PORTUGALIAE MATHEMATICA
Vol. 63 Fasc. 4 – 2006
Nova Série

ON THE EXISTENCE OF QUASI-SELF-SIMILAR SOLUTIONS OF THE WEAKLY SHEAR-THINNING EQUATION

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Recommended by J.F. Rodrigues

Abstract: We prove the existence for solutions of a third order, nonlinear and degenerate ODE boundary value problem. The ODE problem has been derived by analysing a class of quasi-self similar solutions to the weakly shear-thinning equation.

1 – Introduction and results

This paper address the study of the following ODE boundary value problem:

$$(P) \qquad \begin{cases} \quad y = u^2\, u'''\big(1+|\epsilon\, u\, u'''|^{p-2}\big)\,, \qquad u>0, \quad y\in(0,a) \\ u'(0)=0 \\ u(a)=0\,, \quad u'(a)=0 \\ \quad M=\displaystyle\int_0^a u(y)\,dy \end{cases}$$

where M is a positive number fixed and the point a is itself an unknown of the problem. By a solution of (P) we mean a pair (a,u), with $a>0$ and $u\in C^3([0,a))\cap C^1([0,a])$. The ODE problem (P) was derived in [1] by considering the spreading of a thin droplet of viscous liquid on a plane surface driven by capillarity alone in the complete wetting regime. In the lubrication approximation, it is well-known that if the viscosity is constant, the no-slip condition at the liquid-solid interface leads to a force singularity at the moving contact lines. The most common way to remove the impossibility of expanding droplets is to allow for

Received: March 22, 2005; *Revised*: March 20, 2006.

appropriate slip conditions. Here we adopt a different relaxation of the pair constant viscosity/no-slip condition, first proposed by Weidner and Schwartz [25], consisting in keeping the no-slip condition and assuming instead a shear-thinning rheology of the form:

$$\frac{1}{\eta} = \frac{1}{\eta_0}\left(1 + \left|\frac{\tau}{\tilde{\tau}}\right|^{p-2}\right), \tag{1.1}$$

where $p > 2$, η is the viscosity, τ denotes the shear stress, η_0 is the viscosity at zero shear stress and $\tilde{\tau} > 0$ is the shear stress at which viscosity is reduced by a factor 1/2. The difference with respect to similar nonlinear relations between the viscosity and the shear stress, such as "power-law" rheology, is that (1.1) does not have a singularity at zero shear stress for $p > 2$, and therefore allows to recover the Newtonian case:

$$\frac{1}{\eta} = \frac{1}{\eta_0}\left(1 + \left|\frac{\tau}{\tilde{\tau}}\right|^{p-2}\right) \longrightarrow \frac{1}{\eta_0} \qquad \forall \tau \in \mathbb{R} \qquad \text{whenever } \ \tilde{\tau}^{p-2} \to \infty \ . \tag{1.2}$$

This approach leads to the following evolution: a fourth order degenerate parabolic equation for the film rescaled height $h(t, x)$ (the shear-thinning equation) on its positivity set

$$h_t + \kappa\Big[h^3\big(1 + |b\,h\,h_{xxx}|^{p-2}\big)\,h_{xxx}\Big]_x = 0 \ , \tag{1.3}$$

where

$$b = \left(\frac{3}{p+1}\right)^{\frac{1}{p-2}}\frac{1}{\tilde{\tau}} \tag{1.4}$$

t is the time and x is the spatial coordinate. The equation is coupled to conditions of vanishing flux and zero contact angle at triple junctions:

$$h_x\Big|_{\partial\{h>0\}} = 0 \ , \qquad \lim_{x\to\partial\{h>0\}} h^3\big(1 + |b\,h\,h_{xxx}|^{p-2}\big)\,h_{xxx} = 0 \ . \tag{1.5}$$

As worked out by [2], the problem (1.3)–(1.5) admits non-negative mass-conserving solutions whose support is compact for all times and fills the whole real line as time tends to infinity. So the shear-thinning liquids are not affected by the contact-line paradox and this suggests the possibility of adopting weakly shear-thinning rheology in order to describe the macroscopic dynamics of liquid films. Here we analyze a class of quasi–self–similar solutions for an almost newtonian rheology (which corresponds to the smallness of the parameter b) using a method introduced in [3]. This gives a quantitative description of the solution in terms of the macroscopic profile and effective contact angle.

