ANALYTIC MANIFOLDS OF NONPOSITIVE CURVATURE

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Abstract. In this article we construct compact, real analytic Riemannian manifolds of nonpositive sectional curvature which have geometric rank one, but which contain a rich structure of totally geodesic subspaces of higher rank. Topologically the manifolds are obtained by blowing up certain, pairwise intersecting, codimension 2 submanifolds of a hyperbolic manifold. The metric on this blow–up is constructed explicitly by means of some Poincaré series, and appropriate methods for controlling its curvature and its rank are developed.

Résumé. Dans cet article sont construites des variétés riemanniennes analytiques compactes à courbure sectionnelle non-positive de rang géométrique un ayant une structure riche de sous-variétés totalement géodésiques de rangs plus élevés. Topologiquement ces variétés sont obtenues en éclatant certaines sous-variétés de codimension 2 d'une variété hyperbolique se coupant deux à deux. La métrique sur cet espace éclaté est construite explicitement grâce à des séries de Poincaré et des méthodes appropriées pour contrôler sa courbure et son rang sont développées.

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1. INTRODUCTION

In this paper we construct new examples of compact real analytic Riemannian manifolds of nonpositive sectional curvature. The main result is

1.1. Theorem. — *Let $\mathbb{H}^n/\Gamma'$ be a compact manifold with constant curvature $K \equiv -1$ and let $\bar{\varrho}_i \in \mathrm{Iso}(\mathbb{H}^n/\Gamma')$, $1 \le i \le N$, be a family of rotations with fixed point sets*

$$\bar{V}_i := \mathrm{Fix}(\bar{\varrho}_i) = \big\{p \in \mathbb{H}^n/\Gamma' \;\big|\; \bar{\varrho}_i(p) = p\big\}$$

of codimension 2. Suppose that each $\bar{\varrho}_i$ permutes[1] *the N fixed point sets $\bar{V}_{i'}$. Moreover, for any pair of distinct fixed point sets $\bar{V}_{i_1}$ and $\bar{V}_{i_2}$ with $\bar{V}_{i_1} \cap \bar{V}_{i_2} \neq \emptyset$, it is required that $\bar{V}_{i_1} \cap \bar{V}_{i_2}$ has codimension 4 and that the intersection is orthogonal. Let $\pi\colon M \to \mathbb{H}^n/\Gamma'$ be the manifold obtained by blowing up $\bigcup_i \bar{V}_i$.*
Then, M carries a real analytic Riemannian metric g with sectional curvature $K \le 0$ everywhere and with $K < 0$ on the complement of $\pi^{-1}\big(\bigcup_{i=1}^N \bar{V}_i\big)$. The preimages $\hat{V}_i := \pi^{-1}(\bar{V}_i)$ and all their intersections $\hat{V}_I := \bigcap_{i\in I} \hat{V}_i$, $I \subset \{1,\dots,N\}$ are totally geodesic submanifolds of (M,g). Each projection $\pi_{(I)} := \pi_{|\hat{V}_I}$ factors through a Riemannian submersion $\hat{\pi}_{(I)}\colon \hat{V}_I \to \bar{V}_I^\star$ onto a space $\bar{V}_I^\star$ of nonpositive curvature. This submersion is a flat bundle over $\bar{V}_I^\star$ with totally geodesic fibres which are isometric to $\#\,I$–fold products of $\mathbb{RP}^1$'s of equal lengths.

The metric g will be constructed explicitly by means of a Poincaré series. For any subset $I \subset \{1,\dots N\}$ the holonomy of the flat bundle $\hat{\pi}_{(I)}\colon \hat{V}_I \to \bar{V}_I^*$ is determined by the holonomy of the normal bundle of $\bar{V}_I := \bigcap_{i\in I} \bar{V}_i \subset \mathbb{H}^n/\Gamma'$. Moreover, the existence of a single nonempty, totally geodesic submanifold $\hat{V}_i \subset M$ implies that the

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[1] W.l.o.g. we may assume that each $\bar{\varrho}_i$ generates the maximal cyclic subgroup in $\mathrm{Iso}_0(\mathbb{H}^n/\Gamma')$ fixing $\bar{V}_i$. With this normalisation it is equivalent to require that the family $(\langle\bar{\varrho}_i\rangle)_{i=1}^N$ is closed under conjugation : for any pair (i_1, i_2) there exists i_3 such that $\bar{\varrho}_{i_1}\langle\bar{\varrho}_{i_2}\rangle\bar{\varrho}_{i_1}^{-1} = \langle\bar{\varrho}_{i_3}\rangle$.

fundamental group $\pi_1(M)$ is not hyperbolic in the sense of [GhH] and [Gr2]. However $\operatorname{rank}(\pi_1(M)) = 1$, where the rank of a finitely generated group Γ is defined in terms of the word metric d_Γ as follows (see [BE])

$$\operatorname{rank}(\Gamma) \geq k \ :\Leftrightarrow\ \exists C > 0 \ \forall \gamma \in \Gamma \ \exists \text{ a subgroup } A_\gamma \simeq \mathbb{Z}^k \text{with } d_\Gamma(\gamma, A_\gamma) \leq C.$$

Looking at the precise estimates for the curvature in Theorem 5.9 one can see that the metric g has *as little zero curvature as permitted by the fundamental group.* We shall explain this in more detail in Section 7.

To show that the hypotheses of Theorem 1.1 are not void, we quote from [AbSch] :

1.2. Theorem. — *Let Γ' be a torsion–free, normal subgroup of finite index in some cocompact, discrete group $\Gamma \subset \operatorname{Iso}(\mathbb{H}^n) = O^+(n,1)$. Suppose in addition that Γ contains commuting isometries $\varrho_1, \ldots, \varrho_k$, whose fixed point sets are hyperbolic subspaces of codimension 2. If at most one of the ϱ_i's has order 2, then the induced rotations $\bar{\varrho}_i$ on $\mathbb{H}^n/\Gamma'$ satisfy the hypotheses of Theorem 1.1.*

In particular, there are *concrete examples*[2] of such groups $\Gamma' < \Gamma < \operatorname{Iso}(\mathbb{H}^n)$ and of rotations $\varrho_1, \ldots, \varrho_k$ of this type with $n = 2k$. In this case the flat $\hat{V}_1 \cap \ldots \cap \hat{V}_k \subset M$ has the *maximal possible dimension* in view of the following general result proved at the end of Section 2.

1.3. Theorem. — *Let X^n be a simply–connected, real analytic Riemannian manifold with $K \leq 0$, and let $F^k \subset X^n$ be a k–flat of maximal dimension. Moreover, let $\Sigma_1, \ldots, \Sigma_m \subset F^k$ be different singular hyperplanes through a common point p, where singular means that the set P_{Σ_i} of parallels to Σ_i is not contained in the flat F^k. Then,*

$$k + \sum_{i=1}^{m} \bigl(\dim P_{\Sigma_i} - k\bigr) \ \leq \ n \ . \tag{1.1}$$

Since $\dim P_{\Sigma_i} > k$, the number m of singular hyperplanes is estimated by the codimension $n-k$ of the flat F^k. In our example the strata $\hat{V}_{i_1} \cap \ldots \cap \hat{V}_{i_{k-1}}$ are

[2] In [Buy] S. Buyalo has used a slightly different approach to construct an interesting configuration of compact, codimension-2 subspaces $\bar{V}_i^2$ in the hyperbolic 120–cell space $\mathbb{H}^4/\Gamma'$ with the intersection pattern required for Theorem 1.1.

parallel sets of k different singular hyperplanes $\Sigma_{i_k} \subset \hat{V}_1 \cap \ldots \cap \hat{V}_k$. Thus Inequality (1.1) is sharp in this example.

The Weyl chamber structure of the flat $\hat{V}_1 \cap \ldots \cap \hat{V}_k$ is the same as the structure of a flat in the k–fold product $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$. An interesting open question is whether there are also real analytic manifolds of rank 1 with a maximal flat which has the Weyl chamber structure of the flat in an irreducible symmetric space.

We emphasize that the *crucial point* in Theorem 1.1 is the existence of a *real analytic* metric of nonpositive curvature on M. Indeed, it is much easier to obtain a C^∞–metric with $K \leq 0$ on M even *without assuming* that the codimension–2 submanifolds are fixed point sets of isometries. For completeness we state

1.4. Theorem. — *Let* $\big(\bar{V}_i\big)_{i=1}^N$ *be a finite family of compact, totally geodesically immersed submanifolds of codimension* 2 *in some compact hyperbolic space* $\mathbb{H}^n/\Gamma'$. *Suppose that the various sheets of* $\bigcup_i \bar{V}_i$ *intersect pairwise orthogonally in sets of codimension* 4, *if they intersect at all. Then, the blow–up* $\pi\colon M^n \to \mathbb{H}^n/\Gamma'$ *of* $\bigcup_i \bar{V}_i$ *carries a smooth metric with sectional curvature* $K \leq 0$.

The proof of Theorem 1.1 occupies Sections 3–6. The metric g in question is constructed explicitly in Theorem 3.7 and the relevant curvature estimates are the subject of Theorem 5.9.

The proof of Theorem 1.4 is much simpler, since all constructions can be done just locally. One could even give an independent proof based on a multiple warped product structure in the sense of [ONl, p. 210, Theorem 42].

2. REAL ANALYTIC MANIFOLDS OF NONPOSITIVE CURVATURE

The fundamental differences between C^∞– and C^ω–functions actually leads to substantially different phenomena in the theory of manifolds of nonpositive sectional curvature in these two categories. For instance, the graph manifolds constitute a large class of manifolds M with a non–hyperbolic, rank 1 fundamental group which carry a C^∞–smooth but no analytic metric with $K \leq 0$.

In fact, the existence of an analytic metric with $K \leq 0$ on a non–hyperbolic rank 1 manifold M has much stronger consequences for the topology of M than the existence of a C^∞-smooth metric of $K \leq 0$. We illustrate this by the following three points :

(1) if M is compact, real analytic with $K \leq 0$, and $A < \pi_1(M)$ is an abelian subgroup (i.e. $A \simeq \mathbb{Z}^k$ for some $k \in \mathbb{N}$), then the centralizer $Z(A)$ is the fundamental group of a closed manifold with $K \leq 0$; in particular the homology of $Z(A)$ satisfies the Poincaré duality [BGS, p. 121]. This is a strong restriction on $\pi_1(M)$ and rules out the existence of analytic metrics with $K \leq 0$ on many manifolds obtained by cut and paste methods like graph manifolds ;

(2) if $-b^2 \leq K \leq 0$, $\mathrm{vol}(M) < \infty$, and M is real analytic, then M is diffeomorphic to the interior of a compact manifold with boundary. This result is contained in Gromov's finiteness theorem [BGS]. For C^∞–manifolds the topology may be unbounded. In [Gr1] Gromov constructs graph manifolds with $-1 \leq K \leq 0, \quad \mathrm{vol}(M) < \infty$ and infinitely generated homology ;

(3) if M is a compact analytic manifold with $K \leq 0$ whose fundamental group is not hyperbolic, then $\pi_1(M)$ contains a subgroup isomorphic to $\mathbb{Z}^2$. This

follows from the closing theorem of flat subspaces [BaSch]. The analogous question in the C^∞–category is very much open.

The facts above indicate that it is difficult to construct real analytic non–hyperbolic manifolds of rank 1 with $K \leq 0$. To our knowledge there are only three types of examples described in the literature :

(i) (*doubling at a cusp*[3]) take a complete manifold W^n with constant curvature -1 and finite volume with one cusp diffeomorphic to $\mathbb{T}^{n-1} \times [0, \infty)$. Glue two copies of W along the cusp to obtain a compact manifold with a joining cylinder $\mathbb{T}^{n-1} \times (-a, a)$. For a suitable smooth warped product metric, all curvatures are negative except that $\mathbb{T}^{n-1} \times \{0\}$ is a totally geodesic, flat torus ;

(ii) (*cusp closing* [Sch1]) start as in Example (i) by W^n with cusp $\mathbb{T}^{n-1} \times [0, \infty)$ and close the cusp with $\mathbb{T}^{n-2} \times$ *disc*. For a suitable metric all curvatures are negative except that $\mathbb{T}^{n-2} \times \{0\}$ is a totally geodesic, flat torus. The closing of complex hyperbolic cusps has been studied recently in [HuSch].

(iii) (*codimension–*2 *surgery* [Sch2]) consider a compact manifold V^n of constant curvature -1 with a totally geodesic submanifold $V^{n-2} \subset V^n$. Take two copies of $V^n \backslash V^{n-2}$ and glue them together to obtain a compact manifold with joining cylinder $V^{n-2} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$. For a suitable warped product metric all curvatures are negative except that $V^{n-2} \times \mathbb{S}^1 \times \{0\}$ is totally geodesic and isometric to a product.

In these examples one constructs first a C^∞-smooth metric which is analytic in the neighborhood of the submanifold where all the zero curvatures are concentrated. Using an argument from sheaf theory [BuGe], one then gets an approximating analytic metric with $K \leq 0$ in each case.

Our main result generalizes the examples obtained by codimension–2 surgery. However, the examples in Theorem 1.1 are constructed in an entirely explicit fashion. We obtain analytic data using a Poincaré series rather than the full machinery of sheaf theory. The price for the explicit approach are the symmetry requirements as explained in Remark 6.4 (iii).

[3] Due to E. Heintze, unpublished.

We conclude this section with a *proof of Theorem* 1.3. *We actually prove a more general statement* including the case that the flat F is not necessarily maximal.

Let X^n be an n–dimensional, complete, simply–connected, real analytic Riemannian manifold with $K \leq 0$. A *k–flat in X* is a totally geodesic, isometric immersion $F\colon \mathbb{R}^k \to X$. We denote by $Gr_k(X) \to X$ the Grassmann bundle of k–planes in TX and by $F_k(X) \subset Gr_k(X)$ the subset of all $\tau \in Gr_k(X)$ such that $\exp\colon \tau \to X$ is a k–flat. We call $\tau, \tau' \in F_k(X)$ *parallel* and write $\tau \parallel \tau'$, if the subsets $\exp(\tau)$ and $\exp(\tau')$ have finite Hausdorff distance $\varrho_{\tau\tau'}$. By the Sandwich Lemma $\exp(\tau)$ and $\exp(\tau')$ bound a convex subset isometric to $\exp(\tau) \times [0, \varrho_{\tau\tau'}]$. More generally, we define

$$P_\tau^{Gr} \;:=\; \left\{\tau' \in F_k(X) \;\middle|\; \tau' \parallel \tau\right\} .$$

It is well known [BGS, Lemma 2.4], that the image P_τ of P_τ^{Gr} under the standard projection $Gr_k(X) \to X$ is a convex subset which splits isometrically as a product $P_\tau = \mathbb{R}^k \times Q$, where Q is a convex subset. Since the metric is assumed to be real analytic, Q is complete and P_τ is a complete, totally geodesic submanifold of X. We define

$$\operatorname{rank}_P(\tau) \;:=\; \dim P_\tau \;=\; \dim P_\tau^{Gr} .$$

Let us now fix a not necessarily maximal flat $\Sigma = \exp(\sigma)$ with $\sigma \in F_k(X)$. For a linear subspace $\tau \subset \sigma$ we obviously have $P_\sigma \subset P_\tau$. Such a τ is called a *singular subspace of* σ, if P_σ is a proper subset of P_τ, or equivalently, if

$$\operatorname{rank}_P(\tau) \;>\; \operatorname{rank}_P(\sigma) .$$

2.1. Theorem. — *Let $\sigma \in F_k(X)$ and let $\tau_1, \dots, \tau_q$ be different maximal singular subspaces of σ. Then,*

$$\operatorname{rank}_P(\sigma) \;+\; \sum_{i=1}^{q} \bigl(\operatorname{rank}_P(\tau_i) - \operatorname{rank}_P(\sigma)\bigr) \;\leq\; n . \tag{2.1}$$

If $\Sigma = \exp(\sigma)$ is a maximal flat in a symmetric space, then the maximal singular subspaces of σ are precisely those hyperplanes which constitute the walls of the Weyl chambers of σ. Using the root space decomposition of the Lie algebra, it is not hard to see that *Inequality* (2.1) *is optimal for symmetric spaces.*

We first prove the following

2.2. Lemma. — *Let $\sigma \in F_k(X)$ and let τ_1, τ_2 be linear subspaces of σ. Then,*

$$P_{\tau_1} \cap P_{\tau_2} \;=\; P_{\tau_1+\tau_2} \tag{2.2}$$

where $\tau_1 + \tau_2$ denotes the span of τ_1 and τ_2.

Proof. The inclusion $P_{\tau_1+\tau_2} \subset P_{\tau_1} \cap P_{\tau_2}$ is evident. To show the opposite inclusion, we pick a point $x \in P_{\tau_1} \cap P_{\tau_2}$ and consider the linear subspaces $\tau_i(x) \subset T_xX$ parallel to τ_i, $i = 1, 2$. Let $\big(S_{\tau_1+\tau_2}, \sphericalangle\big)$ be the unit sphere in the space $\tau_1 + \tau_2 \subset \sigma$ together with the canonical angular distance function $\sphericalangle$. For any $v \in TX$ let $c_v\colon \mathbb{R} \to X$ be the geodesic with $\dot c_v(0) = v$ .We define a map

$$\varphi_x\colon S_{\tau_1+\tau_2} \to T^1_xX$$

into the unit sphere $T^1_xX \subset T_xX$ such that $\varphi_x(v)$ is the unique vector $\bar v \in T^1_xX$ with $c_{\bar v}(\infty) = c_v(\infty)$. We claim that this map φ_x is contracting

$$\sphericalangle_x\big(\varphi_x(v)\,, \varphi_x(w)\big) \;\le\; \sphericalangle\big(v\,, w\big)\ . \tag{2.3}$$

Here, $\sphericalangle_x$ is the angle measured in T^1_xX. Since $\exp\big(\tau_1(x) + \tau_2(x)\big)$ is a flat, we have $\mathrm{Td}\big(c_v(\infty), c_w(\infty)\big) \;=\; \sphericalangle(v, w)$ for all $v, w \in \big(\tau_1(x) + \tau_2(x)\big) \cap S_{\tau_1+\tau_2}$, where Td is the Tits–distance on $X(\infty)$ defined in [BGS]. By the well–known properties of the Tits–distance we have $\sphericalangle_x(\bar v, \bar w) \le \mathrm{Td}\big(c_{\bar v}(\infty), c_{\bar w}(\infty)\big)$, hence inequality (2.3).

Consider any vector $v \in \tau_1 \cap S_{\tau_1+\tau_2}$. Since $\tau_1(x) \parallel \tau_1$, there exists some $\bar v \in \tau_1(x)$ such that the geodesic $t \mapsto \exp(tv)$ is parallel to $t \mapsto \exp(t\bar v)$. This implies that $\varphi_x(-v) = -\varphi_x(v)$. The same observation holds for any $v \in \tau_2 \cap S_{\tau_1+\tau_2}$, and hence $\varphi_x(-v) = -\varphi_x(v)$ for any $v \in (\tau_1 \cup \tau_2) \cap S_{\tau_1+\tau_2}$.

Now, an elementary argument[4] based on this symmetry property and on the contracting property established before reveals that φ_x *is an isometric embedding of*

[4] cf. the proof of the Sublemma in [BGS, p. 230].

$S_{\tau_1+\tau_2}$ onto a great sphere in $T^1_x X$. Thus $\varphi_x(S_{\tau_1+\tau_2}) = \big(\tau_1(x)+\tau_2(x)\big) \cap T^1_x X$, and the proof of Lemma E in [BGS, p. 229] implies that $\exp\big(\tau_1(x)+\tau_2(x)\big)$ is a flat which is parallel to $\exp(\tau_1+\tau_2)$. □

2.3. Lemma (cf. [BaSch, Lemma 2.1]). — *The spaces P_{τ_1} and P_{τ_2} are orthogonal in the sense that*

$$\pi_{P_{\tau_1}}(P_{\tau_2}) \;=\; P_{\tau_1} \cap P_{\tau_2} \;=\; \pi_{P_{\tau_2}}(P_{\tau_1}) \tag{2.4}$$

where $\pi_{P_{\tau_i}}$ denotes the orthogonal projection onto the convex subset P_{τ_i}.

Proof of Theorem 2.1. Let τ_i, τ_j be different maximal singular subspaces of σ. Then, $P_{\tau_i} \cap P_{\tau_j} = P_{\tau_i+\tau_j}$ by Lemma 2.2. Since τ_i and τ_j are maximal singular subspaces of σ, we have $P_{\tau_i+\tau_j} = P_\sigma$. We pick a point $x \in P_\sigma$ and consider the normal space $\nu_x P_\sigma \subset T_x X$ of P_σ in x . Since $P_{\tau_i+\tau_j} = P_\sigma$, the subspaces $T_x(P_{\tau_i}) \cap \nu_x P_\sigma$, $1 \le i \le k$, have pairwise trivial intersection, and by Lemma 2.3 they are pairwise orthogonal. These facts imply Inequality (2.1), once we observe that

$$\dim\big(T_x(P_{\tau_i})\big) \;=\; \mathrm{rank}_P(\tau_i) - \mathrm{rank}_P(\sigma)\ .$$ □

2.4. Remarks.

(i) The analyticity of the metric is neither required for the proof of Lemma 2.2 nor for the proof of Lemma 2.3. It is only needed in order to guarantee the completeness of the sets P_{τ_i} and P_σ.

(ii) It is not difficult to construct for every $k \in \mathbb{N}$ a 4–dimensional manifold X^4_k with a C^∞–metric of non-positive sectional curvature which is not identically flat. Nevertheless, X^4_k contains a 2–flat $\Sigma = \exp(\sigma)$ which comes with 1–dimensional subspaces $\tau_1, \ldots, \tau_k \subset \sigma$ such that the geodesics $\exp(\tau_i)$ are contained in some 2–flat F_i with $F_i \cap \Sigma = \exp(\tau_i)$. In this case an open neighborhood of Σ is flat.

3. THE BLOW–UP $\pi: M \to \mathbb{H}^n/\Gamma'$

In this section we describe the blow–up $\pi: M \to \mathbb{H}^n/\Gamma'$ and the new metric on M. Our assumption is that we have given a family $\big(\bar V_i\big)_{i=1}^N$ of compact, totally geodesically embedded submanifolds of codimension 2 in a compact hyperbolic space $\mathbb{H}^n/\Gamma'$. The various sheets of $\bigcup_i \bar V_i$ intersect pairwise orthogonally in sets of codimension ≥ 4. We shall work in the universal covering $\mathrm{pr}: \mathbb{H}^n \to \mathbb{H}^n/\Gamma'$. The preimage of $\bigcup_i \bar V_i$ is a divisor in $\mathbb{H}^n$ whose trace is a countable union of hyperbolic subspaces $\mathbb{H}^{n-2}_j$, $j \in J$, of codimension 2.

The collection $\big(\mathbb{H}^{n-2}_j\big)_{j\in J}$ satisfies

3.1. *Axiom.* — There exists a constant $d_0 > 0$ with the following properties :

(i) the index set J decomposes into $\hat N$ subsets $J_1 \cup \ldots \cup J_{\hat N}$ such that for all pairs $(j_1, j_2) \in J_\mu \times J_\mu$, $1 \leq \mu \leq \hat N$, with $j_1 \neq j_2$ one has

$$\mathrm{dist}\big(\mathbb{H}^{n-2}_{j_1}, \mathbb{H}^{n-2}_{j_2}\big) \geq 2\, d_0 \ ;$$

(ii) for any point $p \in \mathbb{H}^n$ there exists some point $q \in \mathbb{H}^n$ such that the subspaces $\mathbb{H}^{n-2}_j$ with $\mathrm{dist}(p, \mathbb{H}^{n-2}_j) < d_0$ contain q and intersect pairwise orthogonally in subspaces of codimension 4.

This axiom describes *all the properties* that we assume for the collection $\mathbb{H}^{n-2}_j$ throughout this section and the next one, where we construct the analytic metric on the blow–up $\pi: M \to \mathbb{H}^n/\Gamma'$, as well as throughout the bulk of Section 5, where the basic curvature computations are done.

As a consequence of Axiom 3.1 a standard packing argument implies the following.

