SYSTOLES AND INTERSYSTOLIC INEQUALITIES

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Abstract. This articles surveys inequalities involving systoles in Riemannian geometry.

Résumé. Cet article présente l'ensemble des inégalités connues sur les systoles en géométrie riemannienne.

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1. PRELIMINARIES

The notion of the *k-dimensional systole* of a Riemannian manifold was introduced by Marcel Berger in 1972 following earlier work by Loewner (around 1949, unpublished), Pu (1952), Accola (1960) and Blatter (1961). Recall that according to Berger *the k-dimensional systole of a Riemannian manifold V is defined as the infimum of the k-dimensional volumes of the k-dimensional cycles (subvarieties) in V which are not homologous to zero in V.*

In fact, the idea of the 1-dimensional systole can be traced back to the classical geometry of numbers as one considers minima of quadratic forms on lattices in $\mathbb{R}^n$. The fundamental result here is an upper bound on such a minimum in terms of the discriminant of the form in question. This can be formulated in geometric language as follows.

1.A. Bound on the 1-systole of a flat torus. — *Let V be a flat Riemannian torus of dimension n. Then, the 1-systole of V can be bounded in terms of the volume of V by*

$$\text{systole} \leq \text{const}_n \, (\text{Volume})^{\frac{1}{n}} \;,$$

where $\text{const}_n = C\sqrt{n}$ *for some universal constant C (which is not far from one).*

Reformulation and proof. The torus V can be isometrically covered by $\mathbb{R}^n$ and so $V = \mathbb{R}^n/\Gamma$ for some lattice Γ, that is a discrete group of parallel translations of $\mathbb{R}^n$. (This group is *isomorphic* to $\mathbb{Z}^n$ but is *not*, in general, equal to the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ consisting of integral points in $\mathbb{R}^n$.) If a point $x \in \mathbb{R}^n$ is moved by some $\gamma \in \Gamma$ to $\gamma(x)$, then the segment $[x, \gamma(x)]$ joining x with $\gamma(x)$ in $\mathbb{R}^n$ projects to a closed curve S in $V = \mathbb{R}^n/\Gamma$ whose length equals $\text{dist}\big(x, \gamma(x)\big)$. Furthermore, if γ is a non-identity element in Γ (i.e., $x \neq \gamma(x)$), then S is non-homologous to zero in V. In fact, S is *non-homotopic* to zero by the elementary theory of covering spaces which implies "non-homologous to zero" since the group $\Gamma = \pi_1(V) = \mathbb{Z}^n$ is Abelian. Thus, the bound on the 1-systole of V is equivalent to the following estimate.

1.A.1. Displacement estimate. — *For the above lattice* Γ *acting on* $\mathbb{R}^n$ *by parallel translations, there exists a point* $x \in \mathbb{R}^n$ *and a non-identity element* $\gamma \in \Gamma$, *such that*

$$\operatorname{dist}\bigl(x, \gamma(x)\bigr) \leq \operatorname{const}_n \operatorname{Vol}(\mathbb{R}^n/\Gamma) \ .$$

Proof. Take a closed ball B of radius R in $\mathbb{R}^n$ such that the volume of B is greater than or equal to that of $V = \mathbb{R}^n/\Gamma$. Then, the projection $p : B \to V$ is *not* one-to-one and we have distinct points x and x' in B with $p(x) = p(x')$. This equality means that $x' = \gamma(x)$ for some $\gamma \in \Gamma$ (by the definition of the quotient space $\mathbb{R}^n/\Gamma$) and, since the diameter of B is $2R$, the distance between x and $\gamma(x) = x'$ is at most $2R$.

Now, we recall that the volume of $B = B(R)$ equals

$$\operatorname{Vol} B = \sigma_n R^n \ ,$$

where σ_n is given by the familiar formula involving the Γ-function (here, Γ has nothing to do with the lattice Γ),

$$\sigma_n = \pi^{n/2} / \Gamma\bigl(\frac{n}{2} + 1\bigr) \ .$$

Then, a pair of points x and x' with $p(x') = p(x)$ necessarily appears for

$$R = (\sigma_n)^{-\frac{1}{n}} (\operatorname{Vol} V)^{\frac{1}{n}} \ .$$

So we obtain the required displacement bound

$$\operatorname{dist}\bigl(x,\ \gamma(x)\bigr) \leq \operatorname{const}_n \operatorname{Vol} V$$

for

$$\operatorname{const}_n = 2\Bigl(\Gamma\bigl(\frac{n}{2} + 1\bigr)\Bigr)^{\frac{1}{n}} \Big/ \sqrt{\pi} \ ,$$

and the number const_n is bounded by $C\sqrt{n}$ according to Stirling's formula $\Gamma(n) \approx n^n$.

Remarks. (a) The above argument is classical, going back to Gauss (to Diophantus ?), Hermite and Minkowski. We dissected the proof in order to make visible the anatomy of our more general systolic inequalities discussed later on.

(b) Since Γ acts by parallel translations, the displacement $\mathrm{dist}\big(x, \gamma(x)\big)$ does not depend on x, and we may take the origin $0 \in \mathbb{R}^n$ for x. Then, our displacement estimate bounds the Euclidean norm on the lattice Γ embedded into $\mathbb{R}^n$ as the Γ-orbit of the origin by

$$\inf \|\gamma\|_{\mathbb{R}^n} \le \mathrm{const}_n (\mathrm{Vol}\, \mathbb{R}^n/\Gamma)^{\frac{1}{n}} \;, \tag{$*$}$$

where inf is taken over $\gamma \in \Gamma - \{0\}$. (The squared Euclidean norm serves as the quadratic form referred to at the beginning of this discussion.)

The above $(*)$ is called the *Minkowski convex body theorem.* It remains valid (by the proof we gave) for an arbitrary Banach (Minkowski) norm on $\mathbb{R}^n$. In traditional language, *every convex centrally symmetric body B in $\mathbb{R}^n$ contains a non-zero point $\gamma \in \Gamma$, provided* $\mathrm{Vol}\, B \ge 2^n \,\mathrm{Vol}(\mathbb{R}^n/\Gamma)$.

(c) The value of const_2 and the extremal lattice $\Gamma \subset \mathbb{R}^2$ are known since Antiquity. Namely, $\mathrm{const}_2 = (2/\sqrt{3})^{\frac{1}{2}}$, and the extremal lattice has a regular hexagon as fundamental domain. (Such an hexagon of unit width has area $\sqrt{3}/2$.) *Thus, for every flat 2-torus one has*

$$\mathrm{systole} \le (2/\sqrt{3})^{\frac{1}{2}} (\mathrm{Area})^{\frac{1}{2}} \;, \tag{$+$}$$

where equality holds if and only if the corresponding lattice $\Gamma \subset \mathbb{R}^2$ is hexagonal.

1.B. Loewner made an amazing discovery around 1949

Loewner torus theorem. — *Let V be the topological 2-torus with an arbitrary Riemannian metric. Then, the 1-systole of V satisfies the same inequality as in the flat case,*

$$\mathrm{systole} \le (2/\sqrt{3})^{\frac{1}{2}} (\mathrm{Area})^{\frac{1}{2}} \;,$$

and equality holds if and only if the metric on V is flat and the corresponding lattice is hexagonal.

Proof. The key argument is the following

The uniformization theorem for tori. — *For every V there exists a flat torus V_0* (which can be normalized by the condition $\mathrm{Area}\, V_0 = \mathrm{Area}\, V$) *admitting a conformal diffeomorphism $\varphi : V_0 \to V$.*

Granted this, the proof is immediate with the following lemma.

Easy Lemma. — *Take a closed geodesic g in V_0 of length ℓ, and let g_s be the family of parallel geodesics parametrized by the circle S^1 of length $\sigma = A_0/\ell$ for $A_0 = \operatorname{Area} V_0$. Then, the average squared length of the φ-images $\varphi(g_s) \in V,\ s \in S^1$, does not exceed ℓ^2, provided $\operatorname{Area} V = A_0 = \operatorname{Area} V_0$. Namely,*

$$\sigma^{-1} \int_{S^1} \big(\operatorname{length} \varphi(g_s)\big)^2 ds \le \ell^2 = (\operatorname{length} g)^2 \ .$$

Easy (length-area) proof. Denote by $a(v),\ v \in V_0$, the implied *conformal factor*, i.e., $\sqrt{\operatorname{Jacobian} \varphi}$, and let $d\lambda$ denote the length element on g_s. Then, observe that

$$\ell\sigma = A_0 = \operatorname{Area} V = \int_{V_0} a^2(v)dv = \int_{S^1} ds \int_{g_s} a^2(v)d\lambda \ge \int_{S^1} ds\, \ell^{-1} \Big(\int_{g_s} a(v)d\lambda\Big)^2 \ .$$

But, since φ is conformal,

$$\int_{g_s} a(v)d\lambda = \operatorname{length} \varphi(g_s) \ ,$$

and the proof follows. $\square$

The lemma implies that φ shortens the lengths of the homology classes of the flat torus and, in particular, it shortens the 1-systole which is the minimum of these lengths. Thus, the Loewner theorem for general Riemannian tori follows from the above inequality (+) for the flat tori.

Let us reformulate the Loewner theorems in terms of displacement.

Let the group $\Gamma = \mathbb{Z}^2$ discretely and isometrically act on a surface X for some Riemannian metric. If X is homeomorphic to $\mathbb{R}^2$, then there exists a point $x \in X$ and an element $\gamma \in \Gamma$ different from the identity such that

$$\operatorname{dist}_X\big(x, \gamma(x)\big) \le (2/\sqrt{3})^{\frac{1}{2}} (\operatorname{Area} X/\Gamma)^{\frac{1}{2}} \ .$$

Warning. In the general case where X is *not isometric* to $\mathbb{R}^2$, the displacement $\operatorname{dist}\big(x,\, \gamma(x)\big)$ does depend on x and may become uncomfortably large for some points in X. Such an X can be obtained, for example, by periodically attaching long thin "fingers" to $\mathbb{R}^2$ at the points of some $\mathbb{Z}^2$-orbit, see Figure 1 below.

