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A VIABLE RESULT FOR NONCONVEX DIFFERENTIAL INCLUSIONS WITH MEMORY

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Abstract: Let X be a separable Banach space, $\sigma > 0$ and $\mathcal{C}_\sigma := \mathcal{C}([-\sigma, 0], X)$ the Banach space of the continuous functions from $[-\sigma, 0]$ into X, K a locally closed set in X and $F\colon [a, b)\times\mathcal{C}_\sigma \to 2^X$ a closed valued and locally integrable bounded multifunction, with $F(., \varphi)$ measurable and $F(t, .)$ Lipschitz continuous in the Hausdorff–Pompeiu metric. In this paper we establish some sufficient conditions in order that, for each $\tau \in [a, b)$ and for each $\varphi \in \mathcal{C}_\sigma$ with $\varphi(0) \in K$, there exist at least one solution $u\colon [\tau-\sigma, T] \to X$ of the differential inclusion $u'(t) \in F(t, u_t)$, such that $u_\tau = \varphi$ on $[-\sigma, 0]$ and $u(t) \in K$ for every $t \in [\tau, T]$.

1 – Introduction

Let X be a separable Banach space, $\sigma > 0$ and $\mathcal{C}_\sigma := \mathcal{C}([-\sigma, 0], X)$ the Banach space of the continuous functions from $[-\sigma, 0]$ into X, endowed with the norm $\|\varphi\|_\sigma := \sup\{\|\varphi(s)\|;\ s \in [-\sigma, 0]\}$. If $u \in \mathcal{C}([\tau-\sigma, T], X)$ is a given function then, for each $t \in [\tau, T]$, we define the function $u_t \in \mathcal{C}_\sigma$ by

$$u_t\colon [-\sigma, 0] \to X\ , \qquad u_t(s) = u(t + s), \quad \text{for every } s \in [-\sigma, 0]\ .$$

If K is a given subset in X then we introduce the following set $\mathcal{K}_0 := \{\varphi \in \mathcal{C}_\sigma;\ \varphi(0) \in K\}$.

Let $\mathcal{I} := [a, b)$ be given, $F\colon \mathcal{I}\times\mathcal{C}_\sigma \to 2^X$ a multifunction with nonempty and closed values and K a nonempty subset in X. We consider the following differential inclusions

$$u'(t) \in F(t, u_t)\ , \qquad t \in \mathcal{I} \tag{1.1}$$

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and we are interested in finding sufficient conditions in order for K to be a *viable domain* for (1.1) i.e. that for each $(\tau,\varphi)\in\mathcal{I}\times\mathcal{K}_0$ there exists at least one solution $u\colon[\tau-\sigma,T]\to X$ of (1.1) satisfying the initial condition

$$u_\tau = \varphi \tag{1.2}$$

and such that $u(t)\in K$ for every $t\in[\sigma,T]$.

We recall that a continuous function $u\colon[\tau-\sigma,T]\to X$, is said to be a *solution* of (1.1) and (1.2) if there exists $f\in L^1([\tau,T],X)$ with $f(t)\in F(t,u_t)$ a.e. on $[\tau,T]$ such that

$$u(t) = \begin{cases} \varphi(t-\tau)\,, & t\in[\tau-\sigma,\tau] \\ \varphi(0)+\displaystyle\int_\tau^t f(s)\,ds\,, & t\in[\tau,T]\,. \end{cases} \tag{1.3}$$

The existence of the viable solutions for the differential inclusion (1.1), in the case in which F is single-valued, were studied by many authors. For result and references in this framework see [1], [3], [11], [12] and [13].

The first viability result for differential inclusions with memory were given by Haddad [8], [9] in the case in which F is upper semi-continuous and with convex compact values and X is a finite dimensional space. The Haddad's result has been extended by Syam [14] and Gavioli and Malaguti [6] in the case in which X is a separable Banach space.

As is well known, any viability result need a tangential conditions in order to keep the trajectory $u(t)$ inside in K. The tangential conditions use in the papers mentioned above are given in terms of classical contingent cone (Bouligand–Severi cone).

The aim of this paper is to established a viable result for non-convex differential inclusion (1.1) using the same kind of tangential condition that in Duc Ha [7], accordingly adapted. Also, the construction method for a sequence of approximate solutions of (1.1), defined on an apriori given interval, is closed to the one used by Cârjă and Vrabie [4].

2 – Preliminaries and main result

In this paper we denote by X a separable Banach space with the norm $\|\,.\,\|$ and by $\mathcal{C}(X)$ the family of nonempty closed subset of X. For the subset $A,B\in\mathcal{C}(X)$ and for $a\in A$ we denote $d(a,B):=\inf\{\|a-b\|;\ b\in B\}$, $d(A,B):=$

$\sup\{d(a,B);\ a\in A\}$ and by $d_{HP}(A,B) := \max\{d(A,B), d(B,A)\}$ the Hausdorff–Pompeiu distance between A and B. Also, we denote by $\mathcal{L}$ the σ-field of the (Lebesque) measurable subset of $\mathcal{I} := [a,b)$.

We recall that a multifunction $G\colon \mathcal{I} \to \mathcal{C}(X)$ is called measurable if $\{t\in\mathcal{I};\ G(t)\cap V \neq \emptyset\} \in \mathcal{L}$ for each open $V\subset X$. Notice that the condition $\{t\in\mathcal{I};\ G(t)\subset V\}\in\mathcal{L}$ for each open V implies the measurability of G. For compact-valued multifunctions the reverse also holds (see Himmelberg [10, Theorem 3.1]).

