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# ON THE COMPUTATION OF WEIERSTRASS GAP SEQUENCES

**Abstract.** In this paper, we develop a technique to compute the Weierstrass Gap Sequence at a given point, no matter if simple or singular, on a plane curve, with respect to any linear system $V \subseteq H^0(C, \mathcal{O}_C(n))$. This technique can be useful to construct examples of curves with Weierstrass points of given weight, or to look for conditions for a sequence to be a Weierstrass Gap Sequence. We use this technique to compute the Weierstrass Gap Sequence at a point of particular curves and of families of curves.

## 1. Introduction

Weierstrass points on curves have been widely studied, in connection with many problems. For example, the moduli space $\mathcal{M}_g$ has been stratified with subvarieties whose points are isomorphism classes of curves with particular Weierstrass points (see [1], [3], [13]).

At first, the theory of the Weierstrass points was developed only for smooth curves, and for their canonical divisors. In the last years, starting from some papers by R. Lax and C. Widland (see [16], [8], [9], [10], [11], [12]), the theory has been reformulated for Gorenstein curves, where the invertible dualizing sheaf substitutes the canonical sheaf. In this contest, the singular points of a Gorenstein curve are always Weierstrass points. In this paper, we shall describe a technique to compute the Weierstrass weight at a point, either smooth or singular, with respect to any linear system $V$. Such a procedure is based on the computation of the sequence of integers which in [4] has been called "$V$-gaps sequence", even at singular points ($V$-WGS for brief).

As better explained in the next section, to define the $V$-gaps it is necessary to distinguish if the point $P$ is smooth on the curve $C$ or if it is singular. In the first case, to compute the $V$-gaps, we need to determine the dimension of the linear systems $V - nP$, for every $n$. In the second case, the $V$-gaps are given by a suitable combination of the $\tilde{V}$-gaps at the points $Q_1, \ldots, Q_s$, $s \geq 1$, over $P$ in a partial normalization $\theta_P : \tilde{C} \to C$ of $C$ at $P$, where $\tilde{V}$ is the pull back of $V$.

In [6], some techniques to compute the WGS with respect to the dualizing sheaf of a Gorenstein curve have been shown, but with some heavy constraints: with those techniques it was possible to compute at most the WGS at ordinary nodes on quartic curves or at cusps on quintic curves. The aim of this paper is to overcome those difficulties. In fact, the technique we describe, consists in performing a fixed sequence of computations, and, for this reason, it can be applied to any curve, at any point, no matter if smooth or singular. If $P$ is smooth, the $V$-gaps are computed by means of the definition, and so some intersection multiplicities must be computed. If $P$ is singular, the $\tilde{V}$-gaps at the points $Q_1, \ldots, Q_s$ over $P$ can be computed as the intersection multiplicities of $C_v$, $v \in V$, and the branches $C'_1, \ldots, C'_s$ of $C$ through $P$, corresponding to

*Partially supported by MURST.

$Q_1, \ldots, Q_s$, respectively, one at a time. In the second case, the study of the branches through $P$ allows to largely simplify the computation of the sequence of the $V$-gaps. This simplification is essentially due to the knowledge of the normalization map in terms of blow-up's, as shown in [4] and [7].

In both cases, the intersection multiplicities are computed by means of the osculating curves of suitable degrees. Moreover, we describe a quick way to compute the osculating curve of assigned degree at a point of a curve, because in the most spread computer algebra systems there is no built in function to perform that computation.

Let us explicitly observe that our technique is "constructive", in some sense, but that its application is based on the factorization of multivariate polynomials, and on the solution in closed form of algebraic equations of arbitrary degree for the computation of the osculating curves. However, all the computation described in the following were made using REDUCE as a computing support.

The plan of the paper is the following. In section 2, we recall definitions and properties of the Weierstrass Gap Sequence and of the Extraweight Sequence. In section 3, we describe the technique and show its correctness, while, in section 4, we let the technique work on various examples. In the last section, we study the families of the quintic curves with a 4-tuple point with respect to the $V$-WGS at the singular point, where $V$ is the linear system of the plane conics.

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## 2. Preliminaries

In this section we shall recall briefly what one should mean by $V$-WGS at a point, singular or not, with respect to any linear system $V$, in a very general situation.

To this purpose, let $C$ be any projective integral curve over the complex field, and let $V$ be an $r$-dimensional linear system (i.e., an $r$-dimensional sub-vector space of $H^0(C, \mathcal{L})$, where $\mathcal{L}$ is a line bundle over $C$). Let $\mathbf{v} = (v_1, \ldots, v_r)$ be a basis of $V$. Then, if $\pi : \widetilde{C} \to C$ is the normalization of $C$, we consider the linear system $\widetilde{V} = \mathrm{span}(\pi^* v_1, \ldots, \pi^* v_r)$ over $\widetilde{C}$.

Let us recall a geometrical definition of $V$-gap at a point $P$ of $C$ ([14], §2).