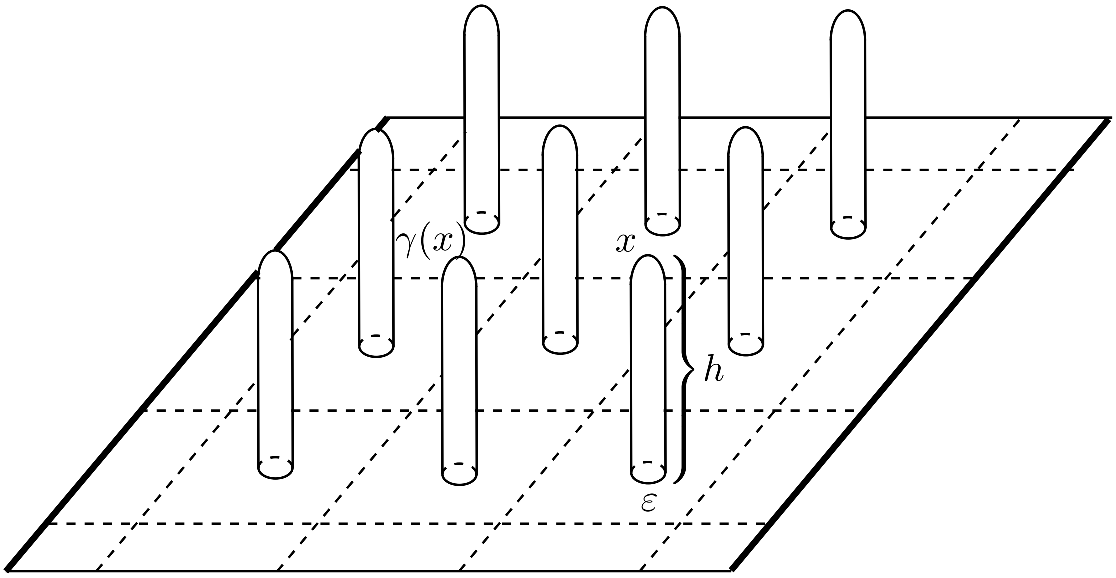

Figure 1

If the fingers have length h, then a point x at the tip of a finger has $\operatorname{dist}\bigl(x,\, \gamma(x)\bigr) \geq 2h$ for all $\gamma \in \Gamma = \mathbb{Z}^2,\ \gamma \neq \mathrm{id}$, and this distance goes to infinity for $h \to \infty$. But the contribution of the fingers to the area of $V = X/\Gamma$ is about εh, for ε being the thickness of the fingers. This is a negligible quantity if ε is chosen small compared to h.

The presence of such fingers has an unpleasant effect on the areas of certain balls in X. If the center of an R-ball $B(R)$ in X is located at the tip of a finger, then $\operatorname{Area} B(R) \approx \varepsilon R$ for $R \in [\varepsilon, h]$. This is significantly less than $\operatorname{Area} B(R) \approx R^2$ needed for the Minkowski argument based on the inequality $\operatorname{Area} B(R) \geq \operatorname{Area} X/\Gamma$ for $R \approx (\operatorname{Area} X/\Gamma)^{\frac{1}{2}}$ (see 1.A.1).

1.C. The purpose of this lecture is an introduction to (and a survey of) (inter)systolic inequalities generalizing the Loewner theorem. Most of these inequalities are rather old and detailed proofs can be found in [Ber1,2] and [Gro2]. Yet, there remain many unresolved problems which we explain along the way.

We also indicate the proof of a (new local) systolic bound for $\mathbb{C}P^2$ using pseudo-holomorphic curves (see § 4) and we explain, following a recent paper by Buser and Sarnak, a relation between systolic inequalities for surfaces of large genus and the geometry of the Jacobian locus in the moduli space of Abelian varieties (see § 2).

2. SYSTOLES OF SURFACES

Let V be a closed connected surface with a Riemannian metric. We want to find a bound on the systole of V, that is the length of the shortest closed curve in V *non-homologous to zero.* An equally interesting question is finding the shortest *non-contractible* curve in V. To distinguish these, we introduce the following

2.A. Notations. a) The length of the shortest non-contractible curve in V is denoted by $\operatorname{syst} \pi_1(V)$.

b) The length of the shortest closed oriented curve in V representing a non-trivial element in the homology $H_1(V) = H_1(V;\mathbb{Z})$ is denoted by $\operatorname{syst} H_1(V)$.

Notice that shortest curves in V do exist : they are certain *simple closed geodesics* in V.

b′) One may replace $H_1(V;\mathbb{Z})$ by $H_1(V;A)$ for an arbitrary domain A of coefficients which leads to the notation $\operatorname{syst} H_1(V;A)$. The most useful A after $\mathbb{Z}$ is $A = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The corresponding systole $\operatorname{syst} H_1(V;\mathbb{Z}_2)$ refers to the shortest closed (non-oriented) curve non-homologous to zero mod 2 in V.

It is obvious that

$$\operatorname{syst} \pi_1(V) \leq \operatorname{syst} H_1(V) \leq \operatorname{syst} H_1(V;\mathbb{Z}_2)$$

and, in fact, this remains true (and obvious) for Riemannian manifolds V of any dimension.

Furthermore, if V is an *oriented surface*, then

$$\operatorname{syst} H_1(V) = \operatorname{syst} H_1(V;\mathbb{Z}_2) \ ,$$

because a *simple* closed curve (realizing $\operatorname{syst}_1 H_1$) in an *oriented surface* V which does not bound in V cannot bound mod 2 in V.

Non-orientable counter-example. Take the connected sums V_ε of two copies of the standard projective plane across an ε-circle (see Figure 2).

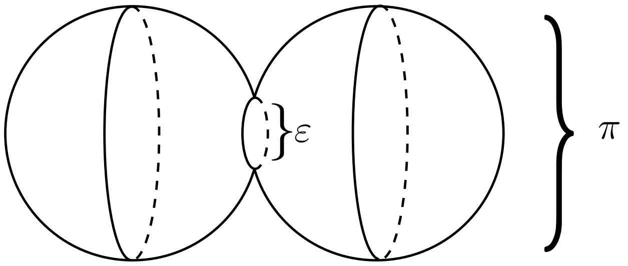

Figure 2

Then, $\operatorname{syst} H_1(V_\varepsilon) = \varepsilon$, that is the length of the joining circle. This can be made arbitrarily small while keeping $\operatorname{syst} H_1(V_\varepsilon; \mathbb{Z}_2) = \pi$ as for the original projective plane.

2.B. Surfaces of constant negative curvature. Our main objective is the inequality

$$\text{systole} \leq \text{const}(\text{Area})^{\frac{1}{2}}$$

where we want to understand the dependence of the constant upon the genus of the surface in question. Here, we are guided by the following

2.B.1. Obvious (Minkowski type) observation. — *If V has constant negative curvature, then*

$$\left.\begin{array}{rcl} \operatorname{syst} \pi_1(V) & \leq & \text{const}(\operatorname{Area} V)^{\frac{1}{2}}, \\ \text{const} & = & C_0 \dfrac{\log \operatorname{genus} V}{\sqrt{\operatorname{genus} V}}\ , \end{array}\right\} \tag{$*$}$$

where C_0 is a universal constant.

Proof. It is convenient to scale the metric in order to have curvature -1. Then, the R-balls in the universal covering $X = \widetilde{V}$ of V have area $\approx \exp R$ for $R \geq 1$. Thus, if $R \geq \log \operatorname{Area} V$, the projection of such a ball to V is *not* one-to-one and so

$$\operatorname{syst} \pi_1(V) \gtrsim \log \operatorname{Area} V\ . \tag{$*'$}$$

On the other hand, since curvature is -1, the area of V approximately equals the genus (in fact, area $= 4\pi(\text{genus} - 1)$), so $(*')$ is equivalent to $(*)$.

Remarks. (a) A slight refinement of the above argument gives a similar bound on $\operatorname{syst} H_1(V)$ (see [Gro2]). In fact, a rather sharp evaluation of the implied constant in the homological version of $(*')$ is due to Buser and Sarnak (see [Bu-Sa]) who prove that

$$\operatorname{syst} H_1(V) \leq 2\log(4\,\text{genus}\, V - 2)\ ,$$

for orientable surfaces V of curvature $K = -1$ (compare 2.D).

(b) F. Jenni, and later C. Bavard, show that every *hyperelliptic* surface V of constant negative curvature $K = -1$ contains a closed non-contractible curve of length $s_1 = \operatorname{syst} \pi_1(V)$ bounded by

$$s_1 \leq 2\log\left(3 + 2\sqrt{3} + 2\sqrt{5+3\sqrt{3}}\right) \approx 5.106\ .$$

In fact, Bavard proves that

$$c\,h\frac{s_1}{4} \leq \left(2\sin\frac{(g+1)\pi}{12g}\right)^{-1}\ ,$$

for $g = \text{genus}\, V$, and exhibits extremal surfaces of genus 2 and 5 for which the latter inequality is sharp. Notice that his inequality for $g = 2$ is equivalent to

$$s_1 \leq 2\log\left(1 + \sqrt{2} + \sqrt{2+2\sqrt{2}}\right) \approx 3.057$$

and, for $g = 5$, to

$$s_1 \leq 2\log\left(1 + \sqrt{5} + 2\sqrt{2+\sqrt{5}}\right) \approx 4.425\ ,$$

(see [Jen], [Bav4]).

2.C. Main systolic inequality for surfaces of large genus. — *Let V be a closed connected surface of genus ≥ 2 with a Riemannian metric. Then,*

$$\left.\begin{aligned} \left(\operatorname{syst} H_1(V;\mathbb{Z}_2)\right)^2 &\leq \text{const Area}\, V\ ,\\ \text{const} &= C\,\frac{(\log \text{genus}\, V)^2}{\text{genus}\, V}\ , \end{aligned}\right\} \qquad (+)$$

where C is a universal constant.

Remarks. (a) Inequality (+) becomes more transparent if the metric in V is normalized by the equality $\operatorname{Area} V = \operatorname{genus} V$. Then, (+) reads

$$\operatorname{syst} H_1(V;\mathbb{Z}_2) \lesssim \log \operatorname{genus} V \ , \tag{$+'$}$$

which means that V contains a simple closed *non-dividing* curve in V of length $\lesssim \log \operatorname{genus} V$.