Let

$$h(t,x) = (7\kappa\, t)^{-\frac{1}{7}}\, u(t,y) \;, \qquad y = x(7\kappa\, t)^{-\frac{1}{7}} \;.$$

At this point the problem (1.3)–(1.5) can be rewritten as the problem (P) where $\epsilon := b\,(7\kappa\, t)^{-\frac{5}{7}}$ and $\epsilon^{p-2} \ll 1$, M is the mass of the droplet and a is the contact point.

Let us state a well-posedness result for problem (P), which will be proved in Section 4:

Theorem (Existence of quasi-self-similar solutions). *For any $M>0$, $p>2$ and $\epsilon > 0$, problem (P) admits a solution (a,u).*

Since this problem is not invariant under rescaling, we will first consider $a > 0$ as fixed and prove existence and uniqueness for the following problem

$$(P_a) \begin{cases} \quad u''' = F(y,u) \quad \text{ in } \;(0,a) \\ u'(0) = 0\,, \\ \;u(a) = 0\,, \quad u'(a) = 0\;. \end{cases}$$

This will be achieved by an argument used by Ferreira and Bernis [12] in a similar context, based on estimates of the Green's function and on Schauder's fixed point theorem. Then we will prove that there exists a positive number a such that $\displaystyle\int_0^a u_a(y)\,dy = M$, where u_a is the solution to (P_a).

2 − Preliminaries

Introducing the function

$$W(y,u,\xi) \;:=\; u^2\,\xi\Big[1+(\epsilon\, u\,\xi)^{p-2}\Big] - y \;,$$

the equation of (P) can be rewritten as

$$W(y,u,\xi) = 0 \tag{2.1}$$

with $u''' = \xi > 0$. Since (2.1) implies

$$\epsilon^{p-2}\, u^p\, \xi^{p-1} = \; y - u^2\, \xi \ ,$$

for any fixed $(y,u) \in (0,\infty)\times(0,\infty)$ there exists a unique value $\xi \in (0,\infty)$ such that $W(y,u,\xi) = 0$. This allows to define the function $\xi = F(y,u)$:

$$\Big\{(y,u,\xi) \in (0,\infty)\times(0,\infty)\times(0,\infty)\colon\; W(y,u,\xi) = 0\Big\} \; = \\ = \; \Big\{(y,u,F(y,u))\colon\; (y,u) \in (0,\infty)\times(0,\infty)\Big\} \ . \tag{2.2}$$

Hence we obtain the explicit form:

$$u''' = F(y,u) \ . \tag{2.3}$$

Since W is continuous, differentiable and strictly increasing with respect to ξ, we see that $F \in C^1((0,\infty)\times(0,\infty))$. Moreover $F \in C([0,\infty)\times(0,\infty))$ and

$$F(y,u) \;\sim\; \begin{cases} \dfrac{y}{u^2} & (\epsilon\, u\, u''')^{p-2} \ll 1 \\[2mm] \left(\dfrac{y}{\epsilon^{p-2}\, u^p}\right)^{\frac{1}{p-1}} & (\epsilon\, u\, u''')^{p-2} \gg 1 \ , \end{cases}$$

that is

$$F(y,u) \;\sim\; \begin{cases} \dfrac{y}{u^2} & \epsilon\, y \ll u \\[2mm] \left(\dfrac{y}{\epsilon^{p-2}\, u^p}\right)^{\frac{1}{p-1}} & \epsilon\, y \gg u \ . \end{cases} \tag{2.4}$$