DEFINITION 3.1. *Let $P$ be a smooth point on the curve $C$. The integer $n$ is a $V$-gap if, and only if, $d(V - (n-1)P) > d(V - nP)$. The sequence of the $V$-gaps is the $V$-WGS at $P$.*

*Let $P$ be a singular point of $C$. The $V$-WGS at $P$ is $\langle a_1^V(P), \ldots, a_r^V(P)\rangle$, where*

$$a_k^V(P) = \sum_{Q \in \pi^{-1}(P)} b_k^{\widetilde{V}}(Q) - k(\#\pi^{-1}(P) - 1) \tag{1}$$

*and $\langle b_1^{\widetilde{V}}(Q), \ldots, b_r^{\widetilde{V}}(Q)\rangle$ is the $\widetilde{V}$-WGS at $Q \in \pi^{-1}(P)$.*

Following [2], we define a function $E_k^V : C \to \mathbb{N}$, for any $1 \le k \le r$: the evaluation of $E_k^V$ at a given point $P \in C$, computed as

$$E_k^V(P) = \sum_{Q \in \pi^{-1}(P)} \omega_k^{\widetilde{V}}(Q) \tag{2}$$

gives the so called $k$-th extraweight at $P$, where $\omega_k^{\widetilde{V}}(Q)$ is the Weierstrass weight at the smooth point $Q$ with respect to the linear system $\widetilde{V}$. Using notation as in [7], $\omega_k^{\widetilde{V}}(Q)$ is nothing but

$\mathrm{ord}_Q \pi^* \mathbf{v} \wedge D\pi^* \mathbf{v} \wedge \ldots \wedge D^{k-1}\pi^* \mathbf{v}$. By definition (see [2]), a point $P \in C$ is a $V$-Weierstrass point if, and only if, either it is singular or its $r$-th extraweight is non zero.

By means of the extraweight sequence, the $V$-WGS at $P$ can be computed as

$$a_k^V(P) = \begin{cases} E_k^V(P) + 1 & \text{if } k = 1 \\ E_k^V(P) - E_{k-1}^V(P) + k & \text{if } 2 \leq k \leq r \end{cases} \tag{3}$$

and hence we have also that

$$E_k^V(P) = \sum_{i=1}^{k} \left( a_i^V(P) - i \right) . \tag{4}$$

Formulas (3) and (4) show that it is equivalent to know the $V$-WGS or the extraweight sequence at $P$, but the first one is easier to compute than the second, because of the geometrical meaning of the $V$-WGS.

REMARK 3.1 (CASE $\mathcal{L} = \mathcal{O}_C(n)$, $C \subset \mathbb{P}^2$). In this case, the so called restriction map $\varphi_n : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \to H^0(C, \mathcal{O}_C(n))$ is surjective for every $n$. If $V$ is spanned by $v_1, \ldots, v_r$, there exist $C_1, \ldots, C_r$ (we use the same symbol for the plane curves and the corresponding elements in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$) such that $\varphi_n(C_i) = v_i$, for each $i$. The elements $C_i$ are not uniquely determined, but if $\varphi_n(C') = \varphi_n(C'')$ then $C' - C'' \in I_C(n) =$ degree $n$ part of the saturated ideal of $C$, which is the kernel of the restriction map, and so $C'$ and $C''$ cannot be distinguished when we consider their behaviour with respect to the curve $C$. Hence, let $W$ be the subspace spanned by $C_1, \ldots, C_r$. Then, the sub-vector space $V - nP$ corresponds to the sub-vector space $W'$ of $W$ spanned by all the curves $C_v \in W$ such that the intersection multiplicity at $P$ of $C$ and $C_v$, $\mu(P, C \cap C_v)$, is not less than $n$, and, in particular, they have the same dimension.

## 3. The technique

As declared in the Introduction, the main purpose of what follows, is to describe a technique which effectively allows to compute the $V$-WGS at a point $P$, no matter if singular or smooth, on a plane curve $C$, with respect to a linear system $V \subseteq H^0(C, \mathcal{O}_C(n))$.

¿From now on, by plane curve $C$ we mean a projective, integral curve $C$ of the projective plane over the complex field.

Now, we state a proposition which allows to compute the $\widetilde{V}$-WGS at a point $Q \in \pi^{-1}(P)$, without explicitly computing the partial normalization of $C$ at $P$.

Let $\theta : \widetilde{C} \to C$ be the partial normalization of $C$ at $P$. It is well known that $\theta$ can be obtained as the restriction to $C$ of a sequence of blow-up' s of an affine neighbourhood $U \simeq A^2$ of $P$. Then, let $\psi : X \to U$ be the map whose restriction is $\theta$: we have that $\widetilde{C} = \psi^{-1}(C \cap U)$. Moreover, set $\psi = \pi_1 \circ \cdots \circ \pi_n$, where the $\pi_i$' s are blow-up' s. Let us choose $Q \in \theta^{-1}(P)$, and let $C'$ be the branch of $C$ through $P$ corresponding to $Q$.

Let us set the following notation.

1. $\pi_0 = \mathrm{id}_U$;
2. $C_i' = (\pi_0 \circ \pi_1 \cdots \circ \pi_i)^{-1}(C')$, $(C_v)_i = (\pi_0 \circ \pi_1 \cdots \pi_i)^{-1}(C_v)$;
3. $P_i = (\pi_{i+1} \circ \cdots \circ \pi_n)(Q)$;
4. $\mu_{P_i}(\ldots)$ is the multiplicity of the curve in parenthesis at $P_i$.