(b) The best (known) constant C in (+) is significantly greater than C_0 in (∗). In fact, one knows that C *should be* greater than C_0 as the extremal surfaces of given genus and area having the maximal possible systole *do not* have constant curvature if genus ≥ 2. This contrasts with the case of V homeomorphic to the torus where extremal metrics *are* flat according to the Loewner torus theorem (see 1.B). In fact, extremal surfaces of genus ≥ 2 tend to have *piecewise flat* metrics. The study of these extremal metrics was conducted by E. Calabi (see [Cal]) and C. Bavard (see [Bav2,3]) who established sharp systolic inequalities for some surfaces of low genus. For example, Bavard finds the sharp bound on $s_1 = \operatorname{syst} H_1(V;\mathbb{Z}_2)$ for V homeomorphic to the *Klein bottle*,

$$s_1^2 \leq (\pi/2\sqrt{2}) \operatorname{Area} V \ .$$

(This result can also be derived from a theorem by Blatter concerning Möbius bound (see [Bla0]), as was pointed out by T. Sakai in a letter to M. Berger. Also recall that, according to Loewner's inequality cited earlier,

$$s_1^2 \leq \frac{\sqrt{3}}{2} \operatorname{Area} V$$

for V homeomorphic to T^2, and that the 1952-inequality by Pu for surfaces homeomorphic to the *real projective plane* P_2 bounds the systole by

$$s_1^2 \leq \frac{\pi}{2} \operatorname{Area} \ ,$$

with the equality for the metrics of constant positive curvature on P^2.)

Recently, Bavard completely solved the systolic problem for all 17 plane cristallographic groups Γ isometrically acting on a surface X homeomorphic to $\mathbb{R}^2$ by finding the sharp bounds for $\inf_{\gamma\in\Gamma_\infty}\inf_{x\in X}\operatorname{dist}\bigl(x,\gamma(x)\bigr)$, where $\Gamma_\infty \subset \Gamma$ denotes the set of the elements of *infinite* order. He also solved the systolic problem for triangular groups (see [Bav5]).

2.C.1. Dividing the proof of (+) into two steps.

Step 1. Let us show that (+) follows from a similar inequality for $\operatorname{syst}\pi_1(V)$, that is the length of the shortest non-contractible curve S in V which may divide V. If S actually divides V, we add round hemispherical cups to the pieces and obtain two closed surfaces V_1 and V_2 satisfying

$$\operatorname{genus} V_1 + \operatorname{genus} V_2 = \operatorname{genus} V$$

and

$$\operatorname{Area} V_1 + \operatorname{Area} V_2 = \operatorname{Area} V + (\operatorname{length} S)^2/\pi \ .$$

By the geometry of the hemisphere, every closed curve in V_i, $i = 1,2$, can be homotoped to the complement of the hemispherical cup without increasing the length. Thus,

$$\operatorname{syst} H_1(V;\mathbb{Z}_2) \le \min_{i=1,2} \operatorname{syst} H_1(V_i;\mathbb{Z}_2) \ ,$$

and a bound on $\operatorname{syst} H_1(V;\mathbb{Z}_2)$ reduces to that for a surface of lower genus since $(\operatorname{length} S)^2$ is bounded by

$$(\operatorname{length} S)^2 \le C_1 \,\frac{(\log \operatorname{genus} V)^2}{\operatorname{genus} V}\, \operatorname{Area} V \ ,$$

according to the π_1-version of (+) which we assume to be valid.

Notice that the division steps necessarily stop if we arrive at a surface V_0 homeomorphic to the torus or the projective plane. For such a V_0 we have

$$\bigl(\operatorname{syst} H_1(V_0;\mathbb{Z}_2)\bigr)^2 \le C_0 \operatorname{Area} V_0 \ .$$

In fact, this inequality for the torus case is covered by the Loewner theorem and for the projective plane this is a result by Pu (see above). (Notice, that the simple topology of V_0 implies $\operatorname{syst} H_1(V_0;\mathbb{Z}_2) = \operatorname{syst}\pi_1(V_0)$.) Thus, by induction on the genus g of V, we have

$$\big(\operatorname{syst} H_1(V;\mathbb{Z}_2)\big)^2 \leq \operatorname{const}(g)\operatorname{Area} V \ .$$

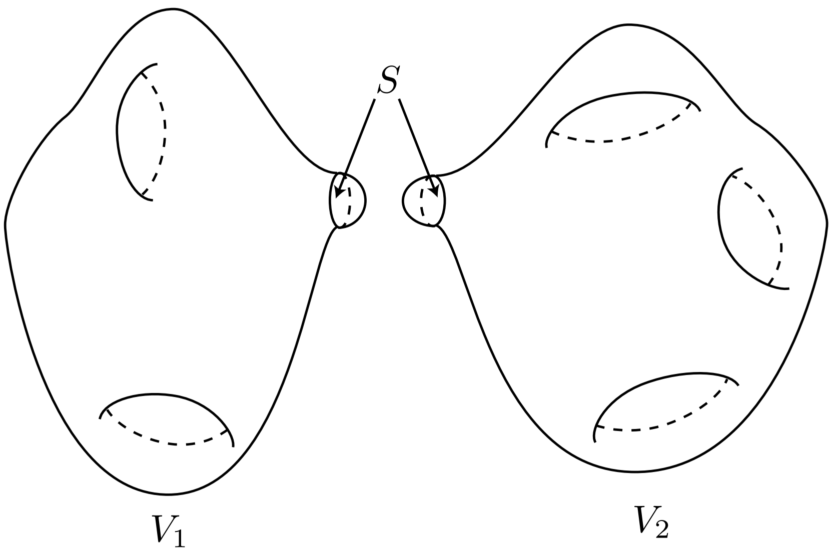

Figure 3

Moreover, by the above discussion, const(g) satisfies the following functional relation for all $g = 2, 3, \ldots$ *either* $\operatorname{const}(g) \leq C_1(\log g)^2/g$ *or there exist natural numbers* g_1 *and* g_2 *with* $g_1+g_2 = g$ *and positive real numbers* a_1+a_2 *having* $a_1+a_2 = 1+\pi^{-1}C_1(\log g)^2/g$, *such that*

$$\operatorname{const}(g) \leq \min\big(a_1 \operatorname{const}(g_1), a_2 \operatorname{const}(g_2)\big) \ .$$

Now, an elementary computation shows that

$$\operatorname{const}(g) \leq C(\log g)^2/g$$

for some universal constant C (depending on C_0 and C_1). □

Step 2. We must estimate $\operatorname{syst}\pi_1$. We normalize the metric of V to have Area = genus and we look for a ball $B(R) \subset V$ of radius $R \approx \log \operatorname{Area}$ which *does not* lift to the universal covering X of V. Unfortunately, we are unable to show that at least some balls in X have area $\approx \exp R$, but it is not hard to find some balls in X of area $\approx R^2$ (see [Gro2,5]). In fact, one can reduce the general case to that where *all* balls $B(R) \subset V$ which lift to X have area $\geq R^2$. This is done by cutting off the "fingers" of V and attaching semispherical cups to the cuts, see Figure 4.

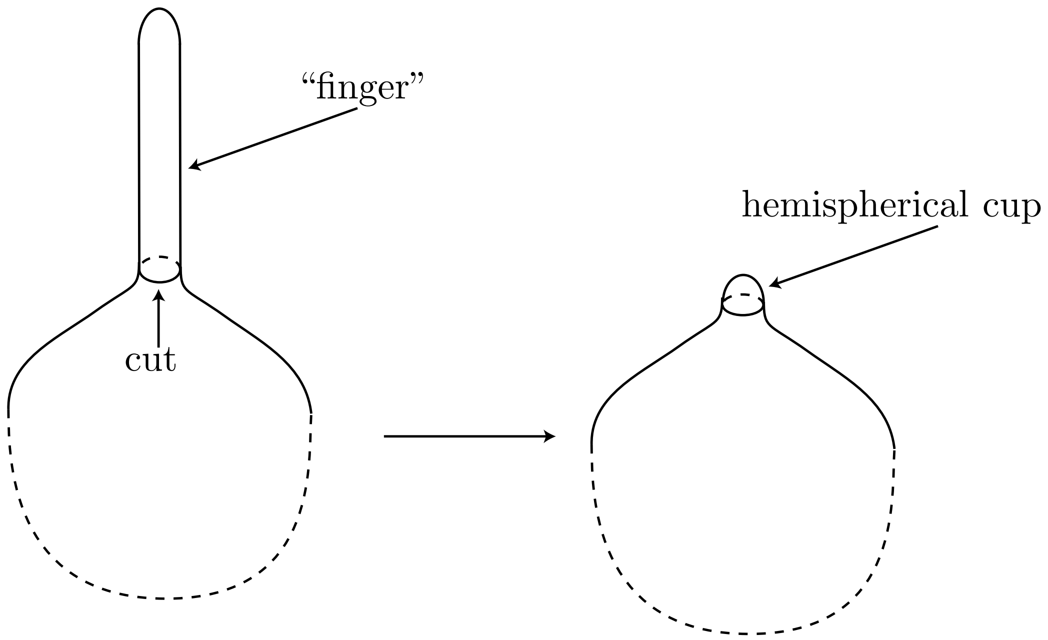

Figure 4

This immediately implies a universal estimate (due to J. Hebda, see [Heb1])

$$(\operatorname{syst} \pi_1)^2 \leq \operatorname{const} \operatorname{Area}$$

with const independent of the genus (see [Gro2] for details).

Now, we want the quadratic bound $\operatorname{Area} B(R) \geq R^2$ to yield the estimate

$$\operatorname{const} \lesssim (\log g)^2 / g$$

for $g =$ genus. For this, we invoke the *simplicial volume* $\|V\|_\Delta$ of V which measures "the optimal number" of triangles needed to represent the fundamental class $[V]$ of V by a real singular cycle. Namely, we assume for the moment that V is orientable and represent $[V] \in H_2(V; \mathbb{R}) = \mathbb{R}$ by real singular cycles $\sum_i r_i \sigma_i$, where $r_i \in \mathbb{R}$ and

σ_i are singular 2-simplices in V. Then, we set

$$\|V\|_\Delta = \mathrm{Inf} \sum_i |r_i|$$

over all representations of $[V]$ by $\sum_i r_i \sigma_i$.