This expansion already shows that the macroscopic behaviour of the solution is governed by the limit equation, whereas the shear–thinning rheology takes over for small values of u. Due to the nonlinearity in the third derivative, such phenomenon is not transparent from the PDE itself. In addition, simple computations show that

$$F(0,u) = 0\,, \quad \frac{\partial F}{\partial y} > 0 \quad \text{and} \quad \frac{\partial F}{\partial u} < 0 \qquad \text{in} \;\; (0,\infty)\times(0,\infty) \tag{2.5}$$

and

$$\lim_{u\to 0^+} F(y,u) = +\infty \qquad \forall\, y > 0 \ . \tag{2.6}$$

3 – Green's function and properties

We consider the following problem:

$$(3.1)\qquad (P_\psi)\begin{cases} u''' = \psi(y) \quad \text{in } (0,a) \\ u'(0) = 0\,, \quad u(a) = 0\,, \quad u'(a) = 0\,. \end{cases}$$

For $t \in (0,a)$, we introduce the parabolas $P_-(y,t)$ defined in $y \in [0,t]$ and $P_+(y,t)$ defined in $y \in [t,a]$ such that

$$(3.2)\qquad P'_-(0,t) \;=\; P_+(a,t) \;=\; P'_+(a,t) \;=\; 0$$

and

$$(3.3)\qquad P_-(t,t) = P_+(t,t)\,, \quad P'_-(t,t) = P'_+(t,t)\,, \quad P''_+(t,t) - P''_-(t,t) = 1$$

where here and throughout the section, $'$ denotes differentiation w.r.t. y. Condition (3.2) and (3.3) give

$$P_-(y,t) = -\frac{(a-t)}{2\,a}\,y^2 + \frac{t}{2}\,(a-t)\,, \qquad P_+(y,t) = \frac{t}{2\,a}\,(a-y)^2\,.$$

Then the Green's function associated to the linear problem (3.1) is defined by the formula

$$(3.4)\qquad G(y,t) \;=\; \begin{cases} \dfrac{t}{2}\,(a-t) - \dfrac{(a-t)}{2\,a}\,y^2 & \text{if } y \le t \\ \dfrac{t}{2\,a}\,(a-y)^2 & \text{if } y \ge t\,. \end{cases}$$

Note that $G(\cdot,t) \in C^1([0,a])$, and we have

$$(3.5)\qquad G'(y,t) \;=\; \begin{cases} -\dfrac{(a-t)}{a}\,y & \text{if } y \le t \\ -\dfrac{t}{a}\,(a-y) & \text{if } y \ge t \end{cases}$$

$$(3.6)\qquad G''(y,t) \;=\; \begin{cases} -\dfrac{(a-t)}{a} & \text{if } y \le t \\ \dfrac{t}{a} & \text{if } y \ge t \end{cases}$$

$$G'''(y,t) = \delta(y-t)\,, \qquad 0<y<a, \quad 0<t<a\,,$$

$$(3.7)\qquad G'(0,t) = G(a,t) = G'(a,t) = 0\,, \qquad 0<t<a\,.$$

We collect some properties of the Green's function in the following Lemma.

Lemma 3.1. *The function defined by (3.4) satisfies the following properties, where C_1 and C_2 are positive constants:*

(**1**) $G(y,t) > 0$ *if* $0 \le y \le a$ *and* $0 < t < a$;

(**2**) $G'(y,t) < 0$ *if* $y, t \in (0,a)$;

(**3**) $G(y,t) \le C_1(a-t)$ *and* $|G'(y,t)| < C_1(a-t)$ *for all* $y, t \in [0,a]$;

(**4**) $\displaystyle\int_y^a G(y,t)\,dt \ge C_2\,(a-y)^3$ *for all* $y \in [0,a]$.

