PORTUGALIAE MATHEMATICA
Vol. 63 Fasc. 4 – 2006
Nova Série

AN INTERPRETATION OF S^1_2 IN Σ^b_1–NIA

Gilda Ferreira and Isabel Oitavem

Abstract: In this paper the theory S^1_2 (of Buss) is interpreted in the theory Σ^b_1–NIA (of Ferreira).

1 – Introduction

Our goal is to interpret Buss's theory S^1_2, [1], in Ferreira's theory Σ^b_1–NIA, originally denoted by Σ^b_1–PIND — see [7]. This correspondence has been mentioned in work of several authors. For instance Cantini [2], Fernandes [4], [5], Ferreira [6], [7], [8], Oliva [9], Strahm [10] and Yamazaki [12]. In spite of the widely acceptance of the result, this is the first time that its proof is formally carried out. Therefore, this is a technical paper which aims to serve as a reference.

In Section 2 we briefly describe the theories and we introduce some elementary properties. The interpretation of the theory S^1_2 in Σ^b_1–NIA is worked out in Section 3. There we start with some general considerations concerning the notion of interpretation, to then enter the proof of the result of this paper. The main statement is established in Theorem 3.1.

2 – The theories S^1_2 and Σ^b_1–NIA

Let $\mathcal{L}_{\mathbb{N}}$ be the first order language, with equality, which has a single constant 0, the function symbols S, $+$, $\cdot$, $|\,.\,|$, $\lfloor \frac{1}{2}.\rfloor$ and $\#$, and the relation symbol $\leq$. $\#$ is usually called the *smash* function and interpreted as $x\,\#\,y = 2^{|x|\cdot|y|}$. By $\mathcal{L}_{\mathbb{W}}$

Received: October 1, 2005; *Revised*: January 26, 2006.
AMS Subject Classification: 03F30, 03F25, 03D20.
Keywords: bounded arithmetic; interpretability; polynomial time.

we denote the first order language, with equality, which has the constants ϵ, 0 and 1, the function symbols $\frown$ and $\times$, and the relation symbol $\subseteq$. In the standard model, ϵ denotes the empty word, $\frown$ stands for the concatenation of 0-1 words, $\times$ for the binary product (i.e. $x \times y = x \frown \ldots \frown x$, $|y|$-times) and $\subseteq$ for the initial subword relation. The symbol $\frown$ is usually omitted. Thus for terms t, r one writes tr instead of $t \frown r$. Moreover, we follow the convention that $\frown$ has precedence over $\times$.

Depending on the language that we consider, the designation "bounded quantification" has different meanings. In $\mathcal{L}_{\mathbb{N}}$, a *bounded quantification* is a quantification of the form $\forall x \leq t \ldots$ or $\exists x \leq t \ldots$, which abbreviates respectively $\forall x \, (x \leq t \rightarrow \ldots)$ or $\exists x \, (x \leq t \wedge \ldots)$, and a *sharply bounded quantification* is a quantification of the form $\forall x \leq |t| \ldots$ or $\exists x \leq |t| \ldots$, which abbreviates respectively $\forall x \, (x \leq |t| \rightarrow \ldots)$ or $\exists x \, (x \leq |t| \wedge \ldots)$, where t is any term not involving x. In $\mathcal{L}_{\mathbb{W}}$, a *bounded quantification* is a quantification of the form $\forall x \preceq t \ldots$ or $\exists x \preceq t \ldots$, which abbreviates $\forall x \, (1 \times x \subseteq 1 \times t \rightarrow \ldots)$ or $\exists x \, (1 \times x \subseteq 1 \times t \wedge \ldots)$ respectively (notice that $x \preceq t$ means that "the length of x is less or equal than the length of t", and a *subword quantification* is a quantification of the form $\forall x \subseteq^* t \ldots$ or $\exists x \subseteq^* t \ldots$, which abbreviates respectively $\forall x \, (\exists w \subseteq t (wx \subseteq t) \rightarrow \ldots)$ or $\exists x \, (\exists w \subseteq t (wx \subseteq t) \wedge \ldots)$, for any term t where x does not occur.

Definition 2.1. S^1_2 is the first order theory in the language $\mathcal{L}_{\mathbb{N}}$ with the following axioms:

- Basic Axioms
  - **(1)** $y \leq x \rightarrow y \leq Sx$
  - **(2)** $x \neq Sx$
  - **(3)** $0 \leq x$
  - **(4)** $x \leq y \wedge x \neq y \leftrightarrow Sx \leq y$
  - **(5)** $x \neq 0 \rightarrow 2 \cdot x \neq 0$
  - **(6)** $y \leq x \vee x \leq y$
  - **(7)** $x \leq y \wedge y \leq x \rightarrow x = y$
  - **(8)** $x \leq y \wedge y \leq z \rightarrow x \leq z$
  - **(9)** $|0| = 0$
  - **(10)** $x \neq 0 \rightarrow |2 \cdot x| = S(|x|) \wedge |S(2 \cdot x)| = S(|x|)$
  - **(11)** $|S0| = S0$
  - **(12)** $x \leq y \rightarrow |x| \leq |y|$
  - **(13)** $|x \# y| = S(|x| \cdot |y|)$

(14) $0 \# y = S0$
(15) $x \neq 0 \;\rightarrow\; 1 \# (2 \cdot x) = 2(1 \# x) \wedge 1 \# (S(2 \cdot x)) = 2(1 \# x)$
(16) $x \# y = y \# x$
(17) $|x| = |y| \rightarrow x \# z = y \# z$
(18) $|x| = |u| + |v| \rightarrow x \# y = (u \# y) \cdot (v \# y)$
(19) $x \leq x + y$
(20) $x \leq y \wedge x \neq y \;\rightarrow\; S(2 \cdot x) \leq 2 \cdot y \wedge S(2 \cdot x) \neq 2 \cdot y$
(21) $x + y = y + x$
(22) $x + 0 = x$
(23) $x + Sy = S(x + y)$
(24) $(x + y) + z = x + (y + z)$
(25) $x + y \leq x + z \leftrightarrow y \leq z$
(26) $x \cdot 0 = 0$
(27) $x \cdot (Sy) = (x \cdot y) + x$
(28) $x \cdot y = y \cdot x$
(29) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
(30) $S0 \leq x \rightarrow (x \cdot y \leq x \cdot z \leftrightarrow y \leq z)$
(31) $x \neq 0 \rightarrow |x| = S(|\lfloor \frac{1}{2} x \rfloor|)$
(32) $x = \lfloor \frac{1}{2} y \rfloor \leftrightarrow (2 \cdot x = y \vee S(2 \cdot x) = y)$

- Axiom Scheme for Induction

  $A(0) \wedge \forall x\big(A(\lfloor \frac{1}{2} x \rfloor) \rightarrow A(x)\big) \rightarrow \forall x A(x)$, where A is a Σ^b_1-formula in $\mathcal{L}_{\mathbb{N}}$.

  By a *Σ^b_1-formula in $\mathcal{L}_{\mathbb{N}}$* we mean a formula belonging to the smallest class of formulas of $\mathcal{L}_{\mathbb{N}}$ containing the set of formulas where all quantifications are sharply bounded and that is closed under $\wedge$, $\vee$, bounded existential quantifications and sharply bounded quantifications. □

Definition 2.2. Σ^b_1–NIA is the first order theory, in the language $\mathcal{L}_{\mathbb{W}}$, with the following axioms:

- Basic Axioms

  (1) $x\epsilon = x$
  (2) $x(y0) = (xy)0$
  (3) $x(y1) = (xy)1$
  (4) $x \times \epsilon = \epsilon$

(5) $x \times y0 = (x \times y)x$
(6) $x \times y1 = (x \times y)x$
(7) $x \subseteq \epsilon \leftrightarrow x = \epsilon$
(8) $x \subseteq y0 \leftrightarrow x \subseteq y \vee x = y0$
(9) $x \subseteq y1 \leftrightarrow x \subseteq y \vee x = y1$
(10) $x0 = y0 \rightarrow x = y$
(11) $x1 = y1 \rightarrow x = y$
(12) $x0 \neq y1$
(13) $x0 \neq \epsilon$
(14) $x1 \neq \epsilon$

- Axiom Scheme for Induction on Notation

$$B(\epsilon) \wedge \forall x \big(B(x) \rightarrow B(x0) \wedge B(x1)\big) \rightarrow \forall x\, B(x) \ ,$$

where B is a Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$, possible with other free variables besides x. By a *Σ_1^b-formula in* $\mathcal{L}_{\mathbb{W}}$ we mean a formula of the form $\exists x\, (x \preceq t(\overline{z}) \wedge A(\overline{z}, x))$, where A is a *sw.q. formula*, i.e. A belongs to the smallest class of formulas of $\mathcal{L}_{\mathbb{W}}$ containing the atomic formulas and which is closed under Boolean operations and subword quantifications. $\square$

We present a list of statements, provable in $\Sigma_1^b\mathrm{-NIA}$, which are used in this paper.

Lemma 2.1. *The following is provable in* $\Sigma_1^b\mathrm{-NIA}$*:*

(1) $\epsilon x = x$
(2) $(xy)z = x(yz)$
(3) $xz = yz \rightarrow x = y$
(4) $\epsilon \times x = \epsilon$
(5) $x \times y = \epsilon \rightarrow x = \epsilon \vee y = \epsilon$
(6) $x \times 0 = x \wedge x \times 1 = x$
(7) $0 \times (x \times y) = 0 \times (y \times x)$
(8) $1 \times (x \times y) = 1 \times (y \times x)$
(9) $(x \times y) \times z = x \times (y \times z)$
(10) $(x \times y)(x \times z) = x \times yz$
(11) $1 \times xy = 1 \times yx$

(**12**) $1 \times x = 1 \times y \,\rightarrow\, 0 \times x = 0 \times y$

(**13**) $1 \times x = 1 \times y \,\rightarrow\, 1 \times 1x = 1 \times 1y$

(**14**) $0 \neq 1$

(**15**) $\epsilon \neq 0 \,\wedge\, \epsilon \neq 1$

(**16**) $x \neq \epsilon \,\rightarrow\, \exists z\,(z0 = x \vee z1 = x)$

(**17**) $x \subseteq y \,\wedge\, y \subseteq x \,\rightarrow\, x = y$

(**18**) $x \subseteq y \,\wedge\, y \subseteq z \,\rightarrow\, x \subseteq z$

(**19**) $x \subseteq xy$

(**20**) $x \subseteq y \,\leftrightarrow\, wx \subseteq wy$

(**21**) $x \subseteq z \,\wedge\, y \subseteq z \rightarrow x \subseteq y \,\vee\, y \subseteq x$

(**22**) $x \subseteq y \,\rightarrow\, \exists z\; xz = y$

(**23**) $x \neq 0 \times x \,\rightarrow\, \exists y \subseteq x\; \exists z \subseteq 0 \times x (x = y1z)$

(**24**) $x \neq 1 \times x \,\rightarrow\, \exists y \subseteq x\; \exists z \subseteq 1 \times x (x = y0z)$.

Proof: All assertions are proved in [6] except (12), (13) and (19). Note that, in [6], results are proved in a theory called PTCA, but similar demonstrations work in Σ_1^b–NIA. (**12**) is a consequence of (6) and (7). (**13**) is a consequence of (11) and Definition 2.2 (6). (**19**) can be obtained by induction on notation on y, using (16) and Definition 2.2 (1), (7), (8) and (9). ∎

Let us consider, in $\mathcal{L}_{\mathbb{W}}$, the class of extended Σ_1^b-formulas (extended Π_1^b-formulas). By an *extended Σ_1^b-formula* (respectively *extended Π_1^b-formula*) we mean a formula that is logically equivalent to a formula that can be constructed in a finite number of steps, starting with sw.q. formulas and permitting conjunctions, disjunctions, subword quantifications and bounded existential quantifications (respectively bounded universal quantifications).

Lemma 2.2. Σ_1^b–NIA $\vdash\; A(\epsilon) \wedge \forall x\,\big(A(x) \rightarrow A(x0) \wedge A(x1)\big) \,\rightarrow\, \forall x\, A(x)$, *for any extended Σ_1^b-formula A.*

Proof: In [7] is proved that, in Σ_1^b–NIA, any extended Σ_1^b-formula is equivalent to a Σ_1^b-formula. This implies our lemma. ∎

Of particular importance is the $\mathcal{L}_{\mathbb{W}}$ formula $x = \epsilon \vee 1 \subseteq x$, that we abbreviate by $x \in \mathbb{W}_1$. $x \in \mathbb{W}_1$ is a Σ_1^b-formula that, in the standard model of Σ_1^b–NIA, corresponds to consider only the empty word or words starting with 1.

Lemma 2.3. *The following is provable in* $\Sigma_1^b\!-\!\mathsf{NIA}$*:*

(1) $x \in \mathbb{W}_1 \wedge y \subseteq x \;\rightarrow\; y \in \mathbb{W}_1$

(2) $x \in \mathbb{W}_1 \wedge x \neq \epsilon \;\rightarrow\; xy \in \mathbb{W}_1$.

Proof: (**1**) Suppose $x \in \mathbb{W}_1$ and $y \subseteq x$. From $x \in \mathbb{W}_1$, we have that $x = \epsilon$ or $1 \subseteq x$. In the first case $y \subseteq \epsilon$ and so, by Definition 2.2 (7), $y = \epsilon$. Hence $y \in \mathbb{W}_1$. In the second case, by Lemma 2.1 (21), we have $1 \subseteq y$ or $y \subseteq 1$. If $1 \subseteq y$ then $y \in \mathbb{W}_1$. If $y \subseteq 1$, then by Lemma 2.1 (1) $y \subseteq \epsilon 1$ and by Definition 2.2 (9), we have $y \subseteq \epsilon$ or $y = \epsilon 1$. If $y \subseteq \epsilon$, by Definition 2.2 (7), $y = \epsilon$, and so $y \in \mathbb{W}_1$. If $y = \epsilon 1$, by Lemma 2.1 (1), $y = 1$. Thus, by Lemma 2.1 (19) and Definition 2.2 (1), $1 \subseteq y$ and so $y \in \mathbb{W}_1$.