If all balls $B(R) \subset V$ liftable to X have $\mathrm{Area}\, B(R) \geq R^2$, then one can show, using a *diffusion of chains* in V, that $\|V\|_\Delta$ is bounded by

$$\|V\|_\Delta \leq C_2 (A/s^2)\big(\log(C_3 A/s^2)\big)^2 \qquad (*)$$

where $A = \mathrm{Area}\, A$, $s = \mathrm{syst}\, \pi_1(V)$, and C_2 and C_3 are universal positive constants. (See 3.B and 6.4.D in [Gro2].)

Warning. The statement of this inequality in 6.4.D$'$ and 6.4.D$''$ in [Gro2] misses the exponent 2 over log for surfaces and $n = \dim V$ in the general case.

Finally we recall that

$$\|V\|_\Delta = 2|\chi|(V)\ ,$$

where χ is the Euler characteristic (see [Gro1]), and so $(*)$ implies the desired bound

$$A/s^2 \gtrsim g/(\log g)^2\ , \qquad (**)$$

for orientable surfaces V since $g = \frac{1}{2}|\chi| + 1$. The non-orientable case follows by applying $(**)$ to the oriented double cover of V.

2.C.2. Remarks. (a) An earlier inequality $(\mathrm{systole})^2 \leq \mathrm{const}_g\, \mathrm{Area}$ due to Accola (see [Acc]) and Blatter (see [Bla]) had $\mathrm{const}_g \approx g = \mathrm{genus}\, V$ rather than $\mathrm{const}_g \approx (\log g)^2/g$ of our inequality. The proof by Accola and Blatter relied on the conformal Abel's embedding of V to the *Jacoby variety* $J(V)$ which is a certain flat torus of dimension $2g = \mathrm{rank}\, H_1(V)$ (here V is assumed orientable). Then, the bound on the systole of V was derived from such a bound for the torus $J(V)$ provided by the Minkowski theorem (see 1.A and 2.D.6). In the next section we explain, following ideas of Buser and Sarnak, how the discrepancy between the two constants serves to distinguish Jacobians among all flat tori (as well as among all principally polarized Abelian varieties).

(b) One can estimate the above const_g by a different method using the more traditional topological meaning of the genus as $\frac{1}{2}\,\mathrm{rank}\, H_1$. At present the result one can obtain this way is weaker than the one provided by the simplicial volume $\|V\|_\Delta$, namely $\mathrm{const}_g \approx g^{-1}\exp(5\sqrt{\log g})$, and one does not know what to make out of it. Notice the $\mathrm{rank}\, H_1$-interpretation of the genus is indispensable in certain situations where the simplicial volume is not available, and so we do need a better understanding of the direct effect of $\mathrm{rank}\, H_1$ on const_g (compare 3.C.3).

(c) One expects further improvements on bounds on const_g under suitable geometric constraints on V. For example, if V admits an isometric involution with about g isolated fixed points, then the present techniques of "cutting fingers" yield the bound $\mathrm{const}_g \approx g^{-1}$, which means $(\mathrm{systole})^2 \lesssim g^{-1}\,\mathrm{Area}$. This generalizes the hyperelliptic result by Jenni and Bavard (who assume that $K(V) = -1$) cited earlier.

(d) One knows that a surface of genus $g \geq 2$ contains a *dividing non-contractible* geodesic of length s_1 bounded by

$$s_1^2 \leq C_\varepsilon\, g^{-1+\varepsilon}\,\mathrm{Area}\, V \ ,$$

for every fixed $\varepsilon > 0$. In fact, this geodesic comes from the commutator of two loops in V with the same kind of bound on their length (see 5.4 in [Gro2]). The question is to decide whether this can be improved to

$$s_1^2 \lesssim g^{-1}(\log g)^2\,\mathrm{Area} \ .$$

(The question becomes simpler if one is content to divide V by a system of *several* non-contractible curves with a bound on their total length.)

2.D. The 2-systoles of the Jacobian tori. We want to refine the notion of the 2-systole of a Riemannian manifold W (which, in the following application, will be a flat Riemannian torus of dimension $2g$). First, we observe that every homology class $h \in H_2(W)$ defines via the cup-product a 2-form on the real cohomology $H^1(W,\mathbb{R})$ denoted

$$h(\alpha,\beta) = \langle \alpha \smile \beta, h\rangle \ , \ \alpha,\beta \in H^1(W;\mathbb{R}) \ .$$

The rank of this form is denoted by rank h. Notice that the form $h(\alpha,\beta)$ is antisymmetric, and so rank h is even.

Example. Let W be an oriented surface of genus g. Then, the fundamental class $[W] \in H_2(W)$ has $\operatorname{rank}[W] = 2g$.

2.D.1. Notation. Consider all two-dimensional cycles in W representing two-dimensional homology classes of a given rank $2r$, and let $\operatorname{syst}_2^{2r}(W)$ denote the infimum of the areas of these cycles.

Example. Let W be the "square" torus, that is, the Cartesian product of d circles, $W = S_1 \times S_2 \times \ldots \times S_d$, with length $S_1 = \ell,\ i = 1, \ldots, d$. Then,

$$\operatorname{syst}_2^{2r} = r\ell^2 \quad \text{for } r = 1, \ldots, [\frac{d}{2}] \ .$$

Proof. Observe that each 2-torus $T_{ij} = S_i \times S_j \subset W$ represents a class of rank two in $H_2(W)$ and $\operatorname{Area} T_{ij} = \ell^2$. Furthermore, the sum (union) $T_{1,2} + T_{3,4} +, \ldots, + T_{2r-1,2r}$ has rank $2r$ and area $r\ell^2$. It follows that $\operatorname{syst}_2^{2r}(W) \le r\ell^2$.

Next, take an arbitrary integral cycle h of rank $2r$ and evaluate its area. This h is homologous to some integral combination of T_{ij}, say

$$h \sim \sum a_{i,j}\, T_{i,j} \ ,$$

where $\{a_{i,j}\}$ is an integral antisymmetric matrix of rank $2r$. Since the determinant of some $2r \times 2r$ submatrix in $\{a_{i,j}\}$ is non-zero, there are r non-zero entries among $a_{i,j}$ where no two have a common index. Thus, we may assume (permuting the indices if necessary) that the entries $a_{1,2},\ a_{3,4}, \ldots a_{2r-1,2r}$ do not vanish. Then, we take the differential form

$$\omega = \pm ds_1 \wedge ds_2 \pm ds_3 \wedge ds_4 \pm, \ldots, \pm ds_{2r-1} \wedge ds_{2r}$$

where the signs are equal to those of the corresponding $a_{i,j}$. It is clear that

$$\left| \int_h \omega \right| = |a_{1,2}| + |a_{3,4}| +, \ldots, + |a_{2r-1,2r}| \ge r \ .$$

Furthermore, for every orthonormal bivector (τ_1, τ_2), one has $|\omega(\tau_1, \tau_2)| \le 1$ by an elementary argument (Wirtinger inequality) and thus,

$$\operatorname{Area} h \ge \left| \int_h \omega \right| \ge r\,\ell^2 \ . \qquad \square$$

Exercise. Evaluate $\operatorname{syst}_2^{2r}$ for the "rectangular" tori which are products of circles of non-equal lengths.

Remark. — A simple (Minkowski type) argument shows that

$$\operatorname{syst}_2^2 \lesssim d(\mathrm{Vol})^{\frac{2}{d}}$$

for every flat d-dimensional torus, but there is no such bound on $\operatorname{syst}_2^{2r}$ for $r \ge 2$, as seen in the example of the "rectangular" tori.

2.D.2. Definition of $\operatorname{syst}_2^{2r,g}$. — *Every 2-cycle in W may be thought of as a surface V in W, and we want to incorporate the genus of V into the definition of* syst_2. *Namely, for given numbers $2r$ and g, we consider smooth maps of a closed connected orientable surface V of genus g into W, such that the image of the fundamental class of V has* $\operatorname{rank} 2r$ *in $H_2(W)$. Then, we take the infimum of areas of these surfaces in W and denote it by* $\operatorname{syst}_2^{2r,g} W$.

It is clear that $\operatorname{syst}_2^{2r,g}$ is monotone decreasing in g and

$$\inf_{g=1,2,\ldots} \operatorname{syst}_2^{2r,g} = \operatorname{syst}_2^{2r} \ .$$

2.D.3. Example : Jacobians. Let V be a closed orientable surface of genus g and $W = J(V) = H_1(V;\mathbb{R})/H_1(V;\mathbb{Z})$. This is a flat affine torus of dimension $2g$ without any Riemannian metric. However, W carries a natural closed (Kähler) 2-form ω_I corresponding to the intersection form on $H_1(V)$, where the implied correspondence comes from the canonical isomorphism $H_1(V) \overset{\sim}{\to} H_1(W)$. Notice that this isomorphism can be realized by a continuous map $\alpha : V \to W$ which induces this isomorphism and which is unique up to homotopy. Also observe that the 2-dimensional homology class $h = \alpha_*[V] \in H_2(W)$ has

$$\operatorname{rank} h = 2g = 2 \int_V \alpha^*(\omega_I) \ .$$

Now, let V be endowed with some Riemannian metric. Then, the space of 1-forms on V acquires a Hilbert space structure which induces such a structure on the (de Rham) cohomology group $H^1(V;\mathbb{R})$, and, using the formal (non-Poincaré) duality between H^1 and H_1, we obtain a Hilbertian (i.e., Euclidean) structure on $H_1(V;\mathbb{R})$ which descends to a flat Riemannian metric ρ on $W = J(V)$. Here is the standard list of properties of this metric ρ.

1) The metric ρ is invariant under conformal changes of the metric on V (since $\dim V = 2$) and thus depends only on the conformal structure of V.

2) The metric ρ agrees with ω_I (by Hodge theory on V) in the following sense. Lift ρ to a quadratic form $\tilde{\rho}$ on the linear space $H = H_1(V;\mathbb{R})$, and lift ω_I to an exterior form $\tilde{\omega}$ on H. (Notice that ρ and ω_I were actually defined via $\tilde{\rho}$ and $\tilde{\omega}$.) Then, there exists a $\tilde{\rho}$-orthonormal basis in H, say $h_1, h_2, \ldots, h_{2g}$, such that

$$\tilde{\omega} = dh_1 \wedge dh_2 + dh_3 \wedge dh_4 + \ldots + dh_{2g-1} \wedge dh_{2g} \ .$$

It follows that $|\omega_I(\tau_1, \tau_2)| \leq 1$ for every orthonormal bivector (τ_1, τ_2) by Wirtinger inequality. In particular,

$$\operatorname{Area} \alpha(V) \geq \int\limits_V \alpha^*(\omega_I) \geq g \ ,$$

for our map $\alpha : V \to W$ as well as for every map homotopic to α.

