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OSCILLATION OF DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract: In this study, under some appropriate conditions over the real sequences $\{p_n\}$ and $\{q_n\}$ we give some sufficient conditions for the oscillation of all solutions of the difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{r} p_{in}\, x_{n-k_i} + q_n\, x_{n-m} = 0 , \qquad m \in \{..., -2, -1, 0\}$$

where $k_i \in \mathbb{N}$ and $k_i \in \{..., -3, -2\}$ $(i = 1, 2, ..., r)$, respectively.

1 – Introduction

For the oscillation of every solution of the difference equation

$$x_{n+1} - x_n + p\, x_{n-k} + q\, x_{n-m} = 0 , \qquad m = -1, 0 , \tag{1.1}$$

necessary and sufficient conditions were given in [9]. The case $q = 0$ was examined in [4] and [7]. In the present paper, under some appropriate conditions, taking the real sequences $\{p_n\}$ and $\{q_n\}$ instead of p and q in equation (1.1) we investigate the oscillatory behaviour of the following difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{r} p_{in}\, x_{n-k_i} + q_n\, x_{n-m} = 0 , \qquad m \in \{..., -2, -1, 0\} \tag{1.2}$$

in cases of $k \in \mathbb{N}$ and $k \in \{..., -3, -2\}$, respectively.

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Note that the case $r = 1$, $q_n = 0$ (for all $n \in \mathbb{N}$) of equation (1.2) has been investigated in [3], [5] and [10]. Furthermore, recently for the oscillatory properties of constant coefficients form of (1.2) has been obtained in [11].

Let $\rho = \max\{k_i, m\}$ for $i = 1, 2, .., r$. Then we recall that a sequence $\{x_n\}$ which is defined for $n \geq -\rho$ and satisfies (1.2) for $n \geq 0$. A solution $\{x_n\}$ of equation (1.2) is called oscillatory if the terms x_n of the sequence $\{x_n\}$ are neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory (see, for details, [1], and also [2], [6]).

2 – Oscillation properties of equation (1.2)

In this section we obtain sufficient conditions for the oscillation of all solutions of the difference equation (1.2) when $m \in \{..., -2, -1, 0\}$, $p_{in}, q_n \in \mathbb{R}$, $k_i \in \mathbb{Z}-\{-1, 0\}$ for $i = 1, 2, ..., r$.

We first have the following result.

Theorem 2.1. *Let $k_i \in \mathbb{N}$, $p_{in} \geq 0$ and $m = -1$ for $i = 1, 2, ..., r$ in equation (1.2), and let $\liminf\limits_{n\to\infty} q_n = q > 0$. Assume further $\liminf\limits_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, ..., r$. If*

$$\sum_{i=1}^{r} p_i \, \frac{(1+q)^{k_i} \, (k_i+1)^{k_i+1}}{k_i^{k_i}} > 1 \,, \tag{2.1}$$

then every solution of (1.2) oscillates.

Proof: Assume that $\{x_n\}$ be an eventually positive solution of equation (1.2). Since $p_{in} \geq 0$ for all $i = 1, 2, ..., r$ and $q > 0$, we get from (1.2) that

$$x_{n+1} - x_n = -\sum_{i=1}^{r} p_{in} \, x_{n-k_i} - q_n \, x_{n+1} < 0 \,.$$

This yields that $\{x_n\}$ eventually decreasing. Now dividing (1.2) by $\{x_n\}$ we obtain

$$\frac{x_{n+1}}{x_n} = 1 - \sum_{i=1}^{r} p_{in} \, \frac{x_{n-k_i}}{x_n} - q_n \, \frac{x_{n+1}}{x_n} \,. \tag{2.2}$$

Let $z_n = \dfrac{x_n}{x_{n+1}}$. So, we have from (2.2) that

$$\frac{1}{z_n} = \frac{1}{1+q_n} \left\{ 1 - \sum_{i=1}^{r} p_{in} \big(z_{n-k_i}\, z_{n-k_i+1} \dots z_{n-1} \big) \right\} . \tag{2.3}$$

