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# A modified Cucconi Test for Location and Scale Change Alternatives

## Un prueba de Cucconi modificada para alternativas de cambio en localización y escala

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**Abstract**

The most common approach to develop a test for jointly detecting location and scale changes is to combine a test for location and a test for scale. For the same problem, the test of Cucconi should be considered because it is an alternative to the other tests as it is based on the squares of ranks and contrary-ranks. It has been previously shown that the Cucconi test is robust in level and is more powerful than the Lepage test, which is the most commonly used test for the location-scale problem. A modification of the Cucconi test is proposed. The idea is to modify this test consistently with the familiar approach which develops a location-scale test by combining a test for location and a test for scale. More precisely, we will combine the Cucconi test with the Wilcoxon rank test for location and a modified Levene test following the theory of the nonparametric combination. A power comparison of this modified Cucconi test with the original one, the Lepage test and the Podgor-Gastwirth $PG2$ test, shows that the modified Cucconi test is robust in size and markedly more powerful than the other tests for every considered type of distributions, from short- to normal- and long-tailed ones. A real data example is discussed.

***Key words*:** Combining tests, Location-scale model, Rank tests.

**Resumen**

La alternativa más común para implementar una prueba que detecta cambios en localización y escala conjuntamente es combinar una prueba de localización con una de escala. Para este problema, la prueba de Cucconi es considerada como una alternativa de otras pruebas que se basan en los cuadrados de los rangos y los contrarangos. Esta prueba es robusta en nivel y es más poderosa que la prueba de Lepage la cual es la más usada para el problema de localización-escala. En este artículo se propone una modificación de la prueba de Cucconi. La idea es modificar la prueba mediante

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la combinación de una prueba de localización y uno de escala. Mas precisamente, se sugiere combinar la prueba de Cucconi con la prueba de rangos de Wilcoxon para localizacion y una prueba modificada de Levene siguiendo la teoría de la combinación no paramétrica. Una comparación de la potencia de esta prueba modificada de Cucconi con la prueba original, la prueba de Lepage y la prueba $PG2$ de Podgor-Gastwirth muestran que la prueba de Cucconi modificada es robusta en tamaño y mucho más poderosa que las anteriores para todas las distribuciones consideradas desde la normal hasta algunas de colas largas. Se hace una aplicación a datos reales.

***Palabras clave*:** combinación de pruebas, modelo de localización y escala, pruebas de rangos.

## 1. Introduction

The two sample Behrens-Fisher problem is to test that the locations, but not necessarily the scales, of the distribution functions associated to the populations behind the samples are equal. There exist situations of practical interest, however, when it is appropriate to jointly test for change in locations and change in scales. For example, Snedecor & Cochran (1989) emphasize that the application of a treatment (e.g. a drug) to otherwise homogeneous experimental units often results in the treated group differing not only in location but also in scales. The practitioner generally has no a prior knowledge about the distribution functions from which the data originate. Therefore, in such situations, an appropriate test does not require distributional assumptions. The test proposed by Perng & Littel (1976) for the equality of means and variances is not appropriate because is a combination of the $t$ test and the $F$ test, as the $F$ test is not $\alpha$ robust for data from heavier than normal tailed distributions. According to Conover, Johnson & Johnson (1981) a test is $\alpha$ robust if its type one error rate is less than $2\alpha$. The cut off point is set to $1.5\alpha$ by Marozzi (2011). As the Perng & Littel (1976) test which uses the Fisher combining function, the tests for the location-scale problem are generally expressed as functions of two tests, one sensitive to location changes and the other to scale changes. The corresponding statistics are generally obtained as direct combination of (i.e. by summing) a standardized statistic sensitive to location changes and a standardized statistic sensitive to scale changes. The most familiar test statistic for the location-scale problem, due to Lepage (1971), which is a direct combination of the squares of the standardized Wilcoxon and Ansari-Bradley statistics. It is important to note that Lepage-type tests can be obtained following Podgor & Gastwirth (1994). Marozzi (2009) compared several Podgor & Gastwirth (1994) efficiency robust tests and found that the $PG2$ test is the most powerful one. To perform the $PG2$ test it is necessary to regress the group indicator on the ranks and on the squares of the ranks of the data and to test that the two regression coefficients are zero. The $PG2$ test can be recast as a quadratic combination of the Wilcoxon test and the Mood squared rank test. For the same problem, the test of Cucconi (1968) should be considered because it is different from the other tests being not based on the combination of a test for location and a test for scale. It is a nonparametric test based on the squares of ranks and contrary-ranks. Marozzi (2009) computed for the very first time exact critical values for this test, compared its power to that of the Lepage and other tests that included several Podgor-Gastwirth tests and showed that the test of Cucconi maintains the size very close to the nominal level and is more powerful than the Lepage test. In this paper we are not interested in the general two sample problem, and therefore we do not consider tests like the Kolmogorov-Smirnov, Cramer-Von Mises or Anderson-Darling tests. In Section 2 we introduce a modification of the Cucconi test developed within the framework of the nonparametric combination of dependent tests (Pesarin 2001). A power comparison of this modified Cucconi test with the original one, the Lepage test and the Podgor-Gastwirth $PG2$ test is carried out in Section 3. These tests are applied to a real data set in Section 4. The conclusions are reported in Section 5.

## 2. The Modified Cucconi Test

In this section we introduce a modification of the Cucconi (Cucconi 1968) test. The idea is to modify this test consistently with the familiar approach which develops a location-scale test by combining a test for location and a test for scale. More precisely, following the theory of the nonparametric combination (Pesarin 2001) we will combine the Cucconi test with the Wilcoxon test for location and the modified Levene test for scale proposed by Brown & Forsythe (1974). We consider the Wilcoxon test and the modified Levene test because they have good properties in addressing the location and the scale problem respectively. Among other things, they are robust against non normality and they have good power, see Hollander & Wolfe (1999) and Marozzi (2011).

