,t)\;\le\;\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)}+\int_0^t e^{\frac{1}{\varepsilon^2}\int_0^s\lambda(\tau)d\tau}\,\frac{\lambda(s)}{\varepsilon^2}\,\|\rho(\cdot,s)\|_{L^\infty(U)}\;ds\;.$$

Now, we take the $L^\infty(U)$-norm, use Gronwall's inequality and find that

$$\|\rho(\cdot,t)\|_{L^\infty(U)}\;\le\;\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)}\;.$$

Gathering both results, we conclude that

$$\|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\;\le\;\max\big\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)},\bar\rho\big\}\;.$$

For $i=2$ the proof is exactly the same. We need only to see that

$$\overset{\circ}{\underset{R}{\nabla}}\tilde{S} = -\overset{\circ}{\underset{R}{\nabla}} S\ .$$

Now, we prove for $i=3$. For simplicity, we define $S := S(x,t)$, $S_+ := S(x+\varepsilon\mu(\rho)v,t)$ and $S'_+ := S(x+\varepsilon\mu(\rho)v',t)$. We define the function

$$\tilde{\psi}(\bar{\rho}t - S, \bar{\rho}t - S_+) := \frac{1}{\bar{\rho}-\rho}\left[\int_V \psi(S,S'_+)\,F(v')\,dv'\,\bar{\rho} - \psi(S,S_+)\rho\right] . \tag{21}$$

We immediately note that

$$\int_V \tilde{\psi}(\bar{\rho}t - S, \bar{\rho}t - S_+)\,F(v)\,dv = \int_V \psi(S,S_+)\,F(v)\,dv\ .$$

We write the kinetic model as

$$\begin{aligned}
&\partial_t(\bar{\rho}F - f) + v\cdot\nabla(\bar{\rho}F - F) = \\
&\quad = -\int_V \psi(S,S'_+)\,F'\,dv'\,(\bar{\rho}F - f) - \psi(S,S_+)F\rho + \int_V \psi(S,S'_+)\,F'\,dv'\,\bar{\rho}F \\
&\quad = \tilde{\psi}(\bar{\rho}t - S, \bar{\rho}t - S_+)\,F(\bar{\rho}-\rho) - \int_V \tilde{\psi}(\bar{\rho}t - S, \bar{\rho}t - S'_+)\,F'\,dv'\,(\bar{\rho}F - f)\ .
\end{aligned}$$

If the kernel defined by Equation (21) is positive, which is true for sufficiently short times, as $\tilde{\psi}|_{t=0} = \lambda(0) \ge \lambda_{\min} > 0$, and $\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)} \le \bar{\rho}$, then the bound for ρ follows. If $\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)} > \bar{\rho}$, we use the same argument as before and the fact that $\psi(S,S) = \lambda$. ∎

Lemma 3. *Consider a time $t_* > 0$ such that $T_\varepsilon[\rho,S] \ge 0$ for all $(x,v,v',t) \in (\mathbb{R}^n \times V \times V \times [0,t_*])$ and consider the assumptions as in Theorem 1 with $i = 1, 2$ or 3. Then*

$$\begin{aligned}
&\sup_{t\in[0,t_*]} \|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \\
&\le \frac{n}{(n-1)^{(n-1)/n}}\left[\frac{\pi^{1/2}\,\Gamma(n)\max\left\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)},\bar{\rho}\right\}}{2^{n-2}}\right]^{(n-1)/n}\left[\frac{\sqrt{2}\,e^{-1/2}\,\|\rho^{\mathrm{I}}\|_{L^1(\mathbb{R}^n)}}{2^{n+1}\,\pi^{n/2}}\right]^{1/n} .
\end{aligned} \tag{22}$$

