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A NOTE ON BASIC FREE MODULES AND THE S_n CONDITION

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Abstract: A new definition of basic free submodule is provided in order to obtain a relationship between it and the S_n condition and it is then used to state an equivalent condition to the Bass–Quillen conjecture be held.

1 – Introduction

Basic element theory plays an important role in the study of several concepts in commutative ring theory (See [2, Chapter 2]). In this note we introduce an equivalent definition of basic free submodule and then explore its relationship to the Serre's S_n condition. Recall that a finitely generated R-module M satisfies the S_n condition if $\operatorname{depth} M_{\mathfrak{p}} \geq \min(n, \dim R_{\mathfrak{p}})$ for every $\mathfrak{p} \in \operatorname{Spec} R$.

Definition 1. Let M be a finitely generated R-module. Then a submodule $M' \subset M$ is called w-fold basic in M at $\mathfrak{p} \in \operatorname{Spec} R$ provided the number of generators of $(M/M')_{\mathfrak{p}}$ is less than or equal to the number of generators of $M_{\mathfrak{p}}$ minus w (see [2, Pag. 26]). □

Here we focus on a free submodule $F \subset M$ and we modify the previous definition in the following sense.

Definition 2. Let $(R, \mathfrak{m})$ denote a local ring and let M be a finitely generated R-module. A free submodule $F \subset M$ is called basic in M at $\mathfrak{p} \in \operatorname{Spec} R$

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provided the natural map

$$F \otimes_R K(\mathfrak{p}) \longrightarrow M \otimes_R K(\mathfrak{p})$$

is injective. We say that $F \subset M$ is a basic submodule up to height k whenever $F \subset M$ is basic in M at $\mathfrak{p} \in \operatorname{Spec} R$ for all $\mathfrak{p}$ such that height $\mathfrak{p} \leq k$. □

Remark. Clearly the notion that $F \subset M$ is basic in M at $\mathfrak{p}$ in Definition 2 is equivalent to the fact that $F \subset M$ is w-fold basic at $\mathfrak{p}$ with $w = \operatorname{rank} F$ in Definition 1. To this end recall that $w = \dim_{K(\mathfrak{p})} F \otimes_R K(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Furthermore, it is easily seen that the induced homomorphism

$$F \otimes_R K(\mathfrak{p}) \longrightarrow M \otimes_R K(\mathfrak{p})$$

is injective if and only if the number of generators of $(M/F)_{\mathfrak{p}}$ is less than or equal to the number of generators of $M_{\mathfrak{p}}$ minus w. □

Our purpose in this paper is to prove that a free submodule F is basic up to height k if and only if the quotient module M/F satisfies the S_k condition. Furthermore, using basic submodules up to height two, the above result allows us to state an equivalent condition to the Bass–Quillen conjecture.

2 – The main result

Theorem 1. *Let $(R, \mathfrak{m})$ be regular local ring and let M be a finitely generated R-module satisfying S_n condition. Then a free submodule $F \subset M$ is basic up to height $k \leq n$ if and only if M/F satisfies the S_k condition.*

Proof: First, assume that F is basic up to height k. Denoting $M' = M/F$ we thus have the short exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow M' \longrightarrow 0 .$$

Then we will prove in two stages that M' satisfies the S_k condition. We first show the assertion for $\mathfrak{p} \in \operatorname{Spec} R$ such that $\operatorname{ht} \mathfrak{p} \leq k$. Because of R is a regular ring, it is well known that the projective dimension of M, denoted by $\operatorname{pd}(M)$, is finite. Moreover, since by hypothesis M satisfies the S_n condition for $k \leq n$, by applying Auslander–Buchsbaum formula to $M_{\mathfrak{p}}$, we obtain $\operatorname{pd}(M_{\mathfrak{p}}) = 0$ since

$$\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq \min(n, \operatorname{ht} \mathfrak{p}) = \operatorname{ht} \mathfrak{p} .$$

Consequently, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module. On the other hand, since M is basic up to height k by virtue of the hypothesis, we can conclude that $F_{\mathfrak{p}}$ is a direct summand of $M_{\mathfrak{p}}$, i.e., $M_{\mathfrak{p}} = F_{\mathfrak{p}} \oplus M'_{\mathfrak{p}}$, where $M'_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module. Hence, it follows that

$$\operatorname{depth} M_{\mathfrak{p}} \ \geq\ \min(k, \dim R_{\mathfrak{p}})\ .$$

