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ON THE CONTRACTED l^1-ALGEBRA OF A POLYCYCLIC MONOID

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Abstract: Let $P(X)$ denote the polycyclic monoid (Cuntz semigroup) on a nonempty set X and let A denote the Banach algebra $l^1(P(X))/Z$, where Z is the (closed) ideal spanned by the zero of $P(X)$. Then A is primitive. Moreover, A is simple if and only if X is infinite.

The l^1-algebra $l^1(S)$ of a semigroup S consists of all functions $a\colon S \to \mathbb{C}$ (the complex field) of finite or countably infinite support and such that $\sum_{x\in S}|a(x)| < \infty$, where addition and scalar multiplication are defined pointwise and multiplication is taken to be convolution. As noted in [1], $l^1(S)$ is a Banach algebra with respect to the norm $\|\ \|$ defined by $\|a\| := \sum_{x\in S} |a(x)|$. By identifying each $x \in S$ with its characteristic function, we can write a typical element of $l^1(S)$ in the form $\sum_{x\in S} \alpha_x x$, where $\sum_{x\in S} |\alpha_x| < \infty$, $(\alpha_x \in \mathbb{C})$.

The semigroup algebra $\mathbb{C}[S]$ is the subalgebra consisting of all functions $a\colon S \to \mathbb{C}$ of finite support. When S is a nontrivial semigroup with zero z, it is often helpful to replace $\mathbb{C}[S]$ by $\mathbb{C}[S]/\mathbb{C}z$, where $\mathbb{C}z$ is the ideal $\{\alpha z : \alpha \in \mathbb{C}\}$. We have thus, in effect, simply identified z with the zero of the algebra. In [4, Chapter 5], $\mathbb{C}[S]/\mathbb{C}z$ is called the 'contracted semigroup algebra' of S over $\mathbb{C}$ and is denoted by $\mathbb{C}_0[S]$. With this in mind, we call the Banach algebra $l^1(S)/\mathbb{C}z$ the *contracted l^1-algebra* of S and denote it by $l^1_0(S)$. A typical element u of $l^1_0(S)$ can be written in the form $u = \sum_{x\in S\backslash 0} \alpha_x x$, where $\sum_{x\in S\backslash 0} |\alpha_x| < \infty$, and we define its *support*, $\operatorname{supp}(u)$, to be $\{x \in S\backslash 0\colon\ \alpha_x \neq 0\}$.

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In this paper, we study $l_0^1(S)$ for the case in which S is the *polycyclic monoid* $P(X)$ on nonempty a set X [13]. It is shown that $l_0^1(S)$ is primitive for all choices of X (Theorem 1) and is simple if and only if X is infinite (Theorem 2).

We begin by recalling the definition of $P(X)$. Let $M(X)$ denote the free monoid on X. For $w = x_1 x_2 \dots x_n \in M(X)$, where each $x_i \in X$, we define the *length* $l(w)$ and the *content* $c(w)$ of w by $l(w) := n$ and $c(w) := \{x_1, x_2, ..., x_n\}$. In addition, we take $l(1) := 0$ and $c(1) := \emptyset$, where 1 denotes the identity of $M(X)$ (the empty word). We say that $u \in M(X)$ is an *initial segment* of $v \in M(X)$, written $u \preceq v$, if and only if $v = uw$ for some $w \in M(X)$. For $u, v \in M(X)$, we write $u \,\|\, v$ if and only if $u \not\preceq v$ and $v \not\preceq u$.

Let $P(X) := (M(X) \times M(X)) \cup \{0\}$ and define a multiplication in $P(X)$ by

$$(a,b)\,(c,d) \;=\; \begin{cases} (au, d) & \text{if } \; c = bu \;\; \text{for some } u \in M(X)\ , \\ (a, dv) & \text{if } \; b = cv \;\; \text{for some } v \in M(X)\ , \\ \quad 0 & \text{if } \; b \,\|\, c\ , \end{cases}$$

$$0(a,b) = (a,b)0 = 0^2 = 0\ .$$

Then $P(X)$ is a monoid with identity $(1,1)$ and zero 0; further, it admits an involution * given by

$$(a,b)^* = (b,a)\ , \qquad 0^* = 0\ .$$

(In fact, $P(X)$ is an example of a 0-bisimple inverse semigroup in which * denotes inversion and in which each subgroup is trivial.) Note that $(a,b)^2 = (a,b)$ if and only if $a = b$. Thus the set $E(X)$ of idempotents of $P(X)$ is

$$\big\{(a,a)\colon\ a \in M(X)\big\} \cup \big\{0\big\}\ .$$

Clearly $E(X)$ is a commutative submonoid of $P(X)$ (the 'semilattice' of $P(X)$) and it is easily seen to be partially ordered by

$$(a,a) \geq (b,b) \iff a \preceq b\ , \qquad (a,a) > 0\ .$$

Observe that $(a,a) \geq (b,b)$ if and only if $(a,a)(b,b) = (b,b)$ $[= (b,b)(a,a)]$.