In what follows we shall use the assumptions:

- (H_0) X is a separable Banach space, K is a locally closed subset in X and $F\colon \mathcal{I}\times\mathcal{K}_0 \to 2^X$ is a nonempty and closed values multifunction;
- (H_1) For each $(\tau,\varphi)\in\mathcal{I}\times\mathcal{K}_0$ there exist $\rho>0$, $r>0$ and $\chi\in L^1([\tau,\tau+\rho],\mathbb{R}_+)$ such that
$$\sup\Big\{|F(t,\psi)|;\ \psi\in\mathcal{K}_0\times B_\sigma(\varphi,r)\Big\} \le \chi(t)$$
a.e. on $[\tau,\tau+\rho]$, where $|F(t,\varphi)| := \sup\{\|y\|;\ y\in F(t,\psi)\}$ and
$$B_\sigma(\varphi,r) := \Big\{\psi\in\mathcal{C}_\sigma;\ \|\psi-\varphi\|\le r\Big\};$$
- (H_2) For each $(\tau,\varphi)\in\mathcal{I}\times\mathcal{K}_0$ there exist $\rho>0$, $r>0$ and $\mu\in L^1([\tau,\tau+\rho],\mathbb{R}_+)$ and a negligible subset $\mathcal{Z}\subset[\tau,\tau+\rho]$ such that
$$d_{HP}\big(F(t,\varphi_1),F(t,\varphi_2)\big) \le \mu(t)\,\|\varphi_1-\varphi_2\|_\sigma$$
for every $t\in[\tau,\tau+\rho]\setminus\mathcal{Z}$ and every $\varphi_1,\varphi_2\in\mathcal{K}_0\times B_\sigma(\varphi,r)$;
- (H_3) For each $\varphi\in\mathcal{K}_0$,the multifunction $F(\cdot,\varphi)\colon \mathcal{I}\to 2^X$ is $\mathcal{L}$-measurable;
- (H_4) For every $(\tau,\varphi)\in\mathcal{I}\times\mathcal{K}_0$ and for every locally integrable selection $f(\cdot)\in F(\cdot,\varphi)$ holds the following tangential condition:
$$\liminf_{h\downarrow 0}\frac{1}{h}\,d\bigg(\varphi(0)+\int_\tau^{\tau+h} f(s)\,ds,\ K\bigg) = 0\ .$$

We are now ready to state the main result of this paper.

Theorem 2.1. *If (H_0)–(H_4) are satisfied, then K is a viable domain for (1.1).*

In order to prove our theorem we need the following technical result, concerning measurable multifunction in Banach spaces, established by Q.I. Zhu [15].

Theorem 2.2. *Let X be a separable Banach space, $\psi\colon [a,b) \to X$ a measurable function and $G(\cdot)\colon [a,b) \to 2^X$ a measurable multifunction with nonempty and closed values. Then for any positive measurable function $\nu\colon [a,b) \to R_+$, there exists a measurable selection $g(\cdot) \in G(\cdot)$ such that*

$$\big\|g(t) - \psi(t)\big\| \;\le\; d\big(\psi(t), G(t)\big) + \nu(t)$$

a.e. on $[a,b)$. ∎

In the following we recall a general principle on ordered sets due to Brézis and Browder [2]. It will be use in the next section in order to obtain some "maximal" elements in an ordered set.

Theorem 2.3. *Let $\preccurlyeq$ be a given preorder on the nonempty set M and $S\colon M \to R \cup \{+\infty\}$ be an increasing function. Suppose that each increasing sequence in M is majorated in M. Then, for each $\xi_0 \in M$, there exists $\overline{\xi} \in M$ with $\xi_0 \preccurlyeq \overline{\xi}$ such that $\overline{\xi} \preccurlyeq \xi$ implies $S(\overline{\xi}) = S(\xi)$.* ∎

In the paper by Brézis and Browder, the function $\mathcal{S}$ is supposed to be finite and bounded from above, but, as remarked in [5], this restriction can be removed by replacing the function $\mathcal{S}$ by $\xi \to \arctan(\mathcal{S}(\xi))$.

Finally, let u a function defined on interval $\mathcal{J}$ of $\mathbb{R}$ with values into X. Thus, for some $\delta > 0$, we denote by $\omega(u, \mathcal{J}_0, \delta)$ the *modulus of continuity* of a function u defined on interval $\mathcal{J}_0 \subset \mathcal{J}$, given by

$$\omega(u, \mathcal{J}_0, \delta) \;=\; \sup\Big\{\big\|u(t) - u(s)\big\|;\;\; t, s \in \mathcal{J}_0,\;\; |t-s| \le \delta\Big\}\;.$$

3 – Proof of the main result

We shall show that the tangential conditions (H_4) and Theorem 2.3 imply that, for each $(\tau, \varphi) \in \mathcal{I}\times\mathcal{K}_0$, there exists one sequence $u^n\colon [\tau-\sigma, T] \to X$ of "approximate solutions" of (1.1) and (1.2), defined on same interval, such that $(u^n)_n$ converges in some sense to a solution of (1.1) satisfying (1.2).

Assume that the hypotheses (H_0)–(H_4) are satisfied and we begin by fixing an arbitrary initial data $(\tau, \varphi) \in \mathcal{I}\times\mathcal{K}_0$. Since the hypotheses (H_1) and (H_2) have a locally character and K is locally closed we can choose $r > 0$, $\rho \in (0, b-\tau)$, χ and μ in $L^1([\tau, \tau+\rho], \mathbb{R}_+)$ such that $K \cap B(\varphi(0), r)$ is closed in X and the relations (2.1) and (2.2) are satisfied on $[\tau, \tau+\rho] \times B_\sigma(\varphi, r)$. We emphasize that this choice for r, ρ, χ and μ is same along of this paper.

Remark 3.1. The following statements hold:

(**i**) If $\alpha \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ then $\alpha(0) \in K \cap B(\varphi(0), r)$,

(**ii**) If $K \cap B(\varphi(0), r)$ is closed in X then $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ is closed in $\mathcal{C}_\sigma$. □

Indeed, the first statement is obvious. For the second statement, we assume that $K \cap B(\varphi(0), r)$ is closed in X and we consider a sequence $(\alpha_n)_n$ in $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ that is convergent (in the norm $\|\,.\,\|_\sigma$) to $\alpha \in \mathcal{C}_\sigma$. Then follows that $\alpha \in B_\sigma(\varphi, r)$, $\alpha_n(0) \to \alpha(0)$ and $\alpha_n(0) \in K \cap B(\varphi(0), r)$; therefore, since $K \cap B(\varphi(0), r)$ is closed, we obtain that $\alpha(0) \in K$ and thus $\alpha \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$.