(**2**) Suppose $x \in \mathbb{W}_1$ and $x \neq \epsilon$, we have $1 \subseteq x$. Lemma 2.1 (19) ensures $x \subseteq xy$. From $x \subseteq xy$ and $1 \subseteq x$, using Lemma 2.1 (18), we have $1 \subseteq xy$. Thus, $xy \in \mathbb{W}_1$. ∎

The next lemma states that, in $\Sigma_1^b\!-\!\mathsf{NIA}$, the scheme of *induction on notation on* $x \in \mathbb{W}_1$, for extended Σ_1^b-formulas, is provable.

Lemma 2.4. $\Sigma_1^b\!-\!\mathsf{NIA} \vdash A(\epsilon) \wedge \forall x \!\in\! \mathbb{W}_1 \big(A(x) \rightarrow (x0 \!\in\! \mathbb{W}_1 \!\rightarrow\! A(x0)) \wedge A(x1)\big) \rightarrow \forall x \in \mathbb{W}_1\, A(x)$, *where* A *is an extended* Σ_1^b*-formula in* $\mathcal{L}_{\mathbb{W}}$*.*

Proof: It results from applying Lemma 2.2 to the extended Σ_1^b-formula: $x \in \mathbb{W}_1 \rightarrow A(x)$. ∎

3 – Interpreting S_2^1 in $\Sigma_1^b\!-\!\mathsf{NIA}$

3.1. Preliminaries

The notion of interpretability between theories was introduced by Tarski, Mostowski and Robinson [11] in 1953 and it has been widely used to prove results on (un)decidability and (relative) consistency. Roughly speaking interpretations are used to show whether a theory is powerful enough to express another.

The notion of interpretability can be formulated in different ways. Here we adopt a formulation similar to the one presented in [3].

Definition 3.1. Let $\mathcal{L}_A$ and $\mathcal{L}_B$ be languages and T_B be a theory in the language $\mathcal{L}_B$. An interpretation of the language $\mathcal{L}_A$ into T_B consists of a formula σ in $\mathcal{L}_B$ and a function ν from the nonlogical symbols of $\mathcal{L}_A$ to expressions (terms, formulas) in $\mathcal{L}_B$ such that:

1. $T_B \vdash \exists x\, \sigma(x)$

2. If c is a constant of $\mathcal{L}_A$, then $\nu(c)$ is a closed term of $\mathcal{L}_B$ and $T_B \vdash \exists x\, (\sigma(x) \wedge \nu(c) = x)$

3. If f is an n-ary function symbol of $\mathcal{L}_A$, then $\nu(f)$ is a formula of $\mathcal{L}_B$ in which at most $n+1$-variables occur free and verify $T_B \vdash \forall x_1 ... \forall x_n \Big(\sigma(x_1) \wedge ... \wedge \sigma(x_n) \;\rightarrow\; \exists y\, \big(\sigma(y) \wedge \forall z\, (\nu(f)(x_1, ..., x_n, z) \leftrightarrow z = y)\big)\Big)$

4. If R is an n-ary relation symbol of $\mathcal{L}_A$, then $\nu(R)$ is a formula of $\mathcal{L}_B$ in which at most n variables occur free. $\square$

The idea is that in any model of T_B, the formula $\sigma(x)$ should define a nonempty set to be used as the universe of an $\mathcal{L}_A$-structure.

Remark 3.1. Having an interpretation (σ, ν) of $\mathcal{L}_A$ into T_B, we can consider a translation I from all formulas of $\mathcal{L}_A$ to expressions of $\mathcal{L}_B$ in the following way:

1) If α is an atomic formula use recursion on the number of places at which function symbols occur in α. If that number is zero $\alpha = R(x_1, ..., x_n, a_1, ..., a_n)$ with R a relation symbol, x_i variables and a_i constants. Then $I(\alpha) = \nu(R)\,(x_1, ..., x_n, \nu(a_1), ..., \nu(a_n))$. Otherwise, take the rightmost place at which a function symbol f occurs. If f is an n-ary symbol, then that place initiates a segment $f x_1 ... x_n$. Replace this segment by some new variable y, obtaining a formula we call $\alpha_y^{f x_1 ... x_n}$. Then $I(\alpha)$ is $\forall y\, \big(\nu(f)(x_1, ..., x_n, y) \rightarrow I(\alpha_y^{f x_1 ... x_n})\big)$;

2) $I(\neg\alpha) := \neg I(\alpha)$, $\quad I(\varphi \,\square\, \psi) := I(\varphi) \,\square\, I(\psi)$ $\quad$ with $\quad \square \in \{\wedge, \vee \rightarrow, \leftrightarrow\}$, $I(\forall x\, \varphi) := \forall x\, (\sigma(x) \rightarrow I(\varphi))$ and $I(\exists x\, \varphi) := \exists x\, (\sigma(x) \wedge I(\varphi))$. $\square$

Definition 3.2. Let T_A be a theory in the language $\mathcal{L}_A$ and T_B be a theory in $\mathcal{L}_B$. T_A is interpretable in T_B (or T_B interprets T_A) if there exists an interpretation (σ, ν) of $\mathcal{L}_A$ into T_B whose translation I (defined as before) verifies: $T_B \vdash I(\varphi)$ for all axioms φ in T_A. $\square$

We also use the following result proved in [7].

Proposition 3.1. *To each function f of* PTIME, *we can assign an extended Σ_1^b-formula G_f and a term b_f of $\mathcal{L}_{\mathbb{W}}$ such that*

$$\Sigma_1^b\text{−}\mathsf{NIA} \vdash \forall \bar{x}\, \exists z \preceq b_f(\bar{x})\, G_f(\bar{x}, z)$$

$$\Sigma_1^b\text{−}\mathsf{NIA} \vdash G_f(\bar{x}, z) \wedge G_f(\bar{x}, y) \rightarrow z = y$$

and

a. *in Σ_1^b−NIA the following is valid,*

(**1**) $G_{C_0}(x, x0)$

(**2**) $G_{C_1}(x, x1)$

(**3**) $G_{P_i^n}(x_1, ..., x_n, x_i),\ 1 \leq i \leq n$

(**4**) $G_Q(x, y, 1) \leftrightarrow x \subseteq y$ *and* $G_Q(x, y, 0) \vee G_Q(x, y, 1)$

b.

(**1**) *if f is defined from $g, h_1, ..., h_k$ by composition then,* Σ_1^b−$\mathsf{NIA} \vdash G_{h_1}(\bar{x}, y_1) \wedge ... \wedge G_{h_k}(\bar{x}, y_k) \wedge G_g(y_1, ..., y_k, z) \rightarrow G_f(\bar{x}, z)$

(**2**) *if f is defined from g, h_0, h_1 by bounded recursion on notation with bound t then,* Σ_1^b−$\mathsf{NIA} \vdash G_g(\bar{x}, z) \rightarrow G_f(\bar{x}, \epsilon, z)$ *and* Σ_1^b−$\mathsf{NIA} \vdash G_f(\bar{x}, y, r) \wedge G_{h_i}(\bar{x}, y, r, u) \wedge z = u_{|t(\bar{x},y)} \rightarrow G_f(\bar{x}, yi, z)$ *with $i = 0, 1$.* ∎

Remark 3.2. If in the previous proposition we replace "extended Σ_1^b-formula G_f" by "extended Π_1^b-formula G_f^π" the result remains true. [Just take $G_f^\pi(\bar{x}, z)$ as being $\forall w \preceq b_f(\bar{x})\, (G_f(\bar{x}, w) \rightarrow w = z)$.] Also note that Σ_1^b−$\mathsf{NIA} \vdash G_f \leftrightarrow G_f^\pi$. □

3.2. Interpreting $\mathcal{L}_{\mathbb{N}}$ in Σ_1^b−NIA

To avoid ambiguity, we often use $\mathbb{N}$ and $\mathbb{W}$ as subscripts. For instance $0_{\mathbb{N}} \in \mathcal{L}_{\mathbb{N}}$ and $0_{\mathbb{W}} \in \mathcal{L}_{\mathbb{W}}$. The interpretation of $\mathcal{L}_{\mathbb{N}}$ into Σ_1^b−NIA is done according to Definition 3.1. The formula $\sigma(x)$, of $\mathcal{L}_{\mathbb{W}}$, is $x \in \mathbb{W}_1$ (i.e. $x = \epsilon \vee 1 \subseteq x$). We interpret $0_{\mathbb{N}}$ as being ϵ. To define the interpretation of the function symbols of $\mathcal{L}_{\mathbb{N}}$ we first introduce some functions, constructed according to the inductive characterization of PTIME given in [7]. There, PTIME is described as the smallest class of functions which includes projections (P_i^n), concatenation with 0 (C_0), concatenation with 1 (C_1) and the characteristic function of "$\subseteq$" (Q) and which is closed under composition and bounded recursion on notation with bound t (a term of $\mathcal{L}_{\mathbb{W}}$). Consider the functions S_{w}, $\lfloor \frac{1}{2} x \rfloor_{\mathrm{w}}$, $+_{\mathrm{w}}$ (also T and U), $\cdot_{\mathrm{w}}$ and $|\ .\ |_{\mathrm{w}}$ defined as follows:

- $S_{\mathrm{w}}(\epsilon) = 1$, $S_{\mathrm{w}}(x0) = x1$ and $S_{\mathrm{w}}(x1) = S_{\mathrm{w}}(x)0$

- $\lfloor \frac{1}{2} . \rfloor_{\text{w}}$ defined by $\lfloor \frac{1}{2} x \rfloor_{\text{w}} = T(x)$, where $T(\epsilon) = \epsilon$, $T(x0) = x$ and $T(x1) = x$
- $\epsilon +_{\text{w}} y = y$ and for $x \neq \epsilon$: $x +_{\text{w}} \epsilon = x$, $x +_{\text{w}} y0 = (T(x) +_{\text{w}} y) \frown U(x)$ and $x +_{\text{w}} y1 = \begin{cases} \big(T(x) +_{\text{w}} y\big)1 & \text{if } U(x) = 0 \\ S\big(T(x) +_{\text{w}} y\big)0 & \text{otherwise} \end{cases}$ where $U(\epsilon) = \epsilon$, $U(x0) = 0$ and $U(x1) = 1$
- $\epsilon \cdot_{\text{w}} y = \epsilon$, and for $x \neq \epsilon$: $x \cdot_{\text{w}} \epsilon = \epsilon$, $x \cdot_{\text{w}} y0 = (x \cdot_{\text{w}} y)0$ and $x \cdot_{\text{w}} y1 = (x \cdot_{\text{w}} y)0 +_{\text{w}} x$
- $|\epsilon|_{\text{w}} = \epsilon$, $|y0|_{\text{w}} = S_{\text{w}}(|y|_{\text{w}})$ and $|y1|_{\text{w}} = S_{\text{w}}(|y|_{\text{w}})$.

In each case it is easy to find a bounding term according to the previous definition of PTIME. Consider, for instance, $t(x) = x11$, $t(x, y) = xy11$, $t(x, y) = (x \times y1)x1$, $t(y) = y1$ for S_{w}, $+_{\text{w}}$, $\cdot_{\text{w}}$ and $| \, . \, |_{\text{w}}$ respectively. Thus the functions above are functions in PTIME. We define the function ν of $\mathcal{L}_{\mathbb{N}}$ into Σ_1^b–NIA by applying the function symbols of $\mathcal{L}_{\mathbb{N}}$ — S_{N}, $\lfloor \frac{1}{2} . \rfloor_{\text{N}}$, $+_{\text{N}}$, $\cdot_{\text{N}}$ and $| \, . \, |_{\text{N}}$ — to the extended Σ_1^b-formulas of $\mathcal{L}_{\mathbb{W}}$ assigned by Proposition 3.1 to the functions S_{w}, $\lfloor \frac{1}{2} . \rfloor_{\text{w}}$, $+_{\text{w}}$, $\cdot_{\text{w}}$ and $| \, . \, |_{\text{w}}$ respectively.

The function symbol $\#_{\text{N}}$ and the relation symbol $\leq_{\text{N}}$, of $\mathcal{L}_{\mathbb{N}}$, are interpreted by ν as being the formulas $G_{\#}(x, y, z) := z = 1 \frown ((0 \times x) \times y)$ and $\leq_{\text{w}} (x, y) := (x \preceq y \wedge \neg(x \equiv y)) \vee (x \equiv y \wedge \exists z \subseteq x(z0 \subseteq x \wedge z1 \subseteq y)) \vee x = y$ respectively, where $x \equiv y$ is an abbreviation of $1 \times x = 1 \times y$. In the sequent we use infix notation for $\leq_{\text{w}}$, i.e. we write $x \leq_{\text{w}} y$ instead of $\leq_{\text{w}} (x, y)$.