For every $v \in V$, let $a_v = \sum_{i=0}^{n-1} \mu_{P_i}(C'_i)\mu_{P_i}((C_v)_i)$.

PROPOSITION 3.1. *$h$ is a $\widetilde{V}$-gap at $Q$ if, and only if, there exists $v \in V$ such that $\mu(P, C' \cap C_v) = h - 1 + a_v$.*

*Proof.* Let us suppose that $h$ is a $\widetilde{V}$-gap at $Q$. Then, by definition, $d(\widetilde{V}-(h-1)Q) > d(\widetilde{V}-hQ)$, and hence there exists $v \in (\widetilde{V}-(h-1)Q)\setminus(\widetilde{V}-hQ)$, that is to say, $\mu(Q, \widetilde{C}\cap\psi^{-1}(C_v)) = h-1$, and so $\mu(P, C\cap C_v) = h-1+a_v$, by ([15], Ch. IV, § 3, Th. 2); the proof of the inverse implication is analogous. $\square$

According to the previous considerations, the technique consists in the following sequence of computations, starting from the equations $f$, $g_v$ of $C$ and $C_v$, $v \in V$, in the chosen affine open subset of $\mathbb{P}^2$, respectively, and from the coordinates of $P$ in the same open subset.

For each branch $C^j$ of $C$ through $P$ do

1. set $V_1 = V$, $r := \dim V$ and $n_1^j := 1 + \mu(P, C^j \cap C_v)$ where $v$ is the general element of $V_1$;
2. compute a condition forcing the general curve of the linear system $V_i$ to have intersection multiplicity with the branch $C^j$ at $P$ not less than $n_i^j$;
3. impose the computed condition on the linear system $V_i$ obtaining the linear sub-system $V_{i+1}$;
4. the computation starts again from 2. until the linear sub-system becomes empty.

The $V$-gaps $(a_1, \ldots, a_r)$ are computed according to the formula

$$a_k := \sum_{i=1}^{\#\text{ branches}} n_k^i - k(\#\text{ branches } - 1)\,.$$

The correctness of the result of this sequence of computations is ensured by Definition 3.1, and Proposition 3.1.

We explicitly observe that this technique can be applied also if $P$ is a base point of $V$ as Example 3.4 below shows.

We observe that, by Proposition 3.1, the knowledge of the intersection multiplicities between the branches $C^j$ and $C_v$, $v \in V_i$, at $P$, allows to compute the $\widetilde{V}$-WGS at any point $Q \in \theta^{-1}(P)$, too.

We remark that the hardest computation to be performed is the one of 2., and so we give some suggestions to perform it. At first, we must analyze the point $P$ as point of $C$. If it is a cusp, it is possible to compute the required condition using the resultant $R(t)$ of $f$, and $g_v$. In fact, the condition is given equating the coefficient of $t^{n_i^j-1}$ in $R(t)$ to zero. If $P$ is not a cusp, the condition can be computed using the osculating curves of sufficiently large degree of $C$ and $C_v$ at $P$, as shown later.

Let us recall the definition and some useful properties of the osculating curves.

DEFINITION 3.2. *Let $P \in C$, and suppose that $P$ is not a cusp for $C$. A curve $C'_n$, smooth at $P$, is said to be osculating of degree $n$ at $P$ if $\mu(P, C\cap C'_n) > n + \mu_P(C) - 1$, and the degree of $C'_n$ is $n$.*

Let us observe that, for a fixed integer $n$, there can be several curves that are osculating. If $P$ is a pluribranch singular point, it is natural to restrict the attention to one branch at a time.

Before we prove some properties of the osculating curves, we shall describe how to compute an osculating curve of a certain degree, in a very quick way.

To this purpose, let us choose a coordinate affine open subset of $\mathbb{P}^2$ containing the point $P$, and let $f(x, y) = 0$ be the equation of the curve $C$ in the chosen open subset, and $(x_0, y_0)$ the coordinates of the point $P$. Moreover, let us suppose that the tangent line at $P$ to $C$ is not the line $x = x_0$.

1. set $P_0(x) := y_0$;
2. for each integer $1 \leq i \leq n$, set $G(x) := P_{i-1}(x) + a_i(x - x_0)^i$ and compute $a_i$ in such a way that the smallest power of $x$ in $f(x, G(x)) = 0$ has coefficient zero;
3. set $P_i(x) := P_{i-1}(x) + a_i(x - x_0)^i$ and start again from 2.

REMARK 3.2. If the point is smooth, the equations to be solved to compute the parameters $a_1, \dots, a_n$ are linear, while, if the point is a pluribranch singularity, there can be equations of higher degree.

Now, let us prove that we can compute the intersection multiplicity of two curves at a given point using the osculating curves.

PROPOSITION 3.2. *Let $P \in C_1 \cap C_2$ be smooth for both curves. Then, $\mu(P, C_1 \cap C_2) = \mu(P, C_2 \cap C_3)$, where $C_3$ is the osculating curve of suitable degree of $C_1$ at $P$, computed as previously explained.*

*Proof.* The point $P$ is a smooth point of $C_1$ and then we can consider the formal power series representing $C_1$ in an affine open neighbourhood $U \simeq A^2$ of $P$: the curve $C_3$ is a cutting off of this series at a suitable point.