In the following, we denote by $\overline{\chi}\colon [\tau, \tau+\rho] \to \mathbb{R}_+$ the function defined by

$$\overline{\chi}(t) = \int_\tau^t \chi(s)\, ds\,, \qquad \text{for every } \ t \in [\tau, \tau+\rho] \tag{3.1}$$

and with $\widetilde{\omega}\colon \mathbb{R}_+ \to \mathbb{R}_+$ the function defined by

$$\widetilde{\omega}(\delta) = \omega\big(\varphi, [-\sigma, 0], \delta\big) + \omega\big(\overline{\chi}, [\tau, \tau+\rho], \delta\big) + \delta\,, \tag{3.2}$$

for every $\delta > 0$.

It is obvious that $\widetilde{\omega}$ is increasing and $\lim_{\delta \downarrow 0} \widetilde{\omega}(\delta) = 0$.

We shall define the "approximate solution" concept.

Definition 3.1. Let $\varepsilon \in (0,1)$ and $\psi \in L^1([\tau, \tau+\rho], X)$ be arbitrary fixed. By the (ε, ψ)-approximate solution of (1.1) and (1.2), defined on an interval $[\tau - \sigma, \nu] \subset [\tau - \sigma, \tau + \rho]$, we mean a 4-tuple (θ, g, f, u) that is compose of the functions $\theta\colon [\tau, \nu] \to [\tau, \nu]$, $g \in L^\infty([\tau, \nu], X)$, $f \in L^1([\tau, \nu], X)$ and of the continuous function $u\colon [\tau - \sigma, \nu] \to X$ defined by

$$u(t) = \begin{cases} \varphi(t-\tau)\,, & t \in [\tau-\sigma, \tau]\,, \\ \varphi(0) + \displaystyle\int_\tau^t f(s)\, ds + \int_\tau^t g(s)\, ds\,, & t \in [\tau, \nu]\,, \end{cases} \tag{3.3}$$

such that:

(A_1) $u_{\theta(t)} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ for every $t \in [\tau, \nu]$;

(A_2) $0 \le t - \theta(t)$ and $\tilde{\omega}(t - \theta(t)) \le \varepsilon$ for every $t \in [\tau, \nu]$;

(A_3) $\|g(t)\| \le \varepsilon$ a.e. on $[\tau, \nu]$;

(A_4) $f(t) \in F(t, u_{\theta(t)})$ a.e. on $[\tau, \nu]$;

(A_5) $\big\|f(t) - \psi(t)\big\| \le d\big(\psi(t), F(t, u_{\theta(t)})\big) + \varepsilon\mu(t)$ a.e. on $[\tau, \nu]$;

(A_6) $u_\nu \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$. □

Remark 3.2. We emphasize that for to define an (ε,ψ)-approximate solution it is sufficiently to indicate the interval $[\tau-\sigma,\nu]\subset[\tau-\sigma,\tau+\rho]$ and the functions θ, g and f. Although the function u is uniquely determined by g and f, for sake of simplicity, we preferred to consider it is a component of (θ,g,f,u). □

Remark 3.3. Let $\nu\in(\tau,\tau+\rho]$, $g\in L^\infty([\tau,\nu],X)$, $f\in L^1([\tau,\nu],X)$ and $u\colon[\tau-\sigma,\nu]\to X$ defined by (3.3). If $\|f(t)\|\le\chi(t)$ and $\|g(t)\|\le 1$ a.e. on $[\tau,\nu]$ then for every $t,s\in[\tau,\nu]$ we have

$$\|u_t-u_s\|_\sigma\le\tilde\omega\big(|t-s|\big)\,. \quad □ \tag{3.4}$$

Indeed, for every $t,s\in[\tau,\nu]$ we have

$$\begin{aligned}\|u_t-u_s\|_\sigma &= \sup_{\alpha\in[-\sigma,0]}\big\|u_t(\alpha)-u_s(\alpha)\big\|\\ &= \sup_{\alpha\in[-\sigma,0]}\big\|u(t+\alpha)-u(s+\alpha)\big\|\\ &\le \omega\big(u,[\tau-\sigma,\nu],|t-s|\big)\\ &\le \omega\big(u,[\tau-\sigma,\tau],|t-s|\big)+\omega\big(u,[\tau,\nu],|t-s|\big)\ .\end{aligned}$$

Further on, from $u_\tau=\varphi$ it follows that

$$\omega\big(u,[\tau-\sigma,\tau],|t-s|\big)=\omega\big(\varphi,[-\sigma,0],|t-s|\big)\ .$$

On the other hand, by definition of u on $[\tau,\nu]$ and (3.1), we have

$$\big\|\varphi(t)-\varphi(s)\big\|\le\left|\int_s^t\chi(\rho)\,d\rho\right|+|t-s|\ <\ \big|\overline{\chi}(t)-\overline{\chi}(s)\big|+|t-s|$$

and so

$$\omega\big(u,[\tau,\nu],|t-s|\big)\le\omega\big(\overline{\chi},[\tau,\tau+\rho],|t-s|\big)+|t-s|\ .$$

Therefore

$$\|u_t-u_s\|_\sigma\ \le\ \omega\big(\varphi,[-\sigma,0],|t-s|\big)+\omega\big(\overline{\chi},[\tau,\tau+\rho],|t-s|\big)+|t-s|\ ,$$

hence (3.4).

Remark 3.4. Let (θ,g,f,u) be an (ε,ψ)-approximate solution of (1.1) and (1.2) defined on $[\tau-\sigma,\nu]\subset[\tau-\sigma,\tau+\rho]$. By ($A_1$), ($A_3$) and ($A_4$) follows that $\|f(t)\|\le\chi(t)$ and $\|g(t)\|\le 1$ a.e. on $[\tau,\nu]$ and by Remark 3.3 and (A_2) we deduce that

$$\big\|u_t-u_{\theta(t)}\big\|_\sigma\le\varepsilon\,,\qquad\text{for every}\quad t\in[\tau,\nu]\ .\quad □ \tag{3.5}$$

Further on, we show how to define an (ε,ψ)-approximate solution of (1.1) and (1.2) defined on an interval $[\tau-\sigma,T]$ with $T\in(\tau,\tau+\rho]$.

