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EXPONENTIAL STABILITY FOR PERIODIC EVOLUTION FAMILIES OF BOUNDED LINEAR OPERATORS

Abstract. We prove that a q -periodic evolution family

$$\mathcal{U} = \{U(t, s) : t \geq s \geq 0\}$$

of bounded linear operators is uniformly exponentially stable if and only if

$$\sup_{t>0} \left\| \int_0^t e^{-i\mu\xi} U(t, \xi) f(\xi) d\xi \right\| = M(\mu, f) < \infty$$

for all $\mu \in \mathbb{R}$ and $f \in P_q(\mathbb{R}_+, X)$, (that is f is a q -periodic and continuous function on \mathbb{R}_+).

Introduction

Let X be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X . We denote by $\|\cdot\|$, the norms of vectors and operators. Let $A \in \mathcal{L}(X)$ and \mathbb{R}_+ , the set of the all non-negative real numbers. It is known, see e.g. [1] that if the Cauchy Problem

$$\dot{x}(t) = Ax(t) + e^{i\mu t} x_0, \quad x(0) = 0,$$

has a bounded solution on \mathbb{R}_+ for every $\mu \in \mathbb{R}$ and any $x_0 \in X$ then the homogenous system $\dot{x} = Ax$, is uniformly exponentially stable. The hypothesis of the above result can be written in the form:

$$\sup_{t>0} \left\| \int_0^t e^{-i\mu\xi} e^{\xi A} x_0 d\xi \right\| < \infty, \quad \forall \mu \in \mathbb{R}, \forall x_0 \in X.$$

This result cannot be extended for C_0 -semigroups (cf. [14], Example 3.1). However, Neerven (cf. [11], Corollary 5) shown that if $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup on X and

$$(1) \quad \sup_{\mu \in \mathbb{R}} \sup_{t>0} \left\| \int_0^t e^{i\mu\xi} T(\xi) x_0 d\xi \right\| < \infty, \quad \forall x_0 \in X,$$

then $\omega_1(\mathbf{T}) < 0$. For details concerning $\omega_1(\mathbf{T})$, we refer to [12] or [9], Theorem A IV.1.4. Moreover, under the hypothesis (1), it results that the resolvent $R(z, A_{\mathbf{T}}) =$

$(z - A_{\mathbf{T}})^{-1}$ of the infinitesimal generator of \mathbf{T} , exists and is uniformly bounded on $\mathbf{C}_+ := \{\lambda \in \mathbf{C} : \operatorname{Re}(\lambda) > 0\}$, see [11]. Combining this with a result of Gearhart [6], (see also Huang [7], Weiss [15] or Pandolfi [13] for other proofs and generalizations), it results that if X is a complex Hilbert space and (1) holds, then \mathbf{T} is uniformly exponentially stable, i.e. its growth bound $\omega_0(\mathbf{T})$ is negative. A similar problem for q -evolution families of bounded linear operators seems to be an open question. In the general case, when X is a Banach space the last result is not true, see e.g. [2], Example 2. However, a weakly result, announced before, holds.

1. Definitions. Preliminary results

Let $q > 0$ and $\Delta = \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\}$. A mapping $\mathcal{U} : \Delta \rightarrow \mathcal{L}(X)$ would be called q -periodic evolution family of bounded linear operators on X , iff:

- (i) $U(t, s) = U(t, r)U(r, s)$ for all $t \geq s \geq r \geq 0$;
- (ii) $U(t, t) = Id$, (Id is the identity on X), for all $t \geq 0$;
- (iii) for all $x \in X$, the map $(t, s) \mapsto U(t, s)x : \Delta \rightarrow X$, is continuous;
- (iv) $U(t+q, s+q) = U(t, s)$ for all $t \geq s \geq 0$.

The operator $\mathcal{U}(t, s)$ was denoted by $U(t, s)$.

If A is a linear operator on X , $\sigma(A)$ will denote the *spectrum* of A , and if $T \in \mathcal{L}(X)$, $r(T)$ will denote the *spectral radius* of T .

The following two lemmas, which would be used later, are essentially known (see [4], Ch.V, Theorem 1.1, Corollary 1.1 or [5], Theorem 6.6).

LEMMA 1. *A q -periodic evolution family \mathcal{U} on X has exponential growth, that is, there exist $\omega \in \mathbb{R}$ and $M > 1$ such that*

$$(2) \quad \|U(t, s)\| \leq Me^{\omega(t-s)} \quad \forall t \geq s \geq 0.$$

We recall that the evolution family \mathcal{U} is called *exponentially stable* if there are $\omega < 0$ and $M > 1$ such that (2) holds. Let $V = U(q, 0) \in \mathcal{L}(X)$.

LEMMA 2. *A q -periodic evolution family \mathcal{U} is exponentially stable if and only if $r(V) < 1$.*

For the proofs of these lemmas we refer to [3].