Set $g(x, y) = 0$ the equation of the curve $C_2$ in $U$. We have that $\mu(P, C_1 \cap C_2)$ is the order of the formal power series obtained from $g(x, y)$ by substituting the series representing $C_1$. Then, the claim follows if we cut off the series at a degree in such a way that the first term of the series obtained from $g(x, y)$ does not change. $\square$

## 4. Examples

In this section we shall discuss five examples. The first two examples have been developed in [7], and have been included to show the different techniques to obtain the WGS. The third and the fourth ones show that the technique works with no constraint on the degree of the curve or on the linear system. In the last example, we show that the technique allows to compute the $V_d$-WGS for every curve zero locus of $x^n + y^n - xy = 0$, where $V_d$ is the linear system of the curves of degree $d$.

EXAMPLE 3.1. Let $C$ be a quartic curve whose equation is $x^4 + x^3 - y^3 + x^2 - y^2 = 0$, in a chosen open affine subset. We want to compute the WGS at the origin $O$ that is a node for

$C$. The osculating curves to the two branches through $O$ have equations

$$\begin{aligned} \text{branch } C^1 &\quad : \quad y = x + \frac{1}{2}x^3 - \frac{3}{4}x^4 + \dots \\ \text{branch } C^2 &\quad : \quad y = -x - x^2 - \frac{3}{2}x^3 + \dots . \end{aligned}$$

The general curve $C_v$ of the linear system, in the same open subset, has equation $b_0x + b_1y + b_2 = 0$. Let us observe that the linear system is base point free, and so we have that $\mu(O, C^1 \cap C_v) = \mu(O, C^2 \cap C_v) = 0$.

By using the technique, we have the following partial results.

Branch $C^1$.

| | Condition | Sub-system | Int. mult. |
|---|---|---|---|
| $n_1^1 = 1$ | $b_2 = 0$ | $b_0x + b_1y = 0$ | 1 |
| $n_2^1 = 2$ | $b_0 = -b_1$ | $-b_1x + b_1y = 0$ | 3 |
| $n_3^1 = 4$ | $b_1 = 0$ | $0 = 0$ | $\infty$ |

Branch $C^2$.

| | Condition | Sub-system | Int. mult. |
|---|---|---|---|
| $n_1^2 = 1$ | $b_2 = 0$ | $b_0x + b_1y = 0$ | 1 |
| $n_2^2 = 2$ | $b_0 = b_1$ | $b_1x + b_1y = 0$ | 2 |
| $n_3^2 = 3$ | $b_1 = 0$ | $0 = 0$ | $\infty$ |

Then, the $V$-WGS at $O$ is $\langle 1, 2, 4\rangle$, computed according to formula (1), while the extraweight sequence is $\langle 0, 0, 1\rangle$, according to formula (4).

EXAMPLE 3.2. Let $C$ be the quintic curve whose equation is $x^5 + x^3 + y^2 = 0$ in a chosen affine open subset. Let us notice that the origin $O$ is a cusp, and then there is only one branch of $C$ through $O$. We want to compute the $V$-WGS at $O$, where the general curve of $V$, in the same open subset, has equation $b_0x^2 + b_1xy + b_2y^2 + b_3x + b_4y + b_5 = 0$. The linear system $V$ is base point free, and so $\mu(O, C \cap C_v) = 0$.

| | Condition | Sub-system | Int. mult. |
|---|---|---|---|
| $n_1^1 = 1$ | $b_5 = 0$ | $b_0x^2 + b_1xy + b_2y^2 + b_3x + b_4y = 0$ | 2 |
| $n_2^1 = 3$ | $b_3 = 0$ | $b_0x^2 + b_1xy + b_2y^2 + b_4y = 0$ | 3 |
| $n_3^1 = 4$ | $b_4 = 0$ | $b_0x^2 + b_1xy + b_2y^2 = 0$ | 4 |
| $n_4^1 = 5$ | $b_0 = 0$ | $b_1xy + b_2y^2 = 0$ | 5 |
| $n_5^1 = 6$ | $b_1 = 0$ | $b_2y^2 = 0$ | 6 |
| $n_6^1 = 7$ | $b_2 = 0$ | $0 = 0$ | $\infty$ |

By applying formulas (1) and (4), we obtain that the $V$-WGS and the extraweight sequence are $\langle 1, 3, 4, 5, 6, 7\rangle$, and $\langle 0, 1, 2, 3, 4, 5\rangle$, respectively.