Proposition 3.2. *The pair (σ, ν) defined above is an interpretation of $\mathcal{L}_{\mathbb{N}}$ into Σ_1^b–NIA.*

Proof: In order to prove that (σ, ν) is, in fact, a valid interpretation of $\mathcal{L}_{\mathbb{N}}$ into Σ_1^b–NIA, we need to ensure the four clauses of Definition 3.1. The only non immediate assertion is 3. The study of $\#_{\text{N}}$ follows immediately from Lemma 2.3 (2). Let us consider any other function symbol f_{N} of $\mathcal{L}_{\mathbb{N}}$. From the definition of $G_{f_{\text{w}}}$, in Proposition 3.1, we know that if f_{N} is an n-ary function symbol then Σ_1^b–NIA $\vdash \forall x_1 ... \forall x_n \, (\sigma(x_1) \wedge ... \wedge \sigma(x_n) \rightarrow \exists y \forall z \, (G_{f_{\text{w}}}(x_1, ..., x_n, z) \leftrightarrow z = y)$. Therefore one just need to prove that Σ_1^b–NIA $\vdash \forall x_1 ... \forall x_n \forall y \, \big(\sigma(x_1) \wedge ... \wedge \sigma(x_n) \wedge G_{f_{\text{w}}}(x_1, ..., x_n, y) \rightarrow \sigma(y)\big)$. The most involving case occurs for $+_{\text{N}}$. We work it out here, assuming the result proved for S_{N}.

To ensure that Σ_1^b–NIA $\vdash \forall x \forall y \forall z \, (x \in \mathbb{W}_1 \wedge y \in \mathbb{W}_1 \wedge G_{+_{\text{w}}}(x, y, z) \rightarrow z \in \mathbb{W}_1)$, we prove that Σ_1^b–NIA $\vdash \forall z \forall x \in \mathbb{W}_1 \forall y \in \mathbb{W}_1 \forall x' \subseteq x \forall z' \subseteq z \, (G_{+}(x', y, z') \rightarrow z' \in \mathbb{W}_1)$, where G_{+} is an abbreviation of $G_{+_{\text{w}}}$. Fix z and x such that $x \in \mathbb{W}_1$. The proof is by induction on notation on $y \in \mathbb{W}_1$ (see Lemma 2.4) considering that, according to Remark 3.2, we can replace G_{+} by G_{+}^{π}. For $y = \epsilon$ we have, by Proposition 3.1 for G_+, that $\forall x' \subseteq x\ \forall z' \subseteq z\ (G_+(x', y, z') \rightarrow z' = x')$, and so, by Lemma 2.3 (1), $z' \in \mathbb{W}_1$. Given $y \in \mathbb{W}_1$ we have, by induction hypothesis, that $\forall x' \subseteq x\ \forall z' \subseteq z\ (G_+(x', y, z') \rightarrow z' \in \mathbb{W}_1)$. Let us prove that $y0 \in \mathbb{W}_1 \rightarrow \forall x' \subseteq x\ \forall z' \subseteq z\ (G_+(x', y0, z') \rightarrow z' \in \mathbb{W}_1)$. Assuming $y0 \in \mathbb{W}_1$, by Lemma 2.1 (1), (14), (15) and Definition 2.2 (7) and (8) we have $y \neq \epsilon$. If $x' = \epsilon$, then $G_+(x', y0, z') \rightarrow z' = y0$ and so $z' \in \mathbb{W}_1$. If $x' \neq \epsilon$ then $G_+(x', y0, z')$ implies that there exists w, u and t such that $z' = w \frown u$, $G_U(x', u)$, $G_T(x', t)$ and $G_+(t, y, w)$, where G_U and G_T are the extended Σ_1^b-formulas of $\mathcal{L}_{\mathbb{W}}$ assigned to the functions U and T that appear in the definition of $+_{\mathrm{w}}$. Provided $x \in \mathbb{W}_1$, by Lemma 2.3 (1), $x' \in \mathbb{W}_1$. Noticing that $t \subseteq x'$ (this is a consequence of Proposition 3.1 for T, Lemma 2.1 (16) and (19)) and that $x' \subseteq x$ we have, by Lemma 2.1 (18), that $t \subseteq x$. By Lemma 2.1 (19), one has that $w \subseteq z'$ and so, by Lemma 2.1 (18), $w \subseteq z$. Thus, by induction hypothesis, $G_+(t, y, w) \rightarrow w \in \mathbb{W}_1$. Moreover, recalling that $y \neq \epsilon$ and $G_+(t, y, w)$, by Proposition 3.1 for $+_{\mathrm{w}}$ and Definition 2.2 (13), (14), one has $w \neq \epsilon$. $w \in \mathbb{W}_1 \wedge w \neq \epsilon$ implies, by Lemma 2.3 (2), $z' \in \mathbb{W}_1$. Now, let us prove that $\forall x' \subseteq x\ \forall z' \subseteq z\ (G_+(x', y1, z') \rightarrow z' \in \mathbb{W}_1)$. If $x' = \epsilon$ then $\forall z' \subseteq z\ (G_+(x', y1, z') \rightarrow z' = y1)$, and so $z' \in \mathbb{W}_1$. If $x' \neq \epsilon$, we consider two cases: $G_U(x', 0)$ and $G_U(x', 1)$. In the first case we have $\forall z' \subseteq z\ (G_+(x', y1, z') \rightarrow z' = w1)$, where $G_+(t, y, w)$ and $G_T(x', t)$. Again, by induction hypothesis, $w \in \mathbb{W}_1$, so $z' \in \mathbb{W}_1$. In the later case we have $\forall z' \subseteq z\ (G_+(x', y1, z') \rightarrow z' = s0)$, where $G_+(t, y, w)$, $G_T(x', t)$ and $G_S(w, s)$. Moreover, by induction hypothesis, $w \in \mathbb{W}_1$. Assuming the result for S_{N}, we have that $s \in \mathbb{W}_1$. Evoking Proposition 3.1 for S_{w}, Lemma 2.1 (15) and Definition 2.2 (13), (14), one ensures that $s \neq \epsilon$. Consequently, by Lemma 2.3 (2), $s0 \in \mathbb{W}_1$, i.e. $z' \in \mathbb{W}_1$. This finishes the proof. ∎

3.3. Main result

Before establish our main result, we present some properties in $\Sigma_1^b\text{-}\mathsf{NIA}$ concerning the formulas involved in the interpretation (σ, ν).

Lemma 3.1. *The following assertions are provable in* $\Sigma_1^b\text{-}\mathsf{NIA}$*:*

(1) $G_U(x, y) \rightarrow y = \epsilon \vee y = 0 \vee y = 1$, $\quad G_U(x, 0) \wedge G_T(x, y) \leftrightarrow x = y0$, $G_U(x, 1) \wedge G_T(x, y) \leftrightarrow x = y1$, $G_U(x, \epsilon) \leftrightarrow x = \epsilon$, $x \neq \epsilon \wedge y \neq \epsilon \wedge G_U(x, a) \wedge G_U(y, b) \wedge G_+(x, y, z) \rightarrow \big[(a = b \rightarrow G_U(z, 0)) \wedge (a \neq b \rightarrow G_U(z, 1))\big]$;

(2) $G_+(x, y, s) \wedge G_T(s, s') \wedge G_+(x', y', r) \wedge G_T(x, x') \wedge G_T(y, y') \rightarrow \big((G_U(x, 1) \wedge G_U(y, 1) \rightarrow G_+(r, 1, s')) \wedge \neg(G_U(x, 1) \wedge G_U(y, 1) \rightarrow s' = r)\big)$;

(3) $\forall x \in \mathbb{W}_1\ \big(G_T(x, y) \wedge G_U(x, z) \rightarrow yz = x\big)$.

Proof: (**1**) The first 4 assertions are immediate by Proposition 3.1 for U and T, and Lemma 2.1 (16). Let us prove the last assertion. There are four relevant cases: $a = b = 0$, $a = b = 1$, $a = 0 \wedge b = 1$ and $a = 1 \wedge b = 0$. If $a = b = 0$ then from $G_+(x, y, z)$, using the assertions before, we have $G_+(x'0, y'0, z)$ where $G_T(x, x')$ and $G_T(y, y')$. By Proposition 3.1 for $+_{\mathrm{w}}$ and T we have $z = z'0$, where $G_+(x', y', z')$. So, we have $G_U(z, 0)$. The other cases are analogous.

(**2**) The proof is easy considering all the possible situations $G_U(x, 1) \wedge G_U(y, 1)$, $G_U(x, 0)$, $G_U(y, 0)$, $G_U(x, \epsilon)$ and $G_U(y, \epsilon)$ and using Lemma 3.1 (1).

(**3**) This assertion is immediate, using Proposition 3.1 for T, U and considering the cases $x = \epsilon, x = z0$ and $x = z1$ — see Lemma 2.1 (16). ∎

Using the abbreviation $x <_{\mathrm{w}} y := x \leq_{\mathrm{w}} y \wedge x \neq y$, we have the following lemma.

Lemma 3.2. *The following is provable in Σ_1^b–NIA:*

(**1**) $\neg x <_{\mathrm{w}} x$

(**2**) $x <_{\mathrm{w}} y \,\vee\, y <_{\mathrm{w}} x \,\vee\, x = y$

(**3**) $x <_{\mathrm{w}} y \,\wedge\, y <_{\mathrm{w}} z \,\rightarrow\, x <_{\mathrm{w}} z$

(**4**) $G_{S_{\mathrm{w}}}\big(1{\times}x, 1(0{\times}x)\big)$

(**5**) $G_{S_{\mathrm{w}}}\big(x0(1{\times}y), x1(0{\times}y)\big)$

(**6**) $x \neq 1{\times}x \,\wedge\, G_{S_{\mathrm{w}}}(x, y) \,\rightarrow\, x \equiv y$

(**7**) $G_{S_{\mathrm{w}}}(x, y) \,\rightarrow\, x <_{\mathrm{w}} y$

(**8**) $\forall x \in \mathbb{W}_1 \; \forall y \in \mathbb{W}_1 \; (x <_{\mathrm{w}} y \,\wedge\, G_{S_{\mathrm{w}}}(x, z) \,\rightarrow\, z \leq_{\mathrm{w}} y)$

(**9**) $x \preceq y \,\wedge\, \neg(x \equiv y) \,\wedge\, G_{|.|_{\mathrm{w}}}(x, z) \,\wedge\, G_{|.|_{\mathrm{w}}}(y, w) \,\rightarrow\, z <_{\mathrm{w}} w$

(**10**) $G_{|.|_{\mathrm{w}}}(x, z) \,\wedge\, G_{|.|_{\mathrm{w}}}(y, w) \,\rightarrow\, (z = w \leftrightarrow x \equiv y)$

(**11**) $x \leq_{\mathrm{w}} y \,\rightarrow\, xz \leq_{\mathrm{w}} yz$

(**12**) $G_{|.|_{\mathrm{w}}}(w, y) \,\wedge\, x \leq_{\mathrm{w}} y \,\rightarrow\, \exists u \subseteq w \; G_{|.|_{\mathrm{w}}}(u, x)$

(**13**) $\forall x \in \mathbb{W}_1 \; \big(G_{+_{\mathrm{w}}}(x, 1, y) \rightarrow G_{S_{\mathrm{w}}}(x, y)\big)$

(**14**) $\forall x \in \mathbb{W}_1 \; \big(G_{\cdot_{\mathrm{w}}}(1, x, x) \wedge G_{\cdot_{\mathrm{w}}}(x, 1, x)\big)$

(**15**) $\forall x {\in} \mathbb{W}_1 \; \forall y {\in} \mathbb{W}_1 \; \big(G_{+_{\mathrm{w}}}(x, y, z) {\wedge} G_{+_{\mathrm{w}}}(y, 1, w) {\wedge} G_{+_{\mathrm{w}}}(z, 1, k) \rightarrow G_{+_{\mathrm{w}}}(x, w, k)\big)$

(**16**) $\forall x {\in} \mathbb{W}_1 \; \forall y {\in} \mathbb{W}_1 \; \big(G_{|.|_{\mathrm{w}}}(x, z) \wedge G_{|.|_{\mathrm{w}}}(y, w) \wedge G_{+_{\mathrm{w}}}(z, w, u) \rightarrow G_{|.|_{\mathrm{w}}}(x \frown y, u)\big)$

(**17**) $a <_{\mathrm{w}} b \leftrightarrow a1 <_{\mathrm{w}} b0$, $a <_{\mathrm{w}} b \rightarrow a0 <_{\mathrm{w}} b1$, $a \leq_{\mathrm{w}} b \leftrightarrow a0 \leq_{\mathrm{w}} b1$, $a \leq_{\mathrm{w}} b \leftrightarrow a0 \leq_{\mathrm{w}} b0$, $a \leq_{\mathrm{w}} b \leftrightarrow a1 \leq_{\mathrm{w}} b1$, *being the two last equivalences also valid if we replace* $\leq_{\mathrm{w}}$ *by* $<_{\mathrm{w}}$.

Proof: The proof of the first three assertions is done in [6], pp. 49–50.

(**4**) Proceed by induction on notation on x. The case $x=\epsilon$ is clear. Suppose $G_{S_\mathrm{w}}(1{\times}x, 1(0{\times}x))$. By Definition 2.2 (6), $1{\times}x1=(1{\times}x)1$ and by Proposition 3.1 for S_w, if $G_{S_\mathrm{w}}((1{\times}x)1,z)$ then $z=w0$ where $G_{S_\mathrm{w}}(1{\times}x,w)$. By induction hypothesis $w=1(0{\times}x)$, and using Definition 2.2 (2) and (6), $z=(1(0{\times}x))0=1((0{\times}x)0)=1(0{\times}x1)$. Thus, $G_{S_\mathrm{w}}(1{\times}x1, 1(0{\times}x1))$. The proof of $G_{S_\mathrm{w}}(1{\times}x0, 1(0{\times}x0))$ is similar because $1{\times}x0=(1{\times}x)1$ and $1(0{\times}x1)=1(0{\times}x0)$.