3.2. Lemma. — *There exists a constant $C > 0$ such that for every $p \in \mathbb{H}^n$*

$$\#\big\{j \in J \;\big|\; \mathrm{dist}\big(p, \mathbb{H}^{n-2}_j\big) \leq r\big\} \ \leq\ C\, \mathrm{e}^{(n-1)r} \ .$$

The blow–up $\pi\colon \hat{M}^n \to \mathbb{H}^n$ along the divisor $\bigcup_{j\in J} \mathbb{H}^{n-2}_j$ is invariant under the group $\Gamma := \{\gamma \in \mathrm{Iso}(\mathbb{H}^n) \mid \gamma(\bigcup_{j\in J}\mathbb{H}^{n-2}_j) = \bigcup_{j\in J}\mathbb{H}^{n-2}_j\}$. This means that for all $\gamma \in \Gamma$ the diagram

$$
\begin{array}{ccc}
\hat{M}^n & \xrightarrow{\gamma} & \hat{M}^n \\
\pi\downarrow & & \downarrow\pi \\
\mathbb{H}^n & \xrightarrow{\gamma} & \mathbb{H}^n
\end{array}
\tag{3.1}
$$

commutes, and the manifold $\hat{M}^n$ is a covering of M^n with deck-transformations in the subgroup $\Gamma' < \Gamma$.

For a more detailed description of $\hat{M}^n$ let us introduce the distance function

$$r_j := \mathrm{dist}\big(\,.\,, \mathbb{H}^{n-2}_j\big)$$

and the one parameter group

$$\vartheta_j\colon \mathbb{R}/2\pi\mathbb{Z} \to \mathrm{Iso}(\mathbb{H}^n)$$

of rotations around $\mathbb{H}^{n-2}_j$. The corresponding Killing field will be denoted by K_j. For every $j \in J$ we choose a hyperplane $W_j \subset \mathbb{H}^n$ containing $\mathbb{H}^{n-2}_j$.

For any (possibly empty) subset $I \subset J$ we consider the sets

$$
\begin{aligned}
U_I \quad &:= \quad \left\{ p \in \mathbb{H}^n \;\middle|\; \begin{array}{ll} r_i(p) < d_0 & \forall i \in I \text{ and} \\ r_j(p) > \frac{1}{2}\, d_0 & \forall j \in J \setminus I \end{array} \right\} \\
W_I^U \quad &:= \quad U_I \cap \bigcap_{i\in I} W_i\ .
\end{aligned}
$$

3.3. Lemma.

(i) $\# I > \frac{n}{2} \;\Rightarrow\; U_I = \emptyset\,.$

(ii) The sets U_I, $I \subset J$, define a locally finite, open covering of $\mathbb{H}^n$.

Proof. The first claim follows directly from Axiom 3.1 (ii). To see that the U_I are open subsets note that by the first part of Axiom 3.1 $\#\{j \in J \,|\, r_j(p) < 2\,d_0\}$ is finite for all $p \in \mathbb{H}^n$. □

We can view W_j as a slice of the 1–parameter groups $\vartheta_j\,(\mathbb{R}/2\pi\mathbb{Z})$. The stabilizer of W_j is the group $Stab_j = \{\vartheta_j(0), \vartheta_j(\pi)\}$.

As a further consequence of Axiom 3.1, we obtain the following detailed description of the blow–up $\pi\colon \hat{M}^n \to \mathbb{H}^n$, which we shall state as a proposition for later reference.

3.4. Proposition. — *Suppose that $U_I \cap \bigcup_{j\in J} \mathbb{H}^{n-2}_j \neq \emptyset$ for some $I \subset J$. Then, I is a finite, nonempty set $\{i_1, \dots, i_k\}$, and moreover*

(i) the rotations $\vartheta_{i'}(\varphi')$ and $\vartheta_{i''}(\varphi'')$ commute for all $i', i'' \in I$ and for all angles $\varphi', \varphi'' \in \mathbb{R}/2\pi\mathbb{Z}$. In particular, $\vartheta_I := \vartheta_{i_1} \circ \dots \circ \vartheta_{i_k}$ defines an injective homomorphism

$$\vartheta_I\colon (\mathbb{R}/2\pi\mathbb{Z})^{\# I} \to \mathrm{Iso}(\mathbb{H}^n)\ ;$$

(ii) the domain $U_I \subset \mathbb{H}^n$ is invariant under the action of ϑ_I, and W_I^U is a slice for this action restricted to U_I. The stabilizer $Stab_I$ of W_I^U is the abelian group

$$Stab_I = \left\{\vartheta_I(\sigma)\middle|\sigma \in \{0,\pi\}^{\# I}\right\}\ ;$$

clearly,

$$\begin{aligned}\pi_I\colon W_I^U \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I} &\to \quad U_I\\ (p\,,\varphi) \qquad &\mapsto \vartheta_I(\varphi)\,p\end{aligned}$$

is a surjective analytic map. The map π_I is invariant under the discrete, fixed point free action of $Stab_I$ on its domain, which is given by

$$\begin{aligned}\vartheta_I(\sigma)\colon W_I^U \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I} &\to W_I^U \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I}\\ (p,\varphi) \qquad &\mapsto (\vartheta_I(\sigma)\,p, \varphi+\sigma)\ ;\end{aligned}$$

(iii) the quotient space

$$\hat{U}_I := {Stab_I}\backslash\big(W_I^U \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I}\big)$$

is an open real analytic manifold with a natural projection $\pi_I\colon \hat{U}_I \to U_I$, which is one to one when restricted to $\pi_I^{-1}\left(U_I \setminus \bigcup_{i\in I} \mathbb{H}^{n-2}_i\right)$;

(iv) for $I' \subset I$ the projection π_I factors over $\pi_{I'}$, provided it is restricted to the preimage of $U_{I'} \cap U_I$.

By definition the manifold $\hat{M}$ is defined by gluing the $\hat{U}_I$ using the maps from Proposition 3.4 (iv). The blow–down map $\pi\colon \hat{M} \to \mathbb{H}^n$ is induced by the π_I. Note that by this description there is a natural action of Γ on $\hat{M}^n$ which commutes with π as stated in diagram (3.1).

We now turn to metric properties. Let $g_0 = \langle . , . \rangle$ be the hyperbolic metric on $\mathbb{H}^n$. The Killing fields $K_j := \frac{d}{dt}_{|t=0}\,\vartheta_j(t)$ have length $|K_j|^2 = \sinh^2 r_j$ where $r_j := \operatorname{dist}\left(. , \mathbb{H}^{n-2}_j \right)$.

3.5. Definition. — *Given $\alpha \in (0,\pi)$ and $\ell \in \mathbb{R}$, we say that a real analytic function $h\colon [0,\infty) \to [0,\infty)$ satisfies the cone condition $\mathcal{C}_\alpha(\ell)$, if and only if*

(i) h can be extended holomorphically to the cone $\mathcal{C}_\alpha := \exp\left(\mathbb{R} + \mathrm{i}\,(-\alpha,\alpha)\right)$;

(ii) for any $\alpha' \in (0,\alpha)$ there exists a constant $c_{\alpha'}$ such that $|h(x)| \le c_{\alpha'}|x|^{-\ell}$ on the subcone $C_{\alpha'} \subset C_\alpha$.

Let us just list some basic properties of the cone condition

$$
\begin{aligned}
h_1 \in \mathcal{C}_{\alpha_1}(\ell_1),\ h_2 \in \mathcal{C}_{\alpha_2}(\ell_2) &\Rightarrow\ h_1 h_2 \in \mathcal{C}_{min\{\alpha_1,\alpha_2\}}(\ell_1+\ell_2) \\
h \in \mathcal{C}_\alpha(\ell) &\Rightarrow\ \frac{d^k h}{dz^k} \in \mathcal{C}_\alpha(k+\ell) \quad \text{for any } k \ge 0\ .
\end{aligned}
\tag{3.2}
$$

3.6. *Examples.* — Let $\delta > 0$. Then,

(i) $h_{\delta,\ell}(x) := (1+\delta x^2)^{-\ell/2}$ lies in $\mathcal{C}_{\pi/2}(\ell)$ for any $\ell \ge 0$, and

(ii) $h_\delta(x) := \exp(-\delta x)$ lies in $\bigcap_{\ell \ge 0} \mathcal{C}_{\pi/2}(\ell)$.

Doubly exponentially decaying functions like $h(x) = \exp(1-\exp(x))$ do however *not* satisfy any cone condition at all.

3.7. Theorem. — *Let $h\colon [0,\infty) \to [0,\infty)$ be a real analytic function with $h(0)=1$, which satisfies the cone condition $\mathcal{C}_\alpha(\ell)$ for some $\alpha > 0$ and some $\ell > \frac{n-1}{2}$, and let $\eta > 0$ be arbitrary. Then,*

(i) the Poincaré series

$$
g(X,Y) = \langle X\,, Y\rangle + \sum_{j\in J} \eta^2 |K_j|^{-2} h\left(|K_j|^2\right)\ \langle X\,, K_j\rangle\langle K_j\,, Y\rangle,
$$

where $g_0 = \langle\,.\,,.\,\rangle$ denotes the standard hyperbolic metric on $\mathbb{H}^n$, converges compactly to a real analytic metric on $\Omega := \mathbb{H}^n \setminus \bigcup_{j\in J} \mathbb{H}_j^{n-2}$;

(ii) $\pi^*(g)$ extends to a Γ-invariant, complete, real analytic metric g on $\hat{M}$, which we shall denote again by g ;

(iii) for any subset $I \subset J$ and any point $p \in S_I := \bigcap_{i\in I} \mathbb{H}_i^{n-2} \setminus \bigcup_{j\in J\setminus I} \mathbb{H}_j^{n-2}$ the preimage $\pi^{-1}\{p\}$ is a totally geodesic, flat, product torus isometric to $(\mathbb{R}/\pi\eta\mathbb{Z})^{\# I}$. Moreover, the stratum $\hat{S}_I := \pi^{-1}S_I$ is intrinsically a flat bundle over S_I with fibres $(\mathbb{R}/\pi\eta\mathbb{Z})^{\# I}$.

3.8. Remarks.

(i) In the general setup it is *not clear* that the strata $\hat{S}_I$ are totally geodesic with respect to the metric g on $\hat{M}$ constructed in the preceding theorem.

(ii) On the other hand, $\hat{S}_I$ *must be totally geodesic*, if $(\hat{M}, g)$ has *nonpositive sectional curvature.* To see this, note that $\hat{S}_I$ is foliated by totally geodesic, flat tori ; these tori are absolutely minimizing in their homotopy class, since $K \le 0$. Since the metric is analytic, $\hat{S}_I$ coincides with the union of all absolutely minimizing tori in this homotopy class.

(iii) Because of these two points we need *an additional assumption in order to deduce Theorem* 1.1. This extra condition is a *symmetry requirement* for the collection $\left(\mathbb{H}_j^{n-2}\right)_{j\in J}$. In Sections 5 and 6 we shall see that the metrics g constructed in this theorem have the curvature properties claimed in Theorem 1.1, provided that η is sufficiently small depending on n, h, d_0, and $\hat{N}$. This *explains the proof of Theorem* 1.1, since for any $\delta > 0$ the function $h_\delta(x) = \exp(-\delta x)$ from the preceding example satisfies all our requirements.

3.9. Remark. — However, some care is necessary when trying to interpret the family of metrics $g \equiv g_{(\eta)}$, $\eta > 0$, from the preceding Theorem as an example for collapsing $\left(M, g_{(\eta)}\right) \underset{\eta\to 0}{\longrightarrow} \left(\mathbb{H}^n/\Gamma', g_0\right)$. The problem is that the *sectional curvatures of* $\left(M, g_{(\eta)}\right)$ *must be unbounded* when η approaches 0.

The reason is that by construction the length of the fibres $\mathbb{RP}^1 \to \hat{S}_i \to S_i$ decreases proportionally to η as $\eta \to 0$. Since the component M_η^{thick} of the thick–thin decomposition of $\big(M, g_{(\eta)}\big)$ is nonempty, it follows that

$$\sup_{E\subset TM_\eta^{\text{thin}}} |K(E)|^{1/2}\,\text{diam}\big(M_\eta^{\text{thin}}\big) \;\gtrapprox\; \ln\frac{c_{\text{Margulis}}}{\text{length}(\mathbb{RP}^1)} \;=\; \ln\frac{c_{\text{Margulis}}}{\pi\eta} \;\xrightarrow[\eta\to 0]{}\infty\ .$$

On the other hand, $\text{diam}\big(M, g_{(\eta)}\big)$ is uniformly bounded for $0<\eta\le 1$.

In fact, the expression for $R^{\#}$ shows directly[5] that for sufficiently small values of η the sectional curvature of any plane $\hat{E}_i$ over the stratum S_i of the divisor which is spanned by the unit normal vector of $\hat{S}_i$ and the tangent vector of the fibration $\hat{S}_i\to S_i$ is approximately $-\eta^{-2}$. Moreover, the region where the sectional curvature gets large in absolute value concentrates more and more along the preimage of the divisor. This behaviour is best understood when considering the Gauß–Bonnet Theorem, figuring out what it means to add a cross–cap of size $\sim\eta$ to a fixed ball orthogonal to $S_i\subset\mathbb{H}_i^{n-2}$.

For the subsequent calculations it is convenient to use the shorthand $x_j := |K_j|^2 \equiv \sinh^2 r_j$. Given $j\in J$, we introduce a bilinear form g_j and its dual endomorphism G_j by means of

$$g_j = \langle\,.\,,G_j\,.\rangle = \eta^2 x_j^{-2}h(x_j)\,\langle\,.\,,K_j\rangle\langle K_j\,,.\rangle \quad . \tag{3.3}$$

Moreover, for any subset $J'\subset J$ we let $g_{J'} := \sum_{j\in J'} g_j$. The convergence of $g_{J'}$ implies that the corresponding series $G_{J'} := \sum_{j\in J'} G_j$ of dual endomorphisms converges as well and that its limit is dual to $g_{J'}$ w.r.t. $g_0 = \langle\ ,\ \rangle$. In particular, the symmetric endomorphism $G := \mathbb{1} + G_J$ is dual to the metric g from the Theorem. When working on some domain $\Omega\cap U_I$, $I\subset J$, it will be convenient to decompose the Poincaré series for g as follows

$$g = g_0 + g_J = g_0 + g_I + g_{J\setminus I} = g_0 + \sum_{j\in J} g_j \quad . \tag{3.4}$$

Proof of Theorem 3.7. (i) Since Ω is covered by the domains $\Omega\cap U_I$ where $I\subset J$ is a *finite* subset, we may refer to Proposition 4.1 for the actual convergence estimates.

[5] cf. formula (5.27)

(ii) We handle each open set $\hat{U}_I$ in our covering of $\hat{M}$ separately. Note that

$$\pi_I^*(g) = \pi_I^*(g_0 + g_I) + \pi_I^*(g_{J\setminus I}) \quad .$$

By Proposition 4.1 $\pi_I^*(g_{J\setminus I})$ is a real analytic, positive semidefinite, bilinear form on $\hat{U}_I$. Proposition 3.4 enables us to compute the term $\pi_I^*(g_0 + g_I)$ on the domain $W_I^U \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I} \cap \pi_I^{-1}(\Omega)$ explicitly. We get

$$\pi_I^*(g_0 + g_I)_{|(p,\varphi)} = g_{0|T_pW_I^U \times T_pW_I^U} + \sum_{i\in I} \big(x_i + \eta^2 h(x_i)\big)_{|p} \, d\varphi_i{}^2 \ . \tag{3.5}$$

Evidently, the right hand side describes a real analytic, $Stab_I$-invariant, Riemannian metric on all of $W_I^U \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I}$.

(iii) Note that $p \in S_I$ is contained in some domain $U_{I'}$ with $I \subset I' \subset J$. By (3.5) it is clear that $\pi_{I'}^{-1}\{p\}$ is a totally geodesic product torus in $W_{I'}^U \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I'}$ equipped with the metric $\pi_{I'}^*\big(g_0 + g_{I'}\big)$. If η is sufficiently small, then the function $x \mapsto x + \eta^2 h(x)$, $x \geq 0$, takes its absolute minimum precisely at $x = 0$. Hence, for these values of η all closed geodesics of the torus are absolutely minimizing elements in their homotopy classes in $W_{I'}^U \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I'}$. In order to pass from the partial metric $\pi_{I'}^*\big(g_0 + g_{I'}\big)$ to $\pi_{I'}^*(g)$, we add a positive semidefinite term which vanishes on the torus. Hence, these curves remain absolutely minimizing, and so the tori remain totally geodesic with respect to g. In order to remove the dependence on the size of η, we observe that g depends analytically on η, and so does the second fundamental form of the torus.

The claimed flat bundle structure follows directly from formula (3.5). □

We conclude this section explaining how the proof of Theorem 1.4 parallels the real analytic case and why the C$^\infty$–case is nevertheless much simpler.

3.10. Remark. — Let us now assume that $h\colon [0,\infty) \to [0,\infty)$ is a C$^\infty$*–function with compact support* such that $h(0) = 1$ rather than a real analytic function which obeys some cone condition. Then, by Axiom 3.1, the Poincaré series $g = g_0 + \sum_{j\in J} g_j$ reduces to a locally finite sum. We therefore obtain a C$^\infty$–metric g on $\hat{M}$ such that each stratum $\hat{S}_I = \pi_I^{-1}(S_I)$ has the (local) product structure described in Theorem 3.7 (iii).

Similarly, all formulae in Section 5 (and in Section 6) carry over literally to the C^{∞}–case. Since all the series in these computations are locally finite, we do not need any convergence estimates. The curvature computations can be simplified even further, if we pick a cut-off function h whose support is contained in the interval $[0, \sinh^2 d_0)$. Here the key point is that by Axiom 3.1 (ii) the given upper bound for the support of h causes many terms in Formula (5.18) to vanish identically. As a result, we get the desired curvature control even without the estimate from Section 6. This is explained in more detail in Remark 5.10 below.

4. COMPLEXIFICATION AND COMPACT CONVERGENCE

The main purpose of this section is to prove the following slight generalization of Theorem 3.7 (i).

4.1. Proposition. — *Let $I \subset J$ be a finite subset. Then, under the assumptions of Theorem 3.7, the series $g_{J\setminus I} := \sum_{j\in J\setminus I} g_j$ converges compactly on U_I to a real analytic, positive semidefinite, bilinear form. In the C^0–topology one has*

$$\|g_{J\setminus I}\| \ \leq\ c_0\eta^2 \quad \text{on } U_I \tag{4.1}$$

where $\|\ \|$ denotes the operator norm with respect to g_0 and where c_0 is a constant depending just on n, h, d_0, and $\hat{N}$.

The C^0–bound (4.1) is a straightforward consequence of Lemma 3.2, since by the cone condition $x^{-1}h(x)$ is bounded by const $\cdot|x|^{(n+1)/2}$ for $x \geq \sinh^2 \frac{1}{2}\, d_0$.

In a similar way one can easily prove uniform convergence of the series $\sum_{j\in J\setminus I} g_j$ on U_I in any C^k–topology with $0 < k < \infty$. The *crucial point* is to establish that the limit is *real analytic and not just* C^∞. By standard results of complex analysis on compact convergence we only have to prove C^0–estimates by passing to a holomorphic extension. Therefore, we first construct a suitable model for this extension. We think of $\mathbb{H}^n$ as a component of the quadric

$$\left\{z \in \mathbb{R}^{n,1} \,\middle|\, \langle\!\langle z\,, z\rangle\!\rangle = -1\right\} \ .$$

Here $\langle\!\langle .\,,.\rangle\!\rangle$ denotes the standard Lorentz inner product. The subspaces $\mathbb{H}^{n-2}_j$ are intersections of this quadric with codimension 2 vector subspaces $E_j \subset \mathbb{R}^{n,1}$. The rotations ϑ_j preserve E_j and act on the space–like planes $E_j^\perp$ in the standard way.

We choose a unit vector $e_j^1 \in E_j^\perp$ such that $W_j = \mathbb{H}^n \cap (e_j^1)^\perp$ and define $e_j^2 := \vartheta_j(\frac{\pi}{2})e_j^1$. Now the Killing field K_j can be expressed as

$$K_{j\,|z} = \langle\!\langle z\,, e_j^1\rangle\!\rangle\; e_j^2 - \langle\!\langle z\,, e_j^2\rangle\!\rangle\; e_j^1\ . \tag{4.2}$$

Evidently, $W_j^\perp = \mathbb{H}^n \cap \left(e_j^2\right)^\perp$ is a totally geodesic hyperplane, which intersects W_j orthogonally along $\mathbb{H}_j^{n-2}$. Using the fact that

$$\sinh\left(\operatorname{dist}(z, W_j)\right) = \left|\langle\!\langle z\,, e_j^1\rangle\!\rangle\right|$$

and by the Law of Sines we can identify the argument of h (i.e. $|K_j|^2$) with a quadratic expression on $\mathbb{R}^{n,1}$

$$\begin{aligned} x_j(z) &= \sinh^2 r_j(z) = \sinh^2 \operatorname{dist}(z, W_j) + \sinh^2 \operatorname{dist}(z, W_j^\perp) \\ &= \langle\!\langle z\,, e_j^1\rangle\!\rangle^2 + \langle\!\langle z\,, e_j^2\rangle\!\rangle^2\ . \end{aligned} \tag{4.3}$$

Thus g_j can be expressed as

$$g_{j\,|z}\,(X, Y) = \eta^2 x_j(z)^{-2} h(x_j(z))\ \langle\!\langle X\,, K_{j\,|z}\rangle\!\rangle\, \langle\!\langle K_{j\,|z}\,, Y\rangle\!\rangle \tag{4.4}$$

for all $z \in \mathbb{H}^n \setminus E_j$ and for all $X, Y \in T_z\mathbb{H}^n$.

By the Formulae (4.2)–(4.4) we have extended the basic geometric objects in a real analytic way to an open neighborhood of $\mathbb{H}^n$ in $\mathbb{R}^{n,1}$. This extension can be complexified in an obvious manner. Let

$$\begin{aligned} \mathbb{C}^{n,1} &:= \ \mathbb{R}^{n,1} \otimes \mathbb{C} \\ \mathbb{H}^n_{\mathbb{C}} &:= \left\{ z \in \mathbb{C}^{n,1} \,\middle|\, \langle\!\langle z\,, z\rangle\!\rangle_{\mathbb{C}} = -1 \right\} \end{aligned}$$

where $\langle\!\langle\ ,\ \rangle\!\rangle_{\mathbb{C}}$ is the complex bilinear extension of $\langle\!\langle\ ,\ \rangle\!\rangle$.

Now, it is natural to extend (4.2) and (4.3) to $\mathbb{C}^{n,1}$ as follows

$$K^{\mathbb{C}}_{j\,|z} := \langle\!\langle z\,, e_j^1\rangle\!\rangle_{\mathbb{C}}\; e_j^2 - \langle\!\langle z\,, e_j^2\rangle\!\rangle_{\mathbb{C}}\; e_j^1 \tag{4.2$'$}$$

$$x^{\mathbb{C}}_j(z) := \langle\!\langle z\,, e_j^1\rangle\!\rangle^2_{\mathbb{C}}\ +\ \langle\!\langle z\,, e_j^2\rangle\!\rangle^2_{\mathbb{C}}\ . \tag{4.3$'$}$$

Furthermore, if h extends holomorphically to a sufficiently large domain in $\mathbb{C}$, then g_j is the restriction of the holomorphic bilinear form

$$g^{\mathbb{C}}_{j\,|z} := \eta^2 x^{\mathbb{C}}_j(z)^{-2} h(x^{\mathbb{C}}_j(z))\ \langle\!\langle .\,, K_{j\,|z}\rangle\!\rangle_{\mathbb{C}}\, \langle\!\langle K_{j\,|z}\,, .\rangle\!\rangle_{\mathbb{C}}\ . \tag{4.4$'$}$$

We remark that the complexified data when restricted to $\mathbb{H}^n_{\mathbb{C}}$ do not depend on the analytic extension from $\mathbb{H}^n$ to $\mathbb{R}^{n,1}$.