Let $\underline{\boldsymbol{X}}_1 = (X_{11}, \dots, X_{1n_1})$ and $\underline{\boldsymbol{X}}_2 = (X_{21}, \dots, X_{2n_2})$ be independent random samples of iid observations. Let $F_1$ and $F_2$ denote the absolutely continuous distribution functions associated to the populations underlying the samples. We wish to test

$$H_0 : F_1(g) = F_2(g) \text{ for all } g \in R \tag{1}$$

versus the location-scale alternative

$$H_1 : F_2(g) = F_1(\frac{g-\vartheta}{\tau}) \text{ with } \vartheta \in R, \tau > 0 \tag{2}$$

Note that for $\vartheta = 0$, $H_1$ reduces to a pure scale alternative and for $\tau = 1$ to a pure location alternative. Let $\mu_j$ and $\sigma_j$ denote the location and scale of $F_j$, $j = 1, 2$. $H_0$ can be equivalently represented as

$$H_0 = H_{0l} \cap H_{0s} \text{ where } H_{0l} : \vartheta = \mu_1 - \mu_2 = 0 \text{ and } H_{0s} : \tau = \sigma_1/\sigma_2 = 1 \tag{3}$$

$H_1$ can be equivalently represented as

$$H_1 = H_{1l} \cup H_{1s} \text{ where } H_{1l} : \mu_1 - \mu_2 \neq 0 \text{ and } H_{1s} : \sigma_1/\sigma_2 \neq 1 \tag{4}$$

This representation of the system of hypotheses emphasizes that it is composed by two partial systems of hypotheses: the location and the scale one.

The test of Cucconi (1968) is based on

$$C = C\,(U, V) = \frac{U^2 + V^2 - 2\rho UV}{2(1-\rho^2)}$$

where

$$U = U(\underline{\boldsymbol{S}}_1) = \frac{6\sum_{i=1}^{n_1} S_{1i}^2 - n_1(n+1)(2n+1)}{\sqrt{n_1 n_2 (n+1)(2n+1)(8n+11)/5}},$$

$$V = V(\underline{\boldsymbol{S}}_1) = \frac{6\sum_{i=1}^{n_1} (n+1-S_{1i})^2 - n_1(n+1)(2n+1)}{\sqrt{n_1 n_2 (n+1)(2n+1)(8n+11)/5}}$$

$$n = n_1 + n_2, \quad \underline{\boldsymbol{S}}_1 = (S_{11}, \ldots, S_{1n_1})$$

$S_{1i}$ denotes the rank of $X_{1i}$ in the pooled sample

$$\underline{\boldsymbol{X}} = (\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2) = (X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}) = (X_1, \ldots, X_{n_1}, X_{n_1+1}, \ldots, X_n)$$

and $\rho = \frac{2(n^2-4)}{(2n+1)(8n+11)} - 1$. Note that $U$ is based on the squares of the ranks $S_{1i}$, while $V$ is based on the squares of the contrary-ranks $(n+1-S_{1i})$ of the first sample. Cucconi (1968) showed that under $H_0$ $(U, V)$ has mean (0,0) because $E(\sum_{i=1}^{n_1} S_{1i}^2) = n_1(n+1)(2n+1)/6$, and that $VAR(U) = VAR(V) = 1$ because $VAR(\sum_{i=1}^{n_1} S_{1i}^2) = n_1 n_2 (n+1)(2n+1)(8n+11)/180$. Of course, it is $E(\sum_{i=1}^{n_1}(n+1-S_{1i})^2) = E(\sum_{i=1}^{n_1} S_{1i}^2)$ and $VAR(\sum_{i=1}^{n_1}(n+1-S_{1i})^2) = VAR(\sum_{i=1}^{n_1} S_{1i}^2)$. $U$ and $V$ are negatively correlated, more precisely, since $CORR(U,V) = COVAR(U,V) = \frac{2(n^2-4)}{(2n+1)(8n+11)} - 1 = \rho$ then $-1 \leq CORR(U,V) < -7/8$, where the minimum occurs when $n = 2$ and the supremum is reached when $n \to \infty$. It has been also shown that under $H_0$ if $n_1, n_2 \to \infty$ and $n_1/n \to \lambda \in ]0,1[$ then $Pr(U \leq u) \to \Phi(u)$ and $Pr(U \leq v) \to \Phi(v)$, where $\Phi$ is the standard normal distribution function, moreover $(U, V)$ converges in distribution to the bivariate normal with mean (0,0) and correlation $\rho_0 = -7/8$

$$\Pr(U \leq u, V \leq v) \to \int_{-\infty}^{u} \int_{-\infty}^{v} \frac{1}{2\pi\sqrt{1-\rho_0^2}} \exp\left(-\frac{q^2 + r^2 - 2\rho_0 qr}{2\left(1-\rho_0^2\right)}\right) dqdr$$

Therefore the points $(u, v)$ outside the rejection region are close to (0,0), i.e. satisfy $\frac{1}{2\pi\sqrt{1-\rho_0^2}} \exp\left(-\frac{u^2+v^2-2\rho_0 uv}{2\left(1-\rho_0^2\right)}\right) \geq k$, where the constant $k$ is chosen so that the type-one error rate is $\alpha$. Let $k = \alpha\left(2\pi\sqrt{1-\rho_0^2}\right)^{-1}$, then it follows that if the point $(u, v)$ is such that $\frac{u^2+v^2-2\rho_0 uv}{2\left(1-\rho_0^2\right)} < -\ln\alpha$ then we failed to have evidence against $H_0$. It is interesting to note that the rejection region $E$ of the test is the set of points $(u, v)$ outside the ellipse $u^2 + v^2 - 2\rho_0 uv = -2(1-\rho_0^2)\ln\alpha$. The test has size $\alpha$ because $\int\int_E \frac{1}{2\pi\sqrt{1-\rho_0^2}} \exp\left(-\frac{q^2+r^2-2\rho_0 qr}{2\left(1-\rho_0^2\right)}\right) dqdr = \alpha$. Note that in practice, unless you have large samples, $\rho_0$ should be replaced by $\rho$. Cucconi (1968) proved also that the test is unbiased and consistent for the location-scale problem.

We develop the modified Cucconi $MC^*$ test following the nonparametric combination of dependent tests theory, which operates within the permutation framework, by combining the permutation version of the Cucconi test with the permutation version of the Wilcoxon $W$ test for comparing locations and the Levene $W50$ test for comparing scales. The Wilcoxon $W$ test is based on

$$W = \frac{\left|\sum_{i=1}^{n_2} S_{2i} - n_2\,(n+1)\,/2\right|}{n_1 n_2\,(n+1)\,/12}$$

The Levene $W50$ test is based on the Student $t$ statistic computed on $R_{ji} = |X_{ji} - \widetilde{X}_j|$ where $\widetilde{X}_j$ is the median of the $j$th sample. Let us denote the mean of $R_{ji}, i = 1, \ldots, n_j$ by $\overline{R}_j$, $j = 1, 2$, the Levene statistic is

$$W50 = \frac{\left|\overline{R}_1 - \overline{R}_2\right|}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\frac{\sum_{i=1}^{n_1}\left(R_{1i}-\overline{R}_1\right)^2 + \sum_{i=1}^{n_2}\left(R_{2i}-\overline{R}_2\right)^2}{n-2}}}$$

Large values of $W$ and $W50$ are evidence of difference in locations and scales respectively. It is desirable that the good performance in detecting separately location and scale changes shown by the $W$ and the $W50$ tests are transferred to the combined test resulting in an improved power for jointly detecting location and scale changes with respect to the original Cucconi test. It has been shown that the nonparametric combination of dependent tests theory is very useful to address the location problem, see Marozzi (2004*b*), Marozzi (2004*a*) and Marozzi (2007), and the scale problem, see Marozzi (2011) and Marozzi (2012). We would like to see whether this theory is also useful to address the location-scale problem.