EXAMPLE 3.3. Let $C$ be the curve whose equation, in a fixed affine open subset, is $x^7 + x^2 - y^2 = 0$. We want to compute the $V$-WGS at the origin $O$ with respect to the linear system of the plane conics. Let us observe that the origin $O$ is a node for $C$, whose branches have osculating curves

$$\begin{aligned} \text{branch } C^1 \quad &: \quad y = x + \frac{1}{2}x^6 - \frac{1}{8}x^{11} + \frac{1}{16}x^{16} + \dots \\ \text{branch } C^2 \quad &: \quad y = -x - \frac{1}{2}x^6 + \frac{1}{8}x^{11} - \frac{1}{16}x^{16} + \dots . \end{aligned}$$

With the same notation of the previous example, the osculating curve to a plane conic through the origin has equation

$$\begin{aligned} y \quad = \quad & -\frac{b_3}{b_4}x + \frac{-b_0b_4^2 + b_1b_3b_4 - b_2b_3^2}{b_4^3}x^2 \\ & + \frac{b_0b_1b_4^3 - 2b_0b_2b_3b_4^2 - b_1^2b_3b_4^2 + 3b_1b_2b_3^2b_4 - 2b_2^2b_3^3}{b_4^5}x^3 . \end{aligned}$$

By the symmetry of the osculating curves at every branch of $C$ through $O$, we can apply the technique to the only first one.

| | Condition | Sub-system | Int. mult. |
|---|---|---|---|
| $n_1^1 = 1$ | $b_5 = 0$ | $b_0x^2 + b_1xy + b_2y^2 + b_3x + b_4y = 0$ | 1 |
| $n_2^1 = 2$ | $b_3 = -b_4$ | $b_0x^2 + b_1xy + b_2y^2 - b_4x + b_4y = 0$ | 2 |
| $n_3^1 = 3$ | $b_2 = -b_0 - b_1$ | $(x-y)[b_0x + (b_0 + b_1)y - b_4] = 0$ | 6 |
| $n_4^1 = 7$ | $b_4 = 0$ | $(x-y)[b_0x + (b_0 + b_1)y] = 0$ | 7 |
| $n_5^1 = 8$ | $b_1 = -2b_0$ | $(x-y)(b_0x - b_0y) = 0$ | 12 |
| $n_6^1 = 13$ | $b_0 = 0$ | $0 = 0$ | $\infty$ |

The two partial $V$-WGS are equal to $\langle 1, 2, 3, 7, 8, 13\rangle$, and hence the $V$-WGS at the origin, according to the formula (1), is $\langle 1, 2, 3, 10, 11, 20\rangle$, while the extraweight sequence is $\langle 0, 0, 0, 6, 12, 26\rangle$.

EXAMPLE 3.4. Let $C$ be the quintic curve whose equation, in a fixed affine open subset, is $x^5 + y^5 - xy = 0$. We want to compute the $V$-WGS at the origin, where $V$ is a sub-linear system of the plane conics, whose general element, in the chosen open subset, has equation $b_0x^2 + b_1y^2 + b_2y = 0$. Let us notice that the origin is the only base point of the linear system, and that no element of the system can be tangent at the origin to the $y$-axis. Moreover, the origin is a node for $C$ and the osculating curves to the two branches of $C$ through $O$ have equations

$$\begin{aligned} \text{branch } C^1 \quad &: \quad y = x^4 + x^{19} + \dots , \\ \text{branch } C^2 \quad &: \quad x = y^4 + y^{19} + \dots , \end{aligned}$$

respectively, while the osculating curve to the general element of $V$, at the origin, is

$$y = -\frac{b_0}{b_2}x^2 - \frac{b_0^2b_1}{b_2^3}x^4 .$$

At last, we observe that $\mu(O, C^1 \cap C_v) = 2$, while $\mu(O, C^2 \cap C_v) = 1$.

Branch $C^1$.

| | Condition | Sub-system | Int. mult. |
|---|---|---|---|
| $n_1^1 = 3$ | $b_0 = 0$ | $b_1 y^2 + b_2 y = 0$ | 4 |
| $n_2^1 = 5$ | $b_2 = 0$ | $b_1 y^2 = 0$ | 8 |
| $n_3^1 = 9$ | $b_1 = 0$ | $0 = 0$ | $\infty$ |

Branch $C^2$.

| | Condition | Sub-system | Int. mult. |
|---|---|---|---|
| $n_1^2 = 2$ | $b_2 = 0$ | $b_0 x^2 + b_1 y^2 = 0$ | 2 |
| $n_2^2 = 3$ | $b_1 = 0$ | $b_0 x^2 = 0$ | 8 |
| $n_3^2 = 9$ | $b_0 = 0$ | $0 = 0$ | $\infty$ |

We can calculate the $V$-WGS at the origin, applying formula (1), and then we have $\langle 4, 6, 15\rangle$, while the extraweight sequence is $\langle 3, 7, 19\rangle$.

EXAMPLE 3.5. Let $C_n$ be the curve whose equation, in an open affine subset, is $x^n + y^n - xy = 0$, $n \geq 4$, and let $V_d$ be the linear system of the curves of degree $d$, with $1 \leq d \leq n-3$. The general curve of $V_d$ has equation $\sum_{i,j\geq 0}^{i+j\leq d} b_{i,j} x^i y^j = 0$.

We want to compute the $V_d$-WGS at the origin $O$ which is a node for the curve $C_n$.

The osculating curves at the two branches of $C_n$ through $O$ can be obtained the one from the other by exchanging the variables, and they are the following:

$$\begin{array}{lcl} \text{Branch } C^1 & : & y = x^{n-1} + x^{n^2-n-1} + \dots \\ \text{Branch } C^2 & : & x = y^{n-1} + y^{n^2-n-1} + \dots \end{array}$$

and hence we shall use only the first branch.