Let $\liminf\limits_{n\to\infty} z_n = z \geq 1$. Therefore, taking limit superior as $n \to \infty$ on the both sides (2.3) and using the fact that

$$\limsup_{n\to\infty} \frac{1}{z_n} = \frac{1}{\liminf\limits_{n\to\infty} z_n} = \frac{1}{z}$$

we have

$$\frac{1}{z} \leq \frac{1}{1+q} \left(1 - \sum_{i=1}^{r} p_i\, z^{k_i} \right) ,$$

which implies that $z \neq q+1$ and that

$$\sum_{i=1}^{r} p_i \, \frac{z^{k_i+1}}{z-q-1} \leq 1 . \tag{2.4}$$

Define the function f by $f(z) = \frac{z^{k_i+1}}{z-q-1}$. So, by (2.4) it is clear that

$$\sum_{i=1}^{r} p_i \, \frac{(1+q)^{k_i}\, (k_i+1)^{k_i+1}}{k_i^{k_i}} \leq 1 \tag{2.5}$$

which contradicts (2.1) and completes the proof. ∎

Since $\inf\limits_{n\in\mathbb{N}} p_n \leq \liminf\limits_{n\to\infty} p_n$, the following result follows from Theorem 2.1 immediately.

Corollary 2.2. *Let $k_i \in \mathbb{N}$, $m=-1$, $q_n > 0$ and $p_{in} \geq 0$ for $n \in \mathbb{N}$ $(i = 1, 2, \dots, r)$. If*

$$\sum_{i=1}^{r} \big(\inf_{n\in\mathbb{N}} p_{in} \big) \, \frac{\big(1+\inf\limits_{n\in\mathbb{N}} q_n\big)^{k_i}\, (k_i+1)^{k_i+1}}{k_i^{k_i}} > 1 ,$$

then every solution of equation (1.2) oscillates. ∎

Theorem 2.3. *Let $k_i \in \{\dots, -3, -2\}$, $p_{in} \leq 0$ and $m=-1$ in equation (1.2), and let $\limsup\limits_{n\to\infty} q_n = q \in (-1, 0)$. Assume further $\limsup\limits_{n\to\infty} p_{in} = p_i$ for $i=1, 2, \dots, r$. If condition (2.1) holds, then every solution of (1.2) oscillates.*

Proof: Assume that $\{x_n\}$ be an eventually positive solution of equation (1.2). Since $p_{in} \le 0$ and $q \in (-1,0)$, by (1.2) we have

$$x_{n+1} - x_n = -\sum_{i=1}^{r} p_{in}\, x_{n-k_i} - q_n\, x_{n+1} > 0 .$$

This yields that $\{x_n\}$ eventually increasing. Now dividing (1.2) by $\{x_n\}$ we get

$$\frac{x_{n+1}}{x_n} = 1 - q_n \frac{x_{n+1}}{x_n} - \sum_{i=1}^{r} p_{in} \frac{x_{n-k_i}}{x_n} . \tag{2.6}$$

Let $z_n = \dfrac{x_{n+1}}{x_n}$. Then, we have from (2.6) that

$$z_n = 1 - q_n z_n - \sum_{i=1}^{r} p_{in}\, z_{n-k_i-1}\, z_{n-k_i-2} \ldots z_n . \tag{2.7}$$

Now let $\liminf\limits_{n\to\infty} z_n = z \ge 1$. Taking limit inferior as $n \to \infty$ on the both sides (2.7), we get

$$z \ge 1 - qz - \sum_{i=1}^{r} p_i\, z^{-k_i} ,$$

which implies that $z \ne \dfrac{1}{q+1}$ and that

$$\sum_{i=1}^{r} p_i \frac{z^{-k_i}}{1-(q+1)z} \le 1 .$$

Then, it is obvious that

$$\sum_{i=1}^{r} p_i \frac{(1+q)^{k_i}\,(k_i+1)^{k_i+1}}{k_i^{k_i}} \le 1 ,$$

which contradicts condition (2.1). ∎

Since $\limsup\limits_{n\to\infty} p_n \le \sup\limits_{n\in\mathbb{N}} p_n$, the following result follows from Theorem 2.3 immediately.