We describe now the permutation version $C^*$ of the Cucconi test. Let $\underline{\boldsymbol{X}}^* = (\underline{\boldsymbol{X}}_1^*, \underline{\boldsymbol{X}}_2^*) = (X_{u_1^*}, \ldots, X_{u_n^*}) = (X_1^*, \ldots, X_n^*)$ denote a random permutation of the combined sample, where $(u_1^*, \ldots, u_n^*)$ is a permutation of $(1, \ldots, n)$, and so $\underline{\boldsymbol{X}}_1^* = (X_{u_1^*}, \ldots, X_{u_{n_1}^*})$ and $\underline{\boldsymbol{X}}_2^* = (X_{u_{n_1+1}^*}, \ldots, X_{u_n^*})$ are the two permuted samples. The permutation version of the $C$ statistic is

$$C^* = C\left(\underline{\boldsymbol{X}}_1^*\right) = C\left(U^*, V^*\right) = \frac{\left(U^*\right)^2 + \left(V^*\right)^2 - 2\rho U^* V^*}{2(1-\rho)}$$

where $U^* = U\left(\underline{\boldsymbol{S}}_1^*\right)$, $V^* = V\left(\underline{\boldsymbol{S}}_1^*\right)$ and $\underline{\boldsymbol{S}}_1^*$ contains the ranks of $\underline{X}_1^*$ elements. The observed value of $C^*$ is ${}_0C = C(U, V)$. To compute the p-value we compute the permutation null distribution of the $C$ statistic as the distribution function of its permutation values: ${}_1C^*, \ldots, {}_kC^*, \ldots, {}_KC^*$ where ${}_kC^* = C\left({}_k\underline{\boldsymbol{X}}_1^*\right)$, ${}_k\underline{\boldsymbol{X}}_1^*$ contains the first $n_1$ elements of the $k$th permutation of $\underline{\boldsymbol{X}}$ and $k = 1, \ldots, K = n!/(n_1!n_2!)$.

Therefore the p-value is

$$L_{C^*}({}_0C) = \frac{1}{K}\sum_{k=1}^{K} I\left({}_kC^* \geq_0 C\right)$$

where $I(.)$ denotes the indicator function.

We briefly describe now the permutation version of the $W$ and $W50$ tests. Let

$$\begin{aligned}\underline{\boldsymbol{Y}} &= \left(\underline{\boldsymbol{X}}_1/SD\left(\underline{\boldsymbol{X}}_1\right), \underline{\boldsymbol{X}}_2/SD\left(\underline{\boldsymbol{X}}_2\right)\right)\\ &= \left(X_1/SD\left(\underline{\boldsymbol{X}}_1\right), \ldots, X_n/SD\left(\underline{\boldsymbol{X}}_2\right)\right)\\ &= \left(Y_1, \ldots, Y_n\right)\end{aligned}$$

be the standardized pooled sample, and let

$$\begin{aligned}\underline{\boldsymbol{Z}} &= \left(\underline{\boldsymbol{X}}_1 - E\left(\underline{\boldsymbol{X}}_1\right), \underline{\boldsymbol{X}}_2 - E\left(\underline{\boldsymbol{X}}_2\right)\right)\\ &= \left(X_1 - E\left(\underline{\boldsymbol{X}}_1\right), \ldots, X_n - E\left(\underline{\boldsymbol{X}}_2\right)\right)\\ &= \left(Z_1, \ldots, Z_n\right)\end{aligned}$$

be the mean aligned pooled sample. Let $\underline{\boldsymbol{Y}}^*$ and $\underline{\boldsymbol{Z}}^*$ be a random permutation of $\underline{\boldsymbol{Y}}$ and $\underline{\boldsymbol{Z}}$ respectively, it is important to emphasize that the $\underline{\boldsymbol{Y}}$ and $\underline{\boldsymbol{Z}}$ elements are not exactly exchangeable under $H_0$ and so the permutation solution is approximate; however it becomes asymptotically exact. $\underline{\boldsymbol{Z}}$ elements would be exchangeable if $\mu_1$ and $\mu_2$ were known and used in place of $E(\underline{\boldsymbol{X}}_1)$ and $E(\underline{\boldsymbol{X}}_2)$, see Pesarin & Salmaso (2010, pp. 73-74) and Good (2000, pp. 38-41). $\underline{\boldsymbol{Y}}$ elements would be exchangeable if $\sigma_1$ and $\sigma_2$ were known and used in place of $SD(\underline{\boldsymbol{X}}_1)$ and $SD(\underline{\boldsymbol{X}}_2)$, see Pesarin & Salmaso (2010, pp. 25 and 166-167). Alternatively, we considered also the median absolute deviation and the median in place of the standard deviation and the mean respectively in transforming $\underline{\boldsymbol{X}}$ and we obtained very similar results to those presented in section 3. It is also to be emphasized that, in order to preserve the within individual dependence on the transformed data $[\underline{\boldsymbol{X}}, \underline{\boldsymbol{Y}}, \underline{\boldsymbol{Z}}]$, the permutations must be carried on the $n$ three-dimensional individual vectors $[(X_1, Y_1, Z_1), \ldots, (X_n, Y_n, Z_n)]$. So that $[\underline{\boldsymbol{X}}^*, \underline{\boldsymbol{Y}}^*, \underline{\boldsymbol{Z}}^*] = [(X_{u_i^*}, Y_{u_i^*}, Z_{u_i^*}), i = 1, \ldots, n]$.