3) $\operatorname{Vol}(W, \rho) = 1$, since ρ agrees with the form ω_I which is unimodular with respect to the lattice $H_1(V;\mathbb{Z}) \subset H = H_1(V;\mathbb{R})$.

4) There is a map α_0 homotopic to α for which

$$\operatorname{Area} \alpha_0(V) = g \ .$$

(In fact, this α_0 is essentially unique and equals the harmonic representative of the homotopy class of α.)

Corollary. — *The "top" 2-systole of the Jacobian (torus) $W = J(V)$ is*

$$\operatorname{syst}_2^{2g,g}(W) = g = \operatorname{genus} V \ . \qquad (*)$$

Now, we invoke our Main Inequality (see 2.C) and conclude to the following relation between the homological 1- and 2-systoles of an arbitrary (not necessarily flat) Riemannian manifold W.

2.D.4. Intersystolic inequality. — *The systoles* $s_1 = \operatorname{syst} H_1(W)$ *and* $s_2(g) = \operatorname{syst}_2^{2g,g}(W)$ *are related for every* $g = 1, 2, \dots$ *by*

$$s_1^2 \le C\, g^{-1}\big(\log(g+1)\big)^2 s_2(g) \qquad (**)$$

for some universal constant C. (We put $g+1$ instead of g to take care of $g=1$.)

Now, we play $(**)$ against $(*)$ and obtain the following

Strengthened Minkowski for Jacobians. — *The Jacobian variety* $W = J(V)$ *of every surface* V *of genus* g *has volume one and*

$$s_1^2 = \big(\operatorname{syst} H_1(W)\big)^2 \le C\big(\log(g+1)\big)^2\ . \qquad (+)$$

This can be strengthened even further by the following

Theorem of Buser-Sarnak. — *The* 1*-systole of the Jacobian of every surface satisfies*

$$s_1^2 \le C_* \log(g+1) \qquad (++)$$

for some universal constant C_*.

We shall prove this later on, but now recall that the original Minkowski theorem bounds s_1 by

$$s_1^2 \le C' g \qquad (-)$$

as $\operatorname{Vol} W = 1$ (see 1.A), and so one is faced with the following alternative :

- *either* $(++)$ *distinguishes Jacobians among all flat tori,*
- *or* $(-)$ *may be improved to* $(++)$ *for all (or at least many) flat tori (which are not Jacobians of anybody).*

This alternative is resolved in favour of $(-)$ (as pointed out in [Bu-Sa]) by the following classical theorem.

2.D.5. Minkowski-Hlawka theorem. (See [Cas].) — *For every $g = 1, 2, \dots$, there exists a flat $2g$-torus W of unit volume and with $(\operatorname{syst} H_1)^2$ about g, namely with*

$$C''g \leq s_1^2 \leq C'g \ ,$$

for some universal positive constant C''.

Remark. — It is not easy to pinpoint an individual $2g$-torus for a large g having the first systole $\gtrsim \sqrt{g}$. For example, the "square" and "rectangular" tori of unit volume have $s_1 \leq 1$ for all g. However, the proof of the Minkowski-Hlawka theorem shows that $s_1 \approx \sqrt{g}$ on the average for a natural measure on the space $\mathcal{T}_{2g}$ of flat tori. (Instead of $\mathcal{T}_{2g}$, one may think of the space of unimodular lattices in $\mathbb{R}^{2g}$, that is $SL_{2g}\mathbb{R}/SL_{2g}\mathbb{Z}$ which comes along with a finite measure associated to the Haar measure on $SL_{2g}\mathbb{R}$.)

2.D.6. Cosystoles and the proof of the Buser-Sarnak theorem (inequality $(++)$). Recall the L^2-norm on 1-forms on V (here, V may be a Riemannian manifold of any dimension ≥ 2), and define *the first L^2-cosystole*, denoted $\operatorname{cosyst} L^2H^1(V)$, as the infimum of the L^2-norm of closed non-exact integral 1-forms λ on V where λ is called *integral* if it represents a cohomology class in $H^1(V; \mathbb{Z})$, i.e., if for every 1-cycle $S \subset V$, the integral $\int_S \lambda$ is an integer.

An equivalent definition appeals to the "dual Jacobian"

$$J^*(V) = H^1(V; \mathbb{R})/H^1(V; \mathbb{Z}) \ ,$$

where the (flat Riemannian) metric on $J^*(V)$ comes from the L^2-norm on the (de Rham) cohomology $H^1(V; \mathbb{R})$. With this metric one sees immediately that

$$\operatorname{cosyst} L^2H^1(V) = \operatorname{syst} H_1\big(J^*(V)\big) \ .$$

Notice that for $\dim V = 2$, the L^2-norm on 1-forms is conformally invariant and so $\operatorname{cosyst} L^2H^1(V)$ is a conformal invariant of surfaces V.

Remark. — The notion of cosystole (though not the word itself) appears in the work by Accola [Acc], Blatter [Bla] and Berger [Ber2] where these authors obtain a bound on the 1-cosystole of a Riemann surface V by applying the Minkowski theorem to the dual Jacobian $J^*(V)$. This gives a bound on the 1-systole of V via the following inequality

$$\operatorname{syst} H_1(V) \leq \operatorname{cosyst} L^2H^1(V)(\operatorname{Area} V)^{\frac{1}{2}} .$$

Proof. (Berger, see [Ber2].) We may assume by scaling the metric of V that $\operatorname{Area} V = 1$ (notice, that the cosystole is scale invariant) and then, the L^2-norm bounds the L^1-norm (on 1-forms). Now, a closed integral 1-form λ on V defines (by integration) a C^1-map μ of V to the circle $\Pi = \mathbb{R}/\mathbb{Z}$ of unit length such that $d\mu = \lambda$. Then, the *coarea formula* expresses the L^1-norm of $\lambda = d\mu$ in terms of the lengths of the pull-backs $\mu^{-1}(t),\ t \in \Pi$, by

$$\|\lambda\|_{L^1} = \int_\Pi \operatorname{length} \mu^{-1}(t)\, dt .$$

Since the form λ is non-exact, the curves $\mu^{-1}(t),\ t \in \Pi$ are not homologous to zero, and some of them are not longer than $\|\lambda\|_{L^1} \leq \|\lambda\|_{L^2}$. $\square$

(Notice that this argument is used by Berger also for $n \geq 3$ where it gives a bound on 1-cosystoles, see [Ber2].)

Cosystolic inequality of Buser-Sarnak. — *The cosystole of a closed oriented Riemann surface V of genus g is bounded by*

$$\operatorname{cosyst} L^2H^1(V) \leq C_* \sqrt{\log(g+1)} , \qquad (++)^*$$

for some universal constant C_*.

Proof. Suppose there is an annulus $A \subset V$ non-homologous to zero, which is conformal to $S_\ell \times [0,1]$ where S_ℓ is the circle of length ℓ. Then,

$$\operatorname{cosyst} L^2H^1(V) \leq \sqrt{\ell} .$$

To see that, start with the form dt on $A = S_\ell \times [0,1],\ t \in [0,1]$, and then slightly perturb it into the differential $d\theta$ for $\theta : (s,t) \mapsto \tau(t)$ where $\tau : [0,1] \to [0,1]$ is a self-mapping of the interval fixing the ends and having *zero derivative* at the ends. Such a perturbed form extends by zero outside $A \subset V$ to a closed non-exact integral form (which is Poincaré dual to S_ℓ in V), say λ on V, which has support in A and whose pointwise norm on A only slightly exceeds $1 = \|dt\|$. Thus, the L^2-norm of λ on V (which, by the conformal invariance, equals such norm on A with the product metric) can be made arbitrarily close to $\sqrt{\ell} = \|dt\|_{L_2}$, and with such λ the inequality cosyst $\leq \sqrt{\ell}$ is ensured.

Now, we need an annulus A in V conformal to $S_\ell \times [0,1]$ with $\ell \lesssim \log g$ and non-homologous to zero. To find this, we first conformally change the metric in V to make the curvature constant -1 and then take the shortest closed non-dividing geodesic γ in V with the new metric. It follows from 2.C that length $\gamma \lesssim \log g$, and it is easy to see (with the classical Zassenhaus-Kazhdan-Margulis lemma) that γ admits a collar neighbourhood A of width $\varepsilon > 0$ for a universal ε, say $\varepsilon = \frac{1}{4}$ (see §5.5.C in [Gro2], but beware of an incorrect claim in Example (a) there). Clearly, A is conformal to $S_\ell \times [0,1]$ with $\ell \lesssim \log g$, and the proof of $(++)^*$ is concluded.

Remark. — The existence of the "short" γ was claimed without proof in §5.5.C′ of [Gro2]. The proof was found by Buser and Sarnak independently of the systolic discussions in [Gro2]. In fact, Buser and Sarnak construct a geodesic γ with length $\leq 2\log(4g-2)$ and with a collar of width $\geq \operatorname{arctan} h(2/3)$. This makes

$$\operatorname{cosyst} L^2H^1(V) \leq \sqrt{\ell} \quad \text{for } \ell = \frac{3}{\pi} \log(4g+2)\ .$$

It remains to show that the systolic bound $(++)^*$ for the dual Jacobian $J^*(V) = H^1(V;\mathbb{R})/H^1(V;\mathbb{Z})$ implies the corresponding bound $(++)$ for the Jacobian $J(V) = H_1(V;\mathbb{R})/H_1(V;\mathbb{Z})$. But this is immediate as $J^*(V)$ is isometric to $J(V)$ for all Riemann surfaces V. In fact, the isometry is given by the Poincaré duality isomorphism $H_1(V,\mathbb{Z}) \overset{\sim}{\to} H^1(V;\mathbb{Z})$.

Remark. — This final step is not truly necessary. The strong bound $(++)^*$ on $\operatorname{syst}_1 J^*(V)$ serves as well to distinguish the Jacobian tori $J^*(V)$ and $J(V)$ (which are, by definition, mutually dual) from general flat tori.