Corollary 2.4. *Let* $k_i \in \{\ldots,-3,-2\}$, $m=-1$, $-1<q_n<0$ *and* $p_{in} \le 0$ *for* $n \in \mathbb{N}$ $(i=1,2,\ldots,r)$. *If*

$$\sum_{i=1}^{r} \bigl(\sup_{n\in\mathbb{N}} p_{in}\bigr) \frac{\bigl(1+\sup\limits_{n\in\mathbb{N}} q_n\bigr)^{k_i}\,(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1 ,$$

then every solution of equation (1.2) oscillates. ∎

Now, taking into consideration the methods of the proofs of preceding theorems one can easily obtain the following results. Hence, we merely state these results without their proofs.

Theorem 2.5. *Let $k_i \in \mathbb{N}$, $p_{in} \geq 0$ and $m = 0$ in equation (1.2), and let $\liminf\limits_{n\to\infty} q_n = q \in (0,1)$. Assume that $\liminf\limits_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, ..., r$. If the condition*

$$\sum_{i=1}^{r} p_i \frac{(k_i+1)^{k_i+1}}{(1-q)^{k_i+1}\, k_i^{k_i}} > 1 \tag{2.8}$$

holds, then every solution of (1.2) oscillates. ∎

Corollary 2.6. *Let $k_i \in \mathbb{N}$, $m = 0$, $0 < q_n < 1$ and $p_{in} \geq 0$ for $n \in \mathbb{N}$ $(i = 1, 2, ..., r)$. If*

$$\sum_{i=1}^{r} \big(\inf_{n\in\mathbb{N}} p_{in}\big) \frac{(k_i+1)^{k_i+1}}{\big(1-\inf\limits_{n\in\mathbb{N}} q_n\big)^{k_i+1} k_i^{k_i}} > 1\ ,$$

then every solution of equation (1.2) oscillates. ∎

Theorem 2.7. *Let $k_i \in \{..., -3, -2\}$, $p_{in} \leq 0$ and $m = 0$ in equation (1.2), and let $\limsup\limits_{n\to\infty} q_n = q < 0$. Assume that $\limsup\limits_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, ..., r$. If condition (2.8) holds, then every solution of (1.2) oscillates.* ∎

Corollary 2.8. *Let $k_i \in \{..., -3, -2\}$, $m = 0$, $q_n < 0$ and $p_{in} \leq 0$ for $n \in \mathbb{N}$ $(i = 1, 2, ..., r)$. If the condition*

$$\sum_{i=1}^{r} \big(\sup_{n\in\mathbb{N}} p_{in}\big) \frac{(k_i+1)^{k_i+1}}{\big(1-\sup\limits_{n\in\mathbb{N}} q_n\big)^{k_i+1} k_i^{k_i}} > 1$$

holds, then every solution of equation (1.2) oscillates. ∎

Theorem 2.9. *Let $k_i \in \mathbb{N}$, $m \in \{...,-3,-2\}$, $q_n > 0$. Assume that $\liminf\limits_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, ..., r$. If the condition*

$$\sum_{i=1}^{r} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1 \tag{2.9}$$

holds, then every solution of (1.2) oscillates. ∎

Theorem 2.10. *Let $k_i \in \{..., -3, -2\}$, $p_{in} \le 0$, $m \in \{..., -3, -2\}$, $q_n < 0$. Assume that $\limsup_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, ..., r$. If the condition (2.9) holds, then every solution of (1.2) oscillates.* ∎

Corollary 2.11. *Let k_i, m, $\{p_{in}\}$, p_i, $\{q_n\}$ and q be the same as in Theorem 2.1. If the condition*

$$r\left(\prod_{i=1}^{r} p_i\right)^{\frac{1}{r}} > \frac{k^k}{(1+q)^k\,(k+1)^{k+1}}\,, \tag{2.10}$$

holds, where $k = \frac{1}{r}\sum_{i=1}^{r} k_i$, then every solution of (1.2) oscillates.