An alternative approach to the monoid described above is as follows. Let $FI(X)$ denote the free monoid with involution* on a nonempty set X. Adjoin a zero 0 to $FI(X)$, take $0^* = 0$ and write $Q(X) := (FI(X) \cup \{0\})/\rho$, where ρ is the congruence determined by the relations $x^*x = 1$ $(x \in X)$ and $x^*y = 0$ $(x, y \in X$ and $x \neq y)$. This monoid is termed the *Cuntz semigroup* on X. Note that every nonzero ρ-class has a unique representative of the form ab^* $(a, b \in M(X))$. We identify this element with its ρ-class and so can write

$Q(X) = \{ab^*\colon\, a, b \in M(X)\} \cup \{0\}$. It is routine to verify that $\theta\colon P(X) \to Q(X)$ is an isomorphism. Various aspects of algebras associated with $Q(X)$ have been studied in [5], [6] and [2]; see also [14]. For an extended discussion of polycyclic monoids, see [9, §9.3].

Next, we review the concept of primitivity. Let A be a complex algebra and let V be a nonzero right A-module under the action $\circ$. A vector $v \in V\backslash 0$ is called *cyclic* if and only if $v \circ A = V$. Recall that V is termed

(**i**) *faithful* if and only if, for all $a \in A$, $V \circ a = 0$ implies $a = 0$,

(**ii**) *strictly irreducible* if and only if every nonzero vector in V is cyclic.

We say that A is *(right) primitive* if and only if there exists a faithful strictly irreducible right A-module.

For the case in which A is a Banach algebra, V a Banach space with norm $\| \;\|_V$ and $\circ$ a right action of A on V with $\|v \circ a\|_V \le \|v\|_V \|a\|$ $(v \in V,\ a \in A)$, we make a further definition. We say that V is *topologically irreducible* if and only if, for all $v \in V\backslash 0$, all $u \in V$ and a given positive real number ϵ, there exists $a \in A$ such that

$$\|v \circ a - u\|_V < \epsilon\ .$$

The following result ([8], [10]) is required below. For convenience, we include a proof.

Lemma. *Let A and V be as in the preceding paragraph. If V is topologically irreducible and possesses a cyclic vector then V is strictly irreducible.*

Proof: Let V be topologically irreducible, with a cyclic vector v_1. Since the mapping $f\colon A \to V$ defined by $f(a) = v_1 \circ a$ is continuous, the open mapping theorem shows that, for some positive real number δ,

$$\Big\{v \in V\colon\ \|v\|_V < \delta\Big\} \subseteq \Big\{f(a)\colon\ a \in A \ \text{ and } \ \|a\| < 1\Big\}\ .$$

Let $v \in V\backslash 0$. Since V is topologically irreducible, there exists $b \in A$ such that $\|v_1 - v \circ b\|_V < \delta$. Hence there exists $a \in A$ with $\|a\| < 1$ such that $v_1 - v \circ b = v_1 \circ a$. Consider $c \in A$ defined by $c = -\sum_{r=1}^{\infty} a^r$. Then $a + c - ac = 0$. Hence

$$\begin{aligned} v \circ (b - bc) \;&=\; (v_1 - v_1 \circ a) - (v_1 - v_1 \circ a) \circ c \\ &=\; v_1 - v_1 \circ (a + c - ac) \;=\; v_1\ . \end{aligned}$$

Consequently, v is cyclic. Thus V is strictly irreducible. ∎

We now come to our first result. Note that since the polycyclic monoid on X admits an involution, so also does its contracted l^1-algebra. Thus the term 'primitive' can be used without qualification.

Theorem 1. *For every nonempty set X, $l_0^1(P(X))$ is primitive.*

Proof: For a given nonempty set X write $S := P(X)$, $E := E(X)$ and $V := l_0^1(E)$.