Lemma 3.1. *Assume that the hypotheses* (H_0)*–*(H_4) *are satisfied. There exists* $T\in(\tau,\tau+\rho]$ *with* $\int_\tau^T\mu(s)\,ds\le\frac{1}{2}$ *such that for every* $\varepsilon\in(0,1)$ *and for every* $\psi\in L^\infty([\tau,\tau+\rho],X)$ *the problem (1.1) and (1.2) have at least one* (ε,ψ)*-approximate solution on* $[\tau-\sigma,T]$.

Proof: We fixed $T\in(\tau,\tau+\rho]$ such that

$$\tilde\omega(T-\tau)\le r \quad\text{ and }\quad \int_\tau^T\mu(s)\,ds\le 1/2\ . \tag{3.6}$$

This choice is always possible because $\mu\in L^1([\tau,\tau+\rho],\mathbb{R}_+)$ and $\lim_{\delta\downarrow0}\tilde\omega(\delta)=0$.

We denote by $\mathcal{M}_T$ the set of all (ε,ψ)-approximate solutions (θ,g,f,u) on $[\tau-\sigma,\nu]\subset[\tau-\sigma,T]$ and we show that $\mathcal{M}_T$ is nonempty set.

Applying Theorem 2.2 to $G(.)=F(.,\varphi)$ on $[\tau,\tau+\rho]$ we obtain that there exists a measurable function $\overline{f}\colon[\tau,\tau+\rho]\to X$ such that $\overline{f}(t)\in F(t,\varphi)$ a.e. on $[\tau,\tau+\rho]$ and

$$\big\|\bar f(t)-\psi(t)\big\|\le d\big(\psi(t),F(t,\varphi)\big)+\varepsilon\,\mu(t)\qquad\text{a.e. on }\ [\tau,\tau+\rho]\ .$$

Moreover, from (H_1) we obtain that $\|\overline{f}(t)\|\le\chi(t)$ a.e. on $[\tau,\tau+\rho]$ and therefore $\bar f\in L^1([\tau,\tau+\rho],X)$. Using tangential condition (H_4) applied at $(\tau,\varphi)\in\mathcal{I}\times\mathcal{K}_0$ for integrable selection $\overline{f}(.)\in F(.,\varphi)$ we obtain that there exist $(h_n)_n$ in $\mathbb{R}_+$ with $h_n\downarrow0$ and $(q_n)_n$ in X with $q_n\to0$ such that

$$\varphi(0)+\int_\tau^{\tau+h_n}\bar f(s)\,ds+h_nq_n\,\in\,K\,,\qquad\text{for every }\ n\in\mathbb{N}\ . \tag{3.7}$$

Since $\lim_{\delta\downarrow0}\tilde\omega(\delta)=0$ we can fix $n_0\in\mathbb{N}$ such that $h_{n_0}\in(0,T-\tau]$, $\tilde\omega(h_{n_0})\le\varepsilon$ and $\|q_{n_0}\|\le\varepsilon$. For n_0 fixed as above, we define: $\nu_0:=\tau+h_{n_0}$, $\theta(t):=\tau$ for every $t\in[\tau,\nu_0]$, $g(t):=q_{n_0}$ and $f(t):=\bar f(t)$ a.e. on $[\tau,\nu_0]$ and we show that (θ,g,f,u), with u defined by (3.3), is an (ε,ψ)-approximate solution on $[\tau-\sigma,\nu_0]\subset[\tau-\sigma,T]$.

Indeed, it is easily to see that the conditions (A_1)–(A_5) are fulfilled. Then $\|f(t)\|\le\chi(t)$ and $\|g(t)\|\le\varepsilon\le1$ a.e. $t\in[\tau,\nu_0]$ and therefore, by (3.4) and (3.6), we have

$$\|u_{\nu_0}-\varphi\|_\sigma\;=\;\|u_{\nu_0}-u_\tau\|_\sigma\;\le\;\tilde\omega(h_{n_0})\;\le\;\tilde\omega(T-\tau)\;\le\;r\ ,$$

hence $u_{\nu_0} \in B_\sigma(\varphi, r)$. Since, by (3.3) and (3.7), we have

$$u_{\nu_0}(0) \;=\; u(\nu_0) \;=\; \varphi(0) + \int_\tau^{\tau+h_{n_0}} \overline{f}(s)\,ds + h_{n_0} q_{n_0} \;\in\; K\;,$$

it follows that $u_{\nu_0} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ and thus (A_6) is also satisfied. Therefore, (θ, g, f, u) is an (ε, ψ)-approximate solution on $[\tau-\sigma, \nu_0]$ and thus we have that $\mathcal{M}_T$ is a nonempty set.

Next, we show that there exists at least one (ε, ψ)-approximate solution of (1.1) and (1.2), defined on the whole interval $[\tau-\sigma, T]$. For this aim we shall use Theorem 2.3 as follows. On $\mathcal{M}_T$ we introduce a preorder as follows.

If $(\theta^1, g^1, f^1, u^1)$ and $(\theta^2, g^2, f^2, u^2)$ are two (ε, ψ)-approximate solutions on $[\tau-\sigma, \nu^1]$ and respectively on $[\tau-\sigma, \nu^2]$, then we say that

$$(\theta^1, g^1, f^1, u^1) \preccurlyeq (\theta^2, g^2, f^2, u^2)$$

if and only if $\nu^1 \le \nu^2$, $\theta^1(t) = \theta^2(t)$, $g^1(t) = g^2(t)$ and $f^1(t) = f^2(t)$ on $[\tau, \nu^1]$.

Let us define the function $\mathcal{S}\colon \mathcal{M}_T \to \mathbb{R}$ by

$$\mathcal{S}\big((\theta, g, f, u)\big) \;=\; \nu\;,$$

for every (ε, ψ)-approximate solution defined on $[\tau-\sigma, \nu] \subset [\tau-\sigma, T]$.