In order to prove Proposition 4.1, we need to establish uniform convergence of the series $g^{\mathbb{C}}_{J\setminus I} = \sum_{j\in J\setminus I} g^{\mathbb{C}}_j$ in an arbitrarily small open neighborhood in $\mathbb{C}^{n,1}$ of any given point $z_0 \in U_I$. Here, we may work with respect to any norm on $\mathbb{C}^{n,1}$ which may even depend on z_0.

By homogeneity, we can assume that $z_0 = (0,\ldots,0,1)$,[6] and as a norm we take the Hermitian inner product $\langle\ ,\ \rangle_h$ from the standard identification of $\mathbb{C}^{n,1}$ with $\mathbb{C}^n \times \mathbb{C}$.

Evidently, for all $z_1, z_2 \in \mathbb{C}^{n,1}$ there is the inequality

$$\left|\langle\!\langle z_1\,,z_2\rangle\!\rangle_{\mathbb{C}}\right|^2 \le \langle z_1\,,z_1\rangle_h\,\langle z_2\,,z_2\rangle_h \quad . \tag{4.5}$$

4.2. Lemma. — *Let $z = z_0 + \xi \in \mathbb{C}^{n,1}$ with $\langle \xi\,,\xi\rangle_h \le a^2$. Then, for any $e \in \mathbb{R}^{n,1}$ with $\langle\!\langle e\,,e\rangle\!\rangle = 1$, the following inequalities hold*

$$\text{(i)} \qquad \left|\langle\!\langle e\,,z\rangle\!\rangle^2_{\mathbb{C}} - \langle\!\langle e\,,z_0\rangle\!\rangle^2_{\mathbb{C}}\right| \le a\left(a+\sqrt{2}\right)\left(1+2\,\langle\!\langle e\,,z_0\rangle\!\rangle^2_{\mathbb{C}}\right)\ ,$$

$$\text{(ii)} \qquad \left|x^{\mathbb{C}}_j(z) - x_j(z_0)\right| \quad \le 2a\left(a+\sqrt{2}\right)\left(1+x_j(z_0)\right)\ .$$

Proof. (i) Write $e = (\bar e, e_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$. Since $e_{n+1} = -\langle\!\langle e\,,z_0\rangle\!\rangle$ and $\langle \bar e\,,\bar e\rangle = 1 + |e_{n+1}|^2$, we have

$$\langle e\,,e\rangle_h = 1 + 2\,\langle\!\langle e\,,z_0\rangle\!\rangle^2 \ . \tag{4.6}$$

Now

$$\left|\langle\!\langle e\,,z\rangle\!\rangle^2_{\mathbb{C}} - \langle\!\langle e\,,z_0\rangle\!\rangle^2_{\mathbb{C}}\right| \le \left(2\,|\,\langle\!\langle e\,,z_0\rangle\!\rangle_{\mathbb{C}}\,| + |\,\langle\!\langle e\,,\xi\rangle\!\rangle_{\mathbb{C}}\,|\right)\left|\langle\!\langle e\,,\xi\rangle\!\rangle_{\mathbb{C}}\right|\ .$$

By (4.5) and (4.6) we obtain

$$\left|\langle\!\langle e\,,\xi\rangle\!\rangle_{\mathbb{C}}\right|^2 \le a^2\left(1+2\,\langle\!\langle e\,,z_0\rangle\!\rangle^2\right) \tag{4.7}$$

hence (i).

[6] This moves of course the collection of subspaces $\mathbb{H}^{n-2}_j$, $j \in J$.

(ii) This inequality is a direct consequence of (i) since e_j^1, e_j^2 are legitimate choices for e. □

As an immediate consequence we obtain the following

4.3. Corollary. — *Suppose that* $2a\big(a+\sqrt{2}\big) < \sin\alpha < 1$, *then it follows for any* $z = z_0 + \xi \in \mathbb{C}^{n,1}$ *with* $\langle \xi\,, \xi\rangle_h \le a^2$ *that*

$$\text{(i)} \qquad x_j(z) \in C_\alpha \qquad \textit{and}$$

$$\text{(ii)} \qquad |x_j(z)| > (1-\sin\alpha)\,|x_j(z_0)|\ ,$$

provided that $|x_j(z)| \in [0,\infty)$ *is greater than some constant depending on* a *and* α.

Furthermore we obtain the following

4.4. Corollary. — *Let* $z = z_0 + \xi \in \mathbb{C}^{n,1}$ *with* $\langle \xi\,, \xi\rangle_h \le a^2$. *Then,*

$$\big|\langle\!\langle .\,, K^{\mathbb{C}}_{j\ |z}\rangle\!\rangle_{\mathbb{C}}\,\langle\!\langle K^{\mathbb{C}}_{j\ |z}\,, .\rangle\!\rangle_{\mathbb{C}}\big|_h \le 2\,\big(1+2a(a+\sqrt{2})\big)\big(1+2x_j(z_0)\big)^2\ .$$

Proof. A straightforward computation using (4.2′) shows that

$$\begin{aligned}
&\langle\!\langle X\,, K^{\mathbb{C}}_{j\ |z}\rangle\!\rangle\,\langle\!\langle K^{\mathbb{C}}_{j\ |z}\,, Y\rangle\!\rangle_{\mathbb{C}}\\
&\qquad =\langle\!\langle z\,, e_j^2\rangle\!\rangle_{\mathbb{C}}^{\ 2}\,\langle\!\langle e_j^1\,, X\rangle\!\rangle_{\mathbb{C}}\,\langle\!\langle e_j^1\,, Y\rangle\!\rangle_{\mathbb{C}} + \langle\!\langle z\,, e_j^1\rangle\!\rangle_{\mathbb{C}}^{\ 2}\,\langle\!\langle e_j^2\,, X\rangle\!\rangle_{\mathbb{C}}\,\langle\!\langle e_j^2\,, Y\rangle\!\rangle_{\mathbb{C}}\\
&\qquad - \langle\!\langle z\,, e_j^1\rangle\!\rangle_{\mathbb{C}}\,\langle\!\langle z\,, e_j^2\rangle\!\rangle_{\mathbb{C}}\,\big[\langle\!\langle e_j^1\,, X\rangle\!\rangle_{\mathbb{C}}\,\langle\!\langle e_j^2\,, Y\rangle\!\rangle_{\mathbb{C}} + \langle\!\langle e_j^2\,, X\rangle\!\rangle_{\mathbb{C}}\,\langle\!\langle e_j^1\,, Y\rangle\!\rangle_{\mathbb{C}}\big]\ .
\end{aligned}$$

By the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\big|\langle\!\langle X\,, K^{\mathbb{C}}_{j\ |z}\rangle\!\rangle_{\mathbb{C}}\,\langle\!\langle K^{\mathbb{C}}_{j\ |z}\,, Y\rangle\!\rangle_{\mathbb{C}}\big|\\
&\qquad \le 2\,\big|\langle\!\langle z\,, e_j^2\rangle\!\rangle_{\mathbb{C}}\big|^2\,\big|\langle\!\langle X\,, e_j^2\rangle\!\rangle_{\mathbb{C}}\big|^2 + 2\,\big|\langle\!\langle z\,, e_j^1\rangle\!\rangle_{\mathbb{C}}\big|^2\,\big|\langle\!\langle X\,, e_j^2\rangle\!\rangle_{\mathbb{C}}\big|^2\ .
\end{aligned}\tag{4.8}$$

By Lemma 4.2 we have

$$\begin{aligned}
\big|\langle\!\langle e_j^\mu\,, z\rangle\!\rangle_{\mathbb{C}}\big|^2 &\le a\,\big(a+\sqrt{2}\big) + \big(1+2a(a+\sqrt{2})\big)\,\langle\!\langle e_j^\mu\,, z_0\rangle\!\rangle_{\mathbb{C}}\\
&\le \frac{1}{2}\big(1+2a(a+\sqrt{2})\big)\big(1+2x_j(z_0)\big)\ .
\end{aligned}$$

Now (4.5) and (4.6) yield

$$\begin{aligned}
\big|\langle\!\langle e_j^\mu\,, X\rangle\!\rangle_{\mathbb{C}}\big|^2 &\le \big(1+2\,\langle\!\langle e_j^\mu\,, z_0\rangle\!\rangle^2\big)\langle X\,, X\rangle_h\\
&\le \big(1+2x_j(z_0)\big)\langle X\,, X\rangle_h\ .
\end{aligned}$$

The claim follows from inserting these inequalities into (4.8). □

Now, we collect the information.

Proof of Proposition 4.1. (*Analyticity of* $g_{J\setminus I}$). W.l.o.g. we may assume that the function h lies in some space $\mathcal{C}_\alpha(\ell)$ with $\alpha < \frac{\pi}{2}$. We pick $a > 0$ so small that

$$2a\big(a + \sqrt{2}\big) < \sin\alpha < 1 \ .$$

Let $z \in U_I \subset \mathbb{H}^n \subset \mathbb{C}^{n,1}$ as above. Then for all but finitely many $j \in J \setminus I$, the value $x_j(z_0)$ is sufficiently large that Corollary 4.3 applies. Let $J' \subset J \setminus I$ be the subset of these indices. It is sufficient to show that the series $\sum_{j\in J'} g_j$ converges uniformly on the ball

$$B(z_0, a) = \big\{ z = z_0 + \xi \in \mathbb{C}^{n,1} \bigm| \langle\!\langle \xi\,, \xi \rangle\!\rangle_h \le a^2 \big\} \ .$$

Then, the Cauchy integral formula implies that the limit is holomorphic and hence real analytic even after restricting to $\mathbb{H}^n$ again. This implies that $g_{J\setminus I}$ is analytic.

To prove the uniform convergence, note that by Corollary 4.3 we have

$$\begin{aligned} |h(x_j(z))| &\le \mathrm{Const}\, |x_j(z)|^{-\ell} \\ |x_j(z)|^{-2} &\le (1 - \sin(\alpha))^{-2}\, |x_j(z_0)|^{-2} \ . \end{aligned}$$

By Corollary 4.4 we see

$$\big| \langle\!\langle .\,, K^{\mathbb{C}}_{j\ |z} \rangle\!\rangle_{\mathbb{C}} \, \langle\!\langle K^{\mathbb{C}}_{j\ |z}\,, . \rangle\!\rangle_{\mathbb{C}} \big|_h \le 4\, \big(1 + 2x_j(z_0)\big)^2 \ .$$

Using again that $|x_j(z_0)|$ is bounded away from 0 for $j \in J'$, we combine these inequalities and obtain

$$\big| g^{\mathbb{C}}_{j\ |z} \big|_h \le \eta^2\, \mathrm{Const}'\, |1 + x_j(z_0)|^{-\ell} \ .$$

Note that $1 + x_j(z_0) = \cosh^2(r_j(z_0)) \ge \frac{1}{4}\exp(2r_j(z_0))$. Hence,

$$\big| g^{\mathbb{C}}_{j\ |z} \big|_h \le \eta^2\ \mathrm{Const}''\, \mathrm{e}^{-2\ell r_j(z_0)} \ .$$

Since by hypothesis $2\ell > n-1$, we conclude from Lemma 3.2 that the right hand side indeed converges. $\square$

5. CURVATURE COMPUTATIONS

Our next goal is to compute the curvatures of the Riemannian manifold $(\hat{M}, g)$ introduced in Theorem 3.7. In this section we restrict ourselves to the open dense domain $\hat{\Omega} := \pi^{-1}(\Omega) \subset \hat{M}$ so that the Poincaré series from (3.4) is at our disposal. Later, in Section 7, we shall determine the curvatures at points $\hat{p} \in \hat{M} \setminus \hat{\Omega}$ by means of limiting arguments.

Whenever there are two Riemannian metrics g and $\langle .\,,.\rangle$ on the same domain Ω, their covariant derivatives ∇ and D, respectively, are related through the equation

$$\nabla_X Y = D_X Y + G^{-1} B(X, Y) \tag{5.1}$$

where $G: T\Omega \to T\Omega$ is the symmetric endomorphism which represents g with respect to $\langle .\,,.\rangle$, i.e. $g = \langle .\,, G \cdot .\rangle$, and where the tensor field $B: T\Omega \times T\Omega \to T\Omega$ is defined by

$$2\langle B(X, Y)\,, Z\rangle = D_X g(Y, Z) + D_Y g(X, Z) - D_Z g(X, Y)\ . \tag{5.2}$$

It is more standard to introduce the Christoffel tensor $\Gamma = G^{-1} B$ right away by replacing the left hand side of (5.2) by $2g\,(\Gamma(X, Y), Z)$. With this notation the $(3,1)$-curvature tensors R and R_0 of the metrics g and $\langle .\,,.\rangle$, respectively, are related by

$$\begin{aligned} R(X,Y)Z = R_0(X,Y)Z &+ D_X\Gamma(Y,Z) - D_Y\Gamma(X,Z) \\ &+ \Gamma(X,\Gamma(Y,Z)) - \Gamma(Y,\Gamma(X,Z))\ . \end{aligned} \tag{5.3}$$

Passing to the $(4,0)$–curvature tensors $R^{\#} = g\big(R(.\,,.).\,,.\big)$ and $R_0^{\#} = \langle R_0(.\,,.).\,,.\rangle$, this formula can be rewritten as

(5.4)

$$\begin{aligned}
&R^{\#}(X,Y;Z,W)\\
&= \frac{1}{4}\Big[R_0^{\#}(GX,Y;Z,W)+R_0^{\#}(X,GY;Z,W)+R_0^{\#}(X,Y;GZ,W)+R_0^{\#}(X,Y;Z,GW)\Big]\\
&-\frac{1}{2}\Big[D^2_{\{X,W\}}g(Y,Z)-D^2_{\{Y,W\}}g(X,Z)-D^2_{\{X,Z\}}g(Y,W)+D^2_{\{Y,Z\}}g(X,W)\Big]\\
&-\Big[\langle B(X,W)\,,G^{-1}B(Y,Z)\rangle-\langle B(Y,W)\,,G^{-1}B(X,Z)\rangle\Big]
\end{aligned}$$

where $D^2_{\{X,W\}}g(Y,Z) := \frac{1}{2}\big[D^2_{X,W}g(Y,Z)+D^2_{W,X}g(Y,Z)\big]$ is symmetrical in the arguments X and W as well as in Y and Z. There are two reasons why the expression given in (5.4) is much more suitable for the curvature computations in our example than formula (5.3) :

(1) the right hand side in (5.4) comes as a *sum of three tensors* which separately obey all the algebraic symmetries of a curvature tensor. Therefore, these pieces can be estimated separately in terms of the eigenvalues of the corresponding bilinear form on $\Lambda^2 T\Omega$;[7]

(2) evidently, the first two lines in the expression for $R^{\#}$ constitute a *linear* differential operator in g of 2[nd] order. Moreover, the map $g \mapsto B$ is a linear differential operator of 1[st] order. So the third term in the expression for $R^{\#}$ depends quadratically on the 1[st] derivatives of g, and the factor G^{-1} in each pairing resembles a common denominator, which depends pointwise linearly on g.

The next step is to *evaluate the various terms* on the right hand side of (5.4) when g is the Poincaré series $g_0 + \sum_{j\in J} g_j$ from (3.4). This is quite straightforward except for a *slight subtlety*[8] due to the last line in (5.4), which we may think of as an essentially quadratic interaction term in an otherwise linear context.

[7] Our sign conventions for the curvature operator $\hat{R}\colon \Lambda^2 T\Omega \to \Lambda^2 T\Omega$ or rather for its associated bilinear form $\hat{R}^{\#}\colon \Lambda^2 T\Omega \times \Lambda^2 T\Omega \to \mathbb{R}$ are explained by the equation $\hat{R}^{\#}(X\wedge Y, Z\wedge W) = -R^{\#}(X,Y;Z,W)$. So, with our conventions, the curvature operator of $\mathbb{H}^n$ is negative definite.

[8] We explain this problem in detail below in Corollary 5.3.

Recall that in our context $g_0 = \langle\,.\,,.\rangle$ denotes the standard hyperbolic metric. The metric g is defined as a Poincaré series where each term g_j corresponds to a particular Killing field K_j on $\mathbb{H}^n$ by means of (3.3). We shall find it convenient to introduce for each $j \in J$ some more objects related to the very special nature of K_j. Since the corresponding group $\vartheta_j(\mathbb{R}/2\pi\mathbb{Z})$ of rotations has a *subspace* $\mathbb{H}^{n-2}_j \subset \mathbb{H}^n$ *of codimension* 2 *as its fixed point set*, it is clear that the endomorphism field DK_j has rank 2 at every point $p \in \mathbb{H}^n$. Recall that $x_j = |K_j|^2 = \sinh^2 r_j$ where $r_j = \operatorname{dist}(\,.\,,\mathbb{H}^{n-2}_j)$. On each domain $\mathbb{H}^n \setminus \mathbb{H}^{n-2}_j$ we introduce the unit vector fields $v_j := \operatorname{grad}_{g_0} r_j$ and $\xi_j := |K_j|^{-1}K_j \equiv x_j^{-1/2}K_j$. These two vector fields form an orthogonal basis for the image of DK_j, which is actually an integrable distribution on $\mathbb{H}^n$. Its leaves are 2–dimensional, totally geodesic subspaces which intersect $\mathbb{H}^{n-2}_j$ orthogonally. It is easy to check that $\langle -D_{K_j}K_j\,,v_j\rangle = \langle D_{v_j}K_j\,,K_j\rangle = \frac{1}{2}\,d_{v_j}x_j = x_j^{1/2}\,(1+x_j)^{1/2}$, and thus

$$DK_j = (1+x_j)^{1/2}\Big(\langle\,.\,,v_j\rangle\xi_j - \langle\,.\,,\xi_j\rangle v_j\Big)\ , \tag{5.5}$$

or equivalently,

$$|K_j|^2\,DK_j = \langle\,.\,,K_j\rangle DK_j - \langle\,.\,,DK_j\rangle K_j\ . \tag{5.5$'$}$$

On $\mathbb{H}^n \setminus \mathbb{H}^{n-2}_j$ the orthogonal projector P_j onto the image of DK_j is given by

$$P_j = \langle\,.\,,v_j\rangle v_j + \langle\,.\,,\xi_j\rangle\xi_j\ . \tag{5.6}$$

The dual bilinear form $p_j = \langle\,.\,,P_j\,.\rangle$ coincides with $(1+x_j)^{-1}\langle DK_j\,,DK_j\rangle$, and thus P_j and p_j are actually real analytic tensor fields on all of $\mathbb{H}^n$. Note, however, that the components

$$\begin{aligned} p_j^v &:= \langle\,.\,,v_j\rangle\langle v_j\,,.\rangle \\ p_j^\xi &:= \langle\,.\,,\xi_j\rangle\langle \xi_j\,,.\rangle \end{aligned} \tag{5.7}$$

of p_j do not even extend continuously across $\mathbb{H}^{n-2}_j$. Still, these fields constitute a useful shorthand notation. The same is true for

$$\begin{aligned} p_j^a &:= p_j^v - p_j^\xi \qquad\qquad \text{and} \\ p_j^b &:= \langle\,.\,,v_j\rangle\langle \xi_j\,,.\rangle + \langle\,.\,,\xi_j\rangle\langle v_j\,,.\rangle\ . \end{aligned} \tag{5.7$'$}$$

5.1. Lemma. — *Let g be the metric on $\Omega \subset \mathbb{H}^n$ defined in Theorem 3.7. Then, the first two lines in the expression for $R^{\#}$ can be evaluated as follows*

$$\begin{aligned}\frac{1}{4}\Big[&R_0^{\#}(GX,Y;Z,W)+R_0^{\#}(X,GY;Z,W)\\ &+R_0^{\#}(X,Y;GZ,W)+R_0^{\#}(X,Y;Z,GW)\Big]\\ &\quad = -(g_0 \owedge g_0)(X,Y;Z,W)-\sum_{j\in J}\eta^2 x_j^{-1}h(x_j)\ (g_0 \owedge p_j^{\xi})(X,Y;Z,W)\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}&\frac{1}{2}\Big[D^2_{\{X,W\}}g(Y,Z)-D^2_{\{Y,W\}}g(X,Z)-D^2_{\{X,Z\}}g(Y,W)+D^2_{\{Y,Z\}}g(X,W)\Big]\\ &\quad =\sum_{j\in J}\Big[T_j(X,Y;Z,W)+\eta^2(2h'(x_j)-x_j^{-1}h(x_j))\,(g_0 \owedge p_j^{\xi})(X,Y;Z,W)\Big]\end{aligned} \tag{5.9}$$

where

$$T_j := \eta^2(1+x_j)\big(2h''(x_j)-x_j^{-1}h'(x_j)+x_j^{-2}h(x_j)\big)\,p_j \owedge p_j\ . \tag{5.9$'$}$$

The basic facts about the $\owedge$–product of symmetric bilinear forms are summarized in Appendix A.

Proof. The curvature tensor of the standard hyperbolic metric is $R_0^{\#} = -g_0 \owedge g_0$. Hence, the first line in the expression for $\mathbb{R}^{\#}$ equals

$$-g_0 \owedge \langle\,.\,,G\,.\rangle(X,Y;Z,W)\,.$$

Since $\langle\,.\,,G\,.\rangle = g \equiv g_0+\sum_{j\in J} g_j$, we can deduce Formula (5.8) by purely formal manipulations using Equation (3.3). The relevant issues of convergence have been dealt with in Proposition 4.1.

In order to obtain Equation (5.9) we note that by the same reasoning as in Section 4 the Poincaré series of Theorem (3.7) may be differentiated term by term. Hence, we get

$$
\begin{aligned}
(5.10)\quad & D_X g(Y,Z) \\
&= \sum_{j\in J} \Big[2\eta^2 x_j^{-2}\big(h'(x_j) - 2x_j^{-1}h(x_j)\big)\,\langle K_j\,, D_X K_j\rangle\langle Y\,, K_j\rangle\langle K_j\,, Z\rangle \\
&\qquad + \eta^2 x_j^{-2} h(x_j)\,\Big(\langle Y\,, D_X K_j\rangle\langle K_j\,, Z\rangle + \langle Y\,, K_j\rangle\langle D_X K_j\,, Z\rangle\Big)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
(5.11)\quad & \\
& D^2_{X,W} g(Y,Z) \\
&= \sum_{j\in J} 4\eta^2 x_j^{-2}\ \big(h''(x_j) - 4x_j^{-1}h'(x_j) + 6x_j^{-2}h(x_j)\big) \\
&\qquad \langle K_j\,, D_X K_j\rangle\langle K_j\,, D_W K_j\rangle\langle K_j\,, Y\rangle\langle K_j\,, Z\rangle \\
&+ \sum_{j\in J} 2\eta^2 x_j^{-2}\big(h'(x_j) - 2x_j^{-1}h(x_j)\big) \\
&\qquad \Big[\langle K_j\,, Y\rangle\langle K_j\,, Z\rangle\ \Big(\langle D_X K_j\,, D_W K_j\rangle - \langle R_0(K_j, X)W\,, K_j\rangle\Big) \\
&\qquad + \langle K_j\,, Y\rangle\ \Big(\langle K_j\,, D_W K_j\rangle\langle D_X K_j\,, Z\rangle + \langle K_j\,, D_X K_j\rangle\langle D_W K_j\,, Z\rangle\Big) \\
&\qquad + \langle K_j\,, Z\rangle\ \Big(\langle K_j\,, D_X K_j\rangle\langle D_W K_j\,, Y\rangle + \langle K_j\,, D_W K_j\rangle\langle D_X K_j\,, Y\rangle\Big)\Big] \\
&+ \sum_{j\in J} \eta^2 x_j^{-2} h(x_j) \\
&\qquad \Big[\langle D_W K_j\,, Y\rangle\langle D_X K_j\,, Z\rangle + \langle D_X K_j\,, Y\rangle\langle D_W K_j\,, Z\rangle \\
&\qquad - \langle R_0(K_j, X)W\,, Y\rangle\langle K_j\,, Z\rangle - \langle K_j\,, Y\rangle\langle R_0(K_j, X)W\,, Z\rangle\Big]\ .
\end{aligned}
$$

In this computation we have used the identity $D^2_{X,W}K_j + R_0(K_j, X)W = 0$ in order to determine the second derivatives $D^2 K_j$.