Proof: Property (2) is evident from (3.5), while (1) follows from (2) and $G(a,t) = 0$. The assertion (3) for G'' and G follows respectively from (3.5) and by integration in y. Since $G(y,t) \ge G(t,t)$ when $y \le t$, and $G(t,t)$ can be rewritten as

$$G(t,t) \;=\; \frac{t}{2\,a}\,(a-t)^2 \;=\; \frac{(a-t)^2}{2} - \frac{(a-t)^3}{2\,a}\;,$$

we have

$$\begin{aligned}
\int_y^a G(y,t)\,dt \;&\ge\; \int_y^a G(t,t)\,dt \\
&= \int_y^a \frac{(a-t)^2}{2}\,dt - \int_y^a \frac{(a-t)^3}{2\,a}\,dt \\
&= \frac{(a-y)^3\,(a+3y)}{24\,a} \;\ge\; C_2\,(a-y)^3
\end{aligned}$$

which is assertion (4). ∎

The solution of (P_ψ) can of course be obtained through the Green's function G, as stated in the following Lemma:

Lemma 3.2. *For any* $\psi \in C([0,a])$ *there exists a unique solution* $u \in C^3([0,a])$ *of problem* (P_ψ). *Furthermore,* u *satisfies*

$$u^{(j)}(y) = \int_0^a G^{(j)}(y,t)\,\psi(t)\;dt\,, \qquad j = 0, 1\;. \tag{3.8}$$

Proof: Let $u(y) = \int_0^a G(y,t)\,\psi(t)\;dt$. Since $G(\cdot,t) \in C^1([0,a])$, by (3) of Lemma 3.1 and (3.7) we obtain

$$u'(y) \;=\; \int_0^a G'(y,t)\,\psi(t)\;dt\;,$$

and $u'(0)=u(a)=u'(a)=0$. Given a test function φ such that $\operatorname{supp}(\varphi)\subset(0,a)$, integrating by parts we obtain

$$\int_0^a u(y)\,\varphi'''(y)\;dy \;=\; -\int_0^a u'''(y)\,\varphi(y)\;dy \;\overset{(3.1)}{=}\; -\int_0^a \psi(y)\,\varphi(y)\;dt\;.$$

This means that $u''' = \psi$ in the sense of distributions. Hence u is a solution of (3.1). Since uniqueness is elementary, the proof is complete. ∎

4 – Existence proof

The proof of the Theorem proceeds along several steps. We first consider $a > 0$ as fixed and prove the following result.

Proposition 4.1. *Let $p > 2$ and F defined by (2.2). For any $a > 0$ there exists $u \in C^3([0,a))\cap C^1([0,a])$, $u>0$ in $[0,a)$ which solves the following problem:*

$$(P_a)\;\begin{cases} u''' = F(y,u) & \text{in}\;\;(0,a)\\ u'(0) = 0\,, & \\ u(a) = 0\,, \quad u'(a) = 0\;. & \end{cases} \tag{4.1}$$

Furthermore,

$$u^{(j)}(y) = \int_0^a G^{(j)}(y,t)\,F(t,u(t))\;dt\,, \qquad j = 0,1\;. \tag{4.2}$$

To this aim, we consider the approximating problem

$$(P_\delta)\;\begin{cases} u''' = F(y,u) \quad \text{in}\;\;(0,a)\\ u'(0) = 0\,, \quad u(a) = \delta\,, \quad u'(a) = 0\;, \end{cases} \tag{4.3}$$

where δ is a positive number.

Remark 4.2. By (2.4), it follows that

$$\frac{y}{2\,u^2} \le F(y,u) \le \frac{y}{u^2} \qquad \text{for} \;\; u \ge \epsilon\, y \tag{4.4}$$

$$\Big(\frac{y}{2\,\epsilon^{p-2}\,u^p}\Big)^{1/p-1} \le\; F(y,u) \;\le \Big(\frac{y}{\epsilon^{p-2}\,u^p}\Big)^{1/p-1} \qquad \text{for} \;\; u \le \epsilon\, y \;. \;\square \tag{4.5}$$