(**5**) Proceed again by induction on notation on y. Once more the case $y=\epsilon$ is clear. If $G_{S_\mathrm{w}}(x0(1{\times}y1),z)$ then, noticing that $x0(1{\times}y1)=x0((1{\times}y)1)=(x0(1{\times}y))1$, $z=w0$ where $G_{S_\mathrm{w}}(x0(1{\times}y),w)$. By induction hypothesis $w=x1(0{\times}y)$. Thus, $z=(x1(0{\times}y))0=x1((0{\times}y)0)=x1(0{\times}y1)$. The case $G_{S_\mathrm{w}}(x0(1{\times}y0),\, x1(0{\times}y0))$ is similar.

(**6**) By Lemma 2.1 (24) and the previous item, it is enough to prove that $1{\times}(x0(1{\times}y))=1{\times}(x1(0{\times}y))$. This uses Lemma 2.1 (10), (7) and (8) among others.

(**7**) Consider x, y verifying $G_{S_\mathrm{w}}(x,y)$. If $x=1{\times}x$ let us prove that $x\preceq y\wedge\neg(x\equiv y)\wedge x\neq y$. Notice that $1{\times}y \overset{(4)}{=} 1{\times}1(0{\times}x) \overset{L2.1(11)}{=} 1{\times}(0{\times}x)1 \overset{D2.2(6)}{=} (1{\times}(0{\times}x))1 \overset{L2.1(9)}{=} ((1{\times}0){\times}x)1 \overset{L2.1(6)}{=} (1{\times}x)1$. Clearly $1{\times}x\subseteq(1{\times}x)1$ and $1{\times}x\neq(1{\times}x)1$. Thus, $1{\times}x\subseteq 1{\times}y\wedge 1{\times}x\neq 1{\times}y$ (which, in particular, implies $x\neq y$). Hence $x\preceq y\wedge\neg(x\equiv y)\wedge x\neq y$. This entails $x<_\mathrm{w} y$. Finally, if $x\neq 1{\times}x$ use Lemma 2.1 (24) to prove that $x=z0(1{\times}w)$, for some z and w. By (5), $y=z1(0{\times}w)$. Now, it is easy to check that $x<_\mathrm{w} y$.

(**8**) Consider x, y, z such that $x\in\mathbb{W}_1\wedge y\in\mathbb{W}_1\wedge x<_\mathrm{w} y\wedge G_{S_\mathrm{w}}(x,z)$. Let us study two cases: $x=1{\times}x$ and $x\neq 1{\times}x$. In the first case it can be proved, using induction on notation on x and Definition 2.2 (7), (13), (5), (9), (12) and (6), that it does not exist w such that $w0\subseteq 1{\times}x=x$ (*), and so, by hypothesis, we have $x\preceq y\wedge\neg(x\equiv y)\wedge x\neq y$. From $x=1{\times}x$, using (4), Lemma 2.1 (11), Definition 2.2 (6) and Lemma 2.1 (9), (6), one has $1{\times}z=(1{\times}x)1$. Now, noticing that $1{\times}x\subseteq 1{\times}y$, one has, by Lemma 2.1 (22), that $1{\times}y=(1{\times}x)k$ for a certain k. By induction on notation, Lemma 2.1 (1), (19) and Definition 2.2 (1), (8), (9), it can be proved that $k=\epsilon\vee 0\subseteq k\vee 1\subseteq k$. Notice that $k\neq\epsilon$ because $\neg(x\equiv y)$. Moreover, if $0\subseteq k$ one would have that $1{\times}y=(1{\times}x)0t$ for a certain t and consequently, by Lemma 2.1 (19), $(1{\times}x)0\subseteq 1{\times}y$ — this is not possible, see (*). Thus $1\subseteq k$, and so, by Lemma 2.1 (20), $(1{\times}x)1\subseteq(1{\times}x)k$, i.e. $z\preceq y$. If $z\preceq y\wedge\neg(z\equiv y)$ then $z\leq_\mathrm{w} y$. Suppose $z\equiv y$. By (4), $z=1(0{\times}x)$.

Noticing that $y \in \mathbb{W}_1$ and $y \neq \epsilon$, one has $1 \subseteq y$, and so there exists r such that $y = 1r$. If $r = 0{\times}x$, then $y = z$, which implies $z \leq_{\mathrm{w}} y$. Otherwise, the existence of t and s such that $r = (0{\times}t)1s$ and $0{\times}t \subseteq 0{\times}x \wedge 0{\times}t \neq 0{\times}x$ can be ensured. By Lemma 2.1 (20), $1(0{\times}t) \subseteq z$. Notice that $1(0{\times}t)0 \subseteq z$ and $1(0{\times}t)1 \subseteq y$, and so $z \leq_{\mathrm{w}} y$. In the second case, $x \neq 1{\times}x$, use Lemma 2.1 (24), in order to prove that $x = u0(1{\times}v)$. Then, by (5), $z = u1(0{\times}v)$. By hypothesis, $x \preceq y \wedge \neg(x \equiv y)$ or $x \neq y \wedge x \equiv y \wedge \exists w \subseteq x(w0 \subseteq x \wedge w1 \subseteq y)$. In the first situation, provided that by (6) $x \equiv z$, one has $z \preceq y \wedge \neg(z \equiv y)$. This implies $z \leq_{\mathrm{w}} y$. In the second situation, let us take $w \subseteq x$ such that $w0 \subseteq x \wedge w1 \subseteq y$. There exist k, k' such that $x = w0k = u0(1{\times}v)$ and $y = w1k'$. By Lemma 2.1 (19), $w0 \subseteq u0(1{\times}v)$ and consequently, by induction on notation, $w0 \subseteq u0$. Then, by Definition 2.2 (8), (10), one has $w = u$ or $w0 \subseteq u$. If $w = u$, then $z = w1(0{\times}v)$. Noticing that $y \equiv x \equiv z$ and $y = w1k'$, one can prove that $z \leq_{\mathrm{w}} y$. If $w0 \subseteq u$ then $w0 \subseteq u \subseteq z$. Noticing that $w1 \subseteq y$, one has $z \leq_{\mathrm{w}} y$.

(**9**) and (**10**) are immediate by [6], p. 67 (in [6] the function $|\,.\,|_{\mathrm{w}}$ is denoted by lh).

(**11**) Consider that $x \leq_{\mathrm{w}} y$. By definition of $\leq_{\mathrm{w}}$, we have three possible cases: $x \preceq y \wedge \neg(x \equiv y)$, $x \equiv y \wedge \exists w \subseteq x(w0 \subseteq x \wedge w1 \subseteq y)$ and $x = y$. In the first case we have $1{\times}x \subseteq 1{\times}y$ and $\neg(x \equiv y)$. Moreover, $1{\times}xz \overset{L2.1(11)}{=} 1{\times}zx \overset{L2.1(10)}{=} (1{\times}z)\,(1{\times}x)$ and, analogously, $1{\times}yz = (1{\times}z)\,(1{\times}y)$. Notice that $1{\times}x \subseteq 1{\times}y \overset{L2.1(20)}{\to} (1{\times}z)(1{\times}x) \subseteq (1{\times}z)(1{\times}y)$. Thus $1{\times}xz \subseteq 1{\times}yz$, i.e. $xz \preceq yz$. If $1{\times}xz = 1{\times}yz$ then $(1{\times}x)\,(1{\times}z) = (1{\times}y)\,(1{\times}z)$ and by Lemma 2.1 (3), $1{\times}x = 1{\times}y$, which is false because $\neg(x \equiv y)$. And so $xz \preceq yz \wedge \neg(xz \equiv yz)$, which implies $xz \leq_{\mathrm{w}} yz$. In the second case, $1{\times}x = 1{\times}y \overset{L2.1(19),D2.2(1)}{\to} 1{\times}x \subseteq 1{\times}y \wedge 1{\times}y \subseteq 1{\times}x \overset{L2.1(20)}{\to} (1{\times}z)(1{\times}x) \subseteq (1{\times}z)(1{\times}y) \wedge (1{\times}z)(1{\times}y) \subseteq (1{\times}z)(1{\times}x) \overset{L2.1(11),(10)}{\to} 1{\times}xz \subseteq 1{\times}yz \wedge 1{\times}yz \subseteq 1{\times}xz \overset{L2.1(17)}{\to} 1{\times}xz = 1{\times}yz \to xz \equiv yz$. We also have that $\exists w \subseteq x\,(w0 \subseteq x \wedge w1 \subseteq y)$. Take such a w. By Lemma 2.1 (19), $x \subseteq xz$. From Lemma 2.1 (18), one has $w0 \subseteq xz$. In a similar way prove $w1 \subseteq yz$. Consequently, $xz \leq_{\mathrm{w}} yz$. The third case is trivial.

(**12**) Proof done in [6], pp. 67–68.

(**13**) Use Lemma 2.1 (16) to ensure that $x = \epsilon \vee \exists z\,(z0 = x \vee z1 = x)$. All the cases are straightforward.

(**14**) It is a consequence of Proposition 3.1, for $\cdot_{\mathrm{w}}$ and $+_{\mathrm{w}}$, together with Lemma 2.1 (1).

(**15**) Fix $x \in \mathbb{W}_1$. It is possible to prove, by induction on notation on $y \in \mathbb{W}_1$, that $\forall y \in \mathbb{W}_1\ \forall x' \subseteq x\ \exists z \preceq b_+(x', y)\ \exists w \preceq b_+(y, 1)\ \exists k \preceq b_+(z, 1)\ \big(G_+(x', y, z) \wedge G_+(y, 1, w) \wedge G_+(z, 1, k) \wedge G_+(x', w, k)\big)$. Note that, by Proposition 3.1, it is enough to prove this assertion. The case $y = \epsilon$ is clear. The case $y0 \in \mathbb{W}_1$ is easy, considering the three possibilities $G_U(x', \epsilon)$, $G_U(x', 0)$ and $G_U(x', 1)$. The case $y1$ (the only that requires the induction hypothesis) is also done considering the cases $G_U(x', \epsilon)$, $G_U(x', 0)$ and $G_U(x', 1)$. In the last two cases use Lemma 3.2 (13) and note that if $x' \subseteq x$ and $G_T(x', \bar{x})$ then $\bar{x} \subseteq x' \wedge x' \subseteq x$. So by Lemma 2.1 (18), $\bar{x} \subseteq x$ and we can apply the induction hypothesis to $\bar{x}$.

(**16**) It is enough to prove that $\Sigma_1^b\mathsf{-NIA} \vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \exists z \preceq b_{|.|}(x)\ \exists w \preceq b_{|.|}(y)\ \exists u \preceq b_+(z, w)\ \big(G_{|.|}(x, z) \wedge G_{|.|}(y, w) \wedge G_+(z, w, u) \wedge G_{|.|}(x \frown y, u)\big)$ — see Proposition 3.1. Fix $x \in \mathbb{W}_1$. The proof is by induction on notation on $y \in \mathbb{W}_1$. The case $y = \epsilon$ is clear. The step cases are immediate by the definition of $|.|_{\mathrm{w}}$ and Lemma 3.2 (13), (15).

(**17**) The proof is easy using the definition of $\leq_{\mathrm{w}}$ and $<_{\mathrm{w}}$. ∎

Theorem 3.1. *S_2^1 is interpretable in $\Sigma_1^b\mathsf{-NIA}$.*

Proof: According to the definition of interpretability between theories and using Proposition 3.2, to prove that S_2^1 is interpretable in $\Sigma_1^b\mathsf{-NIA}$ we have to prove that all axioms of S_2^1 translated by I are valid in $\Sigma_1^b\mathsf{-NIA}$. Notice that I is the translation associated with (σ, ν), presented in Remark 3.1. We analyse the translation of the basic axioms in the following order: 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 16, 17, 21, 22, 23, 24, 25, 19, 26, 28, 29, 27, 5, 10, 13, 15, 18, 20, 30, 31 and 32.

By Proposition 3.1, to each $f \in \mathsf{PTIME}$ we assign a formula G_f which in particular verifies $\Sigma_1^b\mathsf{-NIA} \vdash \forall \bar{x}\ \exists^1 z\ G_f(\bar{x}, z)$. To improve readability in this proof, we sometimes adopt the following abbreviation: for any formula A of $\mathcal{L}_{\mathbb{W}}$, and for any term $\bar{t}$, $A(f(\bar{t}))$ abbreviates $\forall z\ (G_f(\bar{t}, z) \to A(z))$. Namely, for terms $\bar{t}$ and s, $f(\bar{t}) = s$ abbreviates $\forall z (G_f(\bar{t}, z) \to z = s)$. Moreover, $\Sigma_1^b\mathsf{-NIA} \vdash (\forall y\, (G_f(\bar{x}, y) \to y = z)) \leftrightarrow G_f(\bar{x}, z)$. Therefore one may consider, modulo equivalence, that $f(\bar{t}) = s$ abbreviates $G_f(\bar{t}, s)$. In some cases we use infix notation. For instance $x \# y = z$ abbreviates $G_{\#}(x, y, z)$.

We have to prove that:

1) $\Sigma_1^b\mathsf{-NIA} \vdash I\big(\forall x\, \forall y\, (y \leq_{\mathrm{N}} x \to y \leq_{\mathrm{N}} S_{\mathrm{N}} x)\big)$, i.e. $\Sigma_1^b\mathsf{-NIA} \vdash \forall x \in \mathbb{W}_1\, \forall y \in \mathbb{W}_1\ \big(y \leq_{\mathrm{w}} x \to \forall z\, (G_{S_{\mathrm{w}}}(x, z) \to y \leq_{\mathrm{w}} z)\big)$, which is equivalent to prove that $\Sigma_1^b\mathsf{-NIA} \vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z\ (y \leq_{\mathrm{w}} x \wedge G_{S_{\mathrm{w}}}(x, z) \to y \leq_{\mathrm{w}} z)$. The result is immediate using Lemma 3.2 (7) and (3).