Proof: Let us define

$$\bar{t} = \left[\frac{(n-1)\,\sqrt{2}\,e^{-1/2}\|\rho^{\mathrm{I}}\|_{L^1(\mathbb{R}^n)}}{8\,\pi^{(n+1)/2}\,\Gamma(n)\max\left\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)},\bar{\rho}\right\}}\right]^{2/n} ,$$

the value that minimizes the function

$$\frac{\pi^{1/2}\,\Gamma(n)}{2^{n-2}}\,\max\bigl\{\|\rho^{\rm I}\|_{L^\infty(\mathbb{R}^n)},\bar\rho\bigr\}\,t^{1/2}+\frac{\sqrt{2}\,e^{-1/2}\|\rho^{\rm I}\|_{L^1(\mathbb{R}^n)}}{2^{n+1}\,\pi^{n/2}\,t^{(n-1)/2}}\ ,$$

restricted to $t\in\mathbb{R}_+$. If $t_*>\bar t$, then, from Lemma 1, Equation (16), with $t_0=\bar t$, we conclude Equation (22). Now, consider $t_*\le\bar t$. Then, from Equation (15), we have that

$$\sup_{t\in[0,t_*]}\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\ \le\ \frac{\pi^{1/2}\,\Gamma(n)\max\bigl\{\|\rho^{\rm I}\|_{L^\infty(\mathbb{R}^n)},\bar\rho\bigr\}}{2^{n-2}}\,t^{1/2}\ =$$

$$=\ (n-1)^{1/n}\left[\frac{\pi^{1/2}\,\Gamma(n)\max\bigl\{\|\rho^{\rm I}\|_{L^\infty(\mathbb{R}^n)},\bar\rho\bigr\}}{2^{n-2}}\right]^{(n-1)/n}\left[\frac{\sqrt{2}\,e^{-1/2}\,\|\rho^{\rm I}\|_{L^1(\mathbb{R}^n)}}{2^{n+1}\,\pi^{n/2}}\right]^{1/n}.$$

Using that $n/(n-1)^{(n-1)/n}>(n-1)^{1/n}$, we finish the proof. ∎

Lemma 4. *Consider the assumptions of Theorem 1. Then the turning kernel is always positive, i.e.,*

$$T_\varepsilon[S,\rho](x,v,v',t)\ge 0\ ,\qquad\forall\,(x,v,v',t)\in\mathbb{R}^n\times V\times V\times\mathbb{R}_+\ .$$

Proof: Let us fix $\varepsilon<\varepsilon_i$ and apply Lemma 2 to a certain maximum time (that exists, because solutions exist locally in time and $T_\varepsilon[S,\rho](x,v,v',0)=\lambda(0)\,F(v)>0$)

$$t_1\ =\ \sup\Bigl\{t\in\mathbb{R}_+\mid\ T_\varepsilon[S,\rho]\ge 0\ \ \forall\,(x,v,v')\in\mathbb{R}^n\times V\times V\Bigr\}\ >\ 0\ .$$

Now, we prove, by contradiction, that $t_1=\infty$. Let us suppose that $t_1<\infty$.

From Lemma 3, with $i=1$, we see that

$$\varepsilon\,a_{\max}\,v_{\max}\sup_{t\in[0,t_1]}\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\ <\ \varepsilon_1\,a_{\max}\,v_{\max}\sup_{t\in[0,t_1]}\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\ \le\ \lambda\ .$$

This implies that $T_\varepsilon[S,\rho](x,v,v',t_1)>0$ and then $\sup\{t|\ T_\varepsilon[S,\rho]\ge 0\}>t_1$, contradiction.

For $i=2$, we use that, from the Mean Value Theorem,

$$\overset{\circ}{\underset{R}{\nabla}}S(x,t;\varepsilon R)=\frac{n}{\varepsilon R w_{n-1}}\int_{S^{n-1}}\nu\bigl(S(x+\varepsilon R\nu,t)-S(x,t)\bigr)\,d\nu\ \le\ n\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\ ,$$

and the same holds true.

If $i=3$, we prove the positivity of the turning kernel given by Equation (21), for $\varepsilon \le \varepsilon_3$.