Lemma 4.3. *For every $p>2$ problem (P_δ) has at least a positive solution $u_\delta \in C^3([0,a])$, which satisfies*

$$u_\delta(y) \;=\; \delta + \int_0^a G(y,t)\,F(t,u_\delta(t))\;dt\;, \tag{4.6}$$

$$u'_\delta(y) \;=\; \int_0^a G'(y,t)\,F(t,u_\delta(t))\;dt\;. \tag{4.7}$$

Proof: We proceed to apply Schauder's fixed point theorem. Let S be the closed convex set of the Banach space $C([0,a])$ defined by

$$S \;=\; \Big\{v\in C([0,a])\colon\; \delta \le v \le A \;\text{ in } [0,a]\Big\}\;,$$

where A is a constant to be chosen later. We introduce a nonlinear operator T by setting $T(v)=u$ for each $v\in S$, where u is the unique solution (cf. Lemma 3.2) of the problem

$$\begin{cases} \quad u''' = F(y,v) \quad\;\; \text{in} \;\; (0,a) \\ u'(0)=0\,, \qquad u(a)=\delta\,, \qquad u'(a)=0\;. \end{cases}$$

By (3.8),

$$u(y) \;=\; \delta + \int_0^a G(y,t)\,F(t,v(t))\;dt\;, \tag{4.8}$$

$$u'(y) \;=\; \int_0^a G'(y,t)\,F(t,v(t))\;dt\;. \tag{4.9}$$

We claim that $T(S)\subset S$ for A sufficiently large. Indeed, by (2.5), $u'''>0$ in $(0,a)$ implies that u' is a convex function with $u'(0)=u'(a)=0$. Therefore $u'<0$ in $(0,a)$, which means that $u(y)\ge u(a)=\delta$. By (4.8), (2.5) and (3) of Lemma 3.1, for $y\in[0,a]$ and $\delta\le v\le A$ we obtain $u(y)\le \delta+\frac{1}{2}\,F(a,\delta)\,C_1\,a^2 := A$. This proves the claim. Again by (4.8), since $F(t,\cdot)$ is uniformly continuous on $[\delta,A]$, T is continuous. By (4.9) and (3) of Lemma 3.1, $|u'(y)| \le A-\delta$; therefore $T(S)$ is bounded in $C^1([0,a])$ and hence relatively compact in $C([0,a])$. By Schauder's fixed point theorem there exists $u_\delta \in S$ such that $T(u_\delta)=u_\delta$, which is the desired solution. Finally, (4.6) and (4.7) follows respectively from (4.8) and (4.9). ∎

For $y \in (0,a]$, we consider

$$\bar{H}_y(\xi) := H(y,\xi) = \frac{\xi}{F(y,\xi)} \ . \tag{4.10}$$

In view of (2.5), $\frac{d\bar{H}_y}{d\xi} > 0$ in $(0,\infty)$. Hence its inverse $\xi = \bar{H}_y{}^{-1}(r)$ is well-defined and increasing in $(0,\infty)$ for any $y \in (0,a]$.

Lemma 4.4. *The solution $u_\delta(y)$ of problem (P_δ) satisfies for all $y \in (0,a]$:*

(1) $u_\delta(y) \ge \bar{H}_y{}^{-1}(C_2(a-y)^3)$ *where $\bar{H}_y(\xi)$ is defined by (4.10);*

(2) $u_\delta(y) \le C$ *and* $|u'_\delta(y)| \le C$ *independently by δ.*

Proof: By (4.6), (2.5) and (4) of Lemma 3.1, denoting with C a generic positive constant independently by δ, we have

$$u_\delta(y) \ \ge\ F(y,u_\delta(y)) \int_y^a G(y,t)\,dt \ \ge\ C(a-y)^3\, F(y,u_\delta(y)) \ . \tag{4.11}$$

Hence

$$\bar{H}_y(u_\delta(y)) \ = \ H(y,u_\delta(y)) \ = \ \frac{u_\delta(y)}{F(y,u_\delta(y))} \ \ge\ C(a-y)^3 \ . \tag{4.12}$$