PROPOSITION 3.3. *The $V_d$-WGS at the origin $O$ of $C_n$ is*

$$\langle \mathbf{b}_{d+1}, \mathbf{b}_d, \dots, \mathbf{b}_1 \rangle$$

*where* $\mathbf{b}_k = \langle 2(d+1-k)(n-1) + 1 - (d+1)(d+2)/2 + k(k+1)/2, \dots, 2(d+1-k)(n-1) + k - (d+1)(d+2)/2 + k(k+1)/2\rangle$.

*Proof.* We shall prove the claim by induction on $d$.

Set $d = 1$. The general curve of $V_1$ has equation $b_{1,0}x + b_{0,1}y + b_{0,0} = 0$. We have

| | Condition | Sub-system | Int. mult. |
|---|---|---|---|
| $n_1^1 = 1$ | $b_{0,0} = 0$ | $b_{1,0}x + b_{0,1}y = 0$ | 1 |
| $n_2^1 = 2$ | $b_{1,0} = 0$ | $b_{0,1}y = 0$ | $n-1$ |
| $n_3^1 = n$ | $b_{0,1} = 0$ | $0 = 0$ | $\infty$ |

and then, the two partial $V_1$-WGS are $\langle 1, 2, n\rangle$, the $V_1$-WGS at $O$ is $\langle 1, 2, 2n-3\rangle$, and so the claim holds for $d = 1$.

Let us suppose that the claim holds for $d - 1$, and let us prove that it holds for $d$, too.

To compute the osculating curves at the origin $O$ of the curves of $V_d$, we consider a curve of $V_d$ whose equation is

$$\sum_{i\geq 0, j>0}^{i+j\leq d} b_{i,j} x^i y^j + b_{d,0} x^d + \ldots + b_{h,0} x^h = 0$$

where $h > 0$, and so we obtain the curve whose equation is $y = -x^h b_{h,0}/b_{0,1}$.

Let us observe that the equation of the considered curve of $V_d$ can be written as $yP_{d-1}(x, y) + x^h Q_{d-h}(x) = 0$, where $P_{d-1}$ is the equation of the general curve of $V_{d-1}$, with suitable subscripts, while $Q_{d-h}$ is the general polynomial in one variable of degree $d - h$.

We have the following partial results.

| | Condition | Sub-system | Int. mult. |
|---|---|---|---|
| $n_1^1 = 1$ | $b_{0,0} = 0$ | $yP_{d-1} + xQ_{d-1} = 0$ | 1 |
| $n_2^1 = 2$ | $b_{1,0} = 0$ | $yP_{d-1} + x^2 Q_{d-2} = 0$ | 2 |
| $\vdots$ | | | |
| $n_d^1 = d$ | $b_{d-1,0} = 0$ | $yP_{d-1} + b_{d,0}x^d = 0$ | $d$ |
| $n_{d+1}^1 = d+1$ | $b_{d,0} = 0$ | $yP_{d-1} = 0$ | $n - 1$ |
| $n_{d+2}^1 = n$ | $\ldots$ | | |

From now on, the intersection multiplicities are the ones of the linear system $V_{d-1}$ increased of $n - 1 = \mu(O, C^1 \cap V(y))$, and hence, the partial $V_d$-WGS is $\langle \mathbf{c}_{d+1}, \mathbf{c}_d, \ldots, \mathbf{c}_1 \rangle$ where $\mathbf{c}_k = \langle (d + 1 - k)(n - 1) + 1, \ldots, (d + 1 - k)(n - 1) + k \rangle$. By formula (1), we have the claim. □

Now, we can compute the extraweight sequence, using formula (4). This sequence is $\langle \mathbf{e}_{d+1}, \mathbf{e}_d, \ldots, \mathbf{e}_1 \rangle$, where

$$\mathbf{e}_k = \begin{cases} \langle a_k + d_k, \ldots, a_k + kd_k \rangle & \text{if } k < d+1 \\ \langle 0, \ldots, 0 \rangle & \text{if } k = d+1 \end{cases}$$

and

$$\begin{aligned} a_k &= \frac{nd(d^2+3d+2)}{3} - 6\left[\begin{pmatrix} d+3 \\ 4 \end{pmatrix} + \begin{pmatrix} k+3 \\ 4 \end{pmatrix}\right] \\ &\quad + \frac{(k^2+k)[3(d^2+5d+6) - 2n(3d-2k+2)]}{6} \\ d_k &= 2(d+1-k)(n-1) - (d+1)(d+2) + k(k+1)\,. \end{aligned}$$

Let us observe that the last element of the extraweight sequence is $d(d + 1)(d + 2)(4n - 3d - 9)/12$, and that, if $d = n - 3$, it is $n(n - 1)(n - 2)(n - 3)/12$.

## 5. Study of a family of quintic curves

In this section, we want to study the family of the irreducible rational plane quintic curves with a 4-tuple point, with respect to the $V$-WGS at the singular point, where $V$ is the linear system of the plane conics. We shall consider five different cases, according to the tangent lines at the 4-tuple point.

At first, we choose an affine open subset of the plane, and a coordinate system whose origin coincides with the singular point of the quintic curves. Moreover, the $y$-axis of the coordinate system is always one of the tangent lines.