In the permutation version $W^*$ of the $W$ test, the p-value is computed as $L_{W^*}({}_0W) = \frac{1}{K}\sum_{k=1}^{K} I\left({}_kW^* \geq_0 W\right)$, where ${}_0W$ is the observed value of the Wilcoxon statistic (that is computed on $\underline{\boldsymbol{Y}}$) and ${}_kW^*$ is the Wilcoxon statistic computed on the $k$th permutation ${}_k\underline{\boldsymbol{Y}}^*$ of $\underline{\boldsymbol{Y}}$. In the permutation version $W50^*$ of the $W50$ test, the p-value is computed as $L_{W50^*}({}_0W50) = \frac{1}{K}\sum_{k=1}^{K} I\left({}_kW50^* \geq_0 W50\right)$, where ${}_0W50$ is the observed value of the $W50$ statistic (that is computed on $\underline{\boldsymbol{Z}}$) and ${}_kW50^*$ is the $W50$ statistic computed on the $k$th permutation ${}_k\underline{\boldsymbol{Z}}^*$ of $\underline{\boldsymbol{Z}}$.

To obtain the $MC^*$ test we combine the p-values of the $C^*$, $W^*$ and $W50^*$ tests. This is equivalent to combine the test statistics being one to one decreasingly related to the p-values. Pesarin (2001, pp. 147-149) reports several combining functions, with the most familiar being

- the Fisher combining function $\ln\left(1/L_{C^*}\right) + \ln\left(1/L_{W^*}\right) + \ln\left(1/L_{W50^*}\right)$;

- the Tippett combining function $\max(1 - L_{C^*}, 1 - L_{W^*}, 1 - L_{W50^*})$;
- the Liptak combining function

$$\Phi^{-1}\left(1 - L_{C^*}\right) + \Phi^{-1}\left(1 - L_{W^*}\right) + \Phi^{-1}\left(1 - L_{W50^*}\right);$$

and noted that the Tippett combining function has a good power behavior when only one among the partial alternatives is true; that the Liptak combining function is generally good when the partial alternatives are jointly true; that the Fisher combining function has an intermediate behavior with respect to the Tippett and Liptak ones and therefore it is suggested when nothing is expected about the partial alternatives. Since we would like a combined test that is sensitive in all the three alternative situations: that are when $H_{1l}$ alone is true, when $H_{1s}$ alone is true, when $H_{1l}$ and $H_{1s}$ are jointly true, we use the Fisher combining function to obtain the test statistic for the null hypothesis $H_0 = H_{0l} \cap H_{0s}$

$$MC^* = \ln\left(1/L_{C^*}\right) + \ln\left(1/L_{W^*}\right) + \ln\left(1/L_{W50^*}\right)$$

Note that the Fisher combining function is used also by Perng & Littel (1976). The observed value of the $MC^*$ statistic is ${}_0MC = \ln\left(1/L_{C^*}\left({}_0C\right)\right) + \ln\left(1/L_{W^*}\left({}_0W\right)\right) + \ln\left(1/L_{W50^*}\left({}_0W50\right)\right)$. The null distribution of the $MC^*$ statistic is the distribution function of ${}_1MC^*, \ldots, {}_kMC^*, \ldots, {}_KMC^*$ where ${}_kMC^* = \ln\left(1/L_{C^*}\left({}_kC^*\right)\right) + \ln\left(1/L_{W^*}\left({}_kW^*\right)\right) + \ln\left(1/L_{W50^*}\left({}_kW50^*\right)\right)$. Large values of ${}_0MC$ are evidence against $H_0$, that should be rejected if $L_{MC^*}({}_0MC) \leq \alpha$ where $L_{MC^*}({}_0MC) = \frac{1}{K}\sum_{k=1}^{K} I\left({}_kMC^* \geq_0 MC\right)$. According to Pesarin (2001) it is possible to combine even a large, although finite, number of tests. In our case, we limit the number of tests to be combined to avoid the possibility that the type one error rate of the combined test may inflate too much, because under $H_0$ $\underline{\boldsymbol{Y}}$ and $\underline{\boldsymbol{Z}}$ elements are only approximately exchangeable.

## 3. Size and Power Study

We investigate via Monte Carlo simulation (5000 replications) the robustness of the significance level and the power of the modified Cucconi $MC^*$ test in detecting location and scale changes, and we made comparisons with the classical Cucconi $C$ test, the Lepage $L$ test and the $PG2$ test. The Lepage test is based on

$$L = W^2 + \frac{\left(A - E(A)\right)^2}{VAR(A)}$$

where $A = \sum_{i=1}^{n_2} A_{2i}$ is the Ansari-Bradley statistic, $A_{ji}$ denotes the Ansari-Bradley score of $X_{ji}$ in the combined sample. To compute the $A_{ji}$s assign the score 1 to both the smallest and largest observations in the pooled sample, the score 2 to the second smallest and second largest, and so on. $E(A)$ and $VAR(A)$ denote the expected value and variance of $A$ under $H_0$. Since the scoring depends on whether $n$ is even or odd, two cases should be distinguished, $E(A) = n_2(n+2)/4$ and $VAR(A) = n_1 n_2 \left(n+2\right)\left(n-2\right)/\left(48\left(n-1\right)\right)$ when $n$ is even, and

$E(A) = n_2(n+1)^2/(4n)$ and $VAR(A) = n_1 n_2 (n+1)(3+n^2)/\left(48n^2\right)$ when $n$ is odd.

Let $I_i$ $i = 1, \dots, n$ be a group indicator so that $I_i = 1$ when the $i$th element of the combined sample belongs to the first sample, $I_i = 0$ otherwise. The $PG2$ test statistic is the $F$ statistic with 2 and $n-3$ df computed by regressing group indicators $I_i$ on the ranks $S_{ji}$ and the squared ranks $S_{ji}^2$ of the observations in the combined sample

$$PG2 = \frac{\left(\underline{\boldsymbol{b}}^T \underline{\boldsymbol{S}}^T \underline{\boldsymbol{I}} - n_1^2/n\right)/2}{\left(n_1 - \underline{\boldsymbol{b}}^T \underline{\boldsymbol{S}}^T \underline{\boldsymbol{I}}\right)/(n-3)}$$

where $^T$ denotes the transpose operator, $\underline{\boldsymbol{b}}$ is the $3 \times 1$ column vector of the OLS estimate of the intercept term and the regression coefficients, $\underline{\boldsymbol{S}}$ is a $n \times 3$ matrix with the first column of 1s, the second column of $S_{ji}$ and the third column of $S_{ji}^2$, $i = 1, \dots, n_j$, $j = 1, 2$, $\underline{\boldsymbol{I}}$ is the $n$x1 column of the group indicators $I_1, \dots, I_n$.