Modifying just the last line in this display, we pass from $D^2_{X,W}g$ to $D^2_{\{X,W\}}g$. We insert this expression into the left hand side of (5.9). After expressing K_j and DK_j in terms of v_j and ξ_j, we just need to collect terms appropriately, using just the definitions of p_j, p_j^ξ, and the $\owedge$–product. $\square$

5.2. Lemma. — *Let g be the metric on $\Omega \subset \mathbb{H}^n$ defined in Theorem 3.7. Then, the bilinear map B introduced in Equations* (5.1) *and* (5.2) *is given as a sum $B = \sum_{j\in J} B_j$ where*

$$
(5.12)\qquad
\begin{aligned}
B_j(Y,Z) &:= -\eta^2 x_j^{-1/2}(1+x_j)^{1/2} h'(x_j)\, p_j^\xi(Y,Z)\ v_j \\
&\qquad + \eta^2 x_j^{-1/2}(1+x_j)^{1/2}\big(h'(x_j) - x_j^{-1}h(x_j)\big)\, p_j^b(Y,Z)\ \xi_j\ .
\end{aligned}
$$

Proof. The series of the right hand side of (5.10) is locally absolutely convergent, and thus it may be inserted into the right hand side of Formula (5.3). We use the notations and elementary properties expressed in Equations (5.5)–(5.7′) and collect terms appropriately, hence Formula (5.12). □

Let us extend the $\owedge$–product of symmetric bilinear forms to a $\owedge$–product defined for triples consisting of a symmetric bilinear form $l\colon T\Omega \times T\Omega \to \mathbb{R}$ and of two symmetric bilinear maps $B_1, B_2\colon T\Omega \times T\Omega \to T\Omega$

$$(5.13)\qquad \begin{aligned}&(B_1 \owedge_l B_2)(X,Y;Z,W)\\ &\quad := \frac{1}{2}\big[l\big(B_1(X,W),B_2(Y,Z)\big) - l\big(B_1(Y,W),B_2(X,Z)\big)\\ &\qquad - l\big(B_1(X,Z),B_2(Y,W)\big) + l\big(B_1(Y,Z),B_2(X,W)\big)\big]\ .\end{aligned}$$

Again $B_1 \owedge_l B_2$ has all the algebraic symmetries of a curvature tensor. For a symmetric endomorphism $L\colon T\Omega \to T\Omega$ we shall also use the shorthand $B_1 \owedge_L B_2 := B_1 \owedge_{\langle .\,,\,L.\rangle} B_2$.

With this notation we can summarize the content of Equation (5.4) and of the Lemmas 5.1 and 5.2 as follows.

5.3. Corollary. — *Let g be the metric on $\Omega \subset \mathbb{H}^n$ defined in Theorem 3.7. Then, the $(4,0)$–curvature tensor of this metric is given by*

$$(5.14)\qquad R^{\#} = -g_0 \owedge g_0 - \sum_{j\in J}\big[T_j + 2\eta^2 h'(x_j)\, g_0 \owedge p_j^{\xi}\big] - \sum_{\substack{(j_1,j_2)\\ \in J\times J}} B_{j_1} \owedge_{G^{-1}} B_{j_2}$$

where the fields T_j and B_j are given by $(5.9')$ and (5.12), respectively.

Still, Formula (5.14) is *not the expression* for the curvature tensor of g *which we eventually want to have.* The tensor fields $T := \sum_{j\in J}\big[T_j + 2\eta^2 h'(x_j)\, g_0 \owedge p_j^{\xi}\big]$ and $B = \sum_{j\in J} B_j$ blow up too quickly when the footpoint p approaches the boundary of Ω. In particular, they do *not* extend continuously from $\hat{\Omega}$ to all of $\hat{M}$. On the other hand, $R^{\#}$ is the curvature tensor of a real analytic metric g on $\hat{M}$, and thus $R^{\#}$ is a globally defined, real analytic, and hence continuous tensor field on $\hat{M}$.

To be more explicit about this phenomenon, we pick a subspace $\mathbb{H}^{n-2}_{j_0}$ in our collection, a point $p_\infty \in U_{\{j_0\}} \cap \mathbb{H}^{n-2}_{j_0}$, and a sequence $(p_\mu)_{\mu=1}^{\infty} \in U_{\{j_0\}} \cap \Omega$ which converges to p_∞ in $\mathbb{H}^n$. Using just the definition of U_{j_0} and Formulae (3.3), (3.4), and (5.9′), it is easy to compute the asymptotic behaviour of $g \owedge g$ and T in the limit where $p_\mu \to p_\infty$. In particular,

$$(5.15) \qquad (g \owedge g)(v_{j_0}, \xi_{j_0}; \xi_{j_0}, v_{j_0})_{|p_\mu} \sim \eta^2 x_{j_0}(p_\mu)^{-1} \qquad \text{and}$$

$$(5.16) \qquad \begin{aligned} T(v_{j_0}, \xi_{j_0}; \xi_{j_0}, v_{j_0})_{|p_\mu} &\sim T_{j_0}(v_{j_0}, \xi_{j_0}; \xi_{j_0}, v_{j_0})_{|p_\mu} \\ &\sim \eta^2 x_{j_0}(p_\mu)^{-2} \ . \end{aligned}$$

As explained above, Formula (5.15) implies that $R^{\#}(v_{j_0}, \xi_{j_0}; \xi_{j_0}, v_{j_0})_{|p_\mu}$ is bounded by *const* $\eta^2 x_{j_0}(p_\mu)^{-1}$ as $\mu \to \infty$. Hence, the *leading order term* in the expansion of T_{j_0} *must cancel* versus a suitable counterterm in $B \owedge_{G^{-1}} B$.

In order to get reasonable estimates for $R^{\#}$ on a term by term basis, we need to *perform this cancelation explicitly.* Looking at the special case[9] where $J = \{j_0\}$, it seems natural to try *the "self–interaction" term* $B_{j_0} \owedge_{(\mathbb{1}+G_{j_0})^{-1}} B_{j_0}$ for canceling the leading order term of T_{j_0} for each j_0 in the case of a large index set J as well. We are going to handle the various domains $U_I \cap \Omega$, where I runs over all finite subsets of J, separately. For ease of notation, we introduce for any subset $J' \subset J$ the partial sums

$$(5.17) \qquad B_{J'} := \sum_{j \in J'} B_j \qquad \text{and} \qquad T_{J'} := \sum_{j \in J'} T_j \quad .$$

5.4. Lemma. — *Let g be as above. Then, there exist constants c_1, c_2, and c_3 depending on n, d_0, h, and $\hat{N}$ such that on any nonempty set $\Omega \cap U_I$ one has*

$$\text{(i)} \qquad -c_1\, g_0 \owedge g_0 \ \le\ \sum\nolimits_{j \in J} 2h'(x_j)\, g_0 \owedge p_j^\xi \ \le\ c_1\, g_0 \owedge g_0$$

$$\text{(ii)} \qquad -c_2 \eta^2\, g_0 \owedge g_0 \ \le\ T_{J \setminus I} \ \le\ c_2 \eta^2\, g_0 \owedge g_0$$

$$\text{(iii)} \qquad \left\| B_{J \setminus I} \right\| \ \le\ c_3 \eta^2$$

$$\text{(iv)} \qquad -c_3 \eta^4\, g_0 \owedge g_0 \ \le\ B_{J \setminus I} \owedge_{G^{-1}} B_{J \setminus I} \ \le\ c_3^2 \eta^4\, g_0 \owedge g_0 \ .$$

[9] This is compatible with Axiom 3.1. It is only the more special setting covered by the theorems in the introduction which always leads to countably infinite sets J.

Here, $\| \ \|$ stands for the operator norm of symmetric bilinear maps.[10]

Proof. The standard properties of the cone condition imply that the functions $h'(x)$, $x^{-1}h(x)$, $(1+x)\,h'(x)$, $x^{-1}(1+x)\,h''(x)$, and $x^{-2}(1+x)\,h(x)$ are bounded by $const\,|x|^{-\ell-1}$ for $x \geq \sinh^2 \frac{1}{2}\,d_0$. Note that the functions h, h', and h'' themselves are also bounded on $[0, \sinh^2 \frac{1}{2}\,d_0]$. Since by hypothesis $\ell > \frac{n-1}{2}$, we obtain absolute convergence by means of Lemma 3.2. In fact,

$$\begin{aligned}
c_1 &:= \sup_{p\in\mathbb{H}^n} \sum_{j\in J} \big|h'(x_j(p))\big| \\
c_2 &:= \sup_{p\in U_I} \sum_{j\in J\setminus I} (1+x_j)\big|h''(x_j) - x_j^{-1}h'(x_j) + x_j^{-2}h(x_j)\big| \\
c_3 &:= \sup_{p\in U_I} \sum_{j\in J\setminus I} x_j^{-1/2}(1+x_j)\big(\big|h'(x_j) - x_j^{-1}h'(x_j)\big| + \big|h'(x_j)\big|\big)
\end{aligned}$$

are finite numbers. In order to establish (i)–(iii), it remains to point out that $0 \leq 2g_0 \owedge p_j^\xi \leq g_0 \owedge g_0$, that $0 \leq p_j \owedge p_j \leq g_0 \owedge g_0$, and that $\max\{|p_j^\xi(W,W)|, |p_j^b(W,W)|\} \leq \langle W\,, W\rangle$. In order to deduce the bounds for $B_{J\setminus I} \owedge_{G^{-1}} B_{J\setminus I}$ given in (iv), we need to employ in addition the inequality $G \geq \mathbb{1}$ as well as Lemma A.1 (iii).

∎

5.5. Proposition. — *Let g be the metric on $\Omega \subset \mathbb{H}^n$ defined in Theorem 3.7, and let $I \subset J$ be any finite subset. Then, on $\Omega \cap U_I$, the curvature tensor of g is given by*

$$\begin{aligned}
R^{\#} = &- g_0 \owedge g_0 - \sum_{i\in I}\big(1+\eta^2 h'(x_i)\big)\big(\varphi_0(\eta,x_i) - \varphi_1(\eta,x_i)\big)\, p_i \owedge p_i \\
&- T_{J\setminus I} - \sum_{j\in J} 2\eta^2 h'(x_j)\, g_0 \owedge p_j^\xi - \sum_{i\in I} 2\eta^2(1+x_i)h''(x_i)\, p_i \owedge p_i \\
&- B_I \owedge_{G^{-1}-(\mathbb{1}+G_I)^{-1}} B_I - \sum_{i_1,i_2\in I} \eta^4 h'(x_{i_1})h'(x_{i_2})\, p_{i_1}^\xi \owedge p_{i_2}^\xi \\
&- 2B_{J\setminus I} \owedge_{G^{-1}} B_I - B_{J\setminus I} \owedge_{G^{-1}} B_{J\setminus I}
\end{aligned} \tag{5.18}$$

where B, G, and T are as above, and where

$$\varphi_0(\eta,x) := (1+x^{-1})\,\frac{\eta^2 h(x)}{x+\eta^2 h(x)} \quad \text{and} \quad \varphi_1(\eta,x) := (1+x^{-1})\,\frac{\eta^2 x h'(x)}{x+\eta^2 h(x)}\ . \tag{5.19}$$

[10] By definition $\|B\| := \sup_{Y\neq 0} |Y|^{-2}|B(Y,Y)| = \sup_{Y\neq 0, Z\neq 0} |Y|^{-1}|Z|^{-1}|B(Y,Z)|$. Note that throughout this paper we take the pointwise norm of any tangent vector w.r.t. the metric g_0.

Proof. Starting with Equations (B.1) and (B.2), it is straightforward to compute that for any pair of distinct indices $i_1, i_2 \in I$

$$B_{i_1} \otimes_{(\mathbb{1}+G_I)^{-1}} B_{i_2} = \eta^4 h'(x_{i_1}) h'(x_{i_2})\, p^\xi_{i_1} \otimes p^\xi_{i_2}\ . \tag{5.20}$$

In the case where $i_1 = i_2 =: i$ we observe that $p^\xi_i \otimes p^\xi_i = p^\xi_i \otimes p^b_i = 0$ and $p^b_i \otimes p^b_i = -p_i \otimes p_i$ Thus, we get

$$B_i \otimes_{(\mathbb{1}+G_I)^{-1}} B_i = -\frac{\eta^4(1+x_i)}{x_i + \eta^2 h(x_i)} \big(-h'(x_i) + x_i^{-1} h(x_i)\big)^2\, p_i \otimes p_i$$

and hence

$$\begin{aligned} &T_i + B_i \otimes_{(\mathbb{1}+G_I)^{-1}} B_i \\ &\quad = \big[(1+\eta^2 h'(x_i))\big(\varphi_0(\eta, x_i) - \varphi_1(\eta, x_i)\big) + 2\eta^2(1+x_i) h''(x_i)\big]\, p_i \otimes p_i\ . \end{aligned} \tag{5.21}$$

Identities (5.20) and (5.21) and the convergence result from Lemma 5.4 (i) are sufficient to deduce the Equation (5.18) directly from Corollary 5.3. □

The next task is to control the various terms on the right hand side of (5.18). Of course, $-g_0 \otimes g_0$ is negative definite.

5.6. Remarks.

(i) By Lemma 5.4 we may *absorb the third, fourth, and nineth term* in our expression for $R^\#$ into $-g_0 \otimes g_0$, provided $\eta > 0$ is sufficiently small.

(ii) The cone condition in the hypotheses of Theorem 3.7 implies that $c_4 := \sup_{x \geq 0} |h'(x)|$ and $c_5 := \sup_{x\geq 0} (1+x)|h''(x)|$ are finite numbers. Hence, the inequalities

$$\begin{aligned} -c_4^2\, g_0 \otimes g_0 \ &\leq \sum_{i_1, i_2 \in I} h'(x_{i_1})\, h'(x_{i_2})\, p^\xi_{i_1} \otimes p^\xi_{i_2} \ \leq\ c_4^2\, g_0 \otimes g_0 \\ -c_5\, g_0 \otimes g_0 \ &\leq \sum_{i \in I} (1+x_i)\, h''(x_i)\ p_i \otimes p_i \ \leq\ c_5\, g_0 \otimes g_0 \end{aligned}$$

enable us to *absorb the fifth and seventh term* on the right hand side of Formula (5.18) by $-g_0 \otimes g_0$ for small positive values of η, too.

The *second term* in the expression on the right hand side of (5.18) can be approximated for small values of η by

$$\Phi_I := \sum_{i \in I} \varphi_0(\eta, x_i)\, p_i \otimes p_i \tag{5.22}$$

which is a sum of manifestly *positive semidefinite* tensor fields which are defined on the domain $\Omega \subset \mathbb{H}^n$. The precise estimates are stated in the following lemma.

5.7. Lemma. — *Suppose that the function $h\colon [0,\infty) \to [0,\infty)$ satisfies the hypotheses of Theorem 3.7, and let $c_4 := \sup_{x\geq 0} |h'(x)|$ as above. Then,*

$$\big|\varphi_1(\eta,x)\big| \;\leq\; \begin{cases} c_4\eta\,\varphi_0(\eta,x) & \text{for } \; 0<x\leq \frac{\eta}{1+c_4\eta} \\ c_4\eta\,(1+\eta+c_4\eta) & \text{for } x \geq \frac{\eta}{1+c_4\eta} \end{cases} . \tag{5.23}$$

In particular, for any $\varepsilon > 0$ there exists some $\eta_0 \equiv \eta_0(h,\varepsilon) > 0$ such that for $0<\eta<\eta_0$ and for any subset $I \subset J$ with $\Omega \cap U_I \neq \emptyset$ the second term in Formula (5.18) *is pinched as follows*

$$\begin{aligned} &-\varepsilon\, g_0 \owedge g_0 - (1+\varepsilon)\,\Phi_I \\ &\qquad \leq\; -\sum_{i\in I} \big(1+\eta^2 h'(x_i)\big)\big(\varphi_0(\eta,x_i)-\varphi_1(\eta,x_i)\big)\,p_i \owedge p_i \\ &\qquad \leq\; \varepsilon\, g_0 \owedge g_0 - (1-\varepsilon)\,\Phi_I\ . \end{aligned} \tag{5.24}$$

Proof. Since h is nonnegative and $h(0)=1$, it is straightforward to verify Inequality (5.23). In order to deduce chains of inequalities (5.24) from (5.23), we recall that $|1+\eta^2 h'(x)| \leq 1 + c_4\eta^2$ and that $0 \leq \sum_{i\in I} p_i \owedge p_i \leq g_0 \owedge g_0$. □

It will be shown in the next section that the terms $B_I \owedge_{G^{-1}-(\mathbb{1}+G_I)^{-1}} B_I$ and $2\,B_{J\setminus I} \owedge_{G^{-1}} B_I$ in our expression for $R^\#$ can also be absorbed into $-g_0 \owedge g_0 - \Phi_I$, provided that $\eta > 0$ is sufficiently small. The relevant estimates are stated precisely in Proposition 6.1. They require the following additional hypothesis on the collection $\big(\mathbb{H}^{n-2}_j\big)_{j\in J}$ of subspaces which enters our construction.

5.8. *Axiom.* — For any $j \in J$ there is a nontrivial rotation $\varrho_j = \vartheta_j(2\pi/m_j) \in \operatorname{Iso}(\mathbb{H}^n)$ which fixes $\mathbb{H}^{n-2}_j$ and maps the divisor $\bigcup_{j'\in J} \mathbb{H}^{n-2}_{j'}$ into itself. In other words, each $\mathbb{H}^{n-2}_j$ shall have a nontrivial stabilizer in the group Γ introduced in the context of Diagram (3.1).

The result of our considerations can be summarized as follows.

5.9. Theorem. — *Suppose that $\mathbb{H}^{n-2}_j \subset \mathbb{H}^n$, $j \in J$, is a collection of totally geodesic, hyperbolic subspaces of codimension* 2 *which satisfies Axioms* 3.1 *and* 5.8.

Let $h\colon [0,\infty) \to [0,\infty)$ be a real analytic function such that $h(0) = 1$ and that h lies in $\mathcal{C}_\alpha(\ell)$ for some $\alpha > 0$ and some $\ell > \frac{n-1}{2}$. Then, for every $\varepsilon > 0$, there exists some $\eta_1 \equiv \eta_1(n, h, d_0, \hat{N}, \varepsilon) > 0$ such that for any $\eta \in (0, \eta_1]$ the curvature tensor $R^{\#}$ of the metric

$$g = g_0 + \sum_{j \in J} \eta^2 x_j^{-1} h(x_j)\ p_j^{\xi} \tag{5.25}$$

defined in Theorem 3.7 is pinched on each domain $\Omega \cap U_I$ as follows

$$-(1+\varepsilon)\big(g_0 \owedge g_0 + \Phi_I\big) \ \leq\ R^{\#} \ \leq\ -(1-\varepsilon)\big(g_0 \owedge g_0 + \Phi_I\big)\ . \tag{5.26}$$

In particular, the curvature operator of g is negative definite on the domain $\Omega \subset \mathbb{H}^n$, which is as usual identified with $\hat{\Omega} \subset \hat{M}$.

Here, Φ_I is the tensor field introduced in (5.22). We emphasize that this theorem is just a summary of Proposition 5.5, Remarks 5.6, Lemma 5.7, and Proposition 6.1. In view of these facts it is also possible to *interpret the summands of Φ_I geometrically* : for small values of η and x_i the sectional curvature of the plane $E_i = \operatorname{span}\{\xi_i, v_i\}$ is approximately

$$\begin{aligned} K(E_i) \ &\approx\ \frac{(-g_0 \owedge g_0 - \varphi_0(\eta, x_i)\ p_i \owedge p_i)(v_i, \xi_i, \xi_i, v_i)}{g \owedge g(v_i, \xi_i, \xi_i, v_i)} \\ &=\ -1 - \frac{\eta^2 h(x_i)}{x_i + \eta^2 h(x_i)}\ \frac{1 - \eta^2 h(x_i)}{x_i + \eta^2 h(x_i)} \ \approx\ -\frac{1}{\eta^2 + x_i} \ \gtrapprox\ -\eta^{-2}\ . \end{aligned} \tag{5.27}$$

Note that this curvature is unbounded as $\eta \to 0$. Clearly, $-\varphi_0(\eta, x_i)\ p_i \owedge p_i$ is the dominant term in Formula (5.27).

If $\bigcup_{j \in J} \mathbb{H}_j^{n-2}$ is the pre-image under the covering map $\mathrm{pr}\colon \mathbb{H}^n \to \mathbb{H}^n/\Gamma'$ of a set $\bigcup_{i=1}^{N} \bar{V}_i^{n-2}$ which has the symmetry properties assumed in Theorem 1.1, then the collection $\big(\mathbb{H}_j^{n-2}\big)_{j \in J}$ satisfies indeed Axioms 3.1 and 5.8 as required. Recall that for each $\delta > 0$ the function $h_\delta(x) := \exp(-\delta x)$ satisfies the required hypotheses on h as well, and hence *we obtain Theorem* 1.1 *in the introduction* from Theorems 3.7 and 5.9.

5.10. Remark. — Let us assume in addition to the hypotheses stated in Remark 3.10 that $\operatorname{supp}(h) \subset [0, \sinh^2 \frac{1}{2} d_0]$. Then, the fields $B_{J \setminus I}$, $T_{J \setminus I}$, and $G^{-1} - (\mathbb{1} + G_I)^{-1}$ vanish identically on $\Omega \cap U_I$ for each $I \subset J$, and Formula (5.18) reduces to

$$
\begin{aligned}
R^{\#} = &- g_0 \owedge g_0 - \sum_{i \in I} \big(1 + \eta^2 h'(x_i)\big)\big(\varphi_0(\eta, x_i) - \varphi_1(\eta, x_i)\big)\, p_i \owedge p_i \\
&- \sum_{i \in I} 2\eta^2 h'(x_i)\, g_0 \owedge p_i^{\xi} - \sum_{i \in I} 2\eta^2 (1 + x_i)\, h''(x_i)\, p_i \owedge p_i \\
&- \sum_{i_1, i_2 \in I} \eta^4 h'(x_{i_1})\, h'(x_{i_2})\, p_{i_1}^{\xi} \owedge p_{i_2}^{\xi} \, .
\end{aligned}
\tag{5.18$'$}
$$

Thus, we *get the final chain of inequalities for* $R^{\#}$ exactly as stated in (5.26), except that we need not refer to Proposition 6.1 and Axiom 5.8 at all. These considerations conclude the proof of Theorem 1.4.

6. SYMMETRIES AND FURTHER ESTIMATES

The purpose of this section is to analyze the sixth and eighth term in the expression for the curvature tensor $R^{\#}$ as computed in Proposition 5.5. The estimates which have been used to deduce Theorem 5.9 above are summarized in the following

6.1. Proposition. — *Suppose that the collection* $\left(\mathbb{H}_j^{n-2}\right)_{j\in J}$ *of subspaces in* $\mathbb{H}^n$ *and the function* $h\colon[0,\infty)\to[0,\infty)$ *satisfy the same hypotheses as in Theorem 5.9. Then, for any* $\varepsilon>0$*, there exists a constant* $\eta_2=\eta_2\big(n,h,d_0,\hat{N},\varepsilon\big)$ *such that for* $0<\eta<\eta_2$ *the following estimates hold on each domain* $\Omega\cap U_I$

$$\begin{array}{llcccl}
\text{(i)} & -\varepsilon\big(g_0\otimes g_0+\Phi_I\big) & \leq & B_I\otimes_{G^{-1}-(\mathbb{1}+G_I)^{-1}}B_I & \leq & \varepsilon\big(g_0\otimes g_0+\Phi_I\big)\\
\text{(ii)} & -\varepsilon\big(g_0\otimes g_0+\Phi_I\big) & \leq & B_{J\setminus I}\otimes_{G^{-1}}B_I & \leq & \varepsilon\big(g_0\otimes g_0+\Phi_I\big)\ .
\end{array}$$

We emphasize that this proposition requires *all hypotheses of Theorem 5.9*, including in particular the Symmetry Axiom 5.8. Its proof is given as a series of lemmas culminating in 6.8 and 6.9 below.