Since $\bar{H}_y{}^{-1}$ is increasing, (4.12) means that

$$u_\delta(y) \ = \ \bar{H}_y{}^{-1}\big(\bar{H}_y(u_\delta(y))\big) \ge \bar{H}_y{}^{-1}\big(C(a-y)^3\big) \,. \tag{4.13}$$

By Remark 4.2, the following inequalities hold:

$$\frac{\xi^3}{y} \le \bar{H}_y(\xi) \le \frac{2\,\xi^3}{y} \qquad \text{for} \ \ \xi \ge \epsilon\, y \ , \tag{4.14}$$

$$\left(\frac{\epsilon^{p-2}\,\xi^{2p-1}}{y}\right)^{1/p-1} \le\ \bar{H}_y(\xi) \ \le\ \left(\frac{2\,\epsilon^{p-2}\,\xi^{2p-1}}{y}\right)^{1/p-1} \qquad \text{for} \ \ \xi \le \epsilon\, y \ . \tag{4.15}$$

In turn, (4.14) and (4.15) imply that

$$\left(\frac{1}{2}\,y\,r\right)^{1/3} \le {\bar{H}_y}^{-1}(r) \le (y\,r)^{1/3} \qquad \text{for} \quad r \ge \bar{H}_y(\epsilon\,y)\ ,$$

$$\left(\frac{1}{2}\,\epsilon^{2-p}\,y\,r^{p-1}\right)^{1/2p-1} \le {\bar{H}_y}^{-1}(r) \le \left(\epsilon^{2-p}\,y\,r^{p-1}\right)^{1/2p-1} \qquad \text{for} \quad r \le \bar{H}_y(\epsilon\,y)\ .$$

Using also the monotonicity of F, if $u_\delta(y) \le \epsilon\,y$ we see that

$$\begin{aligned} F\big(y, u_\delta(y)\big) &\overset{(4.13)}{\le} F\Big(y,\, {\bar{H}_y}^{-1}\big(C(a-y)^3\big)\Big) \\ &\le F\Big(y,\, C\,y^{\frac{1}{2p-1}}\,(a-y)^{\frac{3(p-1)}{2p-1}}\Big) \\ &\overset{(4.5)}{\le} C\,y^{\frac{1}{2p-1}}\,(a-y)^{\frac{-3p}{2p-1}}\ . \end{aligned} \tag{4.16}$$

Let $y^* \in (0,a)$ such that $\epsilon\,y^* = u_\delta(y^*)$. This point y^* exists and is unique for δ sufficiently small since $u'_\delta < 0$ in $(0,a)$ and as it has been proved in Lemma 4.3, $u_\delta \in S$. Moreover since u_δ is decreasing we observe that $u_\delta(y) \ge u_\delta(y^*) = \epsilon\,y^* \ge \epsilon\,y$ for $0<y\le y^*$ and $u_\delta(y) \le u_\delta(y^*) = \epsilon\,y^* \le \epsilon\,y$ for $y^* \le y \le a$. By (4.6), (3) of Lemma 3.1, (4.4) and (4.16), we obtain

$$\begin{aligned} u_\delta(y) &\le 1 + C\int_0^{y^*} (a-t)\,F\big(y^*, u_\delta(y^*)\big)\,dt \\ &\quad + C\int_{y^*}^a t^{\frac{1}{2p-1}}\,(a-t)^{-\frac{p+1}{2p-1}}\,dt \\ &\le 1 + C\,\frac{a\,{y^*}^2}{{u(y^*)}^2} + C\,a^{\frac{3p-2}{2p-1}} \\ &= 1 + C\,a + C\,a^{\frac{p-1}{2p-1}}\ . \end{aligned} \tag{4.17}$$

Hence $u_\delta(y) \le C$ independently by δ. In the same way one proves that $|u'_\delta(y)| \le C$. ∎

Proof of Proposition 4.1: We wish to pass to the limit as $\delta \downarrow 0$ in the approximating problems. By (2) of Lemma 4.4, there exists a subsequence (still labelled by δ) such that

$$u_\delta \to u \quad \text{uniformly in} \quad [0,a] \qquad \text{as} \quad \delta \downarrow 0\ .$$