It is clear that $\mathcal{S}$ is increasing on $\mathcal{M}_T$. Further on, we show that each increasing sequence $((\theta^i, g^i, f^i, u^i))_{i\in\mathbb{N}}$ in $\mathcal{M}_T$ is majorated in $\mathcal{M}_T$. We define a majorant as follows. We define

$$\nu^* \;=\; \lim_i \nu^i$$

and we observe that $\nu^* \in (\tau, T]$. For each $i \in \mathbb{N}$, we define $\theta^*(t) = \theta^i(t)$ if $t \in [\tau, \nu^i]$ and $\theta^*(\nu^*) = \nu^*$, $g^*(t) = g^i(t)$ and $f^*(t) = f^i(t)$ if $t \in [\tau, \nu^i]$, and we observe that, by the fact that $((\theta^i, g^i, f^i, u^i))_{i\in\mathbb{N}}$ is an increasing sequence in $\mathcal{M}_T$, the functions θ^*, g^*, and f^* are well defined. Moreover, since for every $i \in \mathbb{N}$ we have that $\|f^i(t)\| \le \chi(t)$ and $\|g^i(t)\| \le \varepsilon$ a.e. on $[\tau, \nu^i]$ it follows that

$$\|f^*(t)\| \le \chi(t) \quad\text{and}\quad \|g^*(t)\| \le \varepsilon \qquad \text{a.e. on } [\tau, \nu^*] \tag{3.8}$$

and thus we obtain that $g^* \in L^\infty([\tau, \nu^*], X)$ and $f^* \in L^1([\tau, \nu^*], X)$.

It is obvious that $\theta^*\colon [\tau, \nu^*] \to [\tau, \nu^*]$. Therefore, we can consider the 4-tuple $(\theta^*, g^*, f^*, u^*)$ with the function $u^*\colon [\tau-\sigma, \nu^*] \to X$ defined by (3.3). Now, we show that $(\theta^*, g^*, f^*, u^*) \in \mathcal{M}_T$. For this, we fixed an arbitrary $i \in \mathbb{N}$ and we observe that for every $t \in [\tau-\sigma, \tau]$ we have $u^*(t) = \varphi(t-\tau) = u^i(t)$ and for every $t \in [\tau, \nu^i]$ we have

$$\begin{aligned} u^*(t) &= \varphi(0) + \int_\tau^t f^*(s)\,ds + \int_\tau^t g^*(s)\,ds \\ &= \varphi(0) + \int_\tau^t f^i(s)\,ds + \int_\tau^t g^i(s)\,ds \;=\; u^i(t) \ . \end{aligned}$$

Therefore, $u^*(t) = u^i(t)$ for every $t \in [\tau-\sigma, \nu^i]$. Moreover, since for every $t \in [\tau, \nu^i]$ and every $s \in [-\sigma, 0]$ we have

$$\tau - \sigma \;\le\; \theta^*(t) + s \;=\; \theta^i(t) + s \;\le\; t + s \;\le\; t \;\le\; \nu^i$$

we obtain that

$$u^*_{\theta^*(t)}(s) \;=\; u^*\big(\theta^*(t)+s\big) \;=\; u^*\big(\theta^i(t)+s\big) \;=\; u^i\big(\theta^i(t)+s\big) \;=\; {u^i}_{\theta^i(t)}(s)$$

and thus

$$u^*_{\theta^*(t)} = u^i_{\theta^i(t)} \qquad \text{for every} \;\; t \in [\tau, \nu^i] \ . \tag{3.9}$$

Further on, let us observe that $(\theta^*, g^*, f^*, u^*)$ satisfies (A_2)–(A_5).

Let us verify the conditions (A_1) and (A_6). For any $t \in [\tau, \nu^*)$ there exists $i \in \mathbb{N}$ such that $t \in [\tau, \nu^i]$ and by (3.9) it follows that

$$u^*_{\theta^*(t)} = {u^i}_{\theta^i(t)} \;\in\; \mathcal{K}_0 \cap B_\sigma(\varphi, r) \ .$$

For $t = \nu^*$ we have $\theta^*(\nu^*) = \nu^*$ and $u^*_{\theta^*(\nu^*)} = u^*_{\nu^*}$. Then, by (3.8), we can use the relation (3.4) that, together with (3.6), implies

$$\|u^*_{\nu^*} - \varphi\|_\sigma \;=\; \|u^*_{\nu^*} - u^*_\tau\|_\sigma \;\le\; \tilde{\omega}(\nu^* - \tau) \;\le\; r \ .$$

and thus $u^*_{\theta^*(\nu^*)} = u^*_{\nu^*} \in B_\sigma(\varphi, r)$.

By the continuity of u^* we have

$$u^*_{\nu^*}(0) \;=\; u^*(\nu^*) \;=\; \lim_i u^*(\nu^i) \;=\; \lim_i u^i(\nu^i)$$

and since $u^i_{\nu^i} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ for every $i \in \mathbb{N}$ we have that $u^i_{\nu^i}(0) \in K \cap B(\varphi(0), r)$ for every $i \in \mathbb{N}$. Therefore, since $K \cap B(\varphi(0), r)$ is closed set we obtain that $u^*_{\nu^*}(0) \in K \cap B(\varphi(0), r)$ and hence we have that $u^*_{\theta^*(\nu^*)} = u^*_{\nu^*} \in \mathcal{K}_0$.

Thus we conclude that $(\theta^*, g^*, f^*, u^*) \in \mathcal{M}_T$. In addition $(\theta^i, g^i, f^i, u^i) \preccurlyeq (\theta^*, g^*, f^*, u^*)$ for each $i \in \mathbb{N}$ and thus the sequence $((\theta^i, g^i, f^i, u^i))_{i\in\mathbb{N}}$ is majorated in $\mathcal{M}_T$. Therefore, the set $\mathcal{M}_T$, endowed with the preorder $\preccurlyeq$, and the function $\mathcal{S}$ satisfies the hypotheses of Theorem 2.3.

Before to use the conclusion of Theorem 2.3, we show that every $(\theta, g, f, u) \in \mathcal{M}_T$ with $\mathcal{S}((\theta, g, f, u)) < T$ is majorated in $\mathcal{M}_T$ by an element $(\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u}) \in \mathcal{M}_T$ with $\mathcal{S}((\theta, g, f, u)) < \mathcal{S}((\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u}))$.