We begin by defining a right action of $l_0^1(S)$ on V. First note that, if $x \in S$ and $e \in E$ then xx^*, $x^*ex \in E$. Now define $\circ : E \times S \to E$ by the rule that

$$(\forall\, e \in E)\ \ (\forall\, x \in S) \qquad e \circ x = \begin{cases} x^*ex & \text{if } e \leq xx^* , \\ 0 & \text{otherwise} . \end{cases}$$

Let $e \in E$ and let $x, y \in S$. A straightforward calculation shows that

$$e \leq xx^* \ \text{ and } \ x^*ex \leq yy^* \iff e \leq xy(xy)^* . \tag{1}$$

Using this, we now prove that

$$(e \circ x) \circ y = e \circ (xy) . \tag{2}$$

Suppose that $e \leq xy(xy)^*$. Then $e \circ (xy) = (xy)^*exy$. But, by (1), $e \leq xx^*$ and $x^*ex \leq yy^*$. Hence $(e \circ x) \circ y = (x^*ex) \circ y = y^*(x^*ex)y = (xy)^*exy$. Thus (2) holds in this case. Now suppose that $e \not\leq xy(xy)^*$. Then $e \circ (xy) = 0$. But, by (1), either $e \not\leq xx^*$ or $x^*ex \not\leq yy^*$. If $e \leq xx^*$ and $x^*ex \not\leq yy^*$ then $(e \circ x) \circ y = (x^*ex) \circ y = 0$, while if $e \not\leq xx^*$ then $e \circ x = 0$ and so again $(e \circ x) \circ y = 0$. Thus (2) holds in this case also. Since, for all $e \in E$ and $x \in S$, $\|e \circ x\| \leq \|x^*ex\| \leq 1$ we can extend $\circ$ to a right action, also denoted by $\circ$, of $l_0^1(S)$ on V; and, clearly, for all $v \in V$ and all $u \in l_0^1(S)$, $\|v \circ u\| \leq \|v\|.\|u\|$.

We show next that V is faithful. Let S' and E' denote $S \backslash 0$ and $E \backslash 0$, respectively. Observe first that E' satisfies the maximal condition with respect to $\leq$; for if T is a nonempty subset of $M(X)$ and $s \in T$ is chosen such that $l(s) \leq l(t)$ for all $t \in T$ then (s, s) is maximal in the subset $\{(t, t) : t \in T\}$ of E'. Let $u \in l_0^1(S) \backslash 0$, say $u = \sum_{x \in S'} \alpha_x x$, with $\sum_{x \in S'} |\alpha_x| < \infty$ and not all $\alpha_x = 0$. Choose $e \in E'$ maximal in $\{xx^* : x \in \operatorname{supp}(u)\}$. Then

$$e \circ u = \sum_{xx^* = e} \alpha_x (x^*ex) . \tag{3}$$

Now let $x, y \in S'$ be such that $xx^* = yy^* = e$ and $x^*ex = y^*ey$. We have that $x = (a, b)$ and $y = (c, d)$ for some $a, b, c, d \in M(X)$. Thus $(a, a) = e = (c, c)$ and $(b, b) = x^*ex = y^*ey = (d, d)$. Hence $a = c$, $b = d$ and so $x = y$. It follows from (3) that $e \circ u \neq 0$. This shows that V is faithful.

To complete the proof, we show that V is strictly irreducible. Let $v \in V\backslash 0$ and let $e \in \operatorname{supp}(v)$, with coefficient $\alpha \in \mathbb{C}\backslash 0$. We prove first that, for a given positive real number ϵ, there exist $v' \in V$ and $u \in l_0^1(E)$ $(\subseteq l_0^1(S))$ such that

$$v \circ u = \alpha e + v' \ , \qquad \|v'\| < \epsilon \ . \tag{4}$$

Note that if e is minimal in $\operatorname{supp}(v)$ then $v \circ e = \alpha e$ and so (4) holds with $u = e$ and $v' = 0$. Suppose, therefore, that e is not minimal in $\operatorname{supp}(v)$. Write $v = w + w'$, where $w, w' \in V$ are such that

$$e \in \operatorname{supp}(w), \quad \operatorname{supp}(w) \text{ is finite}, \quad \operatorname{supp}(w) \cap \operatorname{supp}(w') = \emptyset, \quad \|w'\| < \epsilon \ . \tag{5}$$