First of all, note that if $\rho > \bar\rho$, $\tilde\psi(\bar\rho t - S, \bar\rho t - S_+) = \psi(S,S) \ge \lambda_{\min} > 0$. Consider $\rho < \bar\rho$. Then

$$\psi(S,S_+)\rho - \int_V \psi(S,S'_+)\,F'\,dv'\,\bar\rho \;\le$$
$$\le\; \psi(S,S)\,(\rho-\bar\rho) + \varepsilon\,\psi_1\,\mu(\rho)\,(\bar\rho+\rho)\,v_{\max}\,\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;.$$

This implies that

$$\frac{1}{\bar\rho-\rho}\left[\psi(S,S_+)\rho - \int_V \psi(S,S'_+)\,F'\,dv'\,\bar\rho\right] \;\le$$
$$\le\; -\psi(S,S) + \varepsilon\,\psi_1\,\frac{\mu(\rho)}{\bar\rho-\rho}\,(\rho+\bar\rho)\,v_{\max}\,\|\nabla S(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;.$$

From Lemma 3, we conclude that $\tilde\psi \ge 0$. Finally, we define $\tilde\psi$ for $\rho = \bar\rho$ by continuity (from both sides). ∎

Proof of Theorem 1: From Lemma 4 we know that the model is well-defined (i.e., the turning kernel is non-negative) for $t \ge 0$. Then, we apply Lemmas 2 and 3 to conclude the boundedness of ρ and ∇S. For the bound on S, we see that from the Young's inequality

$$\|S(\cdot,t)\|_{L^p(\mathbb{R}^n)} \;\le\; \int_0^t \|\Upsilon(\cdot,s)\|_{L^q(\mathbb{R}^n)}\,\|\rho(\cdot,t-s)\|_{L^\infty(\mathbb{R}^n)}\,ds \;,$$

for $p^{-1}+1 = q^{-1}$. We immediately see that

$$\|\Upsilon(\cdot,t)\|_{L^q(\mathbb{R}^n)} \;=\; \frac{1}{q^{n/(2q)}\,(4\pi t)^{n(q-1)/2}} \;,$$

and then

$$\|S(\cdot,t)\|_{L^p(\mathbb{R}^n)} \;\le\; \frac{\max\left\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)},\bar\rho\right\}}{q^{n/(2q)}\,(4\pi)^{n(q-1)/(2q)}}\int_0^t \frac{ds}{s^{n(q-1)/(2q)}} \;. \tag{23}$$

For $p > n/2$, $q > n/(2+n)$ and, then, the last integral is convergent, as $n(q-1)/(2q) > -1$.

From the Definition (4), we see that

$$\|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;\le\; \left\|\frac{f(\cdot,\cdot,t)}{F}\right\|_{L^\infty(\mathbb{R}^n\times V)}\int_V F(v)\,dv \;=\; \left\|\frac{f(\cdot,\cdot,t)}{F}\right\|_{L^\infty(\mathbb{R}^n\times V)} . \tag{24}$$

Finally, we use Equation (19) and apply Gronwall's Lemma to Equation (20) to conclude that

$$\left\|\frac{f(\cdot,\cdot,t)}{F}\right\|_{L^\infty(\mathbb{R}^n)} \leq \max\left\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)}, \bar{\rho}\right\} . \blacksquare \tag{25}$$

Remark 5. For models of hyperbolic-elliptic type, i.e., with $\delta=0$ in Equation (9), Theorem 1 remains valid, possibly with different ε_i, $i=1,2,3$, as the inequality in Remark 4 continues to be true. □

Remark 6. We can relax the assumption that $S^{\mathrm{I}} \equiv 0$, replacing it for the weaker assumption that $S^{\mathrm{I}} \in L^1_+ \cap W^{1,\infty}(\mathbb{R}^n)$, possibly changing the values of ε_i, $i=1,2$ or 3 in Theorem 1. We need only to add $\|\nabla S^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)}$ on the right hand side of Equations in Lemmas 1 and 3 and redefine, in Lemma 1, $\tilde{S} = \bar{\rho}t + \|S^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)} - S$. The left hand side of Equation (21) should also change to $\tilde{\psi}\big(\bar{\rho}t + \|S^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)} - S,\ \bar{\rho}t + \|S^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)} - S_+\big)$, and we need also to impose that

$$\inf_{S,S'\geq 0} \psi(S,S') \;=\; \psi_0 \;>\; 0 \;. \;\square$$

Remark 7. Assumption (M3) can be relaxed to the weaker case where $\mu(\rho) \to 0$, when $\rho \to \infty$, sufficiently fast. It is proven (see [6]) global existence of solutions and global-in-time convergence to the drift-diffusion limit (with sensitivity decaying to zero for high cell concentration). That limit model was introduced in [37, 36] and its solutions exist globally. □