For this aim let us consider (θ, g, f, u) an (ε, ψ)-approximate solution defined $[\tau-\sigma, \nu]$ with $\nu \in (\tau, T)$. Since $u_\nu \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ we can apply the Theorem 2.3 on $[\nu, \tau+\rho]$ for $G(.) = F(., u_\nu)$. We obtain that there exists a measurable function $\overline{f}\colon [\nu, \tau+\rho] \to X$ such that $\overline{f}(t) \in F(t, u_\nu)$ a.e. on $[\nu, \tau+\rho]$ and

$$\|\overline{f}(t) - \psi(t)\| \le d\big(\psi(t), F(t, u_\nu)\big) + \varepsilon\,\mu(t) \qquad \text{a.e. on } \ [\nu, \tau+\rho]\ .$$

By (H_1) it follows that $\|\bar{f}(t)\| \le \chi(t)$ a.e. on $[\nu, \tau+\rho]$ and hence $\bar{f} \in L^1(\nu, \tau+\rho; X)$. Using tangential condition (H_4) applied at $(\nu, u_\nu) \in \mathcal{I}\times\mathcal{K}_0$ for integrable selection $\overline{f}(.) \in F(., u_\nu)$ we obtain that there exist $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$ and $(q_n)_n$ in X with $q_n \to 0$ such that

$$u_\nu(0) + \int_\nu^{\nu+h_n} \overline{f}(s)\,ds + h_n q_n \ \in\ K\,, \qquad \text{for every } \ n \in \mathbb{N}\ . \tag{3.10}$$

Since $\lim_{\delta\downarrow 0} \widetilde{\omega}(\delta) = 0$ we can fix $\widetilde{n} \in \mathbb{N}$ such that $h_{\widetilde{n}} \in (0, T-\tau]$, $\widetilde{\omega}(h_{\widetilde{n}}) \le \varepsilon$, and $\|q_{\widetilde{n}}\| \le \varepsilon$. Further on, we define $\widetilde{\nu} := \nu + h_{\widetilde{n}}$ and

$$\widetilde{\theta}(t) := \begin{cases} \theta(t) & \text{if } t \in [\tau, \nu]\,, \\ \nu & \text{if } t \in (\nu, \widetilde{\nu}]\ ; \end{cases}$$

$$\widetilde{g}(t) := \begin{cases} g(t) & \text{if } t \in [\tau, \nu]\,, \\ q_{\widetilde{n}} & \text{if } t \in (\nu, \widetilde{\nu}]\ ; \end{cases}$$

$$\widetilde{f}(t) := \begin{cases} f(t) & \text{if } t \in [\tau, \nu]\,, \\ \overline{f}(t) & \text{if } t \in (\nu, \widetilde{\nu}]\ . \end{cases}$$

We show that $(\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$, with $\widetilde{u}$ given by (3.3), is an (ε, ψ)-approximate solution defined on $[\tau-\sigma, \widetilde{\nu}] \subset [\tau-\sigma, T]$. First, we observe that

$$\widetilde{u}(t) = u(t) \qquad \text{for every } \ t \in [\tau-\sigma, \nu]$$

and

$$\begin{aligned}\widetilde{u}(t) &= \varphi(0) + \int_\tau^t \widetilde{f}(s)\,ds + \int_\tau^t \widetilde{g}(s)\,ds \\ &= \widetilde{u}(\nu) + \int_\nu^t \widetilde{f}(s)\,ds + \int_\nu^t \widetilde{g}(s)\,ds \\ &= u_\nu(0) + \int_\nu^t \overline{f}(s)\,ds \;+\; (t-\nu)\,q_{\widetilde{n}}\ ,\end{aligned}$$

for every $t \in [\nu, \widetilde{\nu}]$. Also, it is obvious that $(\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$ satisfies (A_2)–(A_5) on $[\tau, \nu]$ and on $(\nu, \widetilde{\nu}]$ they are satisfies by our choice of $\overline{f}$, $h_{\widetilde{n}}$ and $q_{\widetilde{n}}$. Since for every $t \in [\tau, \nu]$ we have

$$\widetilde{u}_{\widetilde{\theta}(t)} = \widetilde{u}_{\theta(t)} = u_{\theta(t)} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$$

and for every $t \in (\nu, \widetilde{\nu})$ we have

$$\widetilde{u}_{\widetilde{\theta}(t)} = \widetilde{u}_\nu = u_\nu \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)\ ,$$

we deduce that (A_1) is fulfilled.

Let us verify the condition (A_6). By (A_1), (A_3) and (A_4) we have that $\|f(t)\| \le \chi(t)$ and $\|g(t)\| \le \varepsilon \le 1$ a.e. on $[\tau, \widetilde{\nu}]$ and therefore we can use (3.4) that, together with (3.6), implies

$$\|\widetilde{u}_{\widetilde{\nu}} - \varphi\|_\sigma \;=\; \|\widetilde{u}_{\widetilde{\nu}} - \widetilde{u}_\tau\|_\sigma \;\le\; \widetilde{\omega}(\widetilde{\nu} - \tau) \;\le\; \widetilde{\omega}(T - \tau) \;\le\; r$$

and thus $\widetilde{u}_{\widetilde{\nu}} \in B_\sigma(\varphi, r)$. Since by (3.2) and (3.10) we have

$$\widetilde{u}_{\widetilde{\nu}}(0) \;=\; \widetilde{u}(\widetilde{\nu}) \;=\; u_\nu(0) + \int_\nu^{\nu + h_{\widetilde{n}}} \overline{f}(s)\,ds + h_{\widetilde{n}} q_{\widetilde{n}} \;\in\; K\ ,$$

it follows that $\widetilde{u}_{\widetilde{\nu}} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$. Therefore, $(\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$ is an (ε, ψ)-approximate solution defined on $[\tau-\sigma, \widetilde{\nu}]$. Moreover, by construction, we have that $(\theta, g, f, u) \preccurlyeq (\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$ and $\mathcal{S}((\theta, g, f, u)) = \nu < \widetilde{\nu} = \mathcal{S}((\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u}))$.

Now, let $(\theta^0, g^0, f^0, u^0)$ be arbitrary fixed in $\mathcal{M}_T$. By Theorem 2.3 we deduce that there exists $(\theta, g, f, u) \in \mathcal{M}_T$, with $(\theta^0, g^0, f^0, u^0) \preccurlyeq (\theta, g, f, u)$, such that $\mathcal{S}((\theta, g, f, u)) = \mathcal{S}((\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u}))$, for each $(\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u}) \in \mathcal{M}_T$ with $(\theta, g, f, u) \preccurlyeq (\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$.