In this coordinate system, the family we want to study is represented by the parametric equation $a_0x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3+a_4xy^4+a_5y^5+a_6x^4+a_7x^3y+a_8x^2y^2+a_9xy^3=0$, where $\mathbf{a}=(a_0:\ldots:a_9)$ ranges in an open subset of $\mathbb{P}^9$, while the linear system $V$ has, as a general member, the curve $C_v : b_0x^2+b_1xy+b_2y^2+b_3x+b_4y+b_5=0$.

We shall report only the results obtained in each case.

*Case 1.* $a_7=a_8=a_9=0$, $a_6=1$, $a_5\neq 0$

The singular point is a cusp for every curve of the family, and hence it is an unibranch singular point.

We have that the $V$-WGS at the origin is constant for each curve of the family, and it is $\langle 1,5,6,9,10,11\rangle$, and then the extraweight sequence is $\langle 0,3,6,11,16,21\rangle$.

*Case 2.* $a_6=a_8=a_9=0$, $a_7=1$, $a_0a_5\neq 0$

There are two branches through the origin $O$: the one having the $y$-axis as tangent line has a cuspidal nature, while the one having the $x$-axis as tangent line corresponds to a simple point on the partial normalization of every curve of the family.

The partial WGS corresponding to the first branch is $\langle 1,4,5,7,8,9\rangle$, for every curve of the family, while the partial WGS corresponding to the second branch is $\langle 1,2,3,4,5,m\rangle$, where $m$ can assume the following values:

$$m=\begin{cases} 6 & \text{if } a_3\neq 0\\ 7 & \text{if } a_3=0,\ a_4\neq 0\\ 8 & \text{if } a_3=a_4=0\end{cases}$$

Then, there are three admissible $V$-WGS's at the origin, and, of course, three possible extraweight sequences, as the following shows

| $V$-WGS | Extraweights | Condition |
|---|---|---|
| $\langle 1,4,5,7,8,9\rangle$ | $\langle 0,2,4,7,10,13\rangle$ | $a_3\neq 0$ |
| $\langle 1,4,5,7,8,10\rangle$ | $\langle 0,2,4,7,10,14\rangle$ | $a_3=0,\ a_4\neq 0$ |
| $\langle 1,4,5,7,8,11\rangle$ | $\langle 0,2,4,7,10,15\rangle$ | $a_3=a_4=0$. |

*Case 3.* $a_6=a_7=a_9=0$, $a_8=1$, $a_0a_5\neq 0$

In this case, every curve of the family has two branches through the origin, each one of which has a cuspidal nature. Then, the two partial WGS are the same, and they are equal to $\langle 1,3,4,5,6,7\rangle$, and so the $V$-WGS at the origin is $\langle 1,4,5,6,7,8\rangle$, while the extraweight sequence is $\langle 0,2,4,6,8,10\rangle$, for every curve of the family.

*Case 4.* $a_6=a_9=0$, $a_7=1$

Let $P(z)=a_0z^5+a_1z^4+a_2z^3+a_3z^2+a_4z+a_5$. In this case, the irreducibility condition can be written as $a_0a_5P(-a_8)\neq 0$.

¿From the above assumptions, it follows that the origin is, for every curve of the family, a 3-branch singular point: one of them ($x = 0$) has a cuspidal nature, while the others ($y = 0, x + a_8 y = 0$) correspond to simple points on the partial normalization of the curve, not tangent to the exceptional divisor.

The partial WGS corresponding to the branch tangent to the $y$-axis is $\langle 1, 3, 4, 5, 6, 7\rangle$, no matter which curve of the family we are considering.

The partial WGS corresponding to the branch tangent to the $x$-axis depends from the considered curve of the family. In fact, the partial WGS is

$$\begin{array}{lll} \langle 1,2,3,4,5,6\rangle & \text{if} \quad (i) & a_0a_8^3 - a_1a_8^2 + a_2a_8 - a_3 \neq 0\,, \\ \langle 1,2,3,4,5,7\rangle & \text{if} \quad (ii) & a_0a_8^3 - a_1a_8^2 + a_2a_8 - a_3 = 0, a_4 \neq 0\,, \\ \langle 1,2,3,4,5,8\rangle & \text{if} \quad (iii) & a_0a_8^3 - a_1a_8^2 + a_2a_8 - a_3 = a_4 = 0\,. \end{array}$$

For the branch tangent to the line $x + a_8 y = 0$ we have the following partial WGS

$$\begin{array}{lll} \langle 1,2,3,4,5,6\rangle & \text{if} \quad (j) & a_0a_8^4 - a_4 \neq 0\,, \\ \langle 1,2,3,4,5,7\rangle & \text{if} \quad (jj) & a_0a_8^4 - a_4 = 0, a_4a_8 - a_5 \neq 0\,, \\ \langle 1,2,3,4,5,8\rangle & \text{if} \quad (jjj) & a_0a_8^4 - a_4 = a_4a_8 - a_5 = 0\,. \end{array}$$