4 – Convergence to the Drift-diffusion Models

In this section we prove convergence of the kinetic models in the drift-diffusion limit (i.e., $\varepsilon \to 0$) to the Keller–Segel equations. We will not require particular models, like in the previous sections, as the theorems to be presented are quite general. However, for the particular models in Section 3, as a consequence of the global bound of Theorem 1, we may also obtain global existence of solutions to the limit models

Let us define the symmetric and anti-symmetric parts of $T_\varepsilon[S,\rho]F$, respectively, by:

$$\phi^S_\varepsilon[S,\rho] \;=\; \frac{T_\varepsilon[S,\rho]F' + T^*_\varepsilon[S,\rho]F}{2} \;, \tag{26}$$

$$\phi^A_\varepsilon[S,\rho] \;=\; \frac{T_\varepsilon[S,\rho]F' - T^*_\varepsilon[S,\rho]F}{2} \;. \tag{27}$$

Theorem 2. *Let $F \in L^\infty(V)$ be a positive velocity distribution satisfying Assumptions (A1–A5) and let $\phi_\varepsilon^S[S]$ and $\phi_\varepsilon^A[S]$ be defined as above. Assume that there exist $q > 3$, $\lambda_0 > 0$, and a non-decreasing function $\Lambda \in L^\infty_{\rm loc}([0,\infty))$, such that*

$$\frac{f^{\rm I}}{F} \in \mathcal{X}_q := L^1_+ \cap L^q(\mathbb{R}^n \times V;\, F\,dx\,dv) \ , \tag{28}$$

$$\phi_\varepsilon^S[S,\rho] \,\geq\, \lambda_0\big(1 - \varepsilon\,\Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^n)})\big)\, F F' \ , \tag{29}$$

$$\int_V \frac{\phi_\varepsilon^A[S,\rho]^2}{F\,\phi_\varepsilon^S[S,\rho]}\, dv' \,\leq\, \varepsilon^2 \Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^n)}) \ . \tag{30}$$

Then there exists $t^* > 0$, independent of ε, such that the existence time of the local mild solution of (2–7) is bigger than t^*, and the solution satisfies, uniformly in ε,

$$\begin{aligned} &\frac{f_\varepsilon}{F} \,\in\, L^\infty(0,t^*;\mathcal{X}_q) \ , \\ &S_\varepsilon \,\in\, L^\infty(0,t^*;\, L^p \cap C^{1,\alpha}(\mathbb{R}^n)) \ , \qquad \alpha < \frac{q-n}{q} \ , \quad 3 < p < \infty \ , \\ r_\varepsilon \,=\, &\frac{f_\varepsilon - \rho_\varepsilon F}{\varepsilon} \,\in\, L^2\left(\mathbb{R}^n \times V \times (0,t^*);\, \frac{dx\,dv\,dt}{F}\right) \ . \end{aligned} \tag{31}$$

Proof: The proof is the same as in [7] and extended in [21]. ∎

Theorem 3. *Let the assumptions of Theorem 2 hold. Assume further that for families S_ε uniformly bounded (as $\varepsilon \to 0$) in $L^\infty_{\rm loc}(0,\infty;\, C^{1,\alpha}(\mathbb{R}^n))$ for some $0 < \alpha \leq 1$, such that S_ε and ∇S_ε converge to S_0 and ∇S_0, respectively, in $L^p_{\rm loc}(\mathbb{R}^n \times [0,\infty))$ for some $p > 3/2$ and ρ_ε converges to ρ_0 in $L^2_{\rm loc}(\mathbb{R}^n \times [0,\infty))$, we have the convergence*