Minkowski-Hlawka for Abelian varieties. Jacobians are distinguished among all flat tori W not only by strong systolic inequalities but also by the existence of a translation invariant closed integral 2-form ω which agrees with the metric by the property 2 in 2.D.3. But the presence of such an ω does not have much effect on the 1-systole of W. In fact, Buser and Sarnak show that the Minkowski-Hlawka remains valid for the tori admitting such ω. Namely, for every g, there exists such a special torus W of unit volume having

$$\operatorname{syst} H_1(W) \geq C''' g$$

for some universal constant $C''' > 0$. In fact, a majority of special tori W satisfies such an inequality (see [Bu-Sa]).

Problem. Consider a Riemann surface V of genus g canonically embedded into its Jacobian $J(V)$ and recall that the bound on the 1-systole of V with the induced metric provided by our Main Inequality (see 2.C) reads

$$\operatorname{syst}_1 V \lesssim \log g \ ,$$

while the bound for the 1-systole of $J(V) \supset V$ is stronger,

$$\operatorname{syst}_1 J(V) \lesssim \sqrt{\log g} \ .$$

There may be two (not mutually exclusive) explanations for the discrepancy between these two inequalities. First, it might happen that the metric in V induced from $J(V)$ is so special that it satisfies a sublogarithmic (in g) systolic inequality. The second possibility is that V is so much curved in $J(V)$ that certain geodesics of length $\ell \approx \log g$ in V shorten to $\approx \sqrt{\ell}$ in $J(V)$. I am inclined to believe that "generic" V have $\operatorname{syst}_1 \approx \log g$ for the metric induced from $J(V)$ (this is known for the metric of constant curvature -1) and the extra shortening in $J(V)$ is due to large exterior curvature of V in $J(V)$.

Remark. — Buser and Sarnak show that their inequality $\operatorname{syst}_1 J(V) \lesssim \log g$ is sharp. The relevant examples are provided by congruence coverings of a fixed arithmetic Riemann surface (see [Bu-Sa] and 3.C).

2.E. Systolic characterization of Jacobians. Recall the definition of the 2-systole of rank r from 2.D.1, denoted by syst_2^r, and let W be a flat Riemannian torus of dimension $2g$ and unit volume.

Theorem. — *The 2-systole of* $\operatorname{rank} r = 2g$ *of* W *is bounded from below by*

$$\operatorname{syst}_2^{2g} W \geq g \tag{$*$}$$

and equality holds if and only if W *is isometric to the Jacobian of some Riemann surface* V *or to a limit of Jacobians.*

First proof of $(*)$. The (linear) space of bilinear antisymmetric forms ω on $\mathbb{R}^{2g}$ carries two natural $O(2g)$-invariant (non-Euclidean) norms : *mass* and *comass* defined as follows

$$\operatorname{comass} \omega = \sup \omega(x, y) \ ,$$

where "sup" has taken over the pairs of orthonormal vectors $x, y \in \mathbb{R}^{2g}$; then by definition, mass is the dual norm on the space of 2-forms, or better (but not necessary) to say, on the space of *bivectors* dual to forms.

Lemma. — *The mass of a form* ω *bounds its discriminant by*

$$|\operatorname{Discr} \omega|^{\frac{1}{g}} \leq g^{-1} \operatorname{mass} \omega \ .$$

Proof. Recall, that the discriminant of ω is just the determinant of the coefficient matrix of ω in some orthonormal basis. We use an orthonormal basis $x_1, \ldots, x_{2g}$ which diagonalizes ω, that is

$$\omega = \mu_1 \, x_1 \wedge x_2 + \mu_2 \, x_3 \wedge x_4 + \ldots + \mu_g \, x_{2g-1} \wedge x_{2g} \ .$$

The discriminant of this ω is the product $\mu_1 \mu_2 \ldots \mu_g$, and the mass is the sum $\mu_1 + \mu_2 + \ldots + \mu_g$. □

Now, we prove $(*)$ by observing that every homology class $h \in H_2(W; \mathbb{Z}) \subset H_2(W; \mathbb{R}) = \wedge^2 \mathbb{R}^{2g}$ has integral discriminant which is non-zero for $\operatorname{rank} h = 2g$. Hence, $\operatorname{mass} h \geq g$, which means (by the definition of mass) that there exists a closed translation invariant 2-form ω on W having unit comass (on every tangent space of W) such that $\int_h \omega \geq g$. This immediately implies that every cycle C realizing h has $\operatorname{area} C \geq g$.

Digression : *Stability of h.* The above argument reproduces a fragment of the discussion by Lawson in [Law], where the author shows (among many other results) that every 2-dimensional integral homology class h in a flat torus is *stable*. This means there exists an integer N such that the multiple class Nh can be realized by a surface C of area N mass h. In other words,

$$\operatorname{Area} C = \operatorname{mass} Nh \ .$$

Sketch of the proof. First, by restricting to a subtorus, one may assume that $\dim W = 2g$. Then, by linear algebra, there exists a translation invariant complex structure $J : T(W) \to T(W)$ which preserves the metric of W and the form (bivector) corresponding to h and such that the symmetric 2-form corresponding to h (i.e., $h(x,y) = h(x, Jy)$) is positive definite. Now, one knows (this is highly non-trivial) that there exists a subvariety $D \subset W$ of real codimension 2 which is J-*complex* (i.e., the singularity of D has codimension ≥ 2 in D and the tangent subbundle of the non-singular locus of D is J-invariant) and such that the homology class $[D] \in H_{n-2}(W)$ satisfies

$$[D] \frown [D] \frown \ldots \frown [D] = (g-1)!h \ .$$

Finally, the required surface C is obtained by intersecting $(g-1)$ generic translates of D. (Notice that this C may have a singularity.)

Remarks. (a) It is unlikely that there exists a direct proof of Lawson's theorem without using complex analysis (and the Θ-divisor D).

(b) It is, probably, unknown if there are integers N not contained in the subset $(g-1)!\mathbb{Z} \subset \mathbb{Z}$ for which

$$\operatorname{Area} Nh = \operatorname{mass} Nh (= N \operatorname{mass} h)$$

(where the area of a homology class is understood as the infimum of areas of surfaces representing this class).

(c) The above inequality $(*)$ is implicit in Lawson's paper [Law] though it is not stated in such form.

(d) It is shown in [Law] that the homology classes of flat tori are stable in dimensions and codimensions 1 and 2 but not stable, in general, for the other dimensions. (A homology class h of any dimension is called *stable* if $\operatorname{Vol} Nh = \operatorname{mass} Nh$ for some $N \in \mathbb{Z}$.)

Now, we return to our inequality $(*)$ and observe that the above argument bounds from below the mass of h as well as the area. In fact, the mass of every *closed current* representing an integral homology class h of rank $2g$ is bounded from below by g. It is also clear at this stage that the equality $\operatorname{mass} h = g$ implies that W is an Abelian variety principally polarized by h and then the equality $\operatorname{Area} h = h$ makes W a Jacobian by the following characterization of Jacobians (compare [Law]).

Matsusaka criterion. (See [Mat].) — *The Abelian variety W is a Jacobian if and only if it contains an algebraic curve C (i.e., an effective 1-cycle) whose homology class $[C] \in H_2(W)$ is related to the class $[D] \in H_{2g-2}(W)$ of the Θ-divisor $D \subset W$ by the relation*

$$[C] = \frac{1}{(g-1)!} \underbrace{[D] \cap [D] \cap \ldots \cap [D]}_{g-1} .$$

(Notice that the intersection class $[D] \cap [D] \cap \ldots \cap [D] \in H_2(W)$ without the coefficient $1/(g-1)!$ can always be realized by a curve, namely by the intersection of $g-1$ generic translates of D.)

A short proof of the Matsusaka theorem is incorporated into our second proof of $(*)$ as we shall presently see.

Second proof of $(*)$. Let $V \subset W$ be the minimal surface of area $= \operatorname{syst}_2^{2g} W$ representing a class of rank $2g$. Then, the cyclic coordinates of W are *harmonic* functions on V (with the induced metric) and therefore, there exists an affine map of the Jacobian $J(V)$ onto W which conformally maps V canonically embedded into $J(V)$ to V in W. For example, if $\operatorname{genus} V = g$, then this map $p : J(V) \to W$ is an affine isomorphism and, in fact, one obtains $J(V)$ by just modifying the (flat) metric on W as follows. Define the new norm on translation invariant 1-forms λ on W by restricting λ to V and taking the L^2-norm of the restriction on V. This gives a norm on the cotangent bundle of W and hence on W. Clearly, W, with this new norm, equals the Jacobian of V.

The area of V in W equals the energy of the map $p : J(V) \to W$ restricted to $V \subset J(V)$ which equals in this case the energy of the map $p : J(V) \to W$, that is, $\frac{1}{2}$ Trace $D_p^* D_p$ for the differential D_p of p at some point. If genus $V = g$ and p is an isomorphism, this energy must at least be g since $\operatorname{Vol} W = \operatorname{Vol} J(V) = 1 = \operatorname{Det} D_p^*$, and the proof is finished. In the general case, where $\dim J(V) = 2 \operatorname{genus} V > \dim W$, we observe that the fibers of the projection $p : J(V) \to W$ must have volume at least one. In fact, the condition $\operatorname{rank} h = 2g$ forces the Kähler form ω_I of $J(V)$ (see 2.D.3) to be non-singular on the fibers, hence has lower bound on the volume by one as ω_I is integral.

The equality case. It follows from the above discussion that the area of V in W equals g if and only if $J(V)$ splits as a *polarized Abelian variety*, i.e., with respect to the Riemannian metric ρ in $J(V)$ and the form ω_I. In fact, $J(V) = W \oplus W^\perp$, where $W^\perp$ is a fiber of the projection with the structure $(\rho, \omega_I) \mid W^\perp$. For example, if genus $V = g$, we have $J(V)$ isometric to W and so W *is* a Jacobian. In general, we need the following algebro-geometric fact provided to me with proof by Jean-Benoît Bost.