Combining the three computed partial WGS, we obtain the $V$-WGS at the origin, that is one of the following, according to the considered curve of the family:

| $V$-WGS | Extraweights | Conditions |
|---|---|---|
| $\langle 1, 3, 4, 5, 6, 7\rangle$ | $\langle 0, 1, 2, 3, 4, 5\rangle$ | $(i), (j)$ |
| $\langle 1, 3, 4, 5, 6, 8\rangle$ | $\langle 0, 1, 2, 3, 4, 6\rangle$ | $(ii), (j)$ or $(i), (jj)$ |
| $\langle 1, 3, 4, 5, 6, 9\rangle$ | $\langle 0, 1, 2, 3, 4, 7\rangle$ | $(iii), (j)$ or $(ii), (jj)$ or $(i), (jjj)$ |
| $\langle 1, 3, 4, 5, 6, 10\rangle$ | $\langle 0, 1, 2, 3, 4, 8\rangle$ | $(iii), (jj)$ or $(ii), (jjj)$ |
| $\langle 1, 3, 4, 5, 6, 11\rangle$ | $\langle 0, 1, 2, 3, 4, 9\rangle$ | $(iii), (jjj)$ |

*Case 5.* $a_6 = 1$

Let $P(z)$ as in Case 4, and let $Q(z) = 1 + a_7 z + a_8 z^2 + a_9 z^3$. The tangent lines, except $x = 0$, have equations $y - c_i x = 0$, $i = 1, 2, 3$, where the $c_i$' s are the roots of the equation $Q(z) = 0$. Let us suppose that $Q(z)$ has only simple roots, i.e. the origin is an ordinary 4-tuple point for every curve of the family. The irreducibility condition is $a_0 P(c_1)P(c_2)P(c_3) \neq 0$.

The partial WGS corresponding to the branch through the origin tangent to the line $y - c_i x = 0$ is $\langle 1, 2, 3, 4, 5, b\rangle$, where $b$ can assume the following values:

$$b = \begin{cases} 6 & \text{if } 4P'''Q'^3 - 6P''Q''Q'^2 + 6P'Q''^2Q' - 4P'Q'''Q'^2 - 3PQ''^3 \\ & \qquad +4PQ'''Q''Q' \neq 0 \\ 7 & \text{if } 4P'''Q'^3 - 6P''Q''Q'^2 + 6P'Q''^2Q' - 4P'Q'''Q'^2 - 3PQ''^3 \\ & \qquad +4PQ'''Q''Q' = 0\,, \\ & 3P^{iv}Q'^3 - 6P''Q'''Q'^2 + 6P'Q'''Q''Q' - 3PQ'''Q''^2 \\ & \qquad +2PQ'''^2Q' \neq 0 \\ 8 & \text{if } 4P'''Q'^3 - 6P''Q''Q'^2 + 6P'Q''^2Q' - 4P'Q'''Q'^2 - 3PQ''^3 \\ & \qquad +4PQ'''Q''Q' = 0\,, \\ & 3P^{iv}Q'^3 - 6P''Q'''Q'^2 + 6P'Q'''Q''Q' - 3PQ'''Q''^2 \\ & \qquad +2PQ'''^2Q' = 0 \end{cases}$$

where all the involved polynomials are evaluated at $c_i$.

The partial WGS corresponding to the branch tangent to the line $x = 0$ is one of the following

$$\begin{array}{lll} \langle 1,2,3,4,5,6\rangle & \text{if} & a_2a_9^3 - a_5a_9^2 - a_4a_7a_9^2 + 2a_5a_7a_8a_9 - a_3a_8a_9^2 \\ & & \quad +a_4a_8^2a_9 - a_5a_8^3 \neq 0 \\ \langle 1,2,3,4,5,7\rangle & \text{if} & a_2a_9^3 - a_5a_9^2 - a_4a_7a_9^2 + 2a_5a_7a_8a_9 - a_3a_8a_9^2 \\ & & \quad +a_4a_8^2a_9 - a_5a_8^3 = 0\,, \\ & & a_1a_9^3 - a_4a_9^2 + a_5a_8a_9 - a_3a_7a_9^2 + a_4a_7a_8a_9 \\ & & \quad +a_5a_7^2a_9 - a_5a_7a_8^2 \neq 0 \\ \langle 1,2,3,4,5,8\rangle & \text{if} & a_2a_9^3 - a_5a_9^2 - a_4a_7a_9^2 + 2a_5a_7a_8a_9 - a_3a_8a_9^2 \\ & & \quad +a_4a_8^2a_9 - a_5a_8^3 = 0\,, \\ & & a_1a_9^3 - a_4a_9^2 + a_5a_8a_9 - a_3a_7a_9^2 + a_4a_7a_8a_9 \\ & & \quad +a_5a_7^2a_9 - a_5a_7a_8^2 = 0\,, \\ & & a_0a_9^3 - a_3a_9^2 + a_4a_8a_9 + a_5a_7a_9 - a_5a_8^2 \neq 0\,. \end{array}$$

Then, the $V$-WGS at the origin is $\langle 1, 2, 3, 4, 5, b\rangle$, where $b$ is an integer with $6 \leq b \leq 14$, according to the considered curve of the family.

Using the formula (4), the extraweight sequence is $\langle 0, 0, 0, 0, 0, m\rangle$, where $m$ is positive and smaller than 9, according to the $V$-WGS.

¿From these computations, we notice that no quintic curve with a 4-tuple point can have the last extraweight equal to 11, 12, or grater than 16, except 21.

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