$$T_\varepsilon[S_\varepsilon,\rho_\varepsilon] \to T_0[S_0,\rho_0] \quad \text{in} \ \ L^p_{\rm loc}(\mathbb{R}^n \times V \times V \times [0,\infty)) \ ,$$

$$\frac{\mathcal{T}_\varepsilon[S_\varepsilon,\rho_\varepsilon](F)}{\varepsilon} = \frac{2}{\varepsilon}\int_V \phi_\varepsilon^A[S_\varepsilon,\rho_\varepsilon]\, dv' \;\to\; \mathcal{T}_1[S_0,\rho_0](F) \quad \text{in} \ \ L^p_{\rm loc}(\mathbb{R}^n \times V \times [0,\infty)) \ .$$

Then solutions of (2–7) satisfy (possibly after extracting subsequences)

$$\begin{aligned} \rho_\varepsilon \,&\to\, \rho_0 \qquad \text{in} \ \ L^2_{\rm loc}(\mathbb{R}^n \times (0,t^*)) \ , \\ S_\varepsilon \,&\to\, S_0 \qquad \text{in} \ \ L^q_{\rm loc}(\mathbb{R}^n \times (0,t^*)) \, , \quad 1 \leq q < \infty \ , \\ \nabla S_\varepsilon \,&\to\, \nabla S_0 \quad \text{in} \ \ L^q_{\rm loc}(\mathbb{R}^n \times (0,t^*)) \, , \quad 1 \leq q < \infty \ . \end{aligned}$$

The limits are weak solutions of (8–9) subject to the initial condition

$$\begin{aligned} \rho_0(x,0) &= \int_V f^{\mathrm{I}}(x,v)\,dv\ , \\ S_0(x,0) &= S^{\mathrm{I}}(x)\ . \end{aligned}$$

Proof: The proof of the convergence of S_ε and ∇S_ε can be found in [7] and [21]. There, we found also the weak convergence of f_ε. Now, we prove the strong convergence of ρ_ε in $L^2_{\mathrm{loc}}(\mathbb{R}^n\times(0,t^*))$. We have that $f_\varepsilon = \rho_\varepsilon F + \varepsilon r_\varepsilon$, then we take equation (2), multiply by v and integrate over V. We find

$$\begin{aligned} &\partial_t \int_V v f_\varepsilon\,dv + \frac{1}{\varepsilon}\,\lambda[S_0,\rho_0]\,D[S_0,\rho_0]\,\nabla\rho_\varepsilon + \nabla\cdot\int_V v\otimes v r_\varepsilon\,dv = \\ &\quad = \frac{1}{\varepsilon}\,\rho_\varepsilon \int_V \frac{\mathcal{T}_\varepsilon[S_\varepsilon,\rho_\varepsilon](F)}{\varepsilon}\,v\,dv + \frac{1}{\varepsilon}\iint_{V\times V} \big[T_\varepsilon[S_\varepsilon,\rho_\varepsilon]r'_\varepsilon - T^*_\varepsilon[S_\varepsilon,\rho_\varepsilon]r_\varepsilon\big]v\ dv\,dv'. \end{aligned}$$

This implies that

$$\begin{aligned} &\lambda[S_0,\rho_0]\,D[S_0,\rho_0]\,\nabla\rho_\varepsilon = \\ &\qquad = \rho_\varepsilon \int_V \frac{\mathcal{T}_\varepsilon[S_\varepsilon,\rho_\varepsilon](F)}{\varepsilon}\,v\ dv + \iint_{V\times V} \big(T_\varepsilon[S_\varepsilon,\rho_\varepsilon]r'_\varepsilon - T^*_\varepsilon[S_\varepsilon,\rho_\varepsilon]r_\varepsilon\big)\,v\ dv\,dv' \\ &\qquad\quad - \varepsilon\,\nabla\cdot\int_V v\otimes v r_\varepsilon\,dv - \varepsilon\,\partial_t\int_V v f_\varepsilon\,dv\ . \end{aligned}$$