The nominal 5% level is used throughout. We consider the following distributions that cover a wide range from short-tailed to very long-tailed distributions:

1. standard normal N(0,1);
2. uniform between $-\sqrt{3}$ and $\sqrt{3}$;
3. bimodal obtained as a mixture of a N(-1.5,1) with probability 0.5 and a N(1.5,1) with 0.5;
4. Laplace double exponential with scale parameter of $1/\sqrt{2}$;
5. 10% outlier obtained as a mixture of a N(0,1) with probability 0.9 and a N(1,10) with 0.1;
6. 30% outlier obtained as a mixture of a N(0,1) with probability 0.7 and a N(1,10) with 0.3;
7. Student's $t$ with 2 df;
8. standard Cauchy, which corresponds to a Student's $t$ with 1 df.

Note that distributions 7 and 8 have infinite second moment, and that distribution 8 has an undefined first moment. We consider only symmetric distributions because if one considers skewed distributions, a change in location is not qualitatively different with respect to a change in scale and therefore the location-scale alternative is not well specified in terms of $\mu_1 - \mu_2$ and $\sigma_1/\sigma_2$. We consider the balanced cases $(n_1, n_2) = (10, 10)$ and $(30, 30)$ as well as the unbalanced cases $(n_1, n_2) = (10, 30)$ and $(30, 10)$. We emphasize that p-values of the $PG2$ test have been computed exactly for all the sample size settings. p-values of the Lepage and Cucconi tests have been computed exactly for $(n_1, n_2) = (10, 10)$ and have been estimated by considering a random sample of 1 million permutations in the remaining settings. p-values of the $MC^*$ test have been estimated by considering a random sample of 1000 permutations. The results in terms of the proportion of times $H_0$ is rejected are reported in Table 1 and Table 2 for the estimates of the size and power. The first two lines of the tables display the parameter choice: in the first column we are under $H_0$, while in the others we are under $H_1$. Note that all the tests are robust in size because their maximum estimated significance level (MESL) does not exceed 0.07. More precisely the MESL is 0.067, 0.058, 0.057 and 0.058 for the $MC^*$, $L$, $C$ and $PG2$ tests respectively. It is important to note that the MESL of all the tests is greater than .05 and that the MESL of the $MC^*$ test is the greatest one. Note that the cut-off point for the robustness in size is set to 0.1 by Conover et al. (1981) and more stringently to 0.075 by Marozzi (2011). Even if we caution that the results are obtained via simulations, they are very clear and show that the $MC^*$ test is more powerful than the other tests for all distribution and sample size settings considered here. The results show that the combination of the Cucconi test with the Wilcoxon test for location and the modified Levene test for scale markedly improve the power of the Cucconi test in detecting separately location and scale changes, and in jointly detecting location and scale changes, for distributions that range from light-, to normal- and heavy-tailed distributions. The cost to be paid is the slightly liberality of the test that has a MESL of .067 (the other tests have a MESL between .057 and .058).

## 4. Application

Table 3 shows expenditure in Hong Kong dollars of 20 single men and 20 single women on the commodity group housing including fuel and light. This real data example is taken from Hand, Daly, Lunn, McConway & Ostrowski (1994, p. 44). Figure 1 presents the box plots of the data.

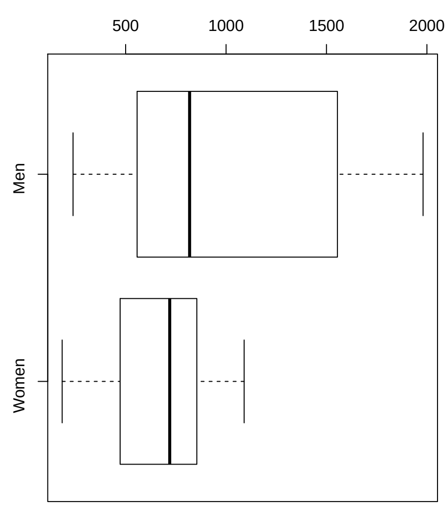


FIGURE 1: Box-plot of household expenditures.

We see from the box plots that the distributions of the data in the two groups seem to have different locations as well as different scales. This example illustrates

Table 1: Size and power of some tests for location and scale changes, $(n_1, n_2) = (10, 10)$ and $(10, 30)$.