2) $\Sigma_1^b{-}\mathsf{NIA} \vdash I(\forall x\,(x \neq Sx))$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1\ \exists y\ (G_S(x,y) \wedge x \neq y)$. This is a consequence of Proposition 3.1 and Lemma 3.2 (7).

3) $\Sigma_1^b{-}\mathsf{NIA} \vdash I(\forall x\,(0 \leq x))$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1\,(\epsilon \leq_{\mathrm{w}} x)$. If $x = \epsilon$, the result is trivial. If $x \neq \epsilon$ then by Definition 2.2 (4), Lemma 2.1 (19), (1), (5) and (15) one has $\epsilon \preceq x \wedge \neg(\epsilon \equiv x)$ and so one has $\epsilon \leq_{\mathrm{w}} x$.

4) $\Sigma_1^b{-}\mathsf{NIA} \vdash I\big(\forall x\ \forall y\ (x \leq y \wedge x \neq y \leftrightarrow Sx \leq y)\big)$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1$ $\forall y \in \mathbb{W}_1\ \big(x \leq_{\mathrm{w}} y \wedge x \neq y\ \leftrightarrow\ \forall z\,(G_S(x,z) \rightarrow z \leq_{\mathrm{w}} y)\big)$. Apply Lemma 3.2 (8) to prove the direct implication. To the other implication i.e. to prove that $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \exists z\ \big((G_S(x,z) \rightarrow z \leq_{\mathrm{w}} y) \rightarrow (x \leq_{\mathrm{w}} y \wedge x \neq y)\big)$, consider $x, y \in \mathbb{W}_1$ and z such that $G_S(x,z)$ (given by Proposition 3.1). If $z \leq_{\mathrm{w}} y$ (the only case to study) then note that $z = y \vee z <_{\mathrm{w}} y$. If $z = y$ use Lemma 3.2 (7). If $z <_{\mathrm{w}} y$ use Lemma 3.2 (7) and (3).

6) $\Sigma_1^b{-}\mathsf{NIA} \vdash I\big(\forall x\ \forall y\ (y \leq x \vee x \leq y)\big)$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1$ $(y \leq_{\mathrm{w}} x \vee x \leq_{\mathrm{w}} y)$. Immediately by Lemma 3.2 (2).

7) $\Sigma_1^b{-}\mathsf{NIA} \vdash I\big(\forall x\ \forall y\ (x \leq y \wedge y \leq x\ \rightarrow\ x = y)\big)$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1$ $\forall y \in \mathbb{W}_1\ (x \leq_{\mathrm{w}} y \wedge y \leq_{\mathrm{w}} x\ \rightarrow\ x = y)$. The proof is straightforward by Lemma 3.2 (3) and (1).

8) $\Sigma_1^b{-}\mathsf{NIA} \vdash I\big(\forall x\,\forall y\,\forall z\ (x \leq y \wedge y \leq z\ \rightarrow\ x \leq z)\big)$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1$ $\forall y \in \mathbb{W}_1\ \forall z \in \mathbb{W}_1\ (x \leq_{\mathrm{w}} y \wedge y \leq_{\mathrm{w}} z\ \rightarrow\ x \leq_{\mathrm{w}} z)$. If $x = y$ or $y = z$ then the result is clear. Otherwise $x <_{\mathrm{w}} y \wedge y <_{\mathrm{w}} z$ and the result follows from Lemma 3.2 (3).

9) $\Sigma_1^b{-}\mathsf{NIA} \vdash I(|0| = 0)$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall y\ \big(G_{|.|_{\mathrm{w}}}(\epsilon, y) \rightarrow y = \epsilon\big)$. This is immediate attending to the definition of $|\,.\,|_{\mathrm{w}}$ and to Proposition 3.1.

11) $\Sigma_1^b{-}\mathsf{NIA} \vdash I(|S0| = S0)$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall y\,\forall z\ \big(G_S(\epsilon, y) \wedge G_{|.|}(y,z) \rightarrow z = y\big)$. The result is immediate using the definitions of S_{w} and $|.|_{\mathrm{w}}$ together with Lemma 2.1 (1).

12) $\Sigma_1^b{-}\mathsf{NIA} \vdash I\big(\forall x\ \forall y\ (x \leq y \rightarrow |x| \leq |y|)\big)$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1$ $\forall z\ \forall w\ \big(x \leq_{\mathrm{w}} y \wedge G_{|.|}(y,z) \wedge G_{|.|}(x,w) \rightarrow w \leq_{\mathrm{w}} z\big)$. Given x, y, z, w such that $x, y \in \mathbb{W}_1$ and $x \leq_{\mathrm{w}} y \wedge G_{|.|}(y,z) \wedge G_{|.|}(x,w)$. From $x \leq_{\mathrm{w}} y$, we have that $x \preceq y \wedge \neg(x \equiv y)$ or $x \equiv y$. In the first case, Lemma 3.2 (9) entails $w <_{\mathrm{w}} z$ and so we have $w \leq_{\mathrm{w}} z$. In the second case, Lemma 3.2 (10) ensures that $w = z$ and so $w \leq_{\mathrm{w}} z$.

14) $\Sigma_1^b{-}\mathsf{NIA} \vdash I\big(\forall y\ (0 \# y = S0)\big)$, i.e. $\Sigma_1^b{-}\mathsf{NIA} \vdash \forall y \in \mathbb{W}_1\,\forall z\,\forall w\ \big(G_S(\epsilon, z) \wedge G_{\#}(\epsilon, y, w)\ \rightarrow\ w = z\big)$. Noticing that $G_S(\epsilon, z)$ implies $z = 1$ and $G_{\#}(\epsilon, y, w) :=$ $w = 1 \frown ((0 \times \epsilon) \times y)$, the result follows immediately from Definition 2.2 (4), Lemma 2.1 (4) and Definition 2.2 (1).

16) Σ_1^b–NIA $\vdash I\big(\forall x\ \forall y\ (x \# y = y \# x)\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z\ \forall w\ \big(G_{\#}(y,x,z) \wedge G_{\#}(x,y,w) \rightarrow w = z\big)$. It is enough to prove that Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \big(1 \frown ((0 \times x) \times y) = 1 \frown ((0 \times y) \times x)\big)$. By Lemma 2.1 (9), $1 \frown ((0 \times x) \times y) = 1 \frown (0 \times (x \times y))$. By Lemma 2.1 (7), we have $1 \frown (0 \times (x \times y)) = 1 \frown (0 \times (y \times x))$. Again by Lemma 2.1 (9), $1 \frown (0 \times (y \times x)) = 1 \frown ((0 \times y) \times x)$. Thus $1 \frown ((0 \times x) \times y) = 1 \frown ((0 \times y) \times x)$.

17) Σ_1^b–NIA $\vdash I\big(\forall x\, \forall y\, \forall z\ (|x| = |y| \rightarrow x \# z = y \# z)\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z \in \mathbb{W}_1\ \exists v\ \exists w\ \big((G_{|.|}(y,v) \wedge G_{|.|}(x,w) \rightarrow w = v) \rightarrow 1 \frown ((0 \times x) \times z) = 1 \frown ((0 \times y) \times z)\big)$. Suppose $x, y, z \in \mathbb{W}_1$. Let v, w be such that $G_{|.|}(y,v)$ and $G_{|.|}(x,w)$. We have to prove that whenever one has $w = v$ we have $1 \frown ((0 \times x) \times z) = 1 \frown ((0 \times y) \times z)$. Lemma 3.2 (10), implies $x \equiv y$, i.e. $1 \times x = 1 \times y$. Then from Lemma 2.1 (12), we have $0 \times x = 0 \times y$. Therefore $1 \frown ((0 \times x) \times z) = 1 \frown ((0 \times y) \times z)$.

21) Σ_1^b–NIA $\vdash I\big(\forall x\ \forall y\ (x + y = y + x)\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z\ \forall w\ \big(G_+(y,x,z) \wedge G_+(x,y,w) \rightarrow w = z\big)$. Take $x \in \mathbb{W}_1$. It is enough to prove, by induction on notation on $y \in \mathbb{W}_1$, that $\forall y \in \mathbb{W}_1\ \forall x' \subseteq x\ \exists z \preceq b_+(y,x')\ \exists w \preceq b_+(x',y)\ \big(G_+(y,x',z) \wedge G_+(x',y,w) \wedge w = z\big)$ — see Proposition 3.1. The case $y = \epsilon$ is clear. For the step cases use the definition of $+_{\mathrm{w}}$ and the induction hypothesis applied to v such that $G_T(x',v)$. Notice that $v \subseteq x$.

22) Σ_1^b–NIA $\vdash I\big(\forall x\, (x + 0 = x)\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y\ \big(G_+(x,\epsilon,y) \rightarrow y = x\big)$. Immediate by definition of $+_{\mathrm{w}}$.

23) Σ_1^b–NIA $\vdash I\big(\forall x\ \forall y\ (x + Sy = S(x + y))\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z\ \forall w\ \forall v\ \big(G_+(x,y,z) \wedge G_S(z,w) \wedge G_S(y,v) \rightarrow G_+(x,v,w)\big)$. This is a consequence of Lemma 3.2 (13) and (15).

24) Σ_1^b–NIA $\vdash I\big(\forall x\ \forall y\ \forall z\ ((x + y) + z = x + (y + z))\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z \in \mathbb{W}_1\ \forall w\ \forall k\ \forall u\ \big(G_+(y,z,w) \wedge G_+(x,w,k) \wedge G_+(x,y,u) \rightarrow G_+(u,z,k)\big)$. The proof is done, by induction on notation on $z \in \mathbb{W}_1$ over the following assertion: $\forall z \in \mathbb{W}_1\ \forall x' \subseteq x\ \forall y' \subseteq y\ \exists w \preceq b_+(y',z)\ \exists k \preceq b_+(x',w)\ \exists u \preceq b_+(x',y')\ \big(G_+(y',z,w) \wedge G_+(x',w,k) \wedge G_+(x',y',u) \wedge G_+(u,z,k)\big)$ and it uses two facts (*) and (**) which result from Lemma 3.2 (15) and Theorem 3.1 (21):

(*****) Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z\ \forall w\ \forall k\ \big(G_+(1,y,z) \wedge G_+(x,z,w) \wedge G_+(x,1,k) \rightarrow G_+(k,y,w)\big)$

(******) Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z\ \forall w\ \forall k\ \big(G_+(x,y,z) \wedge G_+(1,z,w) \wedge G_+(1,x,k) \rightarrow G_+(k,y,w)\big)$.

The case $z = \epsilon$ is clear. Suppose, by induction hypothesis, that we have the above assertion for an arbitrary $z \in \mathbb{W}_1$ and suppose $z0 \in \mathbb{W}_1$. Being $x' \subseteq x$

and $y' \subseteq y$ we want to prove that $\exists w' \preceq b_+(y', z0)\ \exists k' \preceq b_+(x', w')\ \exists u' \preceq b_+(x', y')$ $G_+(y', z0, w') \wedge G_+(x', w', k') \wedge G_+(x', y', u') \wedge G_+(u', z0, k')$. Consider w', k', u' such that $G_+(y', z0, w') \wedge G_+(x', w', k') \wedge G_+(x', y', u')$. Suppose that $G_+(u', z0, r)$. We want to prove that $r = k'$. The cases $x' = \epsilon$ or $y' = \epsilon$ are easily verified. Here we study the case $x', y' \neq \epsilon$. By definition of $+_{\mathrm{w}}$, we have $r = a \frown b$, where $G_T(u', s)$, $G_+(s, z, a)$ and $G_U(u', b)$. We consider the three possible situations: a) $G_U(x', 1) \wedge G_U(y', 1)$, b) $G_U(x', 0) \wedge G_U(y', 0)$, c) not in the previous cases. In situation a), by Lemma 3.1 (1) and (2), considering that $G_T(x', \bar{x})$, $G_T(y', \bar{y})$, $G_+(\bar{x}, \bar{y}, \bar{u})$ and $G_+(\bar{u}, 1, c)$ we have $r = d \frown 0$, with $G_+(c, z, d)$. By Theorem 3.1 (21) (already verified), $G_+(1, \bar{u}, c)$ and so, using fact (**), $r = e \frown 0$, where $G_+(\bar{u}, z, l)$ and $G_+(1, l, e)$. By induction hypothesis applied to $\bar{x}$ and $\bar{y}$, we have $G_+(\bar{x}, h, l)$, where $G_+(\bar{y}, z, h)$, and again by Theorem 3.1 (21) we have $r = f \frown 0$, where $G_+(l, 1, f)$. So, by definition of $+_{\mathrm{w}}$, $G_+(x', h1, r)$. Provided $G_U(y', 1)$, we have $G_+(x', m, r)$ where $G_+(y', z0, m)$. Analogously, we also conclude that in situations b) and c) one obtains $G_+(x', m, r)$, where $G_+(y', z0, m)$. By hypothesis, we have $G_+(y', z0, w')$ and $G_+(x', w', k')$ and so $m = w'$ and $r = k'$. The case $z1$ is analogous to the case $z0$. Use the definition of $+_{\mathrm{w}}$, consider the situations a), b) and c) as before, use the Lemma 3.1 (1) and (2), the fact (*), Theorem 3.1 (21) (already verified), Lemma 3.2 (15) and the induction hypothesis. This finishes the proof.