It follows that $\mathcal{S}((\theta, g, f, u)) = T$ because, contrary, by precedent step, there exists $(\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u}) \in \mathcal{M}_T$ with $(\theta, g, f, u) \preccurlyeq (\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$ and such that $\mathcal{S}((\theta, g, f, u)) < \mathcal{S}((\widetilde{\theta}, \widetilde{g}, \widetilde{f}, \widetilde{u}))$, that is in contradiction with our choice for (θ, g, f, u).

Thus we have proved that there exists an (ε, ψ)-approximate solution of (1.1) and (1.2) defined on the whole interval $[\tau - \sigma, T]$. ∎

We are now prepared to prove Theorem 2.1.

Proof of Theorem 2.1: Let $(\tau,\varphi)\in\mathcal{I}\times\mathcal{K}_0$ be fixed and we consider $T\in(\tau,\tau+\rho]$ given as in Lemma 3.1. We introduce now the *solution operator* $Q\colon L^1([\tau,T],X)\to C([\tau-\sigma,T],X)$ defined by

$$(3.11)\qquad (Qf)(t) = \begin{cases} \varphi(t-\tau) & \text{if } t\in[\tau-\sigma,\tau]\ , \\ \varphi(0)+\displaystyle\int_\tau^t f(s)\,ds & \text{if } t\in(\tau,T]\ . \end{cases}$$

We notice that u is a solution of (1.1) and (1.2) on $[\tau-\sigma,T]$ if there exists $f\in L^1([\tau,T],X)$ such that $u=Qf$ and $f(t)\in F(t,u(t))$ a.e. on $[\tau,T]$.

Let $(\varepsilon_n)_n$ be a decreasing sequence of positive real numbers such that $\sum_{n=1}^\infty \varepsilon_n<+\infty$ and $\varepsilon_n\in(0,1)$ for every $n\in\mathbb{N}$.

Starting with one measurable selection $f_0(.)\in F(.,\varphi)$ in view of Lemma 3.1, we can define inductively the sequence $((\theta^n,g^n,f^n,u^n))_{n\in\mathbb{N}}$ such that (θ^n,g^n,f^n,u^n) is an (ε_n,f^n)-approximate solution on $[\tau-\sigma,T]$ for every $n\in\mathbb{N}$.

Thus, for every $n\in\mathbb{N}$ we have

$$(3.12)\qquad u^n(t) = \begin{cases} \varphi(t-\tau) & \text{if } t\in[\tau-\sigma,\tau]\ , \\ \varphi(0)+\displaystyle\int_\tau^t f^n(s)\,ds+\int_\tau^t g^n(s)\,ds & \text{if } t\in(\tau,T] \end{cases}$$

and

- (B_1) $u^n_{\theta^n(t)}\in\mathcal{K}_0\cap B_\sigma(\varphi,r)$ for every $t\in[\tau,T)$;
- (B_2) $0\le t-\theta^n(t)$ and $\tilde\omega(t-\theta^n(t))\le\varepsilon$ for every $t\in[\tau,T]$;
- (B_3) $\|g^n(t)\|\le\varepsilon_n$ a.e. on $[\tau,T]$;
- (B_4) $f^n(t)\in F(t,u^n_{\theta^n(t)})$ a.e. on $[\tau,T]$;
- (B_5) $\big\|f^n(t)-f^{n-1}(t)\big\|\le d\big(f^{n-1}(t),F(t,u^n_{\theta^n(t)})\big)+\varepsilon_n\,\mu(t)$ a.e. on $[\tau,T]$;
- (B_6) $u^n_T\in\mathcal{K}_0\cap B_\sigma(\varphi,r)$.

We show that $(u^n)_n$ converge uniformly to a function $u\colon[\tau-\sigma,T]\to X$ that is a solution of (1.1) and (1.2).

For this, first we show that for every $n\in\mathbb{N}$ we have

- (C_1) $\|f^n(t)\|\le\chi(t)$ a.e. on $[\tau,T]$;
- (C_2) $\big\|u^n_t-u^n_{\theta^n(t)}\big\|_\sigma\le\varepsilon_n$ for every $t\in[\tau,T]$;
- (C_3) $\big\|u^n(t)-(Qf^n)(t)\big\|\le(T-\tau)\varepsilon_n$ for every $t\in[\tau-\sigma,T]$;

(C_4) $\big\|u^{n+1}_{\theta^{n+1}(t)} - u^n_{\theta^n(t)}\big\|_\sigma \le 2\,\varepsilon_n + \|u^{n+1} - u^n\|_T$, for every $t \in [\tau, T]$, where $\|\cdot\|_T$ is norm in $C([\tau-\sigma, T]; X)$;

(C_5) $\big\|f^{n+1}(t) - f^n(t)\big\| \le \mu(t)\big(\|u^{n+1} - u^n\|_T + 3\,\varepsilon_n\big)$ a.e. on $[\tau, T]$.

Indeed, (C_1) follows from (H_1) and (B_4), (C_2) follows from Remark 3.4, and (C_3) follows from (3.11), (3.12) and (B_3). For to show (C_4) we observe that

$$\begin{aligned}\big\|u^{n+1}_t - u^n_t\big\|_\sigma &= \sup_{-\sigma\le s\le 0} \big\|u^{n+1}(t+s) - u^n(t+s)\big\| \\ &\le \sup_{\tau-\sigma\le\nu\le T} \big\|u^{n+1}(\nu) - u^n(\nu)\big\| \\ &= \big\|u^{n+1} - u^n\big\|_T\end{aligned}$$

and thus by (C_2) we obtain that

$$\begin{aligned}\big\|u^{n+1}_{\theta^{n+1}(t)} - u^n_{\theta^n(t)}\big\|_\sigma &\le \big\|u^{n+1}_{\theta^{n+1}(t)} - u^{n+1}_t\big\|_\sigma + \big\|u^{n+1}_t - u^n_t\big\|_\sigma + \big\|u^n_t - u^n_{\theta^n(t)}\big\|_\sigma \\ &\le \varepsilon_{n+1} + \big\|u^{n+1}_t - u^n_t\big\|_\sigma + \varepsilon_n \\ &\le 2\,\varepsilon_n + \big\|u^{n+1} - u^n\big\|_T\end{aligned}$$

for every $t \in [\tau, T]$.