Without loss of generality, we may assume that e is not minimal in $\operatorname{supp}(w)$. (If need be, transfer a term from w' to w.) Let $F := \{f \in \operatorname{supp}(w) \colon f < e\}$ and define $u \in l_0^1(E)$ by

$$u := \prod_{f \in F} (e - f) \ .$$

We now show that

$$(\forall\, g \in E') \qquad g \circ u = \begin{cases} g & \text{if } g \leq e \text{ and, for all } f \in F, \ \ g \not\leq f \ , \\ 0 & \text{if } g \leq e \text{ and, for some } f \in F, \ \ g \leq f \ , \\ 0 & \text{if } g \not\leq e \ . \end{cases} \tag{6}$$

Suppose first that $g \in E'$ is such that $g \leq e$ and that, for all $f \in F$, $g \not\leq f$. Then, for all $f \in F$, $g \circ (e - f) = g$ and so $g \circ u = g$. Next, suppose that $g \in E'$ is such that $g \leq e$ and that there exists $f \in F$ with $g \leq f$. Then $g \circ (e - f) = g - g = 0$ and so $g \circ u = 0$. Finally, suppose that $g \in E'$ is such that $g \not\leq e$. Then, for any $f \in F$, $g \not\leq f$ and so $g \circ (e - f) = 0$. Hence again $g \circ u = 0$. This establishes (6).

It follows from (6) that $w \circ u = \alpha e$. Write $v' := w' \circ u$. Since, by (6), for all $g \in \operatorname{supp}(w')$, $g \circ u$ is either g or 0 we have that $\|v'\| \leq \|w'\|$. Thus, from (5), we see that (4) holds.

Next, let $f \in E'$. There exist $a, b \in M(X)$ such that $e = (a, a)$ and $f = (b, b)$. Write $x := (a, b)$. Then $xx^* = e$ and

$$e \circ x = f \ . \tag{7}$$

Hence, from (4), $v \circ (ux) = \alpha f + (v' \circ x)$ and, in addition, $\|v' \circ x\| \leq \|v'\| < \epsilon$. Thus

$$\big\| v \circ (ux) - \alpha f \big\| < \epsilon \ ,$$

from which we deduce that V is topologically irreducible. But, from (7), it follows that e is a cyclic vector in V. Hence, by the Lemma, V is strictly irreducible. ∎

The corresponding result for $\mathbb{C}_0[P(X)]$ is a consequence of a theorem of Domanov [7]. A short proof is given in [12]. As already remarked, $P(X)$ is a special case of a 0-bisimple inverse semigroup with only trivial subgroups. In [3], we show that if S is a 0-bisimple inverse semigroup with a nonzero maximal subgroup G such that $l^1(G)$ is primitive then $l^1_0(S)$ is primitive. This generalises Theorem 1 above, but is harder to prove since we have to allow for the presence of nontrivial subgroups and cannot assume that the semilattice of S satisfies the maximal condition under the natural partial ordering.

Our second result gives a necessary and sufficient condition for $l^1_0(P(X))$ to be a simple algebra.

Theorem 2. *Let X be a nonempty set. Then $l^1_0(P(X))$ is simple if and only if X is infinite.*

Proof: Write $S := P(X)$ and $S' := S\backslash 0$. Assume first that X is infinite. Let T be a nonzero ideal of $l^1_0(S)$. We show that $T = l^1_0(S)$.

Let $t \in T\backslash 0$. Choose $a \in M(X)$ such that a has minimal length amongst the first components of the elements of $\operatorname{supp}(t)$; and choose $b \in M(X)$ such that $(a,b) \in \operatorname{supp}(t)$. Then, for some positive integer n, we may write t in the form

$$t = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + v \ , \tag{1}$$

where $u_1, u_2, ..., u_n$ are distinct elements of $\operatorname{supp}(t)$ with $u_1 = (a,b)$, $\alpha_i \in \mathbb{C}\backslash 0$ $(i = 1, 2, ..., n)$ and $v \in l^1_0(S)$ is such that $\|v\| < |\alpha_1|$. Write $u_i = (a_i, b_i) \in M(X) \times M(X)$ $(i = 1, 2, ..., n)$ and assume, without loss of generality, that for some $k \in \{1, 2, ..., n\}$, $(a =)\, a_1 = a_2 = \cdots = a_k$, while $a_i \neq a$ if $k < i \leq n$. Since $u_1, u_2, ..., u_k$ are distinct, it follows that $(b =)\, b_1, b_2, ..., b_k$ are distinct.