The first step is to rewrite the terms under consideration in such a way that they can be estimated more easily. For this purpose, we shall introduce some additional notation. We consider a fixed (finite!) subset $I\subset J$ such that $\Omega\cap U_I\neq\emptyset$. Note that $\mathbb{1}+G_i$, $\mathbb{1}+G_I$, and $G=\mathbb{1}+G_I+G_{J\setminus I}$ are invertible, and thus we may set

$$\hat{B}_i\ :=\ (\mathbb{1}+G_i)^{-1}B_i\ , \tag{6.1}$$

$$L_I\ :=G_{J\setminus I}-G_{J\setminus I}G^{-1}G_{J\setminus I}\ . \tag{6.2}$$

A straightforward computation yields

$$L_I\ =\ G_{J\setminus I}G^{-1}(\mathbb{1}+G_I)\ =\ (\mathbb{1}+G_I)-(\mathbb{1}+G_I)G^{-1}(\mathbb{1}+G_I)\ , \tag{6.3}$$

or equivalently,

$$G^{-1} - (\mathbb{1} + G_I)^{-1} \;=\; -(\mathbb{1} + G_I)^{-1} L_I (\mathbb{1} + G_I)^{-1}\ , \tag{6.3$'$}$$

hence our interest in L_I. The basic inequalities are

6.2. Lemma. — *On $\Omega \cap U_I$, there are the following bounds for the operator norm of L_I and $(\mathbb{1} + G_I)G^{-1}$ in terms of the constant c_0 from Proposition 4.1*

$$\|L_I\| \;\le\; c_0\eta^2\ , \tag{i}$$

$$\|(\mathbb{1} + G_I)G^{-1}\| \;\le\; 1 + c_0\eta^2\ . \tag{ii}$$

Proof. (i) Note that Equation (6.2) implies directly that $0 \le L_I \le G_{J\setminus I}$.

(ii) We apply the bound from Proposition 4.1 to the right hand side of the identity $(\mathbb{1} + G_I)G^{-1} = \mathbb{1} - G_{J\setminus I}G^{-1}$. □

Next, we shall discuss the quantities $\hat{B}_i$. From (5.12) we obtain

$$\begin{aligned} \hat{B}_i(Y,Z) \;=\; & -x_i^{-1/2}(1+x_i)^{1/2}\beta(\eta, x_i)\, p_i^b(Y,Z)\, \xi_i \\ & - \eta^2 x_i^{-1/2}(1+x_i)^{1/2} h'(x_i)\, p_i^\xi(Y,Z)\, v_i \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} \beta(\eta, x) \;:=\; & \frac{\eta^2 h(x)}{x + \eta^2 h(x)} - \frac{x}{x + \eta^2 h(x)}\, \eta^2 h'(x) \\ =\; & x(1+x)^{-1}\big(\varphi_0(\eta, x) - \varphi_1(\eta, x)\big)\ . \end{aligned} \tag{6.4$'$}$$

and φ_0 and φ_1 are as introduced in (5.19). Recall that $|h'(x)| \le c_4$, and hence $|\beta(\eta, x)| \le 1 + c_4\eta^2$. In particular, $\hat{B}_i$ has only a singularity of order $x_i^{-1/2}$ along $\mathbb{H}_i^{n-2}$ which is much milder than the singularity of B_i itself. The following identities are easy to check

$$B_I \otimes_{G^{-1} - (\mathbb{1} + G_I)^{-1}} B_I \;=\; -\sum_{i_1, i_2 \in I} \hat{B}_{i_1} \otimes_{L_I} \hat{B}_{i_2}\ , \tag{6.5}$$

$$B_{J\setminus I} \otimes_{G^{-1}} B_I \;=\; \sum_{i \in I} B_{J\setminus I} \otimes_{G^{-1}} (\mathbb{1} + G_I)\hat{B}_i\ . \tag{6.6}$$

Moreover, for any $i, i_1, i_2 \in I$ we compute that

$$
\begin{aligned}
&x_{i_1}^{1/2} x_{i_2}^{1/2} (1+x_{i_1})^{-1/2} (1+x_{i_2})^{-1/2}\, \hat{B}_{i_1} \otimes_{L_I} \hat{B}_{i_2} \\
&\quad = \beta(\eta, x_{i_1})\, \beta(\eta, x_{i_2})\, \langle \xi_{i_1}\,, L_I\, \xi_{i_2} \rangle p^b_{i_1} \otimes p^b_{i_2} \\
&\qquad + \eta^2 \beta(\eta, x_{i_1})\, h'(x_{i_2})\, \langle \xi_{i_1}\,, L_I\, v_{i_2} \rangle p^b_{i_1} \otimes p^\xi_{i_2} \\
&\qquad + \eta^2 h'(x_{i_1})\, \beta(\eta, x_{i_2})\, \langle v_{i_1}\,, L_I\, \xi_{i_2} \rangle p^\xi_{i_1} \otimes p^b_{i_2} \\
&\qquad + \eta^4 h'(x_{i_1})\, h'(x_{i_2})\, \langle v_{i_1}\,, L_I\, v_{i_2} \rangle p^\xi_{i_1} \otimes p^\xi_{i_2}
\end{aligned}
\tag{6.5$'$}
$$

and

$$
\begin{aligned}
&x_i^{1/2} (1+x_i)^{-1/2}\, B_{J\setminus I} \otimes_{G^{-1}} (\mathbb{1} + G_I) \hat{B}_i \\
&\quad = \beta(\eta, x_i)\, \langle \xi_i\,, (\mathbb{1} + G_I) G^{-1}\, P_i B_{J\setminus I}(\,.\,,\,.\,) \rangle \otimes p^b_i \\
&\qquad + \eta^2 h'(x_i)\, \langle v_i\,, (\mathbb{1} + G_I) G^{-1}\, P_i B_{J\setminus I}(\,.\,,\,.\,) \rangle \otimes p^\xi_i \\
&\qquad + \beta(\eta, x_i)\, \langle \xi_i\,, P_i (\mathbb{1} + G_I) G^{-1} (\mathbb{1} - P_i)\, B_{J\setminus I}(\,.\,,\,.\,) \rangle \otimes p^b_i \\
&\qquad + \eta^2 h'(x_i)\, \langle v_i\,, P_i (\mathbb{1} + G_I) G^{-1} (\mathbb{1} - P_i)\, B_{J\setminus I}(\,.\,,\,.\,) \rangle \otimes p^\xi_i\ .
\end{aligned}
\tag{6.6$'$}
$$

By Lemma 5.4 and 6.2 the tensor fields L_I, $(\mathbb{1}+G_I)G^{-1}$, and $B_{J\setminus I}$ are uniformly bounded on $\Omega \cap U_I$. In fact, the estimates for L_I and $B_{J\setminus I}$ are proportional to η^2. However, because of the factors $x_{i_1}^{1/2} x_{i_2}^{1/2} (1+x_{i_1})^{-1/2} (1+x_{i_2})^{-1/2}$ and $x_i^{1/2}(1+x_i)^{-1/2}$ the *straightforward bounds* for $B_I \otimes_{G^{-1}-(\mathbb{1}+G_I)^{-1}} B_I$ and $B_{J\setminus I} \otimes_{G^{-1}} B_I$ are *still singular* near the divisor. Roughly speaking, our plan is to show that

(1) most terms on the right hand side of (6.5$'$) and (6.6$'$) actually contain a hidden zero of the same order, and so they only make a contribution to $B_I \otimes_{G^{-1}-(\mathbb{1}+G_I)^{-1}} B_I$ resp. $B_{J\setminus I} \otimes_{G^{-1}} B_I$ which is small enough to be absorbed into $-g_0 \otimes g_0$;

(2) the remaining terms are of such a special nature that their contribution can still be dominated by $\varepsilon(g_0 \otimes g_0 + \Phi_I)$ despite the fact that it is singular.

The Symmetry Axiom 5.8 is required precisely for these refined estimates. Note that the terms $g_{J\setminus I}$ and $B_{J\setminus I}$ are actually real analytic tensor fields on all of U_I and not just on $\Omega \cap U_I$.

6.3. Lemma. — *Let $I \subset J$ be as above. Suppose that the collection $\big(\mathbb{H}^{n-2}_j\big)_{j\in J}$ of subspaces in $\mathbb{H}^n$ satisfies Axioms 3.1 and 5.8. Then, the following identities hold* *along* $\mathbb{H}_i^{n-2} \cap U_I$ *for any* $i \in I$

(i) $$P_i G_{J\setminus I}(\mathbb{1} - P_i) \;=\; (\mathbb{1} - P_i) G_{J\setminus I} P_i \;=\; 0$$

(ii) $$P_i B_{J\setminus I}\big((\mathbb{1} - P_i)\,.\,, (\mathbb{1} - P_i)\,.\big) \;=\; 0\ .$$

Similarly, $(\mathbb{1} - P_i) B_{J\setminus I}\big(P_i\,.\,, (\mathbb{1} - P_i)\,.\big)$ *and* $(\mathbb{1} - P_i) B_{J\setminus I}\big((\mathbb{1} - P_i)\,.\,, P_i\,.\big)$ *vanish. Moreover,* $P_i B_{J\setminus I}\big(P_i\,.\,, P_i\,.\big) = 0$, *provided* $m_i \equiv \operatorname{ord} \varrho_i \neq 3$.

Proof. Clearly, the isometry $\varrho_i \colon \mathbb{H}^n \to \mathbb{H}^n$ maps each subspace $\mathbb{H}_{i'}^{n-2}$, $i' \in I$, into itself, and it permutes the other subspaces $\mathbb{H}_j^{n-2}$, $j \in J \setminus I$. Hence, $\varrho_i^*(g_{J\setminus I}) = g_{J\setminus I}$, or equivalently

$$d\varrho_i\, G_{J\setminus I} = G_{J\setminus I}\, d\varrho_i\ .$$

Since $B_{J\setminus I}$ is a linear combination of covariant derivatives of $g_{J\setminus I}$, we get

$$d\varrho_i\, B_{J\setminus I} = B_{J\setminus I}\big(d\varrho_i\,.\,, d\varrho_i\,.\big)\ .$$

Because of these two identities it is sufficient to observe that at any point $p \in \mathbb{H}_i^{n-2}$ the differential $d\varrho_{i|p}$ acts as a rotation of order $m_i = \operatorname{ord} \varrho_i$ on the 2–dimensional subspace $\operatorname{im} P_{i|p} \subset T_p\mathbb{H}_i^n$, whereas it acts as the identity on its orthogonal complement $\ker P_{i|p} \subset T_p\mathbb{H}_i^n$. □

6.4. Remarks.

(i) This lemma is the only place in the proof of Proposition 6.1, and hence in the proof of Theorem 5.9, where we make explicit use of the symmetry requirements from Axiom 5.8.

(ii) The crucical statement in this lemma is that the terms $P_i G_{J\setminus I}(\mathbb{1} - P_i)$ and $P_i B_{J\setminus I}\big((\mathbb{1} - P_i)\,.\,, (\mathbb{1} - P_i)\,.\big)$ vanish along $\mathbb{H}_i^{n-2} \cap U_I$, and this claim is essentially *equivalent* to saying that *all the strata* $\hat S_I \subset \hat M$ *are totally geodesic.* The latter property is actually also a necessary condition for $K_{\hat M} \leq 0$ because of the intrinsic product structure of the S_I as explained in Remark 3.8 (ii).

(iii) This explains that the *symmetry requirements* from Axiom 5.8 are in fact *natural hypotheses* for Theorem 5.9. If one wants to drop them, one has to change the construction of the new metric g in such a way that the strata $\hat S_I \subset \hat M$ are automatically totally geodesic.

The next step is to prove estimates[11] in a neighborhood of the divisor which reflect the vanishing results of Lemma 6.3. For this purpose, we need

6.5. Lemma. — *Let g be as in Theorem 3.7. Then, there exist constants c_6 and c_7 depending just on n, h, d_0, and $\hat{N}$ such that on $U_I \subset \mathbb{H}^n$*

$$\text{(i)} \qquad \|DG_{J\setminus I}\| \;\leq\; c_6\eta^2\,,$$
$$\text{(ii)} \qquad \|D^2G_{J\setminus I}\| \;\leq\; c_7\eta^2\,,$$
$$\text{(iii)} \qquad \|DB_{J\setminus I}\| \;\leq\; \frac{3}{2}\,c_7\eta^2\;.$$

Proof. The expressions for $Dg_{J\setminus I}$ and for $D^2g_{J\setminus I}$ differ from those given in Formulae (5.10) and (5.11) only with respect to the range of indices j in the overall summation. Here, this range is restricted to the subset $J\setminus I \subset J$. Since h satisfies the cone condition $C_\alpha(\ell)$ for some $\alpha > 0$ and some $\ell > \frac{n-1}{2}$, it is clear that the functions $(1+x)\,h''(x)$, $x^{-1}(1+x)\,h'(x)$ and $x^{-2}(1+x)\,h(x)$ are all bounded by $\mathrm{Const}\,|x|^{-(n+1)/2}$ for $x \geq \sinh^2\frac{1}{2}\,d_0$. Now, the first two inequalities are direct consequences of Axiom 3.1 and Lemma 3.2. The third inequality in the Lemma follows from the second one, since

$$\begin{aligned}&2\,\langle W\,, D_XB_{J\setminus I}(Y,Z)\rangle\\ &\quad= D^2_{X,Y}g_{J\setminus I}(Z,W) + D^2_{X,Z}g_{J\setminus I}(Y,W) - D^2_{X,W}g_{J\setminus I}(Y,Z)\;.\end{aligned} \qquad \square$$

6.6. Lemma. — *Under the hypotheses of Theorem 5.9 there exist constants c_8, c_9, and c_{10} depending just on n, h, d_0, and $\hat{N}$ such that on any domain $U_I \subset \mathbb{H}^n$ the following estimates hold for any $i, i_1, i_2 \in I$ with $i_1 \neq i_2$*

$$\text{(i)} \qquad \|P_iG_{J\setminus I}(\mathbb{1}-P_i)\| \;\leq\; c_8\eta^2\,x_i^{1/2}(1+x_i)^{-1/2}\,,$$
$$\text{(ii)} \qquad \|P_iB_{J\setminus I}\big((\mathbb{1}-P_i)g.\,,(\mathbb{1}-P_i)\,.\,\big)\| \;\leq\; c_9\eta^2x_i^{1/2}(1+x_i)^{-1/2}\,,$$
$$\text{(iii)} \qquad \|P_{i_1}G_{J\setminus I}P_{i_2}\| \;\leq\; c_{10}\eta^2x_{i_1}^{1/2}x_{i_2}^{1/2}(1+x_{i_1})^{-1/2}(1+x_{i_2})^{-1/2}\;.$$

[11] We continue to work with the operator norms taken w.r.t. g_0.

Proof. (i) *and* (ii) Any point p in the domain $U_I \subset \mathbb{H}^n$ can be reached by a normal geodesic $c\colon [0, r_i(p)] \to U_I$ with $c(0) \in \mathbb{H}_i^{n-2}$ and $\dot c(0) \perp T_{c(0)}\mathbb{H}_i^{n-2}$. We restrict the tensor fields in question to c and rely on the Fundamental Theorem of Calculus. Lemma 6.3 provides the required initial values. Since P_i is parallel along c, we obtain

$$\begin{aligned} \big\|P_i G_{J\setminus I}(\mathbb{1}-P_i)_{|p}\big\| &\le \int_0^{r_i(p)} \Big\|\frac{D}{dt}\big(P_i G_{J\setminus I}(\mathbb{1}-P_i)\big)_{|c(t)}\Big\|\,dt \\ &\le \int_0^{r_i(p)} \big\|D_{\dot c(t)} G_{J\setminus I}\big\|\,dt \end{aligned} \tag{6.7}$$

and

$$\big\|P_i B_{J\setminus I}\big((\mathbb{1}-P_i)\,.\,,(\mathbb{1}-P_i)\,.\,\big)_{|p}\big\| \;\le\; \int_0^{r_i(p)} \big\|D_{\dot c(t)} B_{J\setminus I}\big\|\,dt\;. \tag{6.8}$$

We set $c_8 := 2\max\{c_0, c_6\}$ and $c_9 := 2\max\{c_3, c_7\}$. If $x_i(p) \le 1$, we observe that

$$r_i(p) \;\le\; 2\tanh r_i(p) \;=\; 2x_i(p)^{1/2}(1+x_i(p))^{-1/2}$$

and deduce Inequalities (i) and (ii) from (6.7) and (6.8) by means of Lemma 6.5. If on the other hand $x_i(p) \ge 1$, we may refer to Proposition 4.1 and Lemma 5.4 (iii) directly.

(iii) Let $i_1, i_2 \in I$ be distinct elements. Since $P_{i_1}P_{i_2} = 0$ along $\mathbb{H}_{i_1}^{n-2} \cup \mathbb{H}_{i_2}^{n-2}$, it follows from Lemma 6.3 (i) that $P_{i_1} G_{J\setminus I} P_{i_2}$ vanishes along $\mathbb{H}_{i_1}^{n-2} \cup \mathbb{H}_{i_2}^{n-2}$ as well. So here the idea is to integrate suitable bounds for the derivatives of $P_{i_1} G_{J\setminus I} P_{i_2}$ over the hyperbolic quadrilateral Q_p which is defined by p and its footpoints in $\mathbb{H}_{i_1}^{n-2}$, $\mathbb{H}_{i_1}^{n-2} \cap \mathbb{H}_{i_2}^{n-2}$, and $\mathbb{H}_{i_2}^{n-2}$. In order to get such bounds we observe that DP_i and D^2P_i are uniformly bounded[12] on $\mathbb{H}^n$. Hence, Proposition 4.1 and Lemma 6.5 imply that on the domain U_I

$$\begin{aligned} \big\|P_{i_1} G_{J\setminus I} P_{i_2}\big\| &\le c_0\eta^2\;, \\ \big\|D_{.}\big(P_{i_1} G_{J\setminus I} P_{i_2}\big)\big\| &\le \hat c_6\eta^2\;, \\ \big\|D^2_{.,.}\big(P_{i_1} G_{J\setminus I} P_{i_2}\big)\big\| &\le \hat c_7\eta^2\;, \end{aligned} \tag{6.9}$$

[12] It is not hard to see that

$$D_X P_i Y \;=\; \tanh(r_i)\big(\langle v_i\,, Y\rangle\,(\mathbb{1}-P_i)X + \langle X\,,(\mathbb{1}-P_i)Y\rangle\, v_i\big).$$

where $\hat{c}_6$ and $\hat{c}_7$ depend only on n, h, d_0, and $\hat{N}$. Now, the claim follows by Lemma B.3 in the Appendix. $\square$

6.7. Corollary. — *Under the hypotheses of Theorem 5.9 there exist continuous functions $\hat{c}_{11}, \hat{c}_{12}\colon [0,\infty) \to [0,\infty)$ such that on any domain $\Omega \cap U_I \subset \mathbb{H}^n$ the following estimates hold for any $i, i_1, i_2 \in I$ with $i_1 \neq i_2$*

(i) $$\big\| P_i(\mathbb{1} + G_I)\, G^{-1}(\mathbb{1} - P_i) \big\| \;\le\; \eta^2 \hat{c}_{11}(\eta)\, x_i^{1/2}(1 + x_i)^{-1/2}\,,$$

(ii) $$\big\| P_{i_1} L_I P_{i_2} \big\| \;\le\; \eta^2 \hat{c}_{12}(\eta)\, x_{i_1}^{1/2} x_{i_2}^{1/2}(1 + x_{i_1})^{-1/2}(1 + x_{i_2})^{-1/2}\ .$$

These estimates are much more suitable for the proof of Proposition 6.1 than the corresponding inequalities in Lemma 6.6. However, the symmetry argument used in the proof of that Lemma does not apply directly, since *neither* $(\mathbb{1} + G_I)\, G^{-1}$ *nor* L_I *extends* even continously from $\Omega \cap U_I$ to all of U_I.

Proof. (i) Since the projector P_i commutes with $\mathbb{1} + G_I$, it is clear that

$$\begin{aligned} GP_i - P_iG \;&=\; G_{J\setminus I}P_i - P_iG_{J\setminus I} \\ &=\; (\mathbb{1} - P_i)\, G_{J\setminus I}P_i - P_iG_{J\setminus I}(\mathbb{1} - P_i)\ . \end{aligned}$$

Hence, Lemma 6.6 (i) implies that

$$\big\| GP_i - P_iG \big\| \;\le\; c_8 \eta^2 x_i^{1/2}(1 + x_i)^{-1/2}\ . \tag{6.10}$$

The following identity is easy to verify

$$P_i(\mathbb{1} + G_I)\, G^{-1}(\mathbb{1} - P_i) \;=\; (\mathbb{1} + G_I)\, G^{-1}\,(GP_i - P_iG)\, G^{-1}\,(\mathbb{1} - P_i)\ .$$

The factors on the right hand side can be controlled separately by means of Inequality (6.10) and Lemma 6.2 (ii), and it is sufficient to set $\hat{c}_{11}(\eta) := c_8(1 + c_0\eta^2)$.

(ii) Expanding $G_{J\setminus I}$ as $G - (\mathbb{1} + G_I)$ and commuting the factors $(\mathbb{1} + G_I)$ and P_{i_μ} where appropriate, it is also straightforward to verify the equation

$$\begin{aligned} P_{i_1} L_I P_{i_2} \;&=\; P_{i_1}\, G_{J\setminus I} G^{-1}(\mathbb{1} + G_I)\, P_{i_2} \\ &=\; G_{J\setminus I} G^{-1}\, P_{i_1} P_{i_2}\, G_{J\setminus I} G^{-1}(\mathbb{1} + G_I) \\ &\quad + (\mathbb{1} + G_I)\, G^{-1}\, P_{i_1} G_{J\setminus I} P_{i_2}\, G^{-1}(\mathbb{1} + G_I) \\ &\quad + (\mathbb{1} + G_I)\, G^{-1}\,(GP_{i_1} - P_{i_1}G)\, G^{-1}\,(GP_{i_2} - P_{i_2}G)\, G^{-1}(\mathbb{1} + G_I)\ . \end{aligned}$$

Using Inequality (6.10), Lemma 6.2, and Lemma 6.6 (iii) we obtain sufficiently good bounds for the second and third term in the preceding sum. In order to deal with the first term, we note that for $i_1 \neq i_2$ the operator norm of the product $P_{i_1} P_{i_2}$ is given by

$$\|P_{i_1} P_{i_2}\| \;=\; \langle v_{i_1}, v_{i_2}\rangle \;=\; x_{i_1}^{1/2} x_{i_2}^{1/2} (1+x_{i_1})^{-1/2} (1+x_{i_2})^{-1/2} \,.$$

Hence Corollary 6.7 (ii) holds with $\hat{c}_{12}(\eta) := \big(1+c_0\eta^2\big)^2 \big(c_{10}+\eta^2(c_0+c_8^2)\big)$. $\square$

It remains to use all these inequalities in order to establish good bounds for the various terms on the right hand side of Formulae (6.5′) and (6.6′), thereby proving Proposition 6.1. For this purpose, it is not sufficient to use just the *crude estimates* $\big|\beta(\eta,x)\big| \leq 1 + c_4\eta^2$ and $\eta^2\big|h'(x)\big| \leq c_4\eta^2$ mentioned above. Note, however, that by Inequality (5.23) from Lemma 5.7 we can do quite a bit better

$$\begin{aligned} (1+x^{-1})\big|\beta(\eta,x)\big| \;&\leq\; \varphi_0(\eta,x) + \big|\varphi_1(\eta,x)\big| \\ &\leq\; (1+c_4\eta)\,\varphi_0(\eta,x) + c_4\eta\,(1+\eta+c_4\eta) \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} (1+x^{-1})\eta^2\big|h'(x)\big| \;&=\; \eta^2\varphi_0(\eta,x)\big|h'(x)\big| + \big|\varphi_1(\eta,x)\big| \\ &\leq\; c_4\eta\,(1+\eta)\,\varphi_0(\eta,x) + c_4\eta\,(1+\eta+c_4\eta)\,. \end{aligned} \tag{6.12}$$

Now, we have finished our preparations and come to the two final lemmata in this section.