25) Σ_1^b–NIA $\vdash I\big(\forall x\ \forall y\ \forall z\ (x + y \leq x + z \leftrightarrow y \leq z)\big)$, i.e. Σ_1^b–NIA $\vdash$ $\forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \forall z \in \mathbb{W}_1\ \big(\forall w\ \forall k\ (G_+(x, z, w) \wedge G_+(x, y, k) \rightarrow k \leq_{\mathrm{w}} w) \leftrightarrow y \leq_{\mathrm{w}} z\big)$. First we need some facts:

Fact 3.1. Σ_1^b–NIA $\vdash \forall a \in \mathbb{W}_1\ \forall b \in \mathbb{W}_1\ \forall c\ \big(G_+(a, b, c) \wedge b \neq \epsilon \rightarrow a <_{\mathrm{w}} c\big)$. □

Fix $a \in \mathbb{W}_1$. To prove this fact we show, by induction on notation on $b \in \mathbb{W}_1$, that $\forall b \in \mathbb{W}_1\ \forall a' \subseteq a\ \exists c \preceq b_+(a', b)\ \big(G_+(a', b, c) \wedge (b \neq \epsilon \rightarrow a' <_{\mathrm{w}} c)\big)$, or simplifying $\forall b \in \mathbb{W}_1\ \forall a' \subseteq a\ (b \neq \epsilon \rightarrow a' <_{\mathrm{w}} a' + b)$. The case $b = \epsilon$ is clear. For $b0 \in \mathbb{W}_1$, fix $a' \subseteq a$. We want to prove that $a' <_{\mathrm{w}} a' + b0$. If $a' = \epsilon$ the result is clear. If $a' \neq \epsilon$, then $a' + b0 = (T(a') + b)\, U(a')$. Notice that $T(a') \subseteq a$. Therefore, by induction hypothesis, $T(a') <_{\mathrm{w}} T(a') + b$. Consequently, by Lemma 3.2 (17), we have that $T(a') \frown U(a') <_{\mathrm{w}} (T(a') + b)\, U(a')$. By Lemma 3.1 (3), $T(a')\, U(a') = a'$, so $a' <_{\mathrm{w}} a' + b0$. For $b1$ proceed in a similar way.

Fact 3.2. Σ_1^b–NIA $\vdash \forall a \in \mathbb{W}_1\ \forall b \in \mathbb{W}_1\ \forall c \in \mathbb{W}_1\ \forall d\ \forall e\ \big(G_+(b, c, d) \wedge G_+(a, c, e)$ $\rightarrow (e \neq d \leftrightarrow a \neq b)\big)$. □

The direct implication is immediate. The other implication is done by induction on notation on $c \in \mathbb{W}_1$. Fix $a, b \in \mathbb{W}_1$ such that $a \neq b$ and prove that

$\forall c \in \mathbb{W}_1\ \forall a' \subseteq a\ \forall b' \subseteq b\ \exists d \preceq b_+(b', c)\ \exists e \preceq b_+(a', c)\ \big(G_+(b', c, d) \wedge G_+(a', c, e) \wedge (a' \neq b' \to e \neq d)\big)$. In an abridged manner, $\forall c \in \mathbb{W}_1\ \forall a' \subseteq a\ \forall b' \subseteq b\ (a' \neq b' \to a' + c \neq b' + c)$. The case $c = \epsilon$ is clear. For $c0 \in \mathbb{W}_1$ we want to prove that $\forall a' \subseteq a\ \forall b' \subseteq b\ (a' \neq b' \to a' + c0 \neq b' + c0)$. Fix $a' \subseteq a$ and $b' \subseteq b$ such that $a' \neq b'$. If $a' = \epsilon$ or $b' = \epsilon$ the result is immediate considering that $a' \neq b'$ and using Fact 3.1. Let us study the case $a' \neq \epsilon$ and $b' \neq \epsilon$. We have $a' \neq b' \overset{L3.1(3)}{\to} T(a')\,U(a') \neq T(b')\,U(b') \to T(a') \neq T(b') \vee U(a') \neq U(b')$. Noticing that $T(a') \subseteq a'$ and $T(b') \subseteq b'$ one has

i) $T(a') \neq T(b') \overset{I.H.}{\to} T(a') + c \neq T(b') + c \overset{L3.2(17)}{\to} (T(a') + c)\,U(a') \neq (T(b') + c)\,U(b') \overset{P3.1 \text{ for } +_{\mathrm{w}}}{\to} a' + c0 \neq b' + c0$

ii) $U(a') \neq U(b') \overset{D2.2(12)}{\to} (T(a') + c)\,U(a') \neq (T(b') + c)\,U(b') \overset{P3.1 \text{ for } +_{\mathrm{w}}}{\to} a' + c0 \neq b' + c0$.

In any case $a' + c0 \neq b' + c0$. The case $c1$ is similar to the one above. The only difference is that while using the definition of $+_{\mathrm{w}}$, we have to consider four situations: $G_U(a', 0)$, $G_U(a', 1)$, $G_U(b', 0)$ and $G_U(b', 1)$.

To prove 25), fix $y \in \mathbb{W}_1$ and $z \in \mathbb{W}_1$ and prove, by induction on notation on $x \in \mathbb{W}_1$, that $\forall x \in \mathbb{W}_1\ \forall y' \subseteq y\ \forall z' \subseteq z\ \exists w \preceq b_+(z', x)\ \exists k \preceq b_+(y', x)\ \big(G_+(z', x, w) \wedge G_+(y', x, k) \wedge (k \leq_{\mathrm{w}} w \leftrightarrow y' \leq_{\mathrm{w}} z')\big)$, which is enough by Proposition 3.1 and Theorem 3.1 (21) already verified. This is equivalent to prove that $\forall x \in \mathbb{W}_1\ \forall y' \subseteq y\ \forall z' \subseteq z\ (y' + x \leq_{\mathrm{w}} z' + x \leftrightarrow y' \leq_{\mathrm{w}} z')$. The case $x = \epsilon$ is clear. Given $x0 \in \mathbb{W}_1$, fix $y' \subseteq y$, $z' \subseteq z$. We want to prove that $y' + x0 \leq_{\mathrm{w}} z' + x0 \leftrightarrow y' \leq_{\mathrm{w}} z'$. If $y' = \epsilon$ and $z' = \epsilon$ the result is trivial. If $y' = \epsilon$ but $z' \neq \epsilon$ then $\epsilon \leq z' \overset{F3.1, T3.1(3)}{\leftrightarrow} x0 < x0 + z' \overset{T3.1(21)}{\leftrightarrow} x0 < z' + x0$. If $y' \neq \epsilon$ and $z' = \epsilon$ then $y' \leq z'$ does not hold (because one would have $y' \leq \epsilon$ but also by Theorem 3.1 (3) $\epsilon \leq y'$, therefore by Theorem 3.1 (7) $y' = \epsilon$). For $y' \neq \epsilon \wedge z' \neq \epsilon$, consider two cases:

a) $U(y') = U(z') \vee \big(U(y') = 0 \wedge U(z') = 1\big)$

b) $U(y') = 1 \wedge U(z') = 0$.

In case a), $y' + x0 \leq_{\mathrm{w}} z' + x0 \overset{P3.1 \text{ for } +_{\mathrm{w}}}{\leftrightarrow} (T(y') + x)\,U(y') \leq_{\mathrm{w}} (T(z') + x)\,U(z') \overset{L3.2(17)}{\leftrightarrow} T(y') + x \leq_{\mathrm{w}} T(z') + x \overset{I.H.}{\leftrightarrow} T(y') \leq_{\mathrm{w}} T(z') \overset{L3.2(17)}{\leftrightarrow} T(y')\,U(y') \leq_{\mathrm{w}} T(z')\,U(z') \overset{L3.1(3)}{\leftrightarrow} y' \leq_{\mathrm{w}} z'$.

In case b), $y' + x0 \leq_{\mathrm{w}} z' + x0 \overset{P3.1 \text{ for } +_{\mathrm{w}}}{\leftrightarrow} (T(y') + x)\,U(y') \leq_{\mathrm{w}} (T(z') + x)\,U(z') \overset{L3.2(17)}{\leftrightarrow} T(y') + x <_{\mathrm{w}} T(z') + x \overset{I.H., F3.2}{\leftrightarrow} T(y') <_{\mathrm{w}} T(z') \overset{L3.2(17)}{\leftrightarrow} T(y')1 \leq_{\mathrm{w}} T(z')0 \overset{L3.1(3)}{\leftrightarrow} y' \leq_{\mathrm{w}} z'$.

The case $z1$ is proved in a similar way, using the definition of $+_{\mathrm{w}}$ (this time divided in the cases: $G_U(y',0)\wedge G_U(z',0)$, $G_U(y',1)\wedge G_U(z',1)$, $G_U(y',0)\wedge G_U(z',1)$ and $G_U(y',1)\wedge G_U(z',0)$), using Lemma 3.2 (17), Fact 3.2, the induction hypothesis and the following result, whose proof does not involve any special difficulty:

Fact 3.3. $\Sigma_1^b\text{–}\mathsf{NIA} \vdash \forall a\in\mathbb{W}_1\,\forall b\in\mathbb{W}_1\,\forall c\,\big(G_S(b,c)\rightarrow(a<_{\mathrm{w}} c\leftrightarrow a\leq_{\mathrm{w}} b)\big)$ and $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall a\in\mathbb{W}_1\,\forall b\in\mathbb{W}_1\,\forall c\,\forall d\,\big(G_+(a,1,c)\wedge G_+(b,1,d)\rightarrow(c\leq_{\mathrm{w}} d\leftrightarrow a\leq_{\mathrm{w}} b)\big)$. □

19) $\Sigma_1^b\text{–}\mathsf{NIA}\vdash I\big(\forall x\,\forall y\,(x\leq x+y)\big)$, i.e. $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall w\,\big(G_+(x,y,w)\rightarrow x\leq_{\mathrm{w}} w\big)$ or $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\; x\leq x+y$. This is an immediate consequence of Theorem 3.1 (25) and (3).

26) $\Sigma_1^b\text{–}\mathsf{NIA}\vdash I\big(\forall x(x\cdot 0=0)\big)$, i.e. $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1\,\forall y\,\big(G_\cdot(x,\epsilon,y)\rightarrow y=\epsilon\big)$. Immediate by definition of $\cdot_{\mathrm{w}}$.

28) $\Sigma_1^b\text{–}\mathsf{NIA}\vdash I\big(\forall x\,\forall y\,(x\cdot y=y\cdot x)\big)$, i.e. $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall z\,\forall k$ $(G_\cdot(y,x,z)\wedge G_\cdot(x,y,k)\rightarrow k=z)$ or $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\; x\cdot y=y\cdot x$. The case $x=\epsilon$ is immediate. If $x\neq\epsilon$ then, by Lemma 2.1 (16), $\exists z\,(z0=x\vee z1=x)$. Fix such a z. We want to prove that $z0\cdot y=y\cdot z0$ and $z1\cdot y=y\cdot z1$. Both assertions can be proved by induction on notation on $y\in\mathbb{W}_1$. The reasoning leading to the second one requires Theorem 3.1 (24) and (21).

29) $\Sigma_1^b\text{–}\mathsf{NIA}\vdash I\big(\forall x\,\forall y\,\forall z\,(x\cdot(y+z)=(x\cdot y)+(x\cdot z))\big)$, i.e. $\Sigma_1^b\text{–}\mathsf{NIA}\vdash$ $\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall z\in\mathbb{W}_1\,\forall w\,\forall k\,\forall l\,\forall r\,\big(G_\cdot(x,z,w)\wedge G_\cdot(x,y,k)\wedge G_+(k,w,l)\wedge G_+(y,z,r)\rightarrow G_\cdot(x,r,l)\big)$ or $\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall z\in\mathbb{W}_1\; x\cdot(y+z)=(x\cdot y)+(x\cdot z)$. By Theorem 3.1 (28) this is equivalent to prove $\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall z\in\mathbb{W}_1$ $(y+z)\cdot x=(y\cdot x)+(z\cdot x)$. The proof is by induction on $x\in\mathbb{W}_1$. The case $x=\epsilon$ is trivial using Theorem 3.1 (28) and (26). For $x0\in\mathbb{W}_1$, let us prove that $(y\cdot z)\cdot x0$ $=(y\cdot x0)+(z\cdot x0)$. If $y=\epsilon\vee z=\epsilon$ the result is immediate. If $y\neq\epsilon\wedge z\neq\epsilon$ then, noticing that $y+z\neq\epsilon$, we have $(y+z)\cdot x0\overset{P3.1\text{ for }\cdot_{\mathrm{w}}}{=}((y+z)\cdot x)0\overset{IH}{=}((y\cdot x)+(z\cdot x))0\overset{P3.1\text{ for }+_{\mathrm{w}},T,U}{=}(y\cdot x)0+(z\cdot x)0\overset{P3.1\text{ for }\cdot_{\mathrm{w}}}{=}(y\cdot x0)+(z\cdot x0)$. For the case $x1$ the reasoning is analogous to the one above, using in addition Theorem 3.1 (24) and (21).

27) $\Sigma_1^b\text{–}\mathsf{NIA}\vdash I\big(\forall x\,\forall y\,(x\cdot(Sy)=(x\cdot y)+x)\big)$, i.e. $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1$ $\forall z\,\forall w\,\forall k\,\big(G_\cdot(x,y,z)\wedge G_+(z,x,w)\wedge G_S(y,k)\rightarrow G_\cdot(x,k,w)\big)$ or $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1$ $\forall y\in\mathbb{W}_1\; x\cdot(Sy)=(x\cdot y)+x$. Given $x,y\in\mathbb{W}_1$ one has $x\cdot(Sy)\overset{L3.2(13)}{=}x\cdot(y+1)\overset{T3.1(29)}{=}(x\cdot y)+(x\cdot 1)\overset{L3.2(14)}{=}(x\cdot y)+x$.