Let Y denote $\bigcup_{i=1}^{n} \big(c(a_i) \cup c(b_i)\big)$. Since Y is a finite subset of the infinite set X, there exists $x \in X\backslash Y$. Write

$$e := (ax, ax) \ , \qquad f := (bx, bx) \ .$$

We shall show that

$$eu_i f = \begin{cases} (ax, bx) & \text{if } \ i = 1 \ , \\ \quad 0 & \text{if } \ 2 \leq i \leq n \ . \end{cases} \tag{2}$$

Suppose first that $1 \leq i \leq k$. Then $eu_i f = (ax, ax)(a, b_i)(bx, bx) = (ax, b_i x)(bx, bx)$. In particular, $eu_1 f = (ax, bx)$. Now consider the case where $2 \leq i \leq k$. Here $b_i x \not\leq bx$; for otherwise, since $x \notin c(b)$, we would have $b_i = b$. Similarly, $bx \not\leq b_i x$.

Hence $eu_i f = 0$. Next, suppose that $k < i \leq n$. Then, by the choice of x, $ax \not\preceq a_i$. Further, $a_i \not\preceq ax$; for otherwise $a_i \preceq a$, which is impossible since $l(a_i) \not< l(a)$ and $a_i \neq a$. Hence $ax \parallel a_i$ and so $eu_i = (ax, ax)(a_i, b_i) = 0$, which gives $eu_i f = 0$. Thus we have established (2).

Take $p := (1, ax)$ and $q := (bx, 1)$. Then, from (1) and (2),

$$p\,e\,t f q \;=\; \alpha_1(1,1) + p\,e\,v f q \;.$$

But, since $p, e, f, q \in S'$, we have that $\|p\,e\,v f q\| \leq \|v\| < |\alpha_1|$. Thus

$$\left\|\alpha_1^{-1}(p\,e\,t f q) - (1,1)\right\| \;<\; 1 \;.$$

Consequently, $\alpha_1^{-1}(p\,e\,t f q)$ is invertible in $l_0^1(S)$; thus there exists $r \in l_0^1(S)$ such that $\alpha_1^{-1}(p\,e\,t f q\,r) = (1,1)$. Since $t \in T$, it follows that $(1,1) \in T$ and so $T = l_0^1(S)$. This shows that $l_0^1(S)$ is simple.

Now assume that X is finite, with elements $x_1, x_2, ..., x_n$. For $(a,b) \in S'$ define $w_{a,b} \in l_0^1(S)$ by

$$w_{a,b} \;:=\; (a,b) - \sum_{i=1}^{n} (ax_i, bx_i) \;.$$

Then $\|w_{a,b}\| = n+1$. Define a subspace T of $l_0^1(S)$ by

$$T \;:=\; \left\{ \sum_{(a,b)\in S'} \alpha_{a,b}\, w_{a,b} \colon \;\; \alpha_{a,b} \in \mathbb{C} \;\text{ and } \sum_{(a,b)\in S'} |\alpha_{a,b}| < \infty \right\} .$$

Let $(a,b), (c,d) \in S'$ and consider the product $w_{a,b}(c,d)$. If $b = cu$ for some $u \in M(X)$ then $w_{a,b}(c,d) = (a, du) - \sum_{i=1}^{n}(ax_i, dux_i) = w_{a,du} \in T$. If $c = bx_r v$ for some r and some $v \in M(X)$ then $w_{a,b}(c,d) = (ax_r v, d) - (ax_r v, d) = 0$. If $b \parallel c$ then $w_{a,b}(c,d) = 0$. Thus $T(c,d) \subseteq T$. This shows that T is a right ideal of $l_0^1(S)$. A similar argument shows that it is a left ideal.

Finally, we prove that the ideal T is proper. Define $\phi\colon S' \to \mathbb{C}$ by $\phi((a,b)) = n^{-(1/2)(l(a)+l(b))}$. Since $|\phi((a,b))| \leq 1$, ϕ extends to a continuous linear functional on $l_0^1(S)$. Now, for all $(a,b) \in S'$,

$$\begin{aligned} \phi(w_{a,b}) \;&=\; \phi\big((a,b)\big) - \sum_{i=1}^{n} \phi\big((ax_i, bx_i)\big) \\ &=\; n^{-(1/2)(l(a)+l(b))} - n\,.\,n^{-(1/2)(l(a)+l(b)+2)} \;=\; 0 \;. \end{aligned}$$

Hence, by continuity, $\phi(t) = 0$ for all $t \in T$. But $\phi((1,1)) = 1$ and so $(1,1) \notin T$. Thus T is proper. ∎

The corresponding result for $\mathbb{C}_0[P(X)]$ was obtained in [11].

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