| $(n_1, n_2) = (10, 10)$ | | | | | | $(n_1, n_2) = (10, 30)$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Normal | | | | | | Normal | | | | | |
| $\mu_1 - \mu_2$ | 0 | 0 | 1 | 1 | 1 | $\mu_1 - \mu_2$ | 0 | 0 | 0.75 | 0.75 | 0.75 |
| $\sigma_1/\sigma_2$ | 1 | 2 | 2 | 1 | 3 | $\sigma_1/\sigma_2$ | 1 | 1.5 | 1.5 | 1 | 2.5 |
| $MC^*$ | 0.055 | 0.423 | 0.646 | 0.595 | 0.821 | $MC^*$ | 0.057 | 0.349 | 0.628 | 0.544 | 0.896 |
| $L$ | 0.050 | 0.249 | 0.383 | 0.415 | 0.585 | $L$ | 0.044 | 0.201 | 0.427 | 0.383 | 0.690 |
| $C$ | 0.052 | 0.281 | 0.414 | 0.410 | 0.639 | $C$ | 0.048 | 0.257 | 0.473 | 0.388 | 0.780 |
| $PG2$ | 0.053 | 0.286 | 0.418 | 0.413 | 0.642 | $PG2$ | 0.046 | 0.253 | 0.467 | 0.381 | 0.775 |
| Uniform | | | | | | Uniform | | | | | |
| $\mu_1 - \mu_2$ | 0 | 0 | 1 | 1 | 1 | $\mu_1 - \mu_2$ | 0 | 0 | 0.75 | 0.75 | 0.75 |
| $\sigma_1/\sigma_2$ | 1 | 2 | 2 | 1 | 3 | $\sigma_1/\sigma_2$ | 1 | 1.5 | 1.5 | 1 | 2.5 |
| $MC^*$ | 0.065 | 0.582 | 0.730 | 0.533 | 0.912 | $MC^*$ | 0.063 | 0.519 | 0.688 | 0.518 | 0.966 |
| $L$ | 0.053 | 0.381 | 0.435 | 0.348 | 0.683 | $L$ | 0.051 | 0.324 | 0.430 | 0.340 | 0.778 |
| $C$ | 0.053 | 0.456 | 0.489 | 0.327 | 0.764 | $C$ | 0.050 | 0.430 | 0.503 | 0.343 | 0.882 |
| $PG2$ | 0.054 | 0.462 | 0.494 | 0.331 | 0.767 | $PG2$ | 0.049 | 0.424 | 0.497 | 0.339 | 0.879 |
| Bimodal | | | | | | Bimodal | | | | | |
| $\mu_1 - \mu_2$ | 0 | 0 | 2.5 | 1.5 | 1.5 | $\mu_1 - \mu_2$ | 0 | 0 | 2 | 1 | 1 |
| $\sigma_1/\sigma_2$ | 1 | 1.5 | 1.5 | 1 | 2.5 | $\sigma_1/\sigma_2$ | 1 | 1.5 | 1.5 | 1 | 1.5 |
| $MC^*$ | 0.062 | 0.285 | 0.718 | 0.431 | 0.824 | $MC^*$ | 0.061 | 0.489 | 0.801 | 0.356 | 0.634 |
| $L$ | 0.048 | 0.174 | 0.453 | 0.261 | 0.587 | $L$ | 0.051 | 0.305 | 0.555 | 0.222 | 0.379 |
| $C$ | 0.047 | 0.203 | 0.441 | 0.251 | 0.652 | $C$ | 0.053 | 0.396 | 0.611 | 0.222 | 0.459 |
| $PG2$ | 0.048 | 0.206 | 0.446 | 0.253 | 0.657 | $PG2$ | 0.050 | 0.389 | 0.605 | 0.216 | 0.453 |
| Laplace | | | | | | Laplace | | | | | |
| $\mu_1 - \mu_2$ | 0 | 0 | 1 | 1 | 1 | $\mu_1 - \mu_2$ | 0 | 0 | 0.75 | 0.75 | 0.75 |
| $\sigma_1/\sigma_2$ | 1 | 2 | 2 | 1 | 3 | $\sigma_1/\sigma_2$ | 1 | 1.5 | 1.5 | 1 | 2.5 |
| $MC^*$ | 0.064 | 0.293 | 0.616 | 0.689 | 0.741 | $MC^*$ | 0.054 | 0.243 | 0.681 | 0.690 | 0.844 |
| $L$ | 0.057 | 0.164 | 0.435 | 0.543 | 0.539 | $L$ | 0.058 | 0.144 | 0.537 | 0.563 | 0.682 |
| $C$ | 0.055 | 0.175 | 0.449 | 0.547 | 0.572 | $C$ | 0.053 | 0.177 | 0.554 | 0.560 | 0.739 |
| $PG2$ | 0.056 | 0.176 | 0.452 | 0.549 | 0.576 | $PG2$ | 0.051 | 0.174 | 0.548 | 0.554 | 0.735 |
| 10% outlier | | | | | | 10% outlier | | | | | |
| $\mu_1 - \mu_2$ | 0 | 0 | 1.5 | 1 | 1 | $\mu_1 - \mu_2$ | 0 | 0 | 1 | 0.75 | 0.75 |
| $\sigma_1/\sigma_2$ | 1 | 2.2 | 2.2 | 1 | 3.5 | $\sigma_1/\sigma_2$ | 1 | 2 | 2 | 1 | 2.2 |
| $MC^*$ | 0.056 | 0.306 | 0.542 | 0.434 | 0.593 | $MC^*$ | 0.063 | 0.355 | 0.606 | 0.443 | 0.557 |
| $L$ | 0.055 | 0.225 | 0.408 | 0.303 | 0.493 | $L$ | 0.054 | 0.331 | 0.513 | 0.289 | 0.482 |
| $C$ | 0.050 | 0.235 | 0.423 | 0.310 | 0.501 | $C$ | 0.053 | 0.370 | 0.549 | 0.294 | 0.526 |
| $PG2$ | 0.051 | 0.238 | 0.427 | 0.312 | 0.505 | $PG2$ | 0.051 | 0.365 | 0.541 | 0.288 | 0.521 |
| 30% outlier | | | | | | 30% outlier | | | | | |
| $\mu_1 - \mu_2$ | 0 | 0 | 3.6 | 1.3 | 1.3 | $\mu_1 - \mu_2$ | 0 | 0 | 1.8 | 1 | 1 |
| $\sigma_1/\sigma_2$ | 1 | 3 | 3 | 1 | 6 | $\sigma_1/\sigma_2$ | 1 | 2.2 | 2.2 | 1 | 3 |
| $MC^*$ | 0.055 | 0.296 | 0.617 | 0.351 | 0.618 | $MC^*$ | 0.057 | 0.306 | 0.608 | 0.350 | 0.569 |
| $L$ | 0.047 | 0.238 | 0.491 | 0.260 | 0.520 | $L$ | 0.052 | 0.242 | 0.503 | 0.239 | 0.464 |
| $C$ | 0.046 | 0.224 | 0.502 | 0.270 | 0.488 | $C$ | 0.052 | 0.259 | 0.506 | 0.243 | 0.480 |
| $PG2$ | 0.047 | 0.227 | 0.504 | 0.271 | 0.494 | $PG2$ | 0.049 | 0.255 | 0.500 | 0.238 | 0.475 |
| Student | | | | | | Student | | | | | |
| $\mu_1 - \mu_2$ | 0 | 0 | 2 | 1 | 1 | $\mu_1 - \mu_2$ | 0 | 0 | 1.1 | 0.8 | 0.8 |
| $\sigma_1/\sigma_2$ | 1 | 2.4 | 2.4 | 1 | 3.6 | $\sigma_1/\sigma_2$ | 1 | 1.8 | 1.8 | 1 | 2.2 |
| $MC^*$ | 0.055 | 0.369 | 0.669 | 0.376 | 0.671 | $MC^*$ | 0.058 | 0.310 | 0.608 | 0.410 | 0.605 |
| $L$ | 0.046 | 0.242 | 0.506 | 0.252 | 0.490 | $L$ | 0.049 | 0.234 | 0.474 | 0.264 | 0.464 |
| $C$ | 0.047 | 0.255 | 0.521 | 0.262 | 0.509 | $C$ | 0.050 | 0.272 | 0.500 | 0.274 | 0.515 |
| $PG2$ | 0.048 | 0.258 | 0.525 | 0.263 | 0.514 | $PG2$ | 0.047 | 0.267 | 0.495 | 0.267 | 0.508 |
| Cauchy | | | | | | Cauchy | | | | | |
| $\mu_1 - \mu_2$ | 0 | 0 | 3 | 1.5 | 1.5 | $\mu_1 - \mu_2$ | 0 | 0 | 1.5 | 1 | 1 |
| $\sigma_1/\sigma_2$ | 1 | 3 | 3 | 1 | 5 | $\sigma_1/\sigma_2$ | 1 | 2 | 2 | 1 | 3 |
| $MC^*$ | 0.053 | 0.320 | 0.591 | 0.425 | 0.577 | $MC^*$ | 0.062 | 0.218 | 0.536 | 0.395 | 0.521 |
| $L$ | 0.046 | 0.255 | 0.490 | 0.318 | 0.495 | $L$ | 0.051 | 0.195 | 0.457 | 0.249 | 0.483 |
| $C$ | 0.048 | 0.250 | 0.494 | 0.321 | 0.488 | $C$ | 0.051 | 0.217 | 0.466 | 0.244 | 0.503 |
| $PG2$ | 0.049 | 0.255 | 0.498 | 0.324 | 0.493 | $PG2$ | 0.050 | 0.214 | 0.460 | 0.239 | 0.496 |