Since $u > 0$ in $[0, a)$ by (1) of Lemma 4.4, then

$$u_\delta''' = F(y, u_\delta) \;\to\; F(y, u) \qquad \text{uniformly in compact subsets of } \;[0, a)\;.$$

On the other hand, $u_\delta''' \to u'''$ in the sense of distributions and hence u satisfies the differential equation of problem (4.1). By (3) of Lemma 3.1 and (4.16), we have

$$|G^{(j)}(y,t)|\, F(t, u_\delta(t)) \;\le\; C\, t^{\frac{1}{2p-1}}\, (a-t)^{-\frac{p+1}{2p-1}}\,, \qquad y^* \le t \le a \quad j=0,1\;.$$

Since $-\frac{p+1}{2p-1} + 1 = \frac{p-2}{2p-1} > 0$, it follows from (4.6) and Lebesgue's dominated convergence theorem that u_δ converges in $C^1([0,a])$ and hence u' satisfies the boundary conditions of problem (4.1). This argument also proves (4.2) and completes the proof of Proposition 4.1. ∎

In the next result we show that the solution u of problem (P_a) obtained in Proposition (4.1) is in fact unique.

Proposition 4.5. *The solution of problem* (P_a) *is unique.*

Proof: Let u and v be two solutions of problem (4.1) and let $w = u - v$; then

$$w'(0) = 0\,, \qquad w(a) = 0\,, \qquad w'(a) = 0\;.$$

Since $w\,w''' = (u-v)\,(u''' - v''') = (u-v)\,(F(y,u) - F(y,v))$ and the function $u \to F(y,u)$ is decreasing, it follows that

$$w\,w''' \le 0\;. \tag{4.18}$$

On the other hand, the following identity holds:

$$y\,w\,w''' \;=\; (y\,w\,w'')' - (w\,w')' - \frac{1}{2}\big(y(w')^2\big)' + \frac{3}{2}(w')^2\;. \tag{4.19}$$

Therefore the function

$$g(y) \;=\; y\,w\,w'' - w\,w' - \frac{1}{2}\,y(w')^2 \tag{4.20}$$

is non-increasing. Clearly $g(0) = 0$. Since g is non-increasing the following limits exists:

$$\lim_{y\to a} g(y) \;=\; \lim_{y\to a} w(y)\,w''(y) \;=\; L\;.$$

Since u' and v' are bounded, and zero in $y = a$, we have that $|w(y)| \le C(a-y)$. If $L \neq 0$ then $|w''(y)| \ge |L|/C(a-y)$ near $y = a$, which contradicts the continuity of w'. Hence $L = 0$. Since $g(0) = 0$ and g is non-increasing, we conclude that $g \equiv 0$. Then by (4.19) and (4.20)

$$g' = y\,w\,w''' - \frac{3}{2}\,(w')^2 \equiv 0 ,$$

and it follows from (4.18) that $w' \equiv 0$. Therefore $w \equiv 0$ and the proof is complete. ∎

Now we are ready to prove the Theorem.

Proof of the Theorem: Let $M_a = \int_0^a u_a(y)\,dy$. In view of Propositions 4.1 and 4.5, it suffices to prove that

$$\lim_{a\to\infty} M_a = \infty \quad \text{and} \quad \lim_{a\to 0} M_a = 0 .$$

Let $\overline{y}_a \in (0,a)$ such that $u_a(\overline{y}_a) = \overline{y}_a{}^{\beta}$, $\beta > 0$. If $\overline{y}_a \ge \frac{a}{4}$, we have

$$M_a \ge \int_0^{\overline{y}_a} u_a(y)\,dy \ge u_a(\overline{y}_a)\,\overline{y}_a \ge C\,a^{\beta+1} \to \infty \qquad \text{as} \quad a \uparrow \infty .$$