Fact. — *Jacobians of non-singular curves do not split in the above sense. If a limit of such Jacobians splits, then the underlying curves converge to a reducible curve.*

Proof. If a polarized Abelian variety splits, $A = A_1 \oplus A_2$, then the Θ-divisor D of A can be represented by $A_1 \times D_2 + D_1 \times A_2$ for the Θ-divisors D_i in A_i. This divisor is reducible, but in a Jacobian $J(V)$ the Θ-divisor is unique up to translations and thus irreducible as it can be represented by the sum of $g-1$ copies of V in $J(V)$. $\square$

Remarks. (a) The minimal surface $V \subset W$ might have singularities, but it can be parametrized by a non-singular (possibly disconnected) Riemann surface (see [Chan]), and this is all what matters for the above proof.

(b) If W is a Minkowski-Hlawka $2g$-dimensional torus with the first systole $\operatorname{syst}_1 W \approx \sqrt{g}$ (see 2.D.5), then, according to 2.D.4,

$$\operatorname{syst}_2^{2g,g}(W) \gtrsim g^2\big(\log(g+1)\big)^2 .$$

We shall see in 3.C.3 that these tori have

$$\mathrm{syst}_2^{2g} \gtrsim g^{2-\varepsilon} ,$$

where ε is an arbitrary positive number and the constant (for the sign $\gtrsim$) depends on ε.

Problem. What is the actual behaviour of the functions $W \mapsto \mathrm{syst}_2^r(W)$ and $W \mapsto \mathrm{syst}_2^{r,g}(W)$ on the space of flat tori ? What are the average values of these functions ?

A more elementary problem concerns the behaviour of the (systolic) $\mathbb{R}$-mass (instead of area) and, more generally, of the eigenvalues μ_i of classes $h \in H_2(W)$ (see the first proof of $(*)$ above).

3. SYSTOLIC INEQUALITIES FOR $K(\Gamma, 1)$-SPACES

We start with a general inequality for closed *aspherical* Riemannian manifolds V, where "aspherical" means that the universal covering X of V is contractible (e.g., homeomorphic to $\mathbb{R}^n$ for $n = \dim V$).

3.A. Basic inequality. — *The length s_1 of the shortest non-contractible curve in V is bounded by*

$$s_1 \underset{\text{def}}{=} \operatorname{syst} \pi_1(V) \leq C_n (\operatorname{Vol} V)^{\frac{1}{n}} \tag{$*$}$$

for $n = \dim V$ and some constant in the interval $0 < C_n < 6(n+1)n^n \sqrt{(n+1)!}$.

Idea of the proof. (See [Gro2] for details.) One can regularize V by a suitable process of chopping away long narrow fingers (see Figure 4) such that $\operatorname{Vol} V_{\text{reg}} \leq \operatorname{Vol} V$ and $\operatorname{syst} \pi_1(V_{\text{reg}}) = \operatorname{syst} \pi_1(V)$, and such that the balls in V_{reg} liftable to the universal covering $\widetilde{V}_{\text{reg}}$ have volumes bounded from below by

$$\operatorname{Vol} B(R) \gtrsim R^n \ .$$

This gives the desired upper bound on R, that is

$$R \lesssim (\operatorname{Vol} V_{\text{reg}})^{\frac{1}{n}} \ .$$

One can reformulate the above theorem in the spirit of Minkowski as the following

3.A.1. Bound on displacement. — *Let a group Γ discretely and isometrically act on a complete Riemannian manifold X. If X is contractible (i.e., homeomorphic to $\mathbb{R}^n$), then there exists a point $x \in X$ and a non-identity element $\gamma \in \Gamma$, such that*

$$\operatorname{dist}\big(x, \gamma(x)\big) \leq C_n (\operatorname{Vol} X/\Gamma)^{\frac{1}{n}}, \ n = \dim X \ . \tag{$*$}'$$

Remarks (a) Notice that we do not assume X/Γ to be compact, but the theorem holds true all the same (see [Gro2]).

(b) We do not need the metric in X to be Riemannian. The space X may be a contractible manifold with an arbitrary metric and then the displacement bound remains valid with the n-dimensional Hausdorff measure of X/Γ instead of the Riemannian volume.

(c) If X is non-contractible, then the above bound $(*)'$ may fail. For example, we may multiply X by a small sphere S with trivial action which makes $\mathrm{Vol}(X \times S/\Gamma)$ small without changing the displacement. Yet we shall see below some instances of $(*)$ and $(*)'$ where X is non-contractible.

(d) Our basic inequality provides a positive answer to one of the conjectures raised by Berger in [Ber1].

3.B. Sharpening the bound on syst_1 by the topology of V. If V is a surface of genus g, then $\mathrm{syst}\,\pi_1(V) \lesssim g^{-\frac{1}{2}}(\log g)(\mathrm{Area}\,V)^{\frac{1}{2}}$ (see 2.C), and this generalizes to manifolds V of dimension ≥ 3 which admit auxiliary Riemannian metrics ρ' of negative sectional curvature $K(\rho') \leq -1$. The topological invariant which we shall use for such V (where V appears in our discussion with an arbitrary metric $\rho \neq \rho'$) is the volume of (V, ρ') playing the role of the genus.

3.B.1. Theorem. — *If a Riemannian manifold V admits a(n) (auxiliary) metric with curvature ≤ -1 and volume g, then*

$$\mathrm{syst}\,\pi_1(V) \leq C_n\, g^{-\frac{1}{n}}\big(\log(1+g)\big)(\mathrm{Vol}\,V)^{\frac{1}{n}}\ . \qquad (**)$$

Idea of the proof. One knows that the simplicial volume $\|V\|$ of V is bounded from below by $\varepsilon_n g$, $\varepsilon_n > 0$, (see [Thu], [Gro1]). On the other hand, a suitable "diffusion of chains" provides an upper bound on $\|V\|_\Delta$ by

$$\|V\|_\Delta \lesssim (V\ell/s^n)\big(\log(C'_n V\ell/s^n)\big)^n$$

for $V\ell = \mathrm{Vol}\,V$ and $s = \mathrm{syst}\,\pi_1 V$. (Compare Step 2 in 2.C.1 and 6.4.D in [Gro2]; we repeat the warning : the exponent $n/\log n$ is missing in 6.4.D$'$ of [Gro2].)

Remark. — The inequality $(**)$ is obvious for the auxiliary metric with $K \leq -1$ as the balls of radius 12 for this metric in the universal covering have $\operatorname{Vol} B(R) \gtrsim \exp R$ for $R \geq 1$.

3.B.2. A bound on syst_1 with Betti numbers. Let $b = b(V)$ denote the sum of the Betti numbers of V, i.e.,

$$b = \sup_F \operatorname{rank} H_*(V;F) \ ,$$

where F runs over all fields.

Theorem. (See 6.4.C″ in [Gro2].) — *The shortest non-contractible curve in a closed aspherical Riemannian manifold V is bounded by*

$$\operatorname{syst} \pi_1(V) \leq C_n b^{-\frac{1}{n}} \left(\exp C'_n \sqrt{\log b}\right) (\operatorname{Vol} V)^{\frac{1}{n}} \qquad (***)$$

for some universal positive constants C_n and C'_n.

Corollary. — *One has*

$$\operatorname{syst} \pi_1(V) \lesssim b^{-\frac{1}{n}+\varepsilon} (\operatorname{Vol} V)^{\frac{1}{n}}$$

for every fixed $\varepsilon > 0$.

Question. Can one replace $\exp\sqrt{\log}$ in $(***)$ by log as in $(**)$?

3.B.3. Simplicial height $h(V)$. Denote by $h = h(V)$ the minimal possible number of simplices of a finite simplicial n-dimensional polyhedron P, for $n = \dim V$, which admits a continuous map $u : P \to V$ surjective on the top dimensional homology. Thus, we want the fundamental class of V (V is assumed connected) to be in the image $u_*\big(H_n(P)\big)$ if V is orientable. In the non-orientable case we pass to the oriented double cover $\widetilde{V} \to V$, take the corresponding double cover $\widetilde{P} \to P$ induced by u from $\widetilde{V} \to V$, and require the fundamental class of $\widetilde{V}$ to be in the image $u_*\big(H_n(\widetilde{P})\big)$.

Theorem. — *The inequality $(***)$ remains valid with $h = h(V)$ in place of b,*

$$\operatorname{syst} \pi_1(V) \leq C_n\, h^{-\frac{1}{n}} \left(\exp C'_n \sqrt{\log h}\right) (\operatorname{Vol} V)^{\frac{1}{n}} \ . \qquad (***)'$$

In fact, the proof of $(***)$ in [Gro2] proceeds via $(***)'$ as $h \le b$ by elementary algebra.

Remarks. (a) The above question applies to $(***)'$ as well as to $(***)$.

(b) One could replace $h(V)$ by an a priori larger number $h_+ = h_+(V)$ by insisting that P were a *pseudomanifold*. This enlarged "height" dominates the simplicial volume, and so the conjectural inequality

$$\operatorname{syst}\pi_1 \lesssim h_+^{-\frac{1}{n}}(\log h_+)(\mathrm{Vol})^{\frac{1}{n}}$$

would imply $(**)$ as well as $(***)$.

Idea of the proof of $(***)$. First, we regularize the manifold V by chopping off long narrow fingers in order to bound from below the volumes of relevant R-balls by $\approx R^n$. Then, the regularized V is covered by a controlled amount of such balls, and the nerve of this covering serves for P (see [Gro2] for details).

3.B.4. Basic inequality for essential manifolds V. Let Γ be the fundamental group $\Gamma = \pi_1(V)$, take the Γ-classifying Eilenberg-MacLane $K(\Gamma,1)$-space W and let $f : V \to W$ be a *classifying map*. Recall that W is an aspherical space with $\pi_1(W) = \Gamma$ and f is uniquely defined up to homotopy by being the identity on the fundamental group,

$$f_* : \pi_1(V) = \Gamma \underset{\mathrm{id}}{\to} \Gamma = \pi_1(W)\ .$$

The space V is called *essential* if its fundamental class does not vanish in $H_n(W)$, $n = \dim V$, i.e., $f_*[V] \ne 0$, and we use the $\mathbb{Z}_2$-homology in the case where V is non-orientable.

Examples. (a) Every closed aspherical manifold V is essential as one may take $W = V$ and f the identity map.

(b) The projective space $\mathbb{R}P^n$ is essential as $W = \mathbb{R}P^\infty$ and the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ is not homologous to zero mod 2.