TABLE 2: Size and power of some tests for location and scale changes, $(n_1, n_2) = (30, 10)$ and $(30, 30)$.

| | $(n_1,n_2)=(30,10)$ | | | | | | $(n_1,n_2)=(30,30)$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Normal | | | | | | Normal | | |
| $\mu_1-\mu_2$ | 0 | 0 | 1 | 0.75 | 0.75 | $\mu_1-\mu_2$ | 0 | 0 | 0.5 | 0.5 | 0.5 |
| $\sigma_1/\sigma_2$ | 1 | 1.8 | 1.8 | 1 | 2.5 | $\sigma_1/\sigma_2$ | 1 | 1.3 | 1.3 | 1 | 1.75 |
| $MC^*$ | 0.059 | 0.416 | 0.726 | 0.512 | 0.854 | $MC^*$ | 0.060 | 0.256 | 0.578 | 0.470 | 0.858 |
| $L$ | 0.046 | 0.240 | 0.427 | 0.391 | 0.579 | $L$ | 0.050 | 0.144 | 0.374 | 0.357 | 0.641 |
| $C$ | 0.045 | 0.240 | 0.431 | 0.397 | 0.612 | $C$ | 0.053 | 0.164 | 0.394 | 0.353 | 0.715 |
| $PG2$ | 0.042 | 0.230 | 0.417 | 0.390 | 0.599 | $PG2$ | 0.053 | 0.165 | 0.396 | 0.355 | 0.716 |
| | | | Uniform | | | | | | Uniform | | |
| $\mu_1-\mu_2$ | 0 | 0 | 1 | 0.75 | 0.75 | $\mu_1-\mu_2$ | 0 | 0 | 0.5 | 0.5 | 0.5 |
| $\sigma_1/\sigma_2$ | 1 | 1.8 | 1.8 | 1 | 2.5 | $\sigma_1/\sigma_2$ | 1 | 1.3 | 1.3 | 1 | 1.75 |
| $MC^*$ | 0.059 | 0.600 | 0.808 | 0.472 | 0.954 | $MC^*$ | 0.053 | 0.450 | 0.662 | 0.464 | 0.957 |
| $L$ | 0.053 | 0.395 | 0.488 | 0.343 | 0.782 | $L$ | 0.049 | 0.272 | 0.425 | 0.340 | 0.796 |
| $C$ | 0.053 | 0.458 | 0.470 | 0.345 | 0.844 | $C$ | 0.051 | 0.376 | 0.478 | 0.327 | 0.896 |
| $PG2$ | 0.052 | 0.446 | 0.458 | 0.339 | 0.836 | $PG2$ | 0.052 | 0.379 | 0.480 | 0.330 | 0.896 |
| | | | Bimodal | | | | | | Bimodal | | |
| $\mu_1-\mu_2$ | 0 | 0 | 2 | 1 | 1 | $\mu_1-\mu_2$ | 0 | 0 | 1.1 | 0.75 | 0.75 |
| $\sigma_1/\sigma_2$ | 1 | 1.5 | 1.5 | 1 | 1.75 | $\sigma_1/\sigma_2$ | 1 | 1.3 | 1.3 | 1 | 1.4 |
| $MC^*$ | 0.067 | 0.328 | 0.740 | 0.318 | 0.671 | $MC^*$ | 0.054 | 0.385 | 0.742 | 0.356 | 0.724 |
| $L$ | 0.057 | 0.218 | 0.487 | 0.217 | 0.403 | $L$ | 0.047 | 0.242 | 0.512 | 0.246 | 0.493 |
| $C$ | 0.056 | 0.216 | 0.421 | 0.212 | 0.391 | $C$ | 0.048 | 0.298 | 0.547 | 0.231 | 0.555 |
| $PG2$ | 0.054 | 0.209 | 0.410 | 0.208 | 0.378 | $PG2$ | 0.049 | 0.301 | 0.549 | 0.234 | 0.558 |
| | | | Laplace | | | | | | Laplace | | |
| $\mu_1-\mu_2$ | 0 | 0 | 1 | 0.75 | 0.75 | $\mu_1-\mu_2$ | 0 | 0 | 0.5 | 0.5 | 0.5 |
| $\sigma_1/\sigma_2$ | 1 | 1.8 | 1.8 | 1 | 2.5 | $\sigma_1/\sigma_2$ | 1 | 1.3 | 1.3 | 1 | 1.75 |
| $MC^*$ | 0.055 | 0.285 | 0.727 | 0.648 | 0.734 | $MC^*$ | 0.055 | 0.184 | 0.637 | 0.632 | 0.783 |
| $L$ | 0.048 | 0.145 | 0.515 | 0.561 | 0.446 | $L$ | 0.050 | 0.108 | 0.481 | 0.519 | 0.593 |
| $C$ | 0.048 | 0.129 | 0.531 | 0.554 | 0.469 | $C$ | 0.052 | 0.118 | 0.482 | 0.514 | 0.624 |
| $PG2$ | 0.046 | 0.124 | 0.522 | 0.550 | 0.455 | $PG2$ | 0.052 | 0.119 | 0.485 | 0.517 | 0.627 |
| | | | 10% outlier | | | | | | 10% outlier | | |
| $\mu_1-\mu_2$ | 0 | 0 | 1.5 | 0.75 | 0.75 | $\mu_1-\mu_2$ | 0 | 0 | 0.75 | 0.5 | 0.5 |
| $\sigma_1/\sigma_2$ | 1 | 2 | 2 | 1 | 3 | $\sigma_1/\sigma_2$ | 1 | 1.5 | 1.5 | 1 | 1.8 |
| $MC^*$ | 0.066 | 0.329 | 0.646 | 0.365 | 0.632 | $MC^*$ | 0.059 | 0.234 | 0.599 | 0.383 | 0.560 |
| $L$ | 0.053 | 0.240 | 0.524 | 0.290 | 0.557 | $L$ | 0.048 | 0.217 | 0.509 | 0.269 | 0.511 |
| $C$ | 0.052 | 0.218 | 0.523 | 0.295 | 0.514 | $C$ | 0.051 | 0.233 | 0.514 | 0.272 | 0.518 |
| $PG2$ | 0.050 | 0.211 | 0.514 | 0.288 | 0.507 | $PG2$ | 0.051 | 0.235 | 0.516 | 0.273 | 0.520 |
| | | | 30% outlier | | | | | | 30% outlier | | |
| $\mu_1-\mu_2$ | 0 | 0 | 3 | 1 | 1 | $\mu_1-\mu_2$ | 0 | 0 | 1.2 | 0.7 | 0.7 |
| $\sigma_1/\sigma_2$ | 1 | 2.5 | 2.5 | 1 | 4.5 | $\sigma_1/\sigma_2$ | 1 | 1.8 | 1.8 | 1 | 2.3 |
| $MC^*$ | 0.054 | 0.283 | 0.664 | 0.324 | 0.617 | $MC^*$ | 0.057 | 0.315 | 0.610 | 0.334 | 0.603 |
| $L$ | 0.047 | 0.249 | 0.513 | 0.258 | 0.541 | $L$ | 0.055 | 0.261 | 0.500 | 0.240 | 0.521 |
| $C$ | 0.047 | 0.189 | 0.513 | 0.260 | 0.454 | $C$ | 0.057 | 0.246 | 0.487 | 0.241 | 0.487 |
| $PG2$ | 0.046 | 0.184 | 0.506 | 0.254 | 0.447 | $PG2$ | 0.058 | 0.247 | 0.491 | 0.244 | 0.489 |
| | | | Student | | | | | | Student | | |
| $\mu_1-\mu_2$ | 0 | 0 | 1.7 | 0.8 | 0.8 | $\mu_1-\mu_2$ | 0 | 0 | 0.8 | 0.6 | 0.6 |
| $\sigma_1/\sigma_2$ | 1 | 2.2 | 2.2 | 1 | 3 | $\sigma_1/\sigma_2$ | 1 | 1.6 | 1.6 | 1 | 1.8 |
| $MC^*$ | 0.060 | 0.414 | 0.712 | 0.344 | 0.703 | $MC^*$ | 0.056 | 0.331 | 0.660 | 0.406 | 0.632 |
| $L$ | 0.048 | 0.278 | 0.506 | 0.260 | 0.515 | $L$ | 0.051 | 0.248 | 0.512 | 0.300 | 0.502 |
| $C$ | 0.050 | 0.244 | 0.527 | 0.263 | 0.500 | $C$ | 0.049 | 0.262 | 0.521 | 0.298 | 0.515 |
| $PG2$ | 0.048 | 0.238 | 0.515 | 0.258 | 0.493 | $PG2$ | 0.049 | 0.265 | 0.525 | 0.299 | 0.518 |
| | | | Cauchy | | | | | | Cauchy | | |
| $\mu_1-\mu_2$ | 0 | 0 | 2.5 | 1 | 1 | $\mu_1-\mu_2$ | 0 | 0 | 1.2 | 0.8 | 0.8 |
| $\sigma_1/\sigma_2$ | 1 | 2.5 | 2.5 | 1 | 4 | $\sigma_1/\sigma_2$ | 1 | 1.8 | 1.8 | 1 | 2.2 |
| $MC^*$ | 0.063 | 0.322 | 0.588 | 0.297 | 0.567 | $MC^*$ | 0.055 | 0.269 | 0.608 | 0.424 | 0.557 |
| $L$ | 0.050 | 0.258 | 0.457 | 0.238 | 0.511 | $L$ | 0.046 | 0.250 | 0.530 | 0.302 | 0.520 |
| $C$ | 0.048 | 0.208 | 0.473 | 0.238 | 0.441 | $C$ | 0.044 | 0.245 | 0.519 | 0.300 | 0.505 |
| $PG2$ | 0.046 | 0.202 | 0.465 | 0.232 | 0.435 | $PG2$ | 0.044 | 0.246 | 0.522 | 0.302 | 0.508 |