Proof: Assume that $m = -1$ and that $\{x_n\}$ is eventually positive solution of equation (1.2). Let $z_n = \dfrac{x_n}{x_{n+1}}$ and $\liminf_{n\to\infty} z_n = z$. Then by using (2.4) and applying the arithmetic-geometric mean inequality, we conclude that

$$\begin{aligned} 1 &\ge \sum_{i=1}^{r} p_i\,\frac{z^{k_i+1}}{z-q-1} \\ &\ge r\left(\prod_{i=1}^{r} p_i\,\frac{z^{k_i+1}}{z-q-1}\right)^{\frac{1}{r}}. \end{aligned}$$

This inequality gives that

$$\begin{aligned} 1 &\ge r\left(\prod_{i=1}^{r} p_i\right)^{\frac{1}{r}} \frac{z^{k+1}}{z-q-1} \\ &\ge r\left(\prod_{i=1}^{r} p_i\right)^{\frac{1}{r}} \frac{(1+q)^k\,(k+1)^{k+1}}{k^k} \end{aligned}$$

which contradicts (2.10). ∎

Using the similar methods in the proof of Corollary 2.11 we have the next results.

Corollary 2.12. *Let k_i, m, $\{p_{in}\}$, p_i, $\{q_n\}$ and q be the same as in Theorem 2.3. If the condition*

$$r\left(\prod_{i=1}^{r} |p_i|\right)^{\frac{1}{r}} > \frac{1}{(1+q)^k}\left|\frac{k^k}{(k+1)^{k+1}}\right|$$

holds, then every solution of (1.2) oscillates. ∎

Corollary 2.13. *Let k_i, m, $\{p_{in}\}$, p_i, $\{q_n\}$ and q be the same as in Theorem 2.5. If*

$$r\left(\prod_{i=1}^{r} p_i\right)^{\frac{1}{r}} > \frac{(1-q)^{k+1}\,k^k}{(k+1)^{k+1}}\ ,$$

then every solution of (1.2) oscillates. ∎

Corollary 2.14. *Let k_i, m, $\{p_{in}\}$, p_i, $\{q_n\}$ and q be the same as in Theorem 2.7. If*

$$r\left(\prod_{i=1}^{r} |p_i|\right)^{\frac{1}{r}} > (1-q)^{k+1}\left|\frac{k^k}{(k+1)^{k+1}}\right|\ ,$$

then every solution of (1.2) oscillates. ∎

Corollary 2.15. *Let k_i, m, $\{p_{in}\}$, p_i, $\{q_n\}$ and q be the same as in Theorem 2.9. If*

$$r\left(\prod_{i=1}^{r} p_i\right)^{\frac{1}{r}} > \frac{k^k}{(k+1)^{k+1}}\ ,$$

then every solution of (1.2) oscillates. ∎

Corollary 2.16. *Let k_i, m, $\{p_{in}\}$, p_i, $\{q_n\}$ and q be the same as in Theorem 2.10. If*

$$r\left(\prod_{i=1}^{r} |p_i|\right)^{\frac{1}{r}} > \left|\frac{k^k}{(k+1)^{k+1}}\right|\ ,$$

then every solution of (1.2) oscillates. ∎

We should finally remark that every solution of equation (1.2) oscillates provided that $1-\sum_{i=1}^{r} p_{in}$ is eventually nonpositive and that $q_n \geq 0$, $k_i = 0$, $m = -1$ $(i=1,2,...,r)$ in (1.2). If $1+q_n$ is eventually nonpositive and that $p_{in} \leq 0$, $k_i = 0$, $m=-1$ $(i = 1,...,r)$ in (1.3), then every solution of equation (1.2) oscillates.

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