6.8. Lemma. — *Under the hypotheses of Theorem 5.9 there exist continuous functions* $\hat{c}_{13}, \hat{c}_{14}, \hat{c}_{15}\colon [0,\infty) \to [0,\infty)$ *such that on any domain* $\Omega \cap U_I \subset \mathbb{H}^n$ *the following inequalities hold for any* $i, i_1, i_2 \in I$ *with* $i_1 \neq i_2$

(i) $$-\,\eta^2\hat{c}_{13}(\eta)\, g_0 \circledwedge g_0 \;\leq\; \hat{B}_{i_1} \circledwedge_{L_I} \hat{B}_{i_2} \;\leq\; \eta^2\hat{c}_{13}(\eta)\, g_0 \circledwedge g_0$$

(ii) $$\begin{aligned} &-\,\eta^2\big(\hat{c}_{14}(\eta)\varphi_0(\eta,x_i) + \hat{c}_{15}(\eta)\big)\, p_i \circledwedge p_i \\ &\qquad \leq \hat{B}_i \circledwedge_{L_I} \hat{B}_i \;\leq\; \eta^2\big(\hat{c}_{14}(\eta)\varphi_0(\eta,x_i) + \eta\hat{c}_{15}(\eta)\big)\, p_i \circledwedge p_i\,. \end{aligned}$$

These inequalities enable us to absorb $B_I \circledwedge_{G^{-1}-(\mathbb{1}+G_I)^{-1}} B_I = \sum_{i_1,i_2\in I} \hat{B}_{i_1} \circledwedge_{L_I} \hat{B}_{i_2}$ for small $\eta > 0$ into $-g_0 \circledwedge g_0 - \Phi_I$ as claimed in Proposition 6.1 (i).

Proof. (i) Since $|h'(x)| \le c_4$ and $|\beta(\eta,x)| \le 1 + c_4\eta^2$ for any $x \in [0,\infty)$, it follows directly from Corollary 6.7 (ii) that

$$\begin{aligned}&\big|\beta(\eta,x_{i_1})\,\beta(\eta,x_{i_2})\,\langle\xi_{i_1}\,,L_I\,\xi_{i_2}\rangle\big| + \eta^2\,\big|\beta(\eta,x_{i_1})\,h'(x_{i_2})\,\langle\xi_{i_1}\,,L_I\,v_{i_2}\rangle\big|\\ &+\eta^2\,\big|h'(x_{i_1})\,\beta(\eta,x_{i_2})\,\langle v_{i_1}\,,L_I\,\xi_{i_2}\rangle\big| + \eta^4\,\big|h'(x_{i_1})\,h'(x_{i_2})\,\langle v_{i_1}\,,L_I\,v_{i_2}\rangle\big|\\ &\qquad\qquad\le\ \eta^2(1+2c_4\eta^2)^2\hat c_{12}(\eta)x_{i_1}^{1/2}x_{i_2}^{1/2}(1+x_{i_1})^{-1/2}(1+x_{i_2})^{-1/2}\end{aligned}$$

hence the claim.

(ii) In this case Equation (6.5′) can be simplified substantially using that the products $p_i^b \otimes p_i^\xi$ and $p_i^\xi \otimes p_i^\xi$ vanish identically and that $p_i^b \otimes p_i^b = -p_i \otimes p_i$

$$\hat B_i \otimes_{L_I} \hat B_i \;=\; \big(1+x_i^{-1}\big)\beta(\eta,x_i)^2\langle\xi_i\,,L_I\xi_i\rangle\,p_i \otimes p_i\ . \tag{6.13}$$

Using the inequality $\big|\beta(\eta,x)\big| \le 1 + c_4\eta^2$, Formula (6.11), and Lemma 6.2 (i), we can finish the proof setting $\hat c_{14} := c_0(1+c_4\eta)(1+c_4\eta^2)$ and $\hat c_{15} := c_0c_4(1+\eta+c_4\eta)(1+c_4\eta^2)$. ☐

It remains to deal with the second part of Proposition 6.1. We decompose the bilinear map $P_iB_{J\setminus I}$ in the expression on the right hand side of (6.6′) as

$$P_i\,B_{J\setminus I} \;=\; B^{0,i}_{J,I} + B^{1,i}_{J,I} + B^{2,i}_{J,I} \tag{6.14}$$

where

$$\begin{aligned}B^{0,i}_{J,I}(Y,Z) \;&:=\; P_i\,B_{J\setminus I}\big(P_i\,Y\,,P_i\,Z\big)\\ B^{1,i}_{J,I}(Y,Z) \;&:=\; P_i\,B_{J\setminus I}\big(P_i\,Y\,,(\mathbb{1}-P_i)Z\big)\\ &\qquad + P_i\,B_{J\setminus I}\big((\mathbb{1}-P_i)Y\,,P_i\,Z\big)\\ B^{2,i}_{J,I}(Y,Z) &:=\; P_i\,B_{J\setminus I}\big((\mathbb{1}-P_i)Y\,,(\mathbb{1}-P_i)Z\big)\ .\end{aligned}$$

It is convenient to introduce

$$\begin{aligned}\Psi^{\mu}_{J,I,i} \;:=\; &\beta(\eta,x_i)\langle\xi_i\,,(\mathbb{1}+G_I)\,G^{-1}B^{\mu,i}_{J,I}(\,.\,,\,.\,)\rangle \otimes p_i^b\\ &+\eta^2h'(x_i)\langle v_i\,,(\mathbb{1}+G_I)\,G^{-1}B^{\mu,i}_{J,I}(\,.\,,\,.\,)\rangle \otimes p_i^\xi\end{aligned} \tag{6.15}$$

$$\begin{aligned}\overline{\Psi}_{J,I,i} := \;&\beta(\eta,x_i)\langle\xi_i\,,P_i(\mathbb{1}+G_I)\,G^{-1}(\mathbb{1}-P_i)B_{J,I}(\,.\,,\,.\,)\rangle \otimes p_i^b\\ &+\eta^2h'(x_i)\langle v_i\,,P_i(\mathbb{1}+G_I)\,G^{-1}(\mathbb{1}-P_i)B_{J,I}(\,.\,,\,.\,)\rangle \otimes p_i^\xi\ .\end{aligned} \tag{6.16}$$

With this notation Formula (6.6′) can be rewritten as

$$(6.6'') \qquad \begin{aligned} &B_{J\setminus I} \otimes (\mathbb{1} + G_I)\hat{B}_i \\ &\quad = -x_i^{-1/2}(1+x_i)^{1/2}\big(\Psi^0_{J,I,i} + \Psi^1_{J,I,i} + \Psi^2_{J,I,i} + \overline{\Psi}_{J,I,i}\big) \ . \end{aligned}$$

The relevant estimates can be summarized as follows.

6.9. Lemma. — *Under the hypotheses of Theorem 5.9 there exist continuous functions $\hat{c}_{16}, \dots, \hat{c}_{20}\colon [0,\infty) \to [0,\infty)$ such that on any domain $\Omega \cap U_I$ the following inequalities hold for any $i \in I$*

$$\text{(i)} \qquad \begin{aligned} &-\eta^2\big(\hat{c}_{16}(\eta)\,\varphi_0(\eta, x_i) + \eta\hat{c}_{17}(\eta)\big)p_i \otimes p_i + \eta^2\hat{c}_{18}(\eta)\,g_0 \otimes g_0 \\ &\qquad \le\ x_i^{-1/2}(1+x_i)^{1/2}\big(\Psi^0_{J,I,i} + \Psi^1_{J,I,i}\big) \\ &\qquad\qquad \le\ \eta^2\big(\hat{c}_{16}(\eta)\,\varphi_0(\eta, x_i) + \eta\,\hat{c}_{17}(\eta)\big)\,p_i \otimes p_i + \eta^2\hat{c}_{18}(\eta)\,g_0 \otimes g_0 \end{aligned}$$

$$\text{(ii)} \qquad -\eta^2\hat{c}_{19}(\eta)\,g_0 \otimes g_0 \ \le\ x_i^{-1/2}(1+x_i)^{1/2}\,\Psi^2_{J,I,i} \ \le\ \eta^2\hat{c}_{19}(\eta)\,g_0 \otimes g_0$$

$$\text{(iii)} \qquad -\eta^4\hat{c}_{20}(\eta)\,g_0 \otimes g_0 \ \le\ x_i^{-1/2}(1+x_i)^{1/2}\,\overline{\Psi}_{J,I,i} \ \le\ \eta^4\hat{c}_{20}(\eta)\,g_0 \otimes g_0 \ .$$

Proof. (i) We consider the subspace $E = \operatorname{span}\{v_i, \xi_i\}$ and the symmetric bilinear forms

$$\begin{aligned} b^{\xi}_{J,I,i}(Y,Z) &:= \langle \xi_i \,, (\mathbb{1} + G_I)\,G^{-1}(B^{1,i}_{J,I}(Y,Z) + B^{2,i}_{J,I}(Y,Z))\rangle \\ b^{v}_{J,I,i}(Y,Z) &:= \langle v_i \,, (\mathbb{1} + G_I)\,G^{-1}(B^{1,i}_{J,I}(Y,Z) + B^{2,i}_{J,I}(Y,Z))\rangle \ . \end{aligned}$$

By the definition of $B^{1,i}_{J,I}$ and $B^{2,i}_{J,I}$ it is clear that Corollary A.4 applies to $b^{\xi}_{J,I,i} \otimes p^b_i$ as well as to $b^{v}_{J,I,i} \otimes p^{\xi}_i$. From Lemma 5.4 (iii) and Lemma 6.2 (ii) we conclude that $\|b^{\xi}_{J,I,i}\| \le c_3\eta^2(1 + c_0\eta^2)$ and $\|b^{v}_{J,I,i}\| \le c_3\eta^2(1 + c_0\eta^2)$. Working with $\delta = x_i^{-1/2}(1+x_i)^{1/2}|\beta(\eta, x_i)|$ (resp. $\delta = x_i^{-1/2}(1+x_i)^{1/2}|\eta^2 h'(x_i)|$), we obtain

$$\begin{aligned} &-c_3\eta^2(1+c_0\eta^2)\,g_0 \otimes g_0 \\ &-c_3\eta^2(1+c_0\eta^2)\big(1+x_i^{-1}\big)\big(\beta(\eta,x_i)^2 + \eta^4 h'(x_i)^2\big)\,p_i \otimes p_i \\ &\qquad \le\ x_i^{-1/2}(1+x_i)^{1/2}\big(\Psi^0_{J,I,i} + \Psi^1_{J,I,i}\big) \\ &\qquad \le\ c_3\eta^2(1+c_0\eta^2)\big(1+x_i^{-1}\big)\big(\beta(\eta,x_i)^2 + \eta^4 h'(x_i)^2\big)\,p_i \otimes p_i \\ &\qquad\qquad\qquad + c_3\eta^2(1+c_0\eta^2)\,g_0 \otimes g_0 \ . \end{aligned}$$

In order to control the coefficients of $p_i \owedge p_i$, we combine Inequalities (6.11) and (6.12) with the crude estimates for $\beta(\eta, x_2)$ and $\eta^2 h'(x_i)$. As a result we get Inequality (i) with

$$\begin{aligned}
\hat{c}_{16}(\eta) &= c_3(1+c_0\eta^2)(1+c_4\eta+c_4\eta^2+2c_4^2\eta^3+c_4^2\eta^4)\,,\\
\hat{c}_{17}(\eta) &= c_3c_4(1+c_0\eta^2)(1+2c_4\eta^2)(1+\eta+c_4\eta)\,,\\
\hat{c}_{18}(\eta) &= c_3(1+c_0\eta^2)\ .
\end{aligned}$$

(ii) Here, we set $\hat{c}_{19}(\eta) := c_9(1+c_0\eta^2)(1+2c_4\eta^2)$ and refer to Lemma 6.2 (ii) and Lemma 6.6 (ii).

(iii) According to Lemma 5.4 (iii) and Corollary 6.7 (i) the result follows when setting $\hat{c}_{20}(\eta) := c_3\hat{c}_{11}(\eta)$. $\square$

7. ZERO CURVATURE

It has been shown in Theorem 5.9 that for sufficiently small η the curvature operator $\hat{R}^{\#}$ of the metric g is negative definite on the open dense set $\hat{\Omega} \subset \hat{M}$ and this defines a metric with negative semi-definite curvature operator on all of M. In this section we are going to describe the 2–planes E with curvature $K(E) = 0$. We will show that our metric has *as little zero curvature as is allowed by the fundamental group* of the blow–up $\pi\colon M \to \mathbb{H}^n/\Gamma'$.

When constructing g in Theorem 3.7 (iii), we have already seen that the submanifolds $\pi^{-1}(\bar{V}_i) \subset \hat{M}$ are totally geodesic, flat $\mathbb{RP}^1$–bundles. In other words, they split off a local $\mathbb{RP}^1$–factor. More generally, if $\bar{V}_{i_1} \cap \ldots \cap \bar{V}_{i_s} \neq \emptyset$ for distinct $\bar{V}_{i_1}, \ldots, \bar{V}_{i_s}$, then $\pi^{-1}(\bar{V}_{i_1} \cap \ldots \cap \bar{V}_{i_s})$ is totally geodesic and splits off a local torus–factor of dimension s. In Theorem 7.3 we show that *all zero curvatures of the metric g come from these product submanifolds.*

In the second part of this chapter, we show that the *existence of the submanifolds $\pi^{-1}(\bar{V}_{i_1} \cap \ldots \cap \bar{V}_{i_s})$ follows from algebraic properties of the fundamental group, and so do the basic geometric properties of $\bar{V}_{i_1}, \ldots, \bar{V}_{i_s}$*. More precisely, if M^* is another compact, real analytic, Riemannian manifold with $K \leq 0$ and $\pi_1(M^*) \cong \pi_1(M)$, then we find in M^* similar totally geodesic product submanifolds. The details are given in Theorem 7.8. Roughly speaking, this result implies that *all zero curvatures of the metric g are enforced by the fundamental group.*

We start with the description of the planes $E \subset T_{\hat{p}}\hat{M}$ with $K(E) = 0$. Since the metric has strictly negative sectional curvature on $\hat{\Omega}$, the footpoint $\hat{p}$ must lie in some singular stratum $\hat{S}_I$, $I \neq \emptyset$. At $\hat{p} \in \hat{S}_I$ the tangent space of $\hat{M}$ splits naturally as an orthogonal sum

(7.1) $$T_{\hat{p}}M = T_{\hat{p}}\mathcal{F}_I \oplus \nu_{\hat{p}}\hat{S}_I \oplus \mathcal{H}_{\hat{S},\hat{p}}$$

where $\mathcal{F}_I$ denotes the foliation of $\hat{S}_I \subset \hat{M}$ by the fibres of the blow–up $\pi_I \colon \hat{M} \to \mathbb{H}^n$, i.e. by flat tori $\mathbb{T}^{\# I} := (\mathbb{R}/\pi\eta\mathbb{Z})^{\# I}$, where $\mathcal{H}_{\hat{S},\hat{p}}$ is the horizontal space of $\pi_I := \pi_{|\hat{S}_I}$, and where $\nu_{\hat{p}}\hat{S}_I$ stands for the normal space of $\hat{S}_I$ at $\hat{p}$. Evidently,

$$T_{\hat{p}}\hat{S}_I = T_{\hat{p}}\mathcal{F}_I \oplus \mathcal{H}_{\hat{S},\hat{p}} \ .$$

We shall find it convenient to introduce the shorthand

$$\mathcal{H}_{\hat{p}} := \nu_{\hat{p}}\hat{S}_I \oplus \mathcal{H}_{\hat{S},\hat{p}} \ . \tag{7.2}$$

In fact, the subbundles $T\mathcal{F}_I$ and $\nu\hat{S}_I$ of $T\hat{M}_{|\hat{S}_I}$ even split naturally as an orthogonal sum of $\# I$ line bundles

$$T\mathcal{F}_I = \textstyle\bigoplus_{i\in I} L_i \qquad \text{and} \qquad \nu\hat{S}_I = \textstyle\bigoplus_{i\in I} L_i^\nu \ . \tag{7.3}$$

This decomposition is characterized by the property that the plane bundles $L_i \oplus L_i^\nu$ extend $(\operatorname{im} P_i)_{|\Omega}$ to an analytic plane bundle $\hat{E}_i$ on $S_i \cup \hat{\Omega}$ for any $i \in I$. More precisely, each $\hat{E}_i$ is the pull-back of the corresponding bundle $E_i = \operatorname{im} P_i \subset T\mathbb{H}^n$ under the blow–up.

7.1. Proposition. — *Let $I \subset J$ and suppose that $\eta > 0$ is sufficiently small* [13]. *Then, at any point $\hat{p} \in \hat{S}_I$, the metric g has strictly negative sectional curvature*

(i) on each plane $\hat{E}_{i|\hat{p}}$, $i \in I$, and

(ii) on any plane $E \subset \mathcal{H}_{\hat{p}}$.

In fact, the restriction of the curvature operator $\hat{R}^{\#}$ to $\Lambda^2\mathcal{H}_{\hat{p}}$ is negative definite.

The proof of this proposition will be given later in this section. The method is to approximate the plane $E \subset T_{\hat{p}}\hat{M}$ in question by a sequence $E^{(\mu)}$ of planes whose footpoints $\hat{p}_\mu$ lie in $\hat{\Omega}$. We then compute the limit

$$K(E) = \lim_{\mu\to\infty} K(E^{(\mu)}) \ . \tag{7.4}$$

By similar limiting arguments we can read off directly that the metric g has in fact zero curvature on at least those planes which are required by Remark 3.8 (ii).

7.2. Proposition. — *Let $\eta > 0$ be sufficently small*[13]. *Then, at a given point $\hat{p} \in \hat{S}_I$, the curvature K of the metric g vanishes on*

(i) any plane $E \subset T_{\hat{p}}\mathcal{F}_I$, and on

(ii) any plane E which intersects $\hat{E}_{i|\hat{p}}$ orthogonally in $L_{i|p}$.

---

[13] cf. Theorem 5.9.

By continuity we know already that the sectional curvature of $(\hat{M}, g)$ is non-positive. This extra piece of information is enough to deduce from Propositions 7.1 and 7.2 the following result by purely algebraic manipulations.

7.3. Theorem. — *Consider a point $\hat{p} \in \hat{S}_I$, $I \subset J$. Then, under the hypotheses of Theorem 5.9, the sectional curvature K of the metric g constructed in Theorem 3.7 vanishes on a plane $E \subset T_{\hat{p}}\hat{M}$, if and only if*

(i) E lies in $T_{\hat{p}}\mathcal{F}_I$, or

(ii) E is the span of a vector $w = \sum_{i \in I} a_i \tilde{K}_i \in T_{\hat{p}}\mathcal{F}_I$ and a vector $\bar{w}$ such that $\bar{w}$ is perpendicular to all planes $\hat{E}_i$, $i \in I$, with $a_i \neq 0$.

7.4. Remark. — Note that $E \subset T_{\hat{p}}\mathcal{F}_I$ means that E is tangent to the torus factor of $\hat{S}_I$. If E is spanned by w and $\bar{w}$ as in the Theorem, let $i_1, \ldots, i_s \in I$ be the indices with $a_i \neq 0$. Then, $\hat{p} \in \pi^{-1}(\mathbb{H}^{n-2}_{i_1} \cap \ldots \cap \mathbb{H}^{n-2}_{i_s})$, and this submanifold splits as $\mathbb{T}^s \times D$ with an s–dimensional torus factor. The vector w is tangent to $\mathbb{T}^s$, and, since $\bar{w}$ is perpendicular to $\hat{E}_{i_k}$, $1 \leq k \leq s$, it follows that $\bar{w}$ is tangent to D. Thus, Theorem 7.3 tells us that *all the zero curvatures come from the totally geodesic product manifolds of the type $\pi^{-1}(\mathbb{H}^{n-2}_{i_1} \cap \ldots \cap \mathbb{H}^{n-2}_{i_s})$.*

Taking into account the stronger statement that the curvature operator of the metric g is negative semi-definite everywhere, we can deduce from Proposition 7.1 and 7.2 by standard polarisation formulae a fairly precise structural statement about the curvature operator along each singular stratum $\hat{S}_I$.

7.5. Theorem. — *For any $I \subset J$ and any $\hat{p} \in \hat{S}_I$ the curvature operator $\hat{R}_{|\hat{p}}$ of the metric g has the following properties :*

(i) the 1–dimensional spaces $\Lambda^2 \hat{E}_{i|\hat{p}}$, $i \in I$, are eigenspaces of $\hat{R}^{\#}_{|\hat{p}}$ with a strictly negative eigenvalue ;

(ii) $\Lambda^2 \mathcal{H}_{\hat{p}}$ is an invariant subspace, and the bilinear form $\hat{R}_{|\hat{p}}$ is negative definite on this subspace ;

(iii) $\ker(\hat{R}_{|\hat{p}}) = \left(\Lambda^2 \mathcal{H}_{\hat{p}} \oplus \bigoplus_{i \in I} \Lambda^2 \hat{E}_{i|p}\right)^{\perp}$.

It remains to prove Proposition 7.1. For this purpose, we use the local formulae for each set $\hat{U}_{I'}$ such that $I \subset I' \subset J$. These domains cover $\hat{S}_I \subset \hat{M}$. For convenience we shall actually work in the $2^{\# I'}$–fold covering $\hat{\hat{U}}_{I'} := W^U_{I'} \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I'} \xrightarrow{pr_{I'}} \hat{U}_I$. Note that $\mathrm{pr}^{-1}_{I'}\,\hat{S}_I$ decomposes into $2^{\# I' - \# I}$ connected components. By slight abuse of notation we continue to write L_i and L^ν_i for the pull–backs $\mathrm{pr}^*_{I'}\,L_i$ and $\mathrm{pr}^*_{I'}\,L^\nu_i$ which describe the decomposition of $T_{\hat{\hat{p}}}\hat{\hat{U}}_I$, at some point $\hat{\hat{p}} \in \mathrm{pr}^{-1}_{I'}(\hat{S}_I)$. For each $i' \in I'$ the field $\hat{\hat{K}}_{i'} = (0; 0, \ldots, 1, \ldots, 0)$ is the unique real analytic vector field such that $d\pi_{I'}\hat{\hat{K}}_{i'} = K_{i'}$. Evidently, $\hat{\hat{K}}_{i_{|\hat{\hat{p}}}}$ lies in the intersection of $T_{\hat{\hat{p}}}\hat{\mathcal{F}}$ with the analytic lift $\hat{\hat{E}}_{i_{|\hat{\hat{p}}}}$. Since $\hat{\hat{K}}_{i_{|\hat{\hat{p}}}} \neq 0$, we can use this vector as a generator for $\mathrm{pr}^*_{I'}\,L_{i\,|\hat{\hat{p}}}$. Hence, the orthogonal sum decomposition stated in the first ormula in (7.3) follows directly from the next lemma.