In finally, by (H_2), (B_5) and (C_4) we have

$$\begin{aligned}\big\|f^{n+1}(t) - f^n(t)\big\| &\le d\Big(f^n(t), F\big(t, u^{n+1}_{\theta^{n+1}(t)}\big)\Big) + \varepsilon_{n+1}\,\mu(t) \\ &\le d_{HP}\Big(F\big(t, u^n_{\theta^n(t)}\big), F\big(t, u^{n+1}_{\theta^{n+1}(t)}\big)\Big) + \varepsilon_{n+1}\,\mu(t) \\ &\le \mu(t)\,\Big(\big\|u^n_{\theta^n(t)} - u^{n+1}_{\theta^{n+1}(t)}\big\| + \varepsilon_{n+1}\Big) \\ &\le \mu(t)\,\Big(\|u^{n+1} - u^n\|_T + 3\,\varepsilon_n\Big)\end{aligned}$$

a.e. on $[\tau, T]$ and hence (C_5) is also checked.

Further on, for every $t \in [\tau, T]$, by (3.6), (3.11), (C_3) and (C_5) we have

$$\begin{aligned}&\big\|u^{n+1}(t) - u^n(t)\big\| \le \\ &\le \big\|u^{n+1}(t) - (Qf^{n+1})(t)\big\| + \big\|(Qf^{n+1})(t) - (Qf^n)(t)\big\| + \big\|(Qf^n)(t) - u^n(t)\big\| \\ &\le (T-\tau)\,(\varepsilon_{n+1} + \varepsilon_n) + \int_\tau^T \big\|f^{n+1}(s) - f^n(s)\big\|\,ds \\ &\le 2\,(T-\tau)\,\varepsilon_n + \Big(3\,\varepsilon_n + \big\|u^{n+1} - u^n\big\|_T\Big) \int_\tau^T \mu(s)\,ds \\ &\le \Big(2\,(T-\tau) + \frac{3}{2}\Big)\,\varepsilon_n + \frac{1}{2}\,\big\|u^{n+1} - u^n\big\|_T\,.\end{aligned}$$

Therefore, since $\|u^{n+1}(t) - u^n(t)\| = 0$ for every $t \in [\tau - \sigma, \tau]$ we obtain

$$\left\|u^{n+1}(t) - u^n(t)\right\| \le \left(2\,(T-\tau) + \frac{3}{2}\right)\varepsilon_n + \frac{1}{2}\left\|u^{n+1} - u^n\right\|_T$$

for every $t \in [\tau - \sigma, T]$ and so, we have

$$\left\|u^{n+1} - u^n\right\|_T \le \left(2\,(T-\tau) + \frac{3}{2}\right)\varepsilon_n + \frac{1}{2}\left\|u^{n+1} - u^n\right\|_T .$$

Thus we have that

$$\left\|u^{n+1} - u^n\right\|_T \le \Big(4\,(T-\tau) + 3\Big)\,\varepsilon_n \tag{3.13}$$

for every $n \in \mathbb{N}^*$ with $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$ and thus we deduce that $(u^n)_n$ converge uniformly to a function $u \colon [\tau - \sigma, T] \to X$.

From (C_5) and (3.13) we deduce that, for almost all $t \in [\tau, T]$, we have

$$\begin{aligned}\left\|f^{n+1}(t) - f^n(t)\right\| &\le \mu(t)\Big(\left\|u^{n+1} - u^n\right\|_T + 3\,\varepsilon_n\Big)\\ &\le \mu(t)\,\Big(4\,(T-\tau) + 6\Big)\,\varepsilon_n\end{aligned}$$

for every $n \in \mathbb{N}^*$. This imply that $(f^n)_n$ converge pointwise almost everywhere to a measurable function f. For any fixed $t \in [\tau - \sigma, T]$, by (C_1) and Lebesgue's Theorem, we obtain that $\lim_{n\to\infty}(Qf^n)(t) = (Qf)(t)$. Consequently, by (C_3), we conclude that $u(t) = (Qf)(t)$ for every $t \in [\tau - \sigma, T]$. For every $t \in [\tau, T]$ and every $n \in \mathbb{N}^*$, by (C_2), we have

$$\left\|u^n_{\theta^n(t)} - u_t\right\|_\sigma \le \left\|u^n_{\theta^n(t)} - u^n_t\right\|_\sigma + \left\|u^n_t - u_t\right\|_\sigma \le \varepsilon_n + \left\|u^n - u\right\|_T$$

and thus $u^n_{\theta^n(t)} \to u_t$ in $\mathcal{C}_\sigma$ as $n \to \infty$.

From (B_1) and Remark 3.1 it follows that $u_t \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ for every $t \in [\tau, T]$.

Now, we observe that, a.e. on $[\tau, T]$, we have

$$\begin{aligned}d\big(f(t), F(t, u_t)\big) &\le \left\|f(t) - f^n(t)\right\| + d\big(F\big(t, u^n_{\theta^n(t)}\big), F(t, u_t)\big)\\ &\le \left\|f(t) - f^n(t)\right\| + \mu(t)\left\|u^n_{\theta^n(t)} - u_t)\right\|_\sigma\end{aligned}$$

for every $n \in \mathbb{N}^*$. Therefore, by letting $n \to \infty$, we obtain that $d(f(t), F(t, u_t)) = 0$ and thus, because the multifunction F has closed values, $f(t) \in F(t, u_t)$ a.e on $[\tau, T]$.

Finally, from $u_t \in \mathcal{K}_0 \cap B_\sigma(\varphi(0), r)$, by Remark 3.1, we deduce that $u(t) \in K \cap B(\varphi(0), r)$ for every $t \in [\tau, T]$.

We have proved that $u\colon [\tau-\sigma, T] \to X$ is a solution of (1.1) and (1.2), and so, (τ, φ) being arbitrarily fixed in $\mathcal{I}\times\mathcal{K}_0$, we have showed that K is a viable domain for (1.1). ∎

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