From now on, let us consider $\mathfrak{p} \in \operatorname{Spec} R$ such that $\operatorname{ht}\mathfrak{p} > k$. Taking into account that

$$\operatorname{depth} M_{\mathfrak{p}} \ \geq\ \min(n, \dim R_{\mathfrak{p}}) \ \geq\ k\ ,$$

we obtain

$$H^i_{\mathfrak{p}}(F_{\mathfrak{p}}) = 0 \qquad \text{for} \quad 0 \leq i \leq k-1\ .$$

Moreover, since $F_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module, it thus follows that

$$H^i_{\mathfrak{p}}(F_{\mathfrak{p}}) = 0 \qquad \text{for} \quad 0 \leq i \leq \dim R_{\mathfrak{p}}-1 = \operatorname{ht}\mathfrak{p} - 1 \geq k\ .$$

From the above we thus conclude

$$H^i_{\mathfrak{p}}(M'_{\mathfrak{p}}) = 0 \qquad \text{for} \quad 0 \leq i \leq k-1\ .$$

Therefore $\operatorname{depth} M'_{\mathfrak{p}} \geq \min(k, \dim R_{\mathfrak{p}})$ so that M' satisfies the S_k condition.

Conversely, suppose now that M' satisfies the S_k condition. Let us consider the exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow M' \longrightarrow 0$$

and let $\mathfrak{p} \in \operatorname{Spec} R$ be a prime ideal with $\operatorname{ht}\mathfrak{p} \leq k$. By virtue of hypothesis

$$\operatorname{depth} M'_{\mathfrak{p}} \ \geq\ \min(k, \dim R_{\mathfrak{p}}) \ =\ \dim R_{\mathfrak{p}}\ .$$

Hence, $M'_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module. Then, the following exact sequence of free $R_{\mathfrak{p}}$-modules

$$0 \longrightarrow F_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M'_{\mathfrak{p}} \longrightarrow 0$$

splits $M_{\mathfrak{p}} = F_{\mathfrak{p}} \oplus M'_{\mathfrak{p}}$. Hence, it follows that the morphism

$$F \otimes_R K(\mathfrak{p}) \ \longrightarrow\ M \otimes_R K(\mathfrak{p})$$

is injective. Therefore F is basic up to height k. ∎

3 – An application

Let R be a d-dimensional local regular ring, let now $R[x]$ be the polynomial ring in one variable over R and let P be a finitely generated projective $R[x]$-module of rank r. Then, taking into account that $R[x]$ is a normal ring, we can conclude by Theorem 2.14 in [2, Pag. 38] that there exists a free submodule $F \subset P$ of rank $r-1$ such that in the next exact sequence

$$0 \longrightarrow F \longrightarrow P \longrightarrow \mathfrak{a} = F/P \longrightarrow 0$$

F/P is isomorphic to an ideal $\mathfrak{a}$.

Theorem 2. *Assume that in the previous exact sequence F is basic up to height two. Then P is free.*

Proof: As is well known, $R[x]$ is a factorial ring. This implies that it suffices to prove that a reflexive ideal is free.

Let $\mathfrak{p}$ be a prime ideal of $R[x]$. According to Proposition 1.4.1 in [1, Pag. 19], first of all we will show that $\mathfrak{a}_\mathfrak{p}$ is free if $\operatorname{ht}\mathfrak{p} = 1$. In effect, since $R[x]_\mathfrak{p}$ is a regular local ring of dimension one and $\mathfrak{a}_\mathfrak{p}$ is torsion-free it follows immediately that $\mathfrak{a}_\mathfrak{p}$ is free.

On the other hand, by applying again the result 1.4.1 in [1], it must be

$$\operatorname{depth}\mathfrak{a}_\mathfrak{p} \geq 2 \qquad \text{for every ideal } \mathfrak{p} \text{ with } \operatorname{ht}\mathfrak{p} \geq 2\ .$$

But as the $R[x]_\mathfrak{p}$-module $\mathfrak{a}_\mathfrak{p}$ satisfies the S_2 condition, by applying the Theorem 1 we conclude that

$$\operatorname{depth}\mathfrak{a}_\mathfrak{p} \;\geq\; \min\bigl(2, \dim R[x]_\mathfrak{p}\bigr) \;\geq\; 2\ .$$

Hence it is deduced that $\mathfrak{a}$ is free. Then the split of above exact sequence yields $P = F \oplus \mathfrak{a}$. This easily implies that P must also be free. ∎

Remark. As it is well-known, the Bass–Quillen conjecture states that P is a free $R[x]$-module (see [3]). Now using Theorem 2 we can give the following version of this conjecture: "Let r be the rank of P. Then there exists a free submodule $F \subset P$ of rank $r-1$ with F/P isomorphic to an ideal such that F is basic up to height 2". □

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