(c) If V admits a map of non-zero degree to an essential (e.g., aspherical) manifold V', then V is essential. In particular, the connected sum $V = V'\#V''$ is essential for an essential V' and an arbitrary V''.

Theorem. — *The basic inequality* $(*)$ *remains valid for essential manifolds* V,

$$\operatorname{syst}\pi_1(V) \leq C_n(\operatorname{Vol} V)^{\frac{1}{n}} . \qquad (*)_{\mathrm{ess}}$$

Example. (Conjectured by Berger in [Ber1], compare [Ber4]). — *Let* $\widetilde{V}$ *be a topological sphere with a Riemannian metric and* $\alpha : \widetilde{V} \to \widetilde{V}$ *be an isometric involution. Then, there exists a point* $\tilde{v} \in \widetilde{V}$ *for which*

$$\operatorname{dist}\big(\tilde{v}, \alpha(\tilde{v})\big) \leq C_n(\operatorname{Vol}\widetilde{V})^{\frac{1}{n}} .$$

This follows from $(*)_{\mathrm{ess}}$ applied to V/α.

Remarks. (a) This result for $n = 2$ is due to Pu (see [Pu]) who obtains the sharp value $C_2 = \sqrt{\pi}/2$ with equality for the metrics of constant positive curvature. It is unknown if metrics of constant curvature are extremal for $\dim \widetilde{V} \geq 3$.

(b) Berger (see [Ber6 $\frac{1}{2}$]) also studied *non*-isometric involutions of S^2 where he obtained the above distance inequality with $C_2 = 2$ and conjectured the sharp inequality (compare (E'_5) in Appendix 1 of [Gro2]).

3.C. Intersystolic inequalities for $W = K(\Gamma, 1)$**.** Let W be a Riemannian manifold with $\pi_1(W) = \Gamma$ for some group Γ and $\pi_2(W) = \pi_3(W) = \ldots = \pi_n(W) = 0$ for some $n \geq 2$. We want to bound the 1-systole of W in terms of the n-systole, thus generalizing the results in 3.A and 3.B. This will give us, in particular, a bound on the homological 1-systole $\operatorname{syst} H_1(V)$ under suitable assumptions by using the classifying map of V to $K(\Gamma, 1)$ for $\Gamma = H_1(V)$.

At this stage we fix the coefficient field F to be $\mathbb{Z}$ or $\mathbb{Z}_p$, and denote by $\operatorname{syst} H_k(W)$ the infimum of the volumes of k-dimensional F-cycles in W non-homologous to zero.

The following inequality generalizes the above $(*)_{\mathrm{ess}}$.

3.C.1. Basic intersystolic inequality. (See [Gro2].) — *The infimum of the length of closed non-contractible curves in* W *is bounded by*

$$\operatorname{syst}\pi_1(W) \leq C_n\big(\operatorname{syst} H_n(W)\big)^{\frac{1}{n}} . \qquad (*)_{\mathrm{inter}}$$

Let us derive $(*)_{\text{ess}}$ from $(*)_{\text{inter}}$. For this, we take $\Gamma = \pi_1(V)$, use an embedding $V \subset W$ for the classifying map, and take such a metric on W for which the embedding is isometric and such that the shortest curves in V are also shortest in W (this is easy to arrange). Then, we get a bound on syst_1 by $(\operatorname{syst} H_n W)^{\frac{1}{n}}$ which clearly is $\leq \operatorname{Vol} V$.

Let us modify the above construction by using $H_1(V)$ for Γ instead of π_1. Then, we get a bound on the *homological* 1-systole of V, by

$$\operatorname{syst} H_1(V) \leq C_n (\operatorname{Vol} V)^{\frac{1}{n}} \ , \qquad (*)_{\text{homol}}$$

provided $[V]$ *does not vanish* in $H_1\big(K(\Gamma;1)\big), \Gamma = H_1(V)$. Thus, $(*)_{\text{homol}}$ *is valid for an n-dimensional Riemannian manifold V, provided there are 1-dimensional cohomology classes $\alpha_1, \ldots, \alpha_n$ in V over some coefficient field whose cup-product does not vanish*,

$$\alpha_1 \smile \alpha_2 \smile \ldots \smile \alpha_n \neq 0 \ .$$

3.C.2. $K(\Gamma;1)$**-spaces with universal metrics.** The inequality $(*)_{\text{ess}}$ can be derived from $(*)_{\text{inter}}$ for the classifying space W with a certain canonical metric which is locally very much similar to the metric in the Banach space $\ell_\infty = \{x_1, x_2, \ldots\}$ with the sup-norm. For example, if $\Gamma = \mathbb{Z}_2$, we first take the unit sphere $S^\infty \subset \ell_\infty$ and $P^\infty = S^\infty/\mathbb{Z}_2$, where P^∞ comes along with the metric induced by S^∞ from ℓ_∞.

Then, the basic systolic inequality for essential n-dimensional manifolds V with $\pi_1(V) = \mathbb{Z}_2$ is equivalent to the lower bound

$$\operatorname{syst} H_n(P^\infty; \mathbb{Z}_2) \geq \varepsilon_n > 0 \ , \qquad (**)$$

with an appropriate convention on how the n-volume is understood in P^∞. In fact, if $\operatorname{syst}_1(V)$ with $\pi_1(V) = \mathbb{Z}_2$ is $\geq 2 + \delta$, then V admits (by an easy argument, see 6.1 in [Gro2]), *a distance decreasing* classifying map $V \to P^\infty$, and $(**)$ yields $\operatorname{Vol} V \geq \operatorname{syst} H_n(P^\infty; \mathbb{Z}_2) \geq \varepsilon_n > 0$.

Now, we can approach $(**)$ by the variational method, i.e., by looking at the minimal n-dimensional $\mathbb{Z}_2$-cycle in P^∞ non-homologous to zero. One can show that such cycles satisfy a version of the classical *monotonicity property* which insures the bound $\operatorname{Vol} B(R) \gtrsim R^n$ for balls in this cycle and thus gives a lower bound on the volume of the cycle.

The monotonicity property is proven on the basis of a suitable *Federer-Fleming* type *isoperimetric inequality* in P^∞ which (implicitly) depends on a possibility to chop away long narrow fingers as is customary in geometric measure theory (see §6 in [Gro2]).

3.C.3. Strenghtening $(*)_{\text{inter}}$ by topology of a cycle in W. Let us introduce the following four characteristics of a homology class $\alpha \in H_n(W)$:

(1) *the simplicial norm* $\|\alpha\|_\Delta$, i.e., the infimum of the sums $\sum |r_i|$ for all representations of α by singular cycles $\sum\limits_i r_i \sigma_i$ with real coefficients r_i (and singular simplices $\sigma_i : \Delta^n \to W$) ;

(2) the rank $b = b(\alpha)$ of the bilinear form on $H^*(W)$ defined by evaluation of the cup product on α, that is $(\omega_1, \omega_2) \mapsto \langle \omega_1 \cup \omega_2, \alpha \rangle$, where this is assumed to be zero if the sum of the degrees of ω_1 and ω_2 is different from n ; (Here one may take arbitrary coefficients. Also, one can use L^2-Betti numbers and the corresponding L^2-rank which may sometimes dominate the ordinary rank.)

(3) *the simplicial height* $h = h(\alpha)$ is the minimal possible number of simplices of a polyhedron P admitting a map into W whose image in H_n contains α ;

(4) *modified height* h_+, that is, the infimum of the sums $\sum\limits_i |r_i|$ over all representations of α by combinations of singular n-simplices σ with *integer* coefficients r_i.

Finally, we denote by $\operatorname{Vol} \alpha$ the infimum of the volumes of n-cycles representing α.

Topological intersystolic inequalities. — *Let W be a Riemannian manifold with $\pi_2(W) = \pi_3(W) = \ldots = \pi_n(W) = 0$ for some $n \geq 2$ and let α be an n-dimensional homology class of W. Then, the infimum of the lengths of non-contractible curves in W, i.e., $s_1 = \operatorname{syst} \pi_1(W)$, is bounded by the following inequalities*

$$s_1 \leq C_n \|\alpha\|_\Delta^{-\frac{1}{n}} (\log \|\alpha\|_\Delta)(\operatorname{Vol} \alpha)^{\frac{1}{n}}, \tag{1}$$

$$s_1 \leq C_n\, h^{-\frac{1}{n}} \Big(\exp C'_n \sqrt{\log h}\Big) (\mathrm{Vol}\, \alpha)^{\frac{1}{n}}, \tag{2}$$

$$s_1 \leq C_n\, b^{-\frac{1}{n}} \Big(\exp C'_n \sqrt{\log b}\Big) (\mathrm{Vol}\, \alpha)^{\frac{1}{n}} \ . \tag{3}$$

It is clear that (2) $\Rightarrow$ (3). It is unclear whether (2) and/or (3) can be brought to the shape of (1). It is desirable to have an inequality with h_+ similar to the above. It is easy to show that the inequalities for aspherical manifolds stated in 3.B are special cases of the above. Namely (1) $\Rightarrow$ $(**)$ in 3.B, (3) $\Rightarrow$ $(***)$ and (2) $\Rightarrow$ $(***)'$. Also notice that inequalities (1) and (3) are already interesting for $n = 2$ and W a flat torus as they generalize the systolic inequalities for surfaces (see 2.D).

3.C.4. An application of (3) to Abelian varieties. Let W be a principally polarized Abelian variety of complex dimension g with Θ-divisor $D \subset W$ and let $\delta(W)$ denote the minimal positive integer such that the homology class

$$\big(\delta/(g-1)!\big)\, \underbrace{[D] \cap [D] \cap \ldots \cap [D]}_{g-1} \in H_2(W)$$

can be realized by an algebraic curve (effective 1-cycle) $V \subset W$ (compare 2.E). Thus, $\delta(W) = 1$ if and only if W is a Jacobian by the Matsusaka theorem (see 2.E). We denote by A_g the moduli space of principally polarized Abelian varieties W of dimension g and let $A_{g,\delta} \subset A_g$ consist of W with $\delta(W) \leq \delta$. We want to show that for δ sufficiently small compared to g the subset $A_{g,\delta}$ is rather thin in A_g, as in the case $\delta = 1$, where this is the Buser-Sarnak theorem (see 2.D.6). First, we recall that, according to the Minkowski-Hlawka theorem for