Let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on X and $A_{\mathbf{T}}$ its infinitesimal generator. In [14], Proposition 3.3, it is shown that if

$$\sup_{t > 0} \left\| \int_0^t e^{i\mu\xi} T(\xi) d\xi \right\| < \infty, \quad \forall x \in X, \forall \mu \in \mathbb{R}$$

then

$$\sigma(A_{\mathbf{T}}) \subset \mathbf{C}_- := \{z \in \mathbf{C} : \operatorname{Re}(z) < 0\}.$$

The discret version of this result is the following:

LEMMA 3. Let $T \in \mathcal{L}(X)$. If

$$\sup_{n \in \mathbf{N}} \left\| \sum_{k=0}^n e^{i\mu k} T^k \right\| = M_\mu < \infty \quad \forall \mu \in \mathbb{R},$$

then $r(T) < 1$.

We mention that the result in Lemma 3 is also known and is, for instance, consequence of the uniform ergodic theorem ([8], Theorems 2.1 and 2.7). For reasons of self-containedness we give the proof of Lemma 3 in detail.

Proof. We will use the identity:

$$(3) \quad \sum_{k=0}^n e^{i\mu k} T^k (e^{i\mu} T - Id) = e^{i\mu(n+1)} T^{n+1} - Id.$$

From (3) it follows:

$$(4) \quad \|e^{i\mu(n+1)} T^{n+1}\| \leq 1 + M_\mu(1 + \|T\|) \quad \forall n \in \mathbf{N},$$

that is $r(T) \leq 1$. Suppose that $1 \in \sigma(T)$. Then for all $m = 1, 2, \dots$, there exists $x_m \in X$ with $\|x_m\| = 1$ and $(Id - T)x_m \rightarrow 0$ as $m \rightarrow \infty$, (see [9], Proposition 2.2, p. 64). From (4) it results that $T^k(Id - T)x_m \rightarrow 0$ as $m \rightarrow \infty$, uniformly for $k \in \mathbf{N}$. Let $N \in \mathbf{N}$, $N > 2M_0$ and $m \in \mathbf{N}$ such that

$$\|T^k(Id - T)x_m\| \leq \frac{1}{2N}, \quad k = 0, 1, \dots, N.$$

Then

$$\begin{aligned} M_0 &\geq \|x_m + \sum_{k=1}^N (x_m + \sum_{j=0}^{k-1} T^j(T - Id)x_m)\| \\ &= \|(N+1)x_m + \sum_{k=1}^N \sum_{j=0}^{k-1} T^j(T - Id)x_m\| \\ &\geq (N+1) - \frac{N(N+1)}{4N} > \frac{N}{2} > M_0. \end{aligned}$$

This contradiction concludes that $1 \notin \sigma(T)$. Now, it is easy to show that $e^{i\mu} \notin \sigma(T)$ for $\mu \in \mathbb{R}$, that is, $r(T) < 1$. □

2. Uniform exponential stability

Let us consider the following spaces:

- $BUC(\mathbf{I}, X)$, $\mathbf{I} \in \{\mathbb{R}, \mathbb{R}_+\}$ is the Banach space of all X -valued bounded uniformly continuous functions on \mathbf{I} , with the sup-norm.
- $AP(\mathbf{I}, X)$ is the linear closed hull in $BUC(\mathbf{I}, X)$ of the set of all functions

$$t \mapsto e^{i\mu t} x : \mathbf{I} \rightarrow X, \quad \mu \in \mathbf{R}, \quad x \in X.$$

- $P_q(\mathbf{I}, X)$ is the set of all continuous functions $f : \mathbf{I} \rightarrow X$ such that $f(t+q) = f(t)$, for any $t \in \mathbf{I}$ and some $q > 0$.

THEOREM 1. Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be a q -periodic evolution family on the Banach space X . If

$$(5) \quad \sup_{t > 0} \left\| \int_0^t e^{-i\mu\xi} U(t, \xi) f(\xi) d\xi \right\| < \infty, \quad \forall \mu \in \mathbb{R}, \forall f \in P_q(\mathbb{R}_+, X),$$

then \mathcal{U} is exponentially stable.

Proof. Let $V = U(q, 0)$, $x \in X$, $n = 0, 1, \dots$ and $g \in P_q(\mathbb{R}_+, X)$, such that

$$g(\xi) = \xi(q - \xi)U(\xi, 0)x, \quad \forall \xi \in [0, q].$$

From (5), for $t = (n + 1)q$, we obtain:

$$(6) \quad \sup_{n \in \mathbb{N}} \left\| \sum_{k=0}^n \int_{kq}^{(k+1)q} U((n+1)q, \xi) e^{-i\mu\xi} g(\xi) d\xi \right\| < \infty, \quad \forall \mu \in \mathbb{R}.$$