From the estimates obtained in Theorem 3 and Rellich's Theorem, we have that $\lambda[S_0,\rho_0]\,D[S_0,\rho_0]\,\nabla\rho_\varepsilon$ is in a compact set of $H^{-1}_{\mathrm{loc}}(\mathbb{R}^n\times(0,t^*))$. Now use that $\lambda[S_0,\rho_0]$ is bounded from below (Assumption (A5)) and $D[S_0,\rho_0]$ is positive definite to conclude that $\nabla\rho_\varepsilon$ lies in a compact set of $H^{-1}_{\mathrm{loc}}(\mathbb{R}^n\times(0,t^*))$. We use the div-curl lemma of L. Tartar [26, 35]. We define

$$J_\varepsilon := \frac{1}{\varepsilon}\int_V v f_\varepsilon\,dv = \int_V v r_\varepsilon\,dv\ .$$

Now, consider

$$\begin{aligned} X_\varepsilon &= (J_\varepsilon,\rho_\varepsilon)\ , \\ Y_\varepsilon &= (0,\rho_\varepsilon)\ . \end{aligned}$$

We have

$$\begin{aligned} \mathrm{div}_{(x,t)} X_\varepsilon &= \nabla\cdot J_\varepsilon + \partial_t\rho_\varepsilon = 0\ , \\ \mathrm{curl}_{(x,t)} Y_\varepsilon &= -\mathrm{curl}_x\,\rho_\varepsilon\ . \end{aligned}$$

The RHS of both equations, lie in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^n\times(0,t^*))$, then from the div-curl lemma, $\rho_\varepsilon^2 = X_\varepsilon\cdot Y_\varepsilon \to X_0\cdot Y_0 = \rho_0^2$, weak-$*$. The convergence is a simple consequence of the bound in f_ε in Theorem 2. See [9] for a similar case. ∎

Corollary 1. *For $i=1,2$ or 3, Models (Mi), subject to Assumptions (A1–5), (Bi) and Remark 2 with regular initial conditions converge to the Keller–Segel model (1) in their drift-diffusion limits, for arbitrarily large existence times (if ε is small enough, according to Theorem 1). The limit model has global existence of its solutions. In particular*

$$\|\rho_0(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;\le\; \max\bigl\{\|\rho^{\mathrm{I}}\|_{L^\infty(\mathbb{R}^n)}, \bar\rho\bigr\}\ . \tag{32}$$

Proof: For these models, we prove Equations (29–30) and we immediately see that assumptions in Theorem 3 are satisfied. Then we find that maximum existence time t^* in Theorem 2 can be arbitrarily large, as, according to Theorem 1, solutions are bounded. It is important to note that the bounds (23), (24) and (25) in Theorem 1 are uniform in ε. From Theorem 3 we have that ρ_ε converges to ρ_0 in $L^2_{\mathrm{loc}}(\mathbb{R}^n\times(0,t^*))$ and, as $\|\rho_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ is uniformly-in-time bounded with a bound uniform in ε, we conclude Equation (32). ∎

Remark 8. It is important to stress the differences between Corollary 1 and the global existence results presented in [16]. In the latter, coefficients β and χ that appear in Equation (1) are supposed to be of class C^3. In models (Mi), $i=1,2,3$, we only need continuity of a, $\mathring{a}$ and μ, resulting in the same assumption for the chemotactical sensitivity in the limit. On the other hand, in order to prove global-in-time existence, we explicitly used assumption (Bi), $i=1,2,3$, imposing that the decay of these constants in the range $\rho\in[0,\bar\rho)$ is at most linear. This was not used in [16]. We also allow the time dependence of the diffusion coefficient D. This was not considered in [16]. Finally, our result holds for the entire space $\mathbb{R}^n$, while in [16] the result is valid on a C^3-differentiable, compact Riemannian manifold with periodic boundary conditions. In [16], they also consider a positive chemical decay for S, while we consider only a non-negative one. Other differences seem to be purely technical. □

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