7.6. Lemma. — *The vector fields* $\big(\frac{1}{\eta}\hat{\hat{K}}_{i_{|\hat{\hat{p}}}}\big)_{i\in I}$ *form an orthonormal basis for* $T_{\hat{\hat{p}}}\hat{\mathcal{F}}$ *with respect to the metric g from Theorem 7.5.*

Proof. For any $i_1, i_2 \in I \subset I'$ the expression $g_{J\setminus I'}(K_{i_1}, K_{i_2})$ vanishes identically on $S_I \subset \mathbb{H}^n$. Thus, it is sufficient to work with the finite piece $g_0 + g_{I'}$ and verify that on $\hat{\hat{U}}_{I'} \cap \hat{\hat{\Omega}}$ the following identity holds

$$(g_0 + g_{I'})\big(\hat{\hat{K}}_{i_1}, \hat{\hat{K}}_{i_2}\big) = \big(x_{i_1} + \eta^2 h(x_{i_1})\big)\,\delta_{i_1 i_2}\ .$$

Here, we have made use of the fact that by the local geometry of the intersection $\langle K_{i_1}, K_{i_2}\rangle = x_{i_1}\delta_{i_1 i_2}$ on $U_{I'}$, provided that $i_1, i_2 \in I'$. To finish the proof, we recall that x_{i_1} vanishes identically on S_I for any $i_1 \in I$ and take into account that $h(0) = 1$ by hypothesis. ☐

The next ingredient into our evaluation of the sectional curvatures along $\hat{S}_I$ are analytic vector fields $\hat{v}_i$ on $\hat{\hat{U}}_{I'}$ whose restrictions to $\mathrm{pr}^{-1}_{I'}(\hat{S}_I)$ generate the line bundles L^ν_i, $i \in I$. For this step it is crucial that we are working on $\hat{\hat{U}}_{I'}$ and not on its quotient $\hat{U}_{I'} = \hat{\hat{U}}_{I'}\big/\,Stab_{I'}$. On each hyperplane $W_{i'} \subset (\mathbb{H}^n, g_0)$ we choose an oriented distance function $\varrho_{i'}\colon W_i \twoheadrightarrow \mathbb{R}$, i.e. $|\varrho_{i'}(p)| = \mathrm{dist}(p, \mathbb{H}^{n-2}_{i'})$. Evidently, its gradient field $v^\varrho_{i'} := \mathrm{grad}_{g_0}\varrho_{i'}$ takes values in the tangent bundle $TW_{i'}$. Moreover, for any $\hat{\hat{p}} \in W^U_{I'} \subset W_{i'}$ it is clear that $v^\varrho_{i'} \in TW^U_{I'}$. It is standard to extend $\varrho_{i'}$ and $v^\varrho_{i'}$ to functions $\hat{\hat{\varrho}}_{i'}\colon \hat{\hat{U}}_{I'} \equiv W^U_{I'} \times (\mathbb{R}/2\pi\mathbb{Z})^{\# I'} \to \mathbb{R}$ and horizontal vector fields $\hat{\hat{v}}^\varrho_{i'}$ on $\hat{\hat{U}}_{I'}$. Note that $(d\pi_{I'}\hat{\hat{v}}^\varrho_{i'})_{|\hat{\hat{p}}} = \pm v_{i\,|\pi_{I'}(\hat{\hat{p}})}$ at any point $\hat{\hat{p}} = (p, \varphi) \in \hat{\hat{U}}_{I'} \cap \pi^{-1}_{I'}(\Omega)$, and hence the pull–back $\mathrm{pr}^*_{I'}\,\hat{E}_i$ is generated by $\frac{1}{\eta}\hat{\hat{K}}_i$ and $\hat{\hat{v}}^\varrho_i$ for any $i \in I \subset I'$.

7.7. Lemma.

(i) The inner products of the vector fields $\hat{\hat{v}}^{\varrho}_{i'}$, $i' \in I'$, *with respect to the metric* $g_0 + g_{I'}$ *on* $\hat{\hat{U}}_{I'}$ *are given by*

$$(7.5)\qquad \begin{aligned} (g_0+g_{I'})\big(\frac{1}{\eta}\hat{\hat{K}}_{i'_1},\hat{\hat{v}}^{\varrho}_{i'_2}\big) &= 0 \\ (g_0+g_{I'})\big(\hat{\hat{v}}^{\varrho}_{i'_1}\,,\hat{\hat{v}}^{\varrho}_{i'_2}\,\big) &= \delta_{i'_1 i'_2} + (1-\delta_{i'_1 i'_2})\tanh\varrho_{i'_1}\,\tanh\varrho_{i'_2}\,. \end{aligned}$$

(ii) At any point $\hat{\hat{p}} \in \mathrm{pr}_{I'}^{-1}\big(\hat{S}_I\big)$ *the vectors* $(\hat{\hat{v}}^{\varrho}_{i\,|\hat{\hat{p}}})_{i\in I}$ *form an orthogonal basis of almost unit vectors for the normal space* $\mathrm{pr}^*_{I'}\big(\nu\hat{S}_I\big)_{|\hat{\hat{p}}}$ *of* $\mathrm{pr}_{I'}^{-1}\big(\hat{S}_I\big)$ *w.r.t. the metric* g *on* $\hat{M}$. *In particular,*

$$(7.6)\qquad 1-c_0\eta^2 \;\le\; g\big(\hat{\hat{v}}^{\varrho}_i,\hat{\hat{v}}^{\varrho}_i\big)_{|\hat{\hat{p}}} \;\le\; 1+c_0\eta^2\,,$$

where c_0 *is the constant from Proposition* 4.1.

(iii) A non–trivial vector $w \in \mathcal{H}_{\hat{p}}$, $\hat{p} \in \hat{S}_I$, *is orthogonal to* $\nu_{\hat{p}}\hat{S}_I$ *with respect to the metric* g *on* $\hat{M}$ *if and only if its image* $d\pi_{I'|\hat{p}}\,w \in T_{\pi_{I'}(\hat{p})}\mathbb{H}^n$ *is orthogonal to all spaces* $E_{i_{|\pi_{I'}(\hat{p})}}$, $i \in I$, *with respect to the hyperbolic metric.*

By construction it is clear that $\mathrm{pr}^*_{I'}\,L^{\nu}_{i\,|\hat{\hat{p}}} = \mathbb{R}\hat{\hat{v}}^{\varrho}_{i\,|\hat{\hat{p}}}$ for any $i \in I$ and any point $\hat{\hat{p}} \in \mathrm{pr}_{I'}^{-1}\,\hat{S}_I$, and thus the lemma implies the orthogonal sum decomposition of $\nu\hat{S}_I$ described in the second formula in (7.3).

Proof. (i) On the open dense set $\mathrm{pr}_{I'}^{-1}(\hat{\Omega})$ these formulae are a mere restatement of the local trigonometric properties of the divisor listed in (B.1). By continuity they can be extended to the entire domain $\hat{\hat{U}}_{I'}$.

(ii) *and* (iii) As explained in Lemma 6.3 (i), the symmetry properties from Axiom 5.8 imply that

$$(7.7)\qquad g_{J\setminus I'}\big(d\pi_{I'}\,v^{\varrho}_{i'_1}\,,d\pi_{I'}\,v^{\varrho}_{i'_2}\big) \;=\; 0$$

for any pair of distinct indices $i'_1, i'_2 \in I'$. However, in the case that $i'_1 = i'_2$ we only have the inequality

$$(7.7')\qquad g_{J\setminus I'}\big(d\pi_{I'}\,v^{\varrho}_{i'_1}\,,d\pi_{I'}\,v^{\varrho}_{i'_1}\big) \;\le\; c_0\eta^2$$

from Proposition 4.1. The claim follows when combining Formulae (7.5), (7.7), and (7.7′). □

Proof of Proposition 7.1. (i) For any $i \in I$ it follows from Theorem 5.9 that

$$
\begin{aligned}
-(1+\varepsilon)\,\eta^{-2}\Big(x_i + \frac{(1+x_i)\eta^2 h(x_i)}{x_i+\eta^2 h(x_i)}\Big)_{|\hat{\hat{p}}_\nu} \\
\leq\ \mathrm{pr}_{I'}^*\big(R^{\#}\big)\big(\tfrac{1}{\eta}\hat{\hat{K}}_i\,,\hat{\hat{v}}_i^{\varrho},\hat{\hat{v}}_i^{\varrho},\tfrac{1}{\eta}\hat{\hat{K}}_i\big)_{|\hat{\hat{p}}_\nu} \\
\leq\ -(1-\varepsilon)\,\eta^{-2}\Big(x_i + \frac{(1+x_i)\eta^2 h(x_i)}{x_i+\eta^2 h(x_i)}\Big)_{|\hat{\hat{p}}_\nu}
\end{aligned}
$$

for any sequence of points $\hat{\hat{p}}_\nu \in \mathrm{pr}_{I'}^{-1}(\hat{\Omega})$ converging to $\hat{\hat{p}}$. In the limit, we obtain

$$
-(1+\varepsilon)\,\eta^{-2}\ \leq\ \mathrm{pr}_{I'}^*\big(R^{\#}\big)\big(\tfrac{1}{\eta}\hat{\hat{K}}_i\,,\hat{\hat{v}}_i^{\varrho},\hat{\hat{v}}_i^{\varrho},\tfrac{1}{\eta}\hat{\hat{K}}_i\big)_{|\hat{\hat{p}}}\ \leq\ -(1-\varepsilon)\,\eta^{-2}\ ,
$$

hence the claim.

(ii) A similar argument based on the horizontal vector fields w_1 and w_2 yields

$$
\begin{aligned}
-(1+\varepsilon)\Big(g_0 \owedge g_0 + \sum_{i'\in I'\setminus I}\Phi_{i'}\Big)(w_1,w_2;w_2,w_1)_{|\hat{p}} \\
\leq\ R^{\#}(w_1,w_2;w_2,w_1)_{|\hat{p}} \\
\leq\ -(1-\varepsilon)\Big(g_0 \owedge g_0 + \sum_{i'\in I'\setminus I}\Phi_{i'}\Big)(w_1,w_2;w_2,w_1)_{|\hat{p}} \quad .
\end{aligned}
$$

Note that $x_{i'}(\hat{p}) \neq 0$ for $i' \in I' \setminus I$ and $\hat{p} \in \hat{S}_I$. Since the area $g \owedge g(w_1,w_2;w_2,w_1)_{|\hat{p}}$ is finite, the inequality proves that the plane $\mathrm{span}\{w_1,w_2\} \subset T_{\hat{p}}\hat{M}$ has bounded, strictly negative sectional curvature. □

Our next goal is to show that the manifold M *has as little zero curvature as allowed by the fundamental group.* In other words, we shall reconstruct all the zero curvatures of M from its fundamental group. By Theorem 7.3 it is sufficient to recover the submanifolds $\pi^{-1}\big(\bar{V}_{i_1}\cap\ldots\cap\bar{V}_{i_s}\big)$ from the fundamental group of M and recognize them as flat $\mathbb{T}^s$–bundles.

More precisely, we shall consider another n–dimensional manifold M^* with a complete, real analytic Riemannian metric of non-positive sectional curvature and with isomorphic fundamental group $\pi_1(M) \cong \pi_1(M^*)$. We represent $M = \tilde{M}/\Delta$ and $M^* = \tilde{M}^*/\Delta^*$, where $\tilde{M}$, $\tilde{M}^*$ are the universal covering spaces, and where Δ and Δ^* are the deck-transformation groups. Note that Δ and Δ^* are isomorphic. In a first step we describe the geometric structure of $\tilde{M}$ and relate it to the algebraic structure of Δ. Afterwards, in Theorem 7.8, we reconstruct the geometric structure of $\tilde{M}^*$ from the algebraic structure of $\Delta^*(\cong \Delta)$.

To begin with, we observe that the pre-image of the embedded submanifolds $\pi^{-1}(\bar{V}_i) \subset M$ under the canonical projection $\tilde{M} \to M$ consists of a collection of totally geodesic hypersurfaces $Y_j \subset \tilde{M}$, $j \in \tilde{J}$. In fact, since the covering $\tilde{M} \to M$ factors over $\hat{M}$, the Y_j can also be described as the connected components of the pre-images of the various singular varieties $\pi^{-1}(\mathbb{H}^{n-2}_{\jmath}) \subset \hat{M}$, $\jmath \in J$. By Theorem 3.7 (iii) Y_j splits isometrically as $Y_j = Y_j' \times \mathbb{R}$. From the description of the zero curvatures of M we conclude that $\mathbb{R}$ is the euclidean de Rham factor, i.e., Y_j does not split off a higher dimensional euclidean factor. Let $\Delta_j := \{\gamma \in \Delta \mid \gamma Y_j = Y_j\}$, then Y_j/Δ_j is compact and can be identified with a component of one of the submanifolds $\pi^{-1}(\bar{V}_i) \subset M$, $i = 1, \ldots, N$.

From the properties of the metric on $\hat{M}$ as described in Theorem 3.7 (iii) we see that there is an element $\alpha_j \in \Delta_j$ operating as *identity* $\times$ *translation* on $Y_j' \times \mathbb{R}$ in such a way that the translational part is minimal. The pair $\alpha_j^{\pm 1}$ is uniquely determined by the minimality condition. The displacement function of each of these two elements is constant on Y_j, and thus α_j is a *Clifford translation.* However, α_j acts as $-\mathbb{1}$ on the 1–dimensional normal space of Y_j. Hence, α_j is an *orientation reversing isometry* of $\tilde{M}$. The group $\langle \alpha_j \rangle \cong \mathbb{Z}$ is a normal subgroup of Δ_j, and by [E1] it can be characterized as the *unique maximal normal abelian subgroup of* Δ_j. Hence, Δ_j is contained in the normalizer $N(\langle \alpha_j \rangle)$. We claim that indeed

$$\Delta_j = N(\langle \alpha_j \rangle) \ . \tag{7.8}$$

To see this, we consider $\gamma \in N(\langle \alpha_j \rangle)$. Then, $\gamma \alpha_j \gamma^{-1} = \alpha_j^{\pm 1}$. Let c be an axis of α_j. The description of the operation of α_j shows that $c = \{p'\} \times \mathbb{R} \subset Y_j$. Since $\alpha_j^{\pm 1}(c) = c$ we have $\alpha_j\big(\gamma^{-1}(c)\big) = \gamma^{-1}(c)$, which means that $\gamma^{-1}(c)$ is an axis of c, too. Thus Y_j is invariant under γ.

We consider now the case that $Y_j \cap Y_k \neq \emptyset$. By the construction of the metric the intersection is orthogonal, i.e., $\pi_{Y_j}(Y_k) = Y_j \cap Y_k$ where π_{Y_j} is the orthogonal projection onto Y_j. Therefore α_j leaves Y_k invariant. Furthermore, α_j operates as a Clifford translation on $Y_j \cap Y_k$. On the normal space of the hypersurface $Y_j \cap Y_k \subset Y_k$ it acts as $-\mathbb{1}$, and thus α_j *is orientation reversing when considered as an isometry of* Y_k. The intersection $Y_j \cap Y_k$ splits isometrically as $Y'_{jk} \times \mathbb{R}^2$, and α_j, α_k operate as translations on the $\mathbb{R}^2$–factor. More generally, if $I \subset \tilde{J}$ is a subset such that $Y_I := \bigcap_{i\in I} Y_i \neq \emptyset$, then $Y_I = Y'_I \times \mathbb{R}^{\# I}$, and the α_i, $i \in I$, leave Y_I invariant. They are translations generating a lattice in $\mathbb{R}^{\# I}$.

Next, we show how to recover all these geometric properties from purely algebraic properties of the fundamental group.

7.8. Theorem. — *Let $M^* = \tilde{M}^*/\Delta^*$ be an n–dimensional, real analytic Riemannian manifold with nonpositive sectional curvature such that there is an isomorphism $\varphi\colon \Delta \to \Delta^*$. Then,*

(i) M^* is compact ;

(ii) for any $\alpha_j^* := \varphi(\alpha_j)$, $j \in \tilde{J}$, there are complete, totally geodesic submanifolds $Y_j^* \subset \tilde{M}^*$, $j \in \tilde{J}$, of codimension 1 which split isometrically as $Y_j^{*\prime} \times \mathbb{R}$ such that the isometry α_j^* leaves Y_j^* invariant and operates as identity $\times$ translation. In the directions perpendicular to Y_j^* the differential of α_j^* acts as $-\mathbb{1}$;

(iii) $\{\gamma^* \in \Delta^* \mid \gamma^* Y_j^* = Y_j^*\} \;=\; \varphi(\Delta_j)$;

(iv) $Y_j^* \cap Y_k^* \neq \emptyset \Leftrightarrow Y_j \cap Y_k \neq \emptyset$ *and in this case the intersection is orthogonal ;*

(v) *for any finite subset $I \subset \tilde{J}$ one has*

$$Y_I^* := \bigcap_{i\in I} Y_i^* \neq \emptyset \quad \Leftrightarrow \quad Y_I := \bigcap_{i\in I} Y_i \neq \emptyset \quad .$$

Moreover, these sets Y_I^* split isometrically as $Y_I^{*\prime} \times \mathbb{R}^{\# I}$, and the α_i^*, $i \in I$, span a lattice in the euclidean factor.

Before we prove the theorem, we recall some results about topological properties which are determined by the fundamental group. In particular, we are concerned with the compactness and orientability.

7.9. Lemma. — *Let Γ be an abstract group. Then,*

(i) a properly discontinuous, fixed point free action of Γ on a manifold X diffeomorphic to $\mathbb{R}^m$ has a compact quotient[14] *X/Γ, if and only if $H_m(\Gamma;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$;*

(ii) there exists a universal homomorphism $w_1\colon \Gamma \to \mathbb{Z}/2\mathbb{Z}$ with the property that for any properly discontinuous, fixed point free, co-compact action of Γ on $\mathbb{R}^m$

$$\gamma \in \Gamma \text{ is orientation preserving} \quad \Leftrightarrow \quad w_1(\gamma) = 0 \quad .$$

Proof. (i) By hypothesis X is an Eilenberg–MacLane space $K(\Gamma,1)$. Therefore, $H_m(\Gamma;\mathbb{Z}/2\mathbb{Z}) \cong H_m(X/\Gamma;\mathbb{Z}/2\mathbb{Z})$, hence the claim.

(ii) By (i) we know that X/Γ is closed. Let $w_1(X/\Gamma) \in H^1(X/\Gamma,\mathbb{Z}/2\mathbb{Z})$ be the first Stiefel–Whitney class of X/Γ. We identify $H^1(X/\Gamma;\mathbb{Z}/2\mathbb{Z})$ with the space $\mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(H_1(X/\Gamma;\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$. By composition with the Hurewicz homomorphism $\pi_1(X/\Gamma) \to H_1(X/\Gamma;\mathbb{Z}/2\mathbb{Z})$ and the identification $\pi_1(X/\Gamma) \cong \Gamma$ we may consider $w_1(X/\Gamma)\colon \Gamma \to \mathbb{Z}/2\mathbb{Z}$. By [Hus, Chap. 16, § 12.1] X/Γ is orientable, if and only if $w_1(X/\Gamma) = 0$. Thus, $\gamma \in \Gamma$ is orientation preserving if and only if $\gamma \in \ker(w_1(X/\Gamma))$. Since the Stiefel–Whitney classes of closed manifolds are homotopy invariants [Hus, Chap. 17, § 8.3], $w_1(X/\Gamma)$ depends only on the homotopy type of X/Γ, hence on Γ. Thus, w_1 defines a homeomorphism $w_1\colon \Gamma \to \mathbb{Z}/2\mathbb{Z}$ with the required properties.

Proof of Theorem 7.8. (i) Since M is compact and n–dimensional, we know that $H_n(\Delta;\mathbb{Z}/2\mathbb{Z}) \neq 0$, and hence $H_n(\Delta^*;\mathbb{Z}/2\mathbb{Z}) = \varphi_* H_n(\Delta;\mathbb{Z}/2\mathbb{Z}) \neq 0$. Since $\tilde{M}^*$ is n–dimensional and contractible, it follows that M^* is compact.

(ii) We consider the subgroup $\Delta_j^* \cong \Delta_j$ containing α_j^*. Since M^* is compact, α_j^* is a hyperbolic isometry [BGS, § 8.2]. We consider the set $Y_j^* := \mathrm{MIN}(\alpha_j^*) \subset \tilde{M}^*$ consisting of all axes of α_j^* [BGS, § 6]. Since the metric on M^* is real analytic, Y_j^* is a complete, totally geodesic submanifold of $\tilde{M}^*$ which splits isometrically as $Y_j^* =$

[14] Note that this criterion depends just on the group Γ and the dimension of X, but not on the particular choice of the action.

${Y_j^*}'\times\mathbb{R}$ such that α_j^* operates as *identity*$\times$*translation*. Since $\langle\alpha_j^*\rangle$ is a normal subgroup of Δ_j^*, we have for every $\gamma^*\in\Delta_j^*$ that $\gamma^*\alpha_j^*{\gamma^*}^{-1}={\alpha_j^*}^{\pm1}$. Hence, $\alpha_j^*({\gamma^*}^{-1}c)={\gamma^*}^{-1}c$ for every axis c of α_j^*. In particular, ${\gamma^*}^{-1}c$ is also an axis of α_j^*. We conclude that Δ_j^* leaves $Y_j^*=\mathrm{MIN}(\alpha_j^*)$ invariant,

$$\Delta_j^*\subset\left\{\gamma^*\in\Delta^*\;\middle|\;\gamma^*Y_j^*=Y_j^*\right\}\quad .$$

Next, we show that Y_j^* is $(n-1)$–dimensional. First, $\dim Y_j^*\le n-1$, since $Y_j^*=\tilde{M}^*$ would imply that $\tilde{M}^*$ has a non-trivial euclidean de Rham factor. But, by [E1] the euclidean de Rham factor can be detected by the fundamental group, and therefore $\tilde{M}$ would have a nontrivial euclidean de Rham factor which is not true. Secondly, $H_{n-1}\big(\Delta_j^*;\mathbb{Z}/2\mathbb{Z}\big)\neq\emptyset$, since $\Delta_j\cong\Delta_j^*$ is the fundamental group of the compact manifold Y_j/Δ_j. Thus, $\dim Y_j\ge n-1$. By Lemma 7.9, α_j^* is orientation reversing, since α_j is orientation reversing. This implies that the isometry α_j^* acts as $-\mathbb{1}$ on the normal space of Y_j^*.

(iii) Recall that Δ_j^* leaves Y_j^* invariant. Since $H_{n-1}\big(\Delta_j^*;\mathbb{Z}/2\mathbb{Z}\big)\neq0$, the group Δ_j^* operates with a compact quotient. As above [E1] tells us that the euclidean de Rham factors of Y_j and Y_j^* are of the same dimension, and hence $Y_j^*={Y_j^*}'\times\mathbb{R}$ is the de Rham splitting. If $\gamma^*\in\Delta^*$ leaves Y_j^* invariant, then $\gamma^*_{|Y_j^*}$ respects the splitting ${Y_j^*}'\times\mathbb{R}$. Since α_j^* only translates in the $\mathbb{R}$–direction, γ^* is contained in the normalizer of $\langle\alpha_j^*\rangle$. □

In order to establish the remaining parts of the Theorem, we need the following

7.10. Lemma.

(i) $$Y_j\cap Y_k\neq\emptyset\quad\Leftrightarrow\quad\alpha_j\textit{ and }\alpha_k\textit{ commute};$$

(ii) $$Y_j^*\cap Y_k^*\neq\emptyset\quad\Leftrightarrow\quad\alpha_j^*\textit{ and }\alpha_k^*\textit{ commute.}$$

Proof. (i) "$\Leftarrow$" Since $Y_j=\mathrm{MIN}(\alpha_j)$ and $Y_k=\mathrm{MIN}(\alpha_k)$, we may refer to [BGS, § 7.1]. "$\Rightarrow$" This implication follows directly from our construction of the metric g.

(ii) "$\Leftarrow$" By our definition of Y_j^* and Y_k^*, we may again refer to [BGS, § 7.1]. "$\Rightarrow$" To prove this direction, we shall first deal with the special case where $Y_j^*=Y_k^*$.

Then, the de Rham splittings of Y_j^* and Y_k^* coincide. It follows that $\alpha_i^* = \alpha_j^{*\pm 1}$, and we are done.

It remains to handle the case where $Y_j^* \neq Y_k^*$. Note that $S := Y_j^* \cap Y_k^*$ is a complete, totally geodesic hypersurface in $Y_j^* = Y_j^{*\prime} \times \mathbb{R}$. We first show that S respects the splitting, i.e., either $S = Y_j^{*\prime} \times \{p\}$ or $S = S_j' \times \mathbb{R}$ where S_j' is a hypersurface in $Y_j^{*\prime}$. To prove this, let N be a unit normal vector field in Y_j^* to S and let $p_1\colon Y_j^{*\prime} \times \mathbb{R} \to Y^{*\prime}$ be the canonical projection. Clearly, N is a global parallel field along S, and hence $p_{1*}N$ is a global parallel field on the convex set $p_1(S) \subset Y^{*\prime}$.

Since $Y_j^{*\prime}$ has no euclidean de Rham factor, $p_{1*}N$ is either trivial, or $p_1(S)$ is a proper subset of $Y_j^{*\prime}$. In the first case $S = Y_j^{*\prime} \times \{p\}$, and in the second case $S = S_j' \times \mathbb{R}$. In the first case S has no euclidean factor, in the second case S has a euclidean factor. Thus, the type of splitting of S is the same in Y_j^* and Y_k^*. Let us assume that $S = Y_j^{*\prime} \times \{p\} = Y_k^{*\prime} \times \{q\}$, then Y_j^* and Y_k^* are foliated by parallels to S. This implies that the set of all parallels to S is n–dimensional, and by real analyticity $\tilde{M}^*$ would split isometrically as $S \times Q$ with $\dim Q = 2$. However, such a splitting could be detected by the fundamental group. In fact, it would follow by [BE] that $\operatorname{rank} \Delta^* \geq 2$, which is not true, since $\tilde{M}$ does not split. We conclude that $S = S_j' \times \mathbb{R} \subset Y_j^{*\prime} \times \mathbb{R}$. By symmetry, we also have an isometric splitting $S = S_k' \times \mathbb{R} \subset Y_k^{*\prime} \times \mathbb{R}$.