If $\overline{y}_a < \frac{a}{4}$ and $t \le 2\,\overline{y}_a \le y$, since $a - 2\overline{y}_a > \frac{a}{2}$, we have

$$M_a \ge \int_{2\overline{y}_a}^{a} dy \int_{\overline{y}_a}^{2\overline{y}_a} G(y,t)\,F(t,u_a(t))\,dt > C\,F\big(\overline{y}_a, u_a(\overline{y}_a)\big)\,\overline{y}_a{}^2\,a^2 . \tag{4.21}$$

From Remark 4.2 and (4.21), it follows that

$$M_a > C\,\overline{y}_a{}^{3-2\beta}\,a^2 \qquad \text{if} \quad u_a(\overline{y}_a) \ge \epsilon\,\overline{y}_a ,$$

and

$$M_a > C\,\overline{y}_a{}^{\frac{2p-1-\beta p}{p-1}}\,a^2 \qquad \text{if} \quad u_a(\overline{y}_a) \le \epsilon\,\overline{y}_a .$$

Then

$$M_a > C\,a^2 \min\Big\{\overline{y}_a{}^{3-2\beta},\ \overline{y}_a{}^{\frac{2p-1-\beta p}{p-1}}\Big\} .$$

Choosing $\beta = 2$ we obtain (since $\overline{y}_a < a/4$)

$$\begin{aligned} M_a &> C\,a^2 \min\left\{ \overline{y}_a{}^{-1},\ \overline{y}_a{}^{-\frac{1}{p-1}} \right\} \\ &> C\,a^2 \min\left\{ a^{-1},\ a^{-\frac{1}{p-1}} \right\} \\ &> C\ \min\left\{ a,\ a^{\frac{2p-3}{p-1}} \right\}\ , \end{aligned}$$

and therefore M_a tends to infinity as $a \to \infty$. In the limit $a \downarrow 0$, we consider

$$\begin{aligned} M_a &= \int_0^a dy \int_0^{y^*} G(y,t)\,F(t,u_a(t))\ dt + \int_0^a dy \int_{y^*}^a G(y,t)\,F(t,u_a(t))\ dt \\ &= I_1 + I_2\ . \end{aligned}$$

As observed before, since $u_a(y) \geq u_a(y^*) = \epsilon\,y^* \geq \epsilon\,y$ for $0 < y \leq y^*$ and $u_a(y) \leq u_a(y^*) = \epsilon\,y^* \leq \epsilon\,y$ for $y^* \leq y \leq a$, by (3) of Lemma 3.1 and (4.4),

$$I_1 \leq C \int_0^a dy \int_0^{y^*} (a-t)\,F\big(y^*, u_a(y^*)\big)\ dt \leq C\,a^2 \left(\frac{y^*}{u(y^*)} \right)^2 = \frac{C\,a^2}{\epsilon^2}\ .$$

By (3) of Lemma 3.1 and passing to the limit $\delta \downarrow 0$ in (4.16),

$$I_2 \leq C \int_0^a dy \int_{y^*}^a t^{\frac{1}{2p-1}}\,(a-t)^{-\frac{p+1}{2p-1}}\ dt \leq C\,a^{\frac{3p-2}{2p-1}}\ ,$$

and the proof is complete. ∎

Remark 4.6. Unfortunately we can not conclude the uniqueness of solution. In fact, it is not difficult to see that the regularity of $u \in C^3([0,a)) \cap C^1([0,a])$ is not sufficient to prove that M_a is monotone in a. Therefore, we refer to the proof in [1] obtained by a standard shooting argument. □

Remark 4.7. It's also interesting to consider solutions of (P) with non-zero contact angle, more precisely, with $u'(a) = 0$ replaced by $u'(a) = -\theta$, where $\theta > 0$ is prescribed. For any $M > 0$, $p > 2$, $\epsilon > 0$ and $\theta > 0$, problem (P) admits a solution. Since the proof is identical to the previous case, we omit it. □

ACKNOWLEDGEMENTS – The author wishes to thank L. Giacomelli and G. Vergara Caffarelli for fruitful discussions and useful comments.

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