5) $\Sigma_1^b\text{–}\mathsf{NIA}\vdash I\big(\forall x\,(x\neq 0\rightarrow 2\cdot x\neq 0)\big)$, i.e. $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1\,\big(x\neq\epsilon\rightarrow\exists y\,(G_\cdot(10,x,y)\wedge y\neq\epsilon)\big)$ or $\Sigma_1^b\text{–}\mathsf{NIA}\vdash\forall x\in\mathbb{W}_1\,(x\neq\epsilon\rightarrow 10\cdot x\neq\epsilon)$. For $x\in\mathbb{W}_1$ such that $x\neq\epsilon$ one has $10\cdot x\overset{T3.1(28)}{=}x\cdot 10\overset{P3.1\text{ for }\cdot_{\mathrm{w}}}{=}(x\cdot 1)0\overset{L3.2(14)}{=}x0\overset{D2.2(13)}{\neq}\epsilon$.

10) Σ_1^b–NIA $\vdash I\big(\forall x\ (x \neq 0 \to |2\cdot x| = S(|x|) \wedge |S(2\cdot x)| = S(|x|))\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ (x \neq \epsilon \to \forall y\, \forall z\, \forall w\, \forall v\, \forall u\, \forall k\ \big(G_{|.|}(x,y) \wedge G_S(y,z) \wedge G_{\cdot}(10,x,w) \wedge G_{|.|}(w,v) \wedge G_S(w,u) \wedge G_{|.|}(u,k) \to v = z \wedge k = z\big)$ or Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \big(x \neq \epsilon \to |10\cdot x| = S(|x|) \wedge |S(10\cdot x)| = S(|x|)\big)$. Given $x \in \mathbb{W}_1$ such that $x \neq \epsilon$, one has that $10\cdot x = x0$ — this uses Theorem 3.1 (28), Proposition 3.1 for $\cdot_{\mathrm{w}}$ and Lemma 3.2 (14). Then, by Proposition 3.1 for $|.|_{\mathrm{w}}$ and S_{w}, the result is immediate.

20) Σ_1^b–NIA $\vdash I\big(\forall x\, \forall y\ (x \leq y \wedge x \neq y \to S(2\cdot x) \leq 2\cdot y \wedge S(2\cdot x) \neq 2\cdot y)\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\, \forall y \in \mathbb{W}_1\ \big(x \leq_{\mathrm{w}} y \wedge x \neq y \to \forall z\, \forall w\, \forall v\ (G_{\cdot}(10,y,z) \wedge G_{\cdot}(10,x,w) \wedge G_S(w,v) \to v \leq_{\mathrm{w}} z \wedge v \neq z)\big)$ or Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\, \forall y \in \mathbb{W}_1\ \big(x < y \to S(10\cdot x) < 10\cdot y\big)$. Notice that, whenever $x \neq \epsilon$, one has $10\cdot x = x0$ (see proof of item 10). If $x = \epsilon$ then $\epsilon < y \overset{L3.2(17)}{\to} \epsilon 1 < y0 \overset{L2.1(1)}{\to} 1 < y0 \overset{P3.1 \text{ for } S_{\mathrm{w}}}{\to} S(\epsilon) < y0$. Thus $\epsilon < y \to S(10\cdot\epsilon) < 10\cdot y$. If $x \neq \epsilon$ the statement is a trivial consequence of Lemma 3.2 (17), attending to Proposition 3.1 for S_{w}.

13) Σ_1^b–NIA $\vdash I\big(\forall x\, \forall y\ (|x \# y| = S(|x|\cdot|y|))\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\, \forall y \in \mathbb{W}_1\, \forall z\, \forall w\, \forall k\, \forall r\, \forall u\, \forall v\ \big(G_{|.|}(x,z) \wedge G_{|.|}(y,w) \wedge G_{\cdot}(z,w,k) \wedge G_S(k,r) \wedge G_{\#}(x,y,u) \wedge G_{|.|}(u,v) \to v = r\big)$. We mean Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\, \forall y \in \mathbb{W}_1\ |1 \frown ((0\times x)\times y)| = S(|x|\cdot|y|)$. We have $1\times((0\times x)\times y) \overset{L2.1(9)}{=} 1\times(0\times(x\times y)) \overset{L2.1(8)}{=} 1\times((x\times y)\times 0) \overset{L2.1(9)}{=} 1\times(x\times(y\times 0)) \overset{L2.1(6)}{=} 1\times(x\times y)$. Noticing that $(0\times x)\times y \equiv x\times y \overset{L2.1(13)}{\to} 1\frown((0\times x)\times y) \equiv 1\frown(x\times y) \overset{L2.1(11)}{\to} 1\frown((0\times x)\times y) \equiv (x\times y)1 \overset{L3.2(10)}{\to} |1\frown((0\times x)\times y)| = |(x\times y)1| \overset{P3.1 \text{ for } |.|_{\mathrm{w}}}{\to} |1 \frown ((0\times x)\times y)| = S(|x\times y|)$, we just need to prove that $|x\times y| = |x|\cdot|y|$. We proceed by induction on $y \in \mathbb{W}_1$. The case $y = \epsilon$ is clear. For $y0$ we have $|x\times y0| \overset{D2.2(5)}{=} |(x\times y) \frown x| \overset{L3.2(16)}{=} |x\times y| + |x| \overset{IH}{=} (|x|\cdot|y|) + |x| \overset{T3.1(29)}{=} |x|\cdot(|y|+1) \overset{L3.2(13)}{=} |x|\cdot S(|y|) \overset{P3.1 \text{ for } |.|_{\mathrm{w}}}{=} |x|\cdot|y0|$. The case $y1$ is analogous. This finishes the proof.

15) Σ_1^b–NIA $\vdash I\big(\forall x\ (x \neq 0 \to 1 \# (2\cdot x) = 2(1 \# x) \wedge 1 \# (S(2\cdot x)) = 2(1 \# x))\big)$, i.e. Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\, \forall y\, \forall z\ \big(x \neq \epsilon \wedge G_{\cdot}(10,x,y) \wedge G_S(y,z) \to x\big), 1 \frown ((0\times 1)\times z)))$ or Σ_1^b–NIA $\vdash \forall x \in \mathbb{W}_1\ \big(x \neq \epsilon \to \big(10\cdot(1 \frown ((0\times 1)\times x)) = 1 \frown ((0\times 1)\times(10\cdot x)) \wedge 10\cdot(1 \frown ((0\times 1)\times x)) = 1 \frown ((0\times 1)\times(S(10\cdot x)))\big)\big)$. Given $x \in \mathbb{W}_1$ such that $x \neq \epsilon$, let us prove the two equalities above. Note that $10\cdot(1 \frown ((0\times 1)\times x)) \overset{L2.1(6)}{=} 10\cdot(1 \frown (0\times x)) \overset{T3.1(28)}{=} (1 \frown (0\times x))\cdot 10 \overset{P3.1 \text{ for } \cdot_{\mathrm{w}}}{=} ((1 \frown (0\times x))\cdot 1) \frown 0 \overset{L3.2(14)}{=} (1 \frown (0\times x)) \frown 0 \overset{L2.1(2)}{=} 1 \frown ((0\times x) \frown 0) \overset{D2.2(5)}{=} 1 \frown (0\times x0)$. Now, for the first equality, just notice that $1 \frown (0\times x0) \overset{L3.2(14)}{=} 1 \frown (0\times(x\cdot 1) \frown 0) \overset{P3.1 \text{ for } \cdot_{\mathrm{w}}}{=} 1 \frown (0\times(x\cdot 10)) \overset{T3.1(28)}{=} 1 \frown (0\times(10\cdot x)) \overset{L2.1(6)}{=} 1 \frown ((0\times 1)\times(10\cdot x))$. To establish the other equality

$1\frown(0\times x0) \overset{D2.2(5),(6)}{=} 1\frown(0\times x1) \overset{L3.2(14)}{=} 1\frown(0\times(x\cdot 1)\frown 1) \overset{P3.1\text{ for } S_{\mathrm{w}}}{=} 1\frown(0\times S((x\cdot 1)\frown 0))$ $\overset{P3.1\text{ for }\cdot_{\mathrm{w}}}{=} 1\frown(0\times S(x\cdot 10)) \overset{T3.1(28)}{=} 1\frown(0\times S(10\cdot x)) \overset{L2.1(6)}{=} 1\frown((0\times 1)\times S(10\cdot x))$.

18) Σ^b_1–NIA $\vdash I\big(\forall x\,\forall y\,\forall u\,\forall v\,(|x| = |u|+|v| \to x\,\#\,y = (u\,\#\,y)\cdot(v\,\#\,y))\big)$, i.e. Σ^b_1–NIA $\vdash \forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall u\in\mathbb{W}_1\,\forall v\in\mathbb{W}_1\,\exists z\,\exists w\,\exists k\,\exists l\,\big(\big(G_{|.|}(v,z)\wedge G_{|.|}(u,w)\wedge G_+(w,z,k)\wedge G_{|.|}(x,l)\to l=k\big)\to G_\cdot\big(1\frown((0\times u)\times y), 1\frown((0\times v)\times y), 1\frown((0\times x)\times y)\big)\big)$ or Σ^b_1–NIA $\vdash \forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall u\in\mathbb{W}_1\,\forall v\in\mathbb{W}_1\,\big(|x|=|u|+|v|\to 1\frown((0\times x)\times y) = (1\frown((0\times u)\times y))\cdot(1\times((0\times v)\times y))\big)$. First notice that (*) Σ^b_1–NIA $\vdash \forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1$ $\big(x\neq\epsilon\to G_\cdot(x, 1\frown(0\times y), x\frown(0\times y))\big)$ can be easily proved by induction on notation on $y\in\mathbb{W}_1$. For $y=\epsilon$ use Lemma 3.2 (14). The cases $y0$ and $y1$ follow straightforwardly using the axioms that define $\times$, $\frown$, the definition of $\cdot_{\mathrm{w}}$ and Lemma 2.1 (2). Now, fix $x,y,u,v\in\mathbb{W}_1$. $|x| = |u|+|v| \overset{L3.2(16)}{\to} |x| = |u\frown v| \overset{L3.2(10)}{\to}$ $x\equiv u\frown v \overset{L2.1(12)}{\to} 0\times x = 0\times(u\frown v) \to 1\frown((0\times x)\times y) = 1\frown((0\times(u\frown v))\times y)$. Now, notice that $1\frown\big((0\times(u\frown v))\times y\big) \overset{L2.1(9)}{=} 1\frown\big(0\times((uv)\times y)\big) \overset{L2.1(7)}{=} 1\frown\big(0\times(y\times uv)\big)$ $\overset{L2.1(10)}{=} 1\frown\big(0\times((y\times u)(y\times v))\big) \overset{L2.1(10)}{=} 1\frown\big((0\times(y\times u))\frown(0\times(y\times v))\big) \overset{L2.1(7)}{=}$ $1\frown\big((0\times(u\times y))\frown(0\times(v\times y))\big) \overset{L2.1(9)}{=} 1\frown\big(((0\times u)\times y)\frown(0\times(v\times y))\big) \overset{L2.1(2)}{=}$ $\big(1\frown((0\times u)\times y)\big)\frown(0\times(v\times y)) \overset{(*)}{=} \big(1\frown((0\times u)\times y)\big)\cdot\big(1\frown(0\times(v\times y))\big) \overset{L2.1(9)}{=}$ $\big(1\frown((0\times u)\times y)\big)\cdot\big(1\frown((0\times v)\times y)\big)$.

30) Σ^b_1–NIA $\vdash I\big(\forall x\,\forall y\,\forall z\,(S0\le x\to(x\cdot y\le x\cdot z\leftrightarrow y\le z))\big)$, i.e. Σ^b_1–NIA $\vdash$ $\forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall z\in\mathbb{W}_1\,\big(\forall w\,(G_S(\epsilon,w)\to w\le_{\mathrm{w}} x)\to(\forall k\,\forall l\,(G_\cdot(x,z,k)\wedge G_\cdot(x,y,l)$ $\to l\le_{\mathrm{w}} k)\leftrightarrow y\le_{\mathrm{w}} z)\big)$ or equivalently, using Theorem 3.1 (28) and Proposition 3.1 for S_{w}, Σ^b_1–NIA $\vdash \forall x\in\mathbb{W}_1\,\forall y\in\mathbb{W}_1\,\forall z\in\mathbb{W}_1\,\big(1\le x\to(y\cdot x\le z\cdot x\leftrightarrow y\le z)\big)$. The proof is by induction on $x\in\mathbb{W}_1$. The case $x=\epsilon$ is clear. For $x0\in\mathbb{W}_1$ we can ensure that $1\le x$. If $y=\epsilon$ or $z=\epsilon$ the result is immediate. Otherwise, $y\cdot x0\le z\cdot x0 \overset{P3.1\text{ for }\cdot_{\mathrm{w}}}{\leftrightarrow} (y\cdot x)0\le(z\cdot x)0 \overset{L3.2(17)}{\leftrightarrow} y\cdot x\le z\cdot x \overset{IH}{\leftrightarrow} y\le z$. For $x1\in\mathbb{W}_1$ notice that $x=\epsilon$ or $1\le x$. The case $x=\epsilon$ is trivial. For $1\le x$ $y\le z \overset{IH}{\leftrightarrow} y\cdot x\le z\cdot x \overset{L3.2(17)}{\leftrightarrow} (y\cdot x)0\le(z\cdot x)0 \overset{T3.1(21),(25)}{\leftrightarrow} (y\cdot x)0+y\le(z\cdot x)0+y$. But also, $y\le z \overset{T3.1(25)}{\leftrightarrow} (z\cdot x)0+y\le(z\cdot x)0+z$. Thus, by Theorem 3.1 (8), $y\le z\to(y\cdot x)0+y\le(z\cdot x)0+z$ and so $y\le z\to y\cdot x1\le z\cdot x1$. The other implication is proved by contraposition. The reasoning is similar to the one above, but it also uses Lemma 2.1 (3) and Theorem 3.1 (7). This finishes the proof.