Table 3: Household expenditures (Honk Kong dollars) of a group of men and a group of women.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Men | | | | | | | | | |
| 497 | 839 | 798 | 892 | 1585 | 755 | 388 | 617 | 248 | 1641 |
| 1180 | 619 | 253 | 661 | 1981 | 1746 | 1865 | 238 | 1199 | 1524 |
| Women | | | | | | | | | |
| 820 | 184 | 921 | 488 | 721 | 614 | 801 | 396 | 864 | 845 |
| 404 | 781 | 457 | 1029 | 1047 | 552 | 718 | 495 | 382 | 1090 |

that in practice we may have situations where $F_1$ and $F_2$ are different in both location and scale. With the aim at finding out whether household expenditures differ from men to women, we use the modified Cucconi test. By considering a random sample of 1 million permutations, the estimated p-value of the $MC^*$ test is 0.0105, that suggests to reject the null hypothesis at level 5%. This result is consistent with the results obtained using the original Cucconi test and the $PG2$ test whose p-values are 0.0446 (estimated by considering a random sample of 1 million permutations) and 0.0441 (exact computation) respectively. The estimated p-value of the Lepage test is 0.0896 and suggests to reject $H_0$ at level 10%. At the basis of these results we conclude that household expenditures of men and women differ. It is worth noting that, with respect to the $MC^*$ test, the other tests need a higher level in order to reject $H_0$. This might suggest a gain in power of the modified Cucconi test with respect to the original one and to the other tests.

## 5. Conclusion

We introduced a modification of the Cucconi test. The main objetive was to modify this test consistently with the familiar approach which develops a location-scale test by combining a test for location and a test for scale. More precisely we combined the Cucconi test with the Wilcoxon test for location and the modified Levene test for scale proposed by Brown & Forsythe (1974) following the theory of the nonparametric combination (Pesarin 2001). We compared the performance of the modified Cucconi test with the original one, the Lepage test and the Podgor-Gastwirth $PG2$ test in separately detecting location and scale changes as well as in jointly detecting location and scale changes. The results show that the combination of the Cucconi test with the Wilcoxon test for location and the modified Levene test for scale gives rise to a test which is slightly more liberal and markedly more powerful than the other tests for all the considered distributions, from short- to normal- and long-tailed ones. In the light of our findings, we recommend the practitioner to use the modified Cucconi test to address the location-scale problem, with caution on its type-one error rate.

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