In the view of definition of q -periodic evolution family **(iv)**, it follows:

$$U(pq + q, pq + u) = U(q, u), \quad \forall p \in \mathbb{N}, \quad \forall u \in [0, q]$$

and

$$U(pq, jq) = U((p - j)q, 0) = V^{p-j}, \quad \forall p \in \mathbb{N}, \forall j \in \mathbb{N}, \quad p \geq j.$$

Now, for every $k = 0, 1, \dots$, we have:

$$\begin{aligned} & \int_{kq}^{(k+1)q} U((n+1)q, \xi) e^{-i\mu\xi} g(\xi) d\xi = \\ &= \int_{kq}^{(k+1)q} U((n+1)q, (k+1)q) U((k+1)q, \xi) e^{-i\mu\xi} g(\xi) d\xi \\ &= V^{n-k} \int_0^q U((k+1)q, u + kq) e^{-i\mu(u+kq)} g(kq + u) du \\ &= e^{-i\mu kq} V^{n-k} \int_0^q e^{-i\mu u} U(q, u) g(u) du \\ &= e^{-i\mu kq} V^{n-k} \int_0^q e^{-i\mu u} u(q - u) U(q, u) U(u, 0) x du \\ &= e^{-i\mu kq} \left(\int_0^q e^{-i\mu u} u(q - u) du \right) V^{n-k+1} x \\ &= M(\mu, q) e^{-i\mu(n+1)q} e^{i\mu(n-k+1)q} V^{n-k+1} x, \end{aligned}$$

where

$$M(\mu, q) = \int_0^q u(q - u) e^{-i\mu u} du \neq 0.$$

We return in (6) and obtain

$$\sup_{n \in \mathbb{N}} \left\| \sum_{j=0}^{n+1} e^{i\mu jq} V^j \right\| < \infty,$$

that is, $r(V) < 1$ and \mathcal{U} is exponentially stable. \square

REMARK 1. It is clear that the converse statement from Theorem 1 is also true. Moreover if we denote by $P_2^0(\mathbb{R}_+, X)$ the set of all functions $f \in P_2(\mathbb{R}_+, X)$ for which $f(0) = 0$, then (5) holds (with $P_2(\mathbb{R}_+, X)$ replaced by $P_2^0(\mathbb{R}_+, X)$ if and only if the family \mathcal{U} is exponentially stable).

COROLLARY 1. *A q -periodic evolution family \mathcal{U} on X is uniformly exponentially stable if and only if*

$$\sup_{t>0} \left\| \int_0^t U(t, \xi) f(\xi) d\xi \right\| < \infty, \quad \forall f \in AP(\mathbb{R}_+, X).$$

For the other proofs of Corollary 1, see e.g. [2] and [14]. In the end we give a result about evolution families on the line. In this context,

$$\mathcal{U} = \{U(t, s) : t \geq s \in \mathbb{R}\}$$

will be a q -periodic evolution family on \mathbb{R} . We shall use the same notations as in Section 1, with \mathbb{R}_+ replaced by \mathbb{R} and variables such as s and t taking any value in \mathbb{R} . Let us consider the evolution semigroup \mathbf{T}_{ap} associated to \mathcal{U} on the space $AP(\mathbb{R}, X)$. This semigroup is strongly continuous, see Naito and Minh ([10], Lemma 2).

COROLLARY 2. *Let $\mathcal{U} = \{U(t, s), t \geq s\}$ be a q -periodic evolution family of bounded linear operators on X and \mathbf{T}_{ap} the evolution semigroup associated to \mathcal{U} on the space $AP(\mathbb{R}, X)$. Then \mathcal{U} is uniformly exponentially stable if and only if*

$$\sup_{t \geq 0} \left\| \left(\int_0^t e^{i\mu\xi} T_{ap}(\xi) f d\xi \right)(t) \right\| < \infty \quad \forall \mu \in \mathbb{R}, \quad \forall f \in P_q(\mathbb{R}_+, X).$$

Proof. For $t > 0$, we have

$$\begin{aligned} \left(\int_0^t e^{i\mu\xi} T_{ap}(\xi) f d\xi \right)(t) &= \int_0^t e^{i\mu\xi} U(t, t - \xi) f(t - \xi) d\xi \\ &= e^{i\mu t} \int_0^t e^{-i\mu\tau} U(t, \tau) f(\tau) d\tau. \end{aligned}$$

Now, from Theorem 1, it follows that the restriction \mathcal{U}_0 of \mathcal{U} to the set $\{(t, s) : t \geq s \geq 0\}$ is uniformly exponentially stable. Let $N > 0$ and $\nu > 0$ such that

$$\|U(t, s)\| \leq N e^{-\nu(t-s)}, \quad \forall t \geq s \geq 0.$$

Then for all real numbers u and v with $u \geq v$, we have

$$\|U(u, v)\| = \|U(u + nq, v + nq)\| \leq N e^{-\nu(u-v)},$$

where $n \in \mathbf{N}$ is such that $v + nq \geq 0$, that is, \mathcal{U} is uniformly exponentially stable. \square

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