31) Σ^b_1–NIA $\vdash I\big(\forall x\,(x\neq 0\to|x|=S(|\lfloor\frac{1}{2}x\rfloor|))\big)$, i.e. Σ^b_1–NIA $\vdash \forall x\in\mathbb{W}_1\,\forall y\,\forall z$ $\forall w\,\forall k\,\big(x\neq\epsilon\wedge G_{\lfloor\frac{1}{2}.\rfloor}(x,y)\wedge G_{|.|}(y,z)\wedge G_S(z,w)\wedge G_{|.|}(x,k)\to k=w\big)$ or Σ^b_1–NIA $\vdash$ $\forall x\in\mathbb{W}_1\,\big(x\neq\epsilon\to|x|=S(|\lfloor\frac{1}{2}x\rfloor|)\big)$. Immediate by Proposition 3.1 for $|.|_{\mathrm{w}}$ and $\lfloor\frac{1}{2}.\rfloor_{\mathrm{w}}$.

32) $\Sigma_1^b\text{−}\mathsf{NIA} \vdash I\big(\forall x\,\forall y\,(x{=}\lfloor\frac{1}{2}y\rfloor \leftrightarrow (2\cdot x = y \vee S(2\cdot x) = y))\big)$, i.e. $\Sigma_1^b\text{−}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \big(G_{\lfloor\frac{1}{2}.\rfloor}(y,x) \leftrightarrow G_.(10,x,y) \vee (\forall w\ (G_.(10,x,w) \rightarrow G_S(w,y)))\big)$ or $\Sigma_1^b\text{−}\mathsf{NIA} \vdash \forall x \in \mathbb{W}_1\ \forall y \in \mathbb{W}_1\ \big(x = \lfloor\frac{1}{2}y\rfloor \leftrightarrow (10\cdot x = y \vee S(10\cdot x) = y)\big)$. Immediate using Theorem 3.1 (28) and Proposition 3.1 for $\cdot_{\text{w}}$, $\lfloor\frac{1}{2}.\rfloor_{\text{w}}$, S_{w}.

• Finally, let us study the induction scheme. We want to prove that $\Sigma_1^b\text{−}\mathsf{NIA} \vdash I\big(A(0) \wedge \forall x\,(A(\lfloor\frac{1}{2}x\rfloor) \rightarrow A(x)) \rightarrow \forall x\, A(x)\big)$, with A a Σ_1^b-formula in $\mathcal{L}_{\mathbb{N}}$, i.e. $\Sigma_1^b\text{−}\mathsf{NIA} \vdash I(A(0)) \wedge \forall x \in \mathbb{W}_1\,\big(I(A(\lfloor\frac{1}{2}x\rfloor)) \rightarrow I(A(x))\big) \rightarrow \forall x \in \mathbb{W}_1\, I(A(x))$, which is equivalent to prove that $\Sigma_1^b\text{−}\mathsf{NIA} \vdash I(A)(\epsilon) \wedge \forall x \in \mathbb{W}_1\,\big(\forall z\,(G_T(x,z) \rightarrow I(A)(z)) \rightarrow I(A)(x)\big) \rightarrow \forall x \in \mathbb{W}_1\, I(A)(x)$ (see the definition of I and Proposition 3.1).

First we prove the following facts.

Fact 3.4. The following formulas are equivalent in $\Sigma_1^b\text{−}\mathsf{NIA}$:
a) $\forall y\,\big(G_{|.|}(w,y) \rightarrow \forall x \leq_{\text{w}} y\ \varphi(x)\big)$, i.e. $\forall x \leq_{\text{w}} |w|\ \varphi(x)$
b) $\forall x \subseteq w\ \forall z\,\big(G_{|.|}(x,z) \rightarrow \varphi(z)\big)$, i.e. $\forall x \subseteq w\ \varphi(|x|)$,
where φ is a formula in $\mathcal{L}_{\mathbb{W}}$. □

Noticing that $x \subseteq w \rightarrow x \preceq w$ (by induction on notation on w) and using Lemma 3.2 (9), (10), we have that a) implies b). The other implication is straightforward using Lemma 3.2 (12).

Fact 3.5. If A is a Σ_1^b-formula in $\mathcal{L}_{\mathbb{N}}$, then $I(A)$ is equivalent, in $\Sigma_1^b\text{−}\mathsf{NIA}$, to an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$. □

The proof of this fact is by induction on the complexity of the formula A. We assume $\rightarrow$ defined, as usually, based on $\neg$ and $\vee$.

If A is an atomic formula in $\mathcal{L}_{\mathbb{N}}$, we have $A := t_1 = t_2$ or $A := t_1 \leq t_2$, where t_1, t_2 are terms of $\mathcal{L}_{\mathbb{N}}$.

If no function symbols occur in the terms, i.e. they are just variables or the constant 0, then $I(A) := t_1' = t_2'$ or $I(A) := t_1' \leq_{\text{w}} t_2'$, where $t_i' = t_i$ if t_i is a variable and $t_i' = \epsilon$ if t_i is the constant 0 ($i{=}1,2$). In both cases $I(A)$ is an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$. Note that $I(A)$ is also an extended Π_1^b-formula in $\mathcal{L}_{\mathbb{W}}$ (this is used later on while studying the negation case).

If there are n function symbols occurring in A then $I(A)$ is equivalent to a formula of the form $\forall y_1 ... \forall y_n\,\big(G_1(...,y_1) \wedge ... \wedge G_n(...,y_n) \rightarrow B\big)$, where B is the atomic formula $a = b$ or $a \leq_{\text{w}} b$, with a and b variables or the constant ϵ, and G_i's are the extended Σ_1^b-formulas of $\mathcal{L}_{\mathbb{W}}$ assign by ν to the functions symbols of $\mathcal{L}_{\mathbb{N}}$ in A. By Proposition 3.1, we know that $I(A)$ is equivalent to $\exists y_1 \preceq b_1(...) ... \exists y_n \preceq b_n(...)\,\big(G_1(...,y_1) \wedge ... \wedge G_n(...,y_n) \wedge B\big)$, so $I(A)$ is equivalent to an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$. Notice that this is also equivalent to the formula $\forall y_1 \preceq b_1(...) \dots \forall y_n \preceq b_n(...) \big(G_1(..., y_1) \wedge \dots \wedge G_n(..., y_n) \rightarrow B\big)$ which is an extended Π_1^b-formula in $\mathcal{L}_{\mathbb{W}}$.

If A and B are formulas in $\mathcal{L}_{\mathbb{N}}$ such that $I(A)$ and $I(B)$ are equivalent to extended Σ_1^b-formulas in $\mathcal{L}_{\mathbb{W}}$, then $I(A \wedge B)$ and $I(A \vee B)$ are respectively $I(A) \wedge I(B)$ and $I(A) \vee I(B)$, which are equivalent to extended Σ_1^b-formulas in $\mathcal{L}_{\mathbb{W}}$.

If $A := \forall x \leq |t|\ B(x)$, such that $I(B)$ is equivalent to an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$, then $I(A)$ is the formula $\forall x \in \mathbb{W}_1 \big(I(x \leq |t|) \rightarrow I(B)(x)\big)$. Suppose that t has no function symbols, otherwise the result is similar just adding $\forall y_i's$ and $G_i's$. So, $I(A)$ has the form $\forall x \in \mathbb{W}_1 \big(\forall y\, (G_{|.|}(t, y) \rightarrow x \leq_{\mathrm{w}} y) \rightarrow I(B)(x)\big)$. This is equivalent to $\forall y \big(G_{|.|}(t, y) \rightarrow \forall x \leq_{\mathrm{w}} y\ (x \in \mathbb{W}_1 \rightarrow I(B)(x))\big)$, which by Fact 3.4, is equivalent to $\forall x \subseteq t\ \forall z \big(G_{|.|}(x, z) \rightarrow (z \in \mathbb{W}_1 \rightarrow I(B)(z))\big)$. Now using Proposition 3.1 the formula above can be rewritten as $\forall x \subseteq t\ \exists z \preceq b_{|.|}(x) \big(G_{|.|}(x, z) \wedge (z \in \mathbb{W}_1 \rightarrow I(B)(z))\big)$ which by its turn is equivalent to an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$. Note that it is also equivalent to $\forall x \subseteq t\ \forall z \preceq b_{|.|}(x) \big(G_{|.|}(x, z) \rightarrow (z \in \mathbb{W}_1 \rightarrow I(B)(z))\big)$.

If $A := \exists x \leq |t|\ B(x)$ such that $I(B)$ is equivalent to an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$, then the proof is similar to the previous case. One just need to prove the following analogue of Fact 3.4: in $\Sigma_1^b\!-\!\mathsf{NIA}$, $\forall y \big(G_{|.|}(w, y) \rightarrow \exists x \leq_{\mathrm{w}} y\ \varphi(x)\big)$ is equivalent to $\exists x \subseteq w\ \forall z \big(G_{|.|}(x, z) \rightarrow \varphi(z)\big)$.

Consider $A := \neg B$, for B a formula of $\mathcal{L}_{\mathbb{N}}$ where all quantifications are sharply bounded. By the remarks we have been doing along the proof $I(B)$ is equivalent to an extended Π_1^b-formula. Now noticing that the negation of an extended Π_1^b-formula is equivalent to an extended Σ_1^b-formula, we finish this case.

Consider now the case $A := \exists x \leq t\ B(x)$, where $I(B)$ is equivalent to an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$. If t is a variable or the constant 0 then $I(\exists x \leq t\ B(x)) := \exists x \in \mathbb{W}_1 \big(x \leq_{\mathrm{w}} t' \wedge I(B)(x)\big)$, where $t' = t$ or $t' = \epsilon$ respectively. This formula is equivalent to $\exists x \preceq t' \big(x \in \mathbb{W}_1 \wedge x \leq_{\mathrm{w}} t' \wedge I(B)(x)\big)$ which, noticing that $x \leq_{\mathrm{w}} t'$ is here an abbreviation of a sw.q. formula, is equivalent to an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$. If t has function symbols, use Proposition 3.1 to deal with the quantifications and the formulas G_f we have to introduce. This finishes the proof of Fact 3.5.

Now, to prove in $\Sigma_1^b\!-\!\mathsf{NIA}$ the translation of the induction scheme, suppose that we have $I(A)(\epsilon)$ and $\forall x \in \mathbb{W}_1 \big(\forall z\, (G_T(x, z) \rightarrow I(A)(z)) \rightarrow I(A)(x)\big)$. We want to prove that $\forall x \in \mathbb{W}_1\, I(A)(x)$. Taking $x = y0 \in \mathbb{W}_1$ we get $I(A)(y) \rightarrow I(A)(y0)$. For $x = y1 \in \mathbb{W}_1$ we have $I(A)(y) \rightarrow I(A)(y1)$. Putting all together it comes $I(A)(\epsilon) \wedge \forall y \in \mathbb{W}_1 \big[I(A)(y) \rightarrow \big((y0 \in \mathbb{W}_1 \rightarrow I(A)(y0)) \wedge I(A)(y1)\big)\big]$. By Fact 3.5, we know that $I(A)$ is equivalent to an extended Σ_1^b-formula in $\mathcal{L}_{\mathbb{W}}$, so applying Lemma 2.4 to that formula, we have that $\forall x \in \mathbb{W}_1\, I(A)(x)$. ∎

ACKNOWLEDGEMENTS – The first author was supported by a PhD grant from FCT, POCTI/FCT, FEDER and CMAF.

The second author thanks FCT-UNL, CMAF and FCT (grant SFRH/BPD/9455/2002 and project POCI/MAT/61720/2004).

REFERENCES

[**1**] Buss, S. – *Bounded Arithmetic*, Bibliopolis, Napoli (1986).

[**2**] Cantini, A. – Asymmetric interpretations for bounded theories, *Mathematical Logic Quarterly,* 42 (1996), 270–288.

[**3**] Enderson, H. – *A Mathematical Introduction to Logic*, Academic Press (1991).

[**4**] Fernandes, A. – *Investigações em Sistemas de Análise Exequível*, Ph.D. Dissertation, Faculdade de Ciências da Universidade de Lisboa (2001).

[**5**] Fernandes, A. and Ferreira, F. – Groundwork for weak analysis, *The Journal of Symbolic Logic,* 67(2) (2002), 557–578.

[**6**] Ferreira, F. – *Polynomial Time Computable Arithmetic and Conservative Extensions*, Ph.D. Dissertation, Pennsylvania State University (1988).

[**7**] Ferreira, F. – *Polynomial time computable arithmetic*, in "Logic and Computation" (W. Sieg, Ed.), Contemporary Mathematics (AMS), vol. 106 (1990), 137–156.

[**8**] Ferreira, F. – A feasible theory for analysis, *The Journal of Symbolic Logic,* 59(3) (1994), 1001–1011.

[**9**] Oliva, P. – *Polynomial-time algorithms from ineffective proofs*, in "LICS", 18$^{\text{th}}$ Annual IEEE Symposium on Logic in Computer Science (2003), 128–137.

[**10**] Strahm, T. – Polynomial time operations in explicit mathematics, *The Journal of Symbolic Logic,* 62(2), 575–594.

[**11**] Tarski, A.; Mostowski, A. and Robinson, R. – *Undecidable Theories*, North-Holland Publishing Company, third printing (1971).

[**12**] Yamazaki, T. – *Reverse mathematics and weak systems of 0-1 strings for feasible analysis*, in "Reverse Mathematics 2001", (S. Simpson, Ed.), Lecture Notes in Logic 21 (2005).

Gilda Ferreira,
CMAF – Universidade de Lisboa,
1649-003 Lisboa – PORTUGAL
E-mail: `gildafer@cii.fc.ul.pt`

and

Isabel Oitavem,
Dept. Matemática da FCT-UNL and CMAF-UL,
PORTUGAL
E-mail: `isarocha@ptmat.fc.ul.pt`