Since the euclidean factors of Y_j^* and Y_k^* do not point in the same direction, S indeed splits as $S_{jk}'' \times \mathbb{R}^2$, and the elements $\alpha_j^*, \alpha_k^* \in \operatorname{Iso}(\tilde{M}^*)$ operate as translations on the $\mathbb{R}^2$–factor. Hence they commute. ☐

Proof of Theorem 7.8 (*continuation*)

(iv) Lemma 7.10 implies that $Y_j^* \cap Y_k^* \neq \emptyset \;\Leftrightarrow\; Y_j \cap Y_k \neq \emptyset$. We have to show that Y_j^* and Y_k^* intersect orthogonally. Since α_j^* and α_k^* commute, α_j^* leaves Y_k^* invariant. Thus, Y_k^* is a closed convex α_j^*–invariant subset of $\tilde{M}^*$, and by [BGS, § 6.4] the orthogonal projection $\pi_{Y_k^*}$ onto Y_k^* does not increase the displacement function $d_{\alpha_j^*}(x) := d(x, \alpha_j^*(x))$. In other words, $d_{\alpha_j^*}(\pi_{Y_k^*}(x)) \leq d_{\alpha_j^*}(x)$ for all $x \in \tilde{M}^*$. Since by definition $Y_j^* = \left\{ x \in \tilde{M}^* \;\middle|\; d_{\alpha_j^*}(x) \text{ minimal } \right\}$, we see that $\pi_{Y_k^*}(Y_j^*) \subset Y_j^* \cap Y_k^*$, as desired.

(v) If $Y_I = \bigcap_{i \in I} Y_i \neq \emptyset$, then the α_i commute pairwise by Lemma 7.10. Hence, the α_i^* commute, and $Y_I^* = \bigcap Y_i \; = \bigcap \mathrm{MIN}(\alpha_i^*) \neq \emptyset$ by [BGS, § 7.1]. Because of the

symmetry of the argument we see that $Y_I \neq \emptyset \ \Leftrightarrow \ Y_I^* \neq \emptyset$.
By Theorem 3.7 (iii) Y_I splits isometrically as $Y_I' \times \mathbb{R}^{\# I}$. On the other hand, $Y_I^* = \bigcap \mathrm{MIN}(\alpha_i^*)$ splits off a euclidean factor $Y_I^* \times \mathbb{R}^s$ such that the α_i^* operate as *identity*$\times$ *translation* and span a lattice in $\mathbb{R}^s$ by [BGS, § 7.1]. Thus $s = \# I$, and we have the desired splitting. □

APPENDIX A. Basic Properties of the $\owedge$–Product of Bilinear Forms

Recall that the $\owedge$–product of two bilinear forms $b_1, b_2\colon \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$, the so–called Kulkarni–Nomizu product, is the $(4,0)$–tensor defined by

$$
\begin{aligned}
&b_1 \owedge b_2(X,Y;Z,W) \\
\text{(A.1)} \qquad &:= \frac{1}{2}\big[b_1(X,W)\, b_2(Y,Z) - b_1(Y,W)\, b_2(X,Z) \\
&\qquad - b_1(X,Z)\, b_2(Y,W) + b_1(Y,Z)\, b_2(X,W)\big] \ .
\end{aligned}
$$

From now on, we concentrate on symmetric bilinear forms b_1 and b_2. Then, it is clear that $b_1 \owedge b_2$ can be considered as a *symmetric bilinear form* on $\Lambda^2\mathbb{R}^n$ through the equation

$$
\text{(A.2)} \qquad (b_1 \owedge b_2)(X \wedge Y\, , Z \wedge W) := -(b_1 \owedge b_2)(X,Y;Z,W) \ .
$$

The *first Bianchi identity*

$$
\text{(A.3)} \qquad 0 = (b_1 \owedge b_2)(X,Y;Z,W) + (b_1 \owedge b_2)(Y,Z;X,W) + (b_1 \owedge b_2)(Z,X;Y,W)
$$

is easy to verify. In other words, $b_1 \owedge b_2$ satisfies all the *algebraic symmetries of a curvature tensor*. Note that $g_0 \owedge g_0$ represents the metric on $\Lambda^2\mathbb{R}^n$ induced by the euclidean metric $g_0 = \langle\, .\, ,\, .\, \rangle$ on $\mathbb{R}^n$. In particular, $g_0 \owedge g_0$ is positive definite.

A.1. Lemma.

(i) Suppose that $b_1, b_2 \geq 0$. Then, $b_1 \owedge b_2 \geq 0$.

(ii) Suppose that $0 \leq b_1 \leq \bar{b}_1$ and $0 \leq b_2 \leq \bar{b}_2$. Then,

$$
b_1 \owedge b_2 \leq b_1 \owedge \bar{b}_2 \leq \bar{b}_1 \owedge \bar{b}_2 \ .
$$

(iii) Let b_1 and b_2 be possibly indefinite, generic symmetric bilinear form on $\mathbb{R}^n$, and let $\|b_i\|$ denote their operator norm w.r.t. the euclidean inner product g_0. Then

$$
-\|b_1\|\, \|b_2\|\, g_0 \owedge g_0 \leq b_1 \owedge b_2 \leq \|b_1\|\, \|b_2\|\, g_0 \owedge g_0 \ .
$$

Proof. In order to establish (i), it is sufficient to spell out what the inequality means when using a basis in which b_1 and b_2 diagonalize simultaneously. Part (ii) of the lemma is an immediate consequence of (i) anyway. In order to obtain Part (iii), we consider the positive and negative semi-definite parts $b_i^{\pm}$ of $b_i = b_i^+ - b_i^-$. Since this decomposition is taken w.r.t. the euclidean inner product on $\mathbb{R}^n$, we find that $\|b_i\| = \max\{\|b_i^+\|, \|b_i^-\|\}$, and hence $0 \leq b_i^{\pm} \leq b_i^+ + b_i^- \leq \|b_i\|\, g_0$. Using (i) and (ii), it is now straightforward to see that

$$b_1 \owedge b_2 \leq b_1^+ \owedge b_2^+ + b_1^- \owedge b_2^- \leq \|b_1\|\, g_0 \owedge (b_2^+ + b_2^-) \leq \|b_1\|\, \|b_2\|\, g_0 \owedge g_0$$

and similarly

$$b_1 \owedge b_2 \geq -b_1^+ \owedge b_2^- - b_1^- \owedge b_2^+ \geq -\|b_1\|\, g_0 \owedge (b_2^+ + b_2^-) \geq -\|b_1\|\, \|b_2\| g_0 \owedge g_0 \,. \qquad \square$$

A.2. *Application* (*in Sections 5 and 6*). — Note that the fields p_j^{ξ} and p_j^{v} have rank 1 each. Hence, $p_j^{\xi} \owedge p_j^{\xi} = p_j^{v} \owedge p_j^{v} = 0$ and thus

$$0 \leq p_j \owedge p_j = 2\, p_j^{\xi} \owedge p_j^{v} \leq 2\, p_j^{\xi} \owedge \big(g_0 - p_j^{\xi}\big) \leq g_0 \owedge g_0 \leq g \owedge g \;. \tag{A.4}$$

Next we want to estimate the symmetric bilinear forms $q_{ij} \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which are defined by

$$q_{ij}(x, y) := \frac{1}{2}\big(\langle w_i\,, x\rangle\langle w_j\,, y\rangle + \langle w_j\,, x\rangle\langle w_i\,, y\rangle\big) \tag{A.5}$$

where $(w_i)_i$ are some fixed vectors in $\mathbb{R}^n$. Note that by the first Bianchi identity we have

$$0 = q_{ij} \owedge q_{kl} + q_{jk} \owedge q_{il} + q_{ki} \owedge q_{jl} \;. \tag{A.6}$$

In particular,

$$q_{ii} \owedge q_{jk} = -2\, q_{ij} q_{ik} \;, \text{ and} \tag{A.7}$$

$$q_{ii} \owedge q_{ik} = 0 \quad . \tag{A.8}$$

In this context Lemma A.1 implies the following Cauchy–Schwarz inequality for curvature operators.

A.3. Corollary. — *Let q_{ij} be as above, and let $b \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an arbitrary, positive semi-definite, symmetric bilinear form. Then, for any $\delta > 0$*

$$-\frac{\delta}{2}\, b \owedge q_{ii} - \frac{1}{2\delta}\, b \owedge q_{jj} \;\leq\; b \owedge q_{ij} \;\leq\; \frac{\delta}{2}\, b \owedge q_{ii} + \frac{1}{2\delta}\, b \owedge q_{jj} \;. \tag{A.9}$$

Proof. Note that q_{ii} and q_{jj} are both positive semi-definite. Hence,

$$\frac{\delta}{2}\, q_{ii} + \frac{1}{2\delta}\, q_{jj} \pm q_{ij} \geq 0$$

for any $\delta > 0$, and we apply Lemma A.1 (i) to the $\owedge$–product of b and the left hand side of the preceding inequality. □

A.4. Corollary. — *Let $E \subset \mathbb{R}^n$ be a 2–dimensional subspace, and let (w_i, w_j) be an orthonormal basis for E. Suppose that $b \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a symmetric bilinear form such that $b|_{E^\perp \times E^\perp} = 0$. Then,*

(i) $$-\delta\, \|b\|\, (q_{ii}+q_{jj}) \owedge (q_{ii}+q_{jj}) - \frac{1}{\delta} \|b\|\, g_0 \owedge g_0 \leq q_{ii} \owedge b \leq \delta\, \|b\|\, (q_{ii}+q_{jj}) \owedge (q_{ii}+q_{jj}) + \frac{1}{\delta} \|b\|\, g_0 \owedge g_0$$

(ii) $$-\delta\, \|b\|\, (q_{ii}+q_{jj}) \owedge (q_{ii}+q_{jj}) - \frac{1}{\delta} \|b\|\, g_0 \owedge g_0 \leq 2\, q_{ij} \owedge b \leq \delta\, \|b\|\, (q_{ii}+q_{jj}) \owedge (q_{ii}+q_{jj}) + \frac{1}{\delta} \|b\|\, g_0 \owedge g_0 \ .$$

Here, $\|b\|$ denotes the operator norm of b taken w.r.t. the metric $g_0 = \langle .\,, . \rangle$.

In the proof we actually obtain slightly better constants, but this does not make any difference when applying the corollary in Section 6 in order to establish Lemma 6.9.

Proof. For clarity it is best to decompose b as a sum $b_0 + b_1$ where b_0 has the property that $E^\perp \subset \ker b_0$ and where b_1 has the property that $b_1|_{E \times E} = 0$. Introducing suitable unit vectors $w_k, w_l \in E^\perp$, we may write[15]

$$\begin{aligned} b_0 &= b_{00}\, q_{ii} + 2\, b_{01}\, q_{ij} + b_{02}\, q_{jj}\ , \\ b_1 &= 2\, b_{11}\, q_{ik} + 2\, b_{12}\, q_{jl}\ . \end{aligned} \tag{A.10}$$

We shall deal with b_0 and b_1 separately.

(1) By Formulae (A.7) and (A.8) we compute that

$$\begin{aligned} q_{ii} \owedge b_0 &= b_{02}\, q_{ii} \owedge q_{jj} = \frac{1}{2} b_{02}\, (q_{ii} + q_{jj}) \owedge (q_{ii} + q_{jj}) \\ 2\, q_{ij} \owedge b_0 &= 4\, b_{01}\, q_{ij} \owedge q_{ij} = -b_{01}\, (q_{ii} + q_{jj}) \owedge (q_{ii} + q_{jj})\ . \end{aligned}$$

[15] w_k and w_l are not necessarily perpendicular.

Since $|b_{00}|$, $|b_{01}|$, and $|b_{02}|$ are all bounded by $\|b_0\|$, we conclude that

$$-\frac{1}{2}\|b_0\|(q_{ii}+q_{jj})\owedge(q_{ii}+q_{jj}) \;\leq\; q_{ii}\owedge b_0 \;\leq\; \frac{1}{2}\|b_0\|(q_{ii}+q_{jj})\owedge(q_{ii}+q_{jj}) \;,$$

$$-\|b_0\|(q_{ii}+q_{jj})\owedge(q_{ii}+q_{jj}) \;\leq\; 2\,q_{ij}\owedge b_0 \;\leq\; \|b_0\|(q_{ii}+q_{jj})\owedge(q_{ii}+q_{jj}) \;.$$

(2) In this case, our computation makes in addition use of the inequality from Corollary A.3. We obtain

$$\begin{aligned} q_{ii}\owedge b_1 &= 2\,b_{12}\,q_{ii}\owedge q_{jl} \;\leq\; |b_{12}|\,\big(\delta\,q_{ii}\owedge q_{jj}+\frac{1}{\delta}\,q_{ii}\owedge q_{ll}\big)\\ 2\,q_{ij}\owedge b_1 &= -2\,b_{11}\,q_{ii}\owedge q_{jk}-2\,b_{12}\,q_{jj}\owedge q_{il}\\ &\leq \frac{\delta}{2}\big(|b_{11}|+|b_{12}|\big)\,q_{ii}\owedge q_{jj}+\frac{2}{\delta}\big(|b_{11}|\,q_{ii}\owedge q_{kk}+|b_{12}|\,q_{jj}\owedge q_{ll}\big)\end{aligned}$$

These inequalities continue to hold if we replace b_1 by $-b_1$. Since $|b_{11}|\leq\|b_1\|$ and $|b_{12}|\leq\|b_1\|$, we conclude that

$$\begin{aligned} &-\frac{\delta}{2}\|b_1\|\,(q_{ii}+q_{jj})\owedge(q_{ii}+q_{jj})-\frac{1}{2\delta}\|b_1\|\,g_0\owedge g_0\\ &\qquad\leq\;\; q_{ii}\owedge b_1 \;\leq\; \frac{\delta}{2}\|b_1\|\,(q_{ii}+q_{jj})\owedge(q_{ii}+q_{jj})+\frac{1}{2\delta}\|b_1\|\,g_0\owedge g_0\\ &-\frac{\delta}{2}\|b_1\|\,(q_{ii}+q_{jj})\owedge(q_{ii}+q_{jj})-\frac{1}{\delta}\|b_1\|\,g_0\owedge g_0\\ &\qquad\leq\;\; 2q_{ij}\owedge b_1 \;\leq\; \frac{\delta}{2}\|b_1\|\,(q_{ii}+q_{jj})\owedge(q_{ii}+q_{jj})+\frac{1}{\delta}\|b_1\|\,g_0\owedge g_0 \;.\end{aligned}$$

Note that $\max\{\|b_0\|,\|b_1\|\}\leq\|b\|$. Hence, the claim follows from the two partial results above. $\square$

APPENDIX B. On Hyperbolic Quadrilaterals

Throughout this appendix $\gamma_1, \gamma_2 \colon \mathbb{R} \to \mathbb{H}^2$ will be two normal geodesics in the hyperbolic plane which intersect each other orthogonally in $p_0 := \gamma_1(0) = \gamma_2(0)$. Let $\tilde{r}_\mu \colon \mathbb{H}^2 \to \mathbb{R}$ be the oriented distance function for $\gamma_\mu \subset \mathbb{H}^2$, $\mu = 1, 2$. This means that

$$|\tilde{r}_\mu(p)| = \operatorname{dist}(p, \gamma_\mu(\mathbb{R})) \ , \qquad \forall p \in \mathbb{H}^2$$

and

$$\tilde{r}_u \circ \gamma_{\mu+1}(t) = t \quad \text{with indices taken} \bmod 2 \ .$$

The distance $|\tilde{r}_\mu(p)|$ is represented by the length of an integral curve $c_{\mu,p}$ of the unit vector field

$$\tilde{v}_\mu := \operatorname{grad} \tilde{r}_\mu \ .$$

All these integral curves are geodesics which intersect γ_μ perpendicularly. We may assume that $c_{\mu,p}(0) \in \gamma_\mu(\mathbb{R})$. For any $p \in \mathbb{H}^2 \setminus \big(\gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})\big)$ the geodesics γ_1, γ_2, $c_{1,p}$, and $c_{2,p}$ bound a hyperbolic quadrilateral Q_p, which has precisely one acute angle γ at p. Note that Q_p has angles equal to $\frac{\pi}{2}$ at its other vertices.

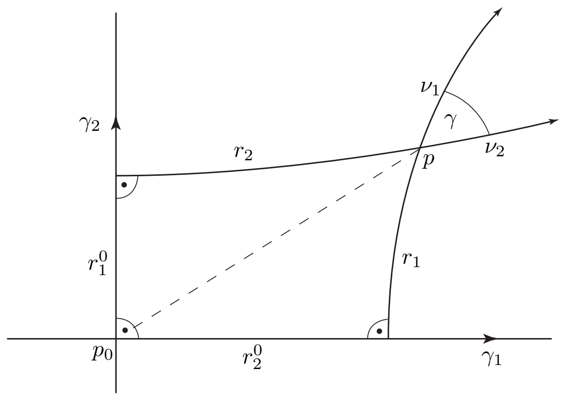

FIGURE 1. The Quadrilateral Q_p in the Hyperbolic Plane

In addition to the (signed) lengths r_1, r_2, r_1^0, and r_2^0 of the four edges we introduce the length $r := \mathrm{dist}(p, p_0)$ of the diagonal which ends at the acute angle.

B.1. *Facts from Planar Hyperbolic Trigonometry.*

(i) $$\sinh^2 r = \sinh^2 r_1 + \sinh^2 r_2 \qquad \text{(by the Law of Sines)}$$

(ii) $$\sinh r_\mu = \sinh r_\mu^0 \, \cosh r_{\mu+1}$$

(iii) $$\cos\gamma \; \equiv \langle \tilde v_1 \, , \tilde v_2 \rangle = \sinh r_1^0 \, \sinh r_2^0 = \tanh r_1 \, \tanh r_2 \ .$$

B.2. *Application* (*Local Geometry of the Divisor*). — We consider a point in $\Omega \cap U_I$ where $U_I \subset \mathbb{H}^n$ is a domain as in Proposition 3.4. Let $i_1, i_2 \in I$ are two distinct elements. Then, there is a unique, totally geodesic, hyperbolic plane in $\mathbb{H}^n$ which contains p and the geodesics $c_\mu = \exp(\mathbb{R}\, v_{i_\mu}{}_{|p})$, $\mu = 1, 2$. This plane clearly intersects $\mathbb{H}^{n-2}_{i_1}$ and $\mathbb{H}^{n-2}_{i_2}$ in geodesics, which we shall denote by γ_1 and γ_2, respectively. Thus, the analysis of *the local geometry of* $\bigcup_{j\in J} \mathbb{H}^{n-2}_j$ gets reduced to the 2–dimensional configuration discussed above. Hence, we conclude from Property (iii) in the preceding list and from Axiom 3.1 that for any i_1, $i_2 \in I$

(B.1) $$\begin{aligned} &\langle v_{i_1} \, , \xi_{i_2} \rangle = 0 \\ &\langle \xi_{i_1} \, , \xi_{i_2} \rangle = \delta_{i_1 i_2} \\ &\langle v_{i_1} \, , v_{i_2} \rangle = \tanh r_{i_1} \, \tanh r_{i_2} + \delta_{i_1 i_2} \, \cosh^{-2} r_{i_1} \ . \end{aligned}$$

Now, it requires just the definition of G_I in the paragraph below formula (3.3) to see that for any $i \in I$

(B.2) $$\begin{aligned} (\mathbb{1} + G_I)^{-1} v_i &= \qquad v_i \\ (\mathbb{1} + G_I)^{-1} \xi_i &= \frac{x_i}{x_i + \eta h(x_i)} \, \xi_i \ . \end{aligned}$$

Furthermore, the local geometry of the divisor leads to some useful estimates for smooth sections of f of euclidean vector bundles F over $\mathbb{H}^2$.

B.3. Lemma. — *Let $F \to \mathbb{H}^2$ be a euclidean vector bundle with metric connection ∇. There exists a constant $c_{\mathbb{H}^2} > 0$ such that, for any point $p \in \mathbb{H}^2 \setminus (\gamma_1(\mathbb{R}) \cup (\gamma_2(\mathbb{R}))$ and any section $f \in \mathrm{C}^2(\mathbb{H}^2, F)$ which vanishes on $\gamma_1(\mathbb{R}) \cup (\gamma_2(\mathbb{R})$, one has the following pointwise upperbound for $f|_{Q_p}$:*

(B.3) $$\|f(q)\| \;\le\; c_{\mathbb{H}^2} \, \|f\|_{\mathrm{C}^2(\bar Q_p)} \, \tanh|r_1(q)| \, \tanh|r_2(q)| \ , \quad \forall q \in \bar Q_p \ .$$

Here, $\|f\|_{C^2(\bar{Q}_p)} := \sup\{\|f(q)\| + \|\nabla f|_q\| + \|\nabla^2 f|q\| \mid q \in \bar{Q}_p\}$ stands for the C^2–norm of the restriction $f|_{\bar{Q}_p}$.

Proof. For short we set $b_k := \sup\{\|\nabla^k f|_q\| \mid q \in \bar{Q}_p\}$ for $k = 0, 1, 2$. Integrating ∇f along the geodesics $c_{\mu,q}$, we get :

$$\|f(q)\| \ \le\ \min\{b_0\,, b_1\,|r_1(q)|\,, b_1\,|r_2(q)|\}\ , \quad \forall q \in \bar{Q}_p\ . \tag{B.4}$$

Note that $\nabla_{v_1} f$ vanishes along γ_2. Integrating the inequality

$$\frac{d}{dt}\|\nabla_{v_1} f\| \ \le\ \|\nabla^2_{\dot{c}_{2,q},v} f\| + \|\frac{\nabla}{dt} v_1\|\,\|\nabla f\|$$

along the geodesic $c_{2,q}$, we get

$$\begin{aligned} \|\nabla_{v_1} f|_q\| \ &\le\ \int_0^{|r_2(q)|} \big(b_2 + b_1\,\big\|\frac{\nabla}{dt} v_1|_{c_{2,q}(t)}\big\|\big)\,dt \\ &=\ b_2\,|r_2(q)| + b_1\,|\frac{\pi}{2} - \gamma(q)| \\ &\le\ b_2\,|\gamma_2(q)| + b_1 \arcsin\big(\tanh|r_1(q)|\,\tanh|r_2(q)|\big)\ . \end{aligned} \tag{B.5}$$

Here we have used the identity $\big\|\frac{\nabla}{dt}\,v_1|_{c_{2,q}(t)}\big\| = \big|\frac{d}{dt}\,\gamma|_{c_{2,q}(t)}\big|$, which is due to the fact that v_1 is parallel along $c_{2,q}$. For the second line of (B.5) we employ in addition the monotonicity of γ, and for the final inequality we make use of the trigonometric identity for $\cos\gamma = \sin(\frac{\pi}{2} - \gamma)$.

The next step is to integrate (B.5) along the geodesic $c_{1,p}$. Since $|r_2 \circ c_{1,q}(t)| \le |r_2(q)|$ on the segment in question, and since $\arcsin\alpha \le \frac{\pi}{2}\,\alpha$ for $0 \le \alpha \le \frac{\pi}{2}$, this step yields the inequality

$$\|f(g)\| \ \le\ b_2\,|r_1(q)|\,|r_2(q)| + \frac{\pi}{2}\ b_1\,|r_1(q)|\,\tanh|r_2(q)|\ . \tag{B.6}$$

It is clear how to improve (B.6) using the symmetry of the whole set-up with respect to interchanging the roles of γ_1 and γ_2. □

Inequality (B.6) is better than (B.4), provided $|r_1(q)|$ and $|r_2(q)|$ are both small. The bound claimed in (B.3) is obtained by combining (B.4) and (B.6) into a single estimate, which is slightly weaker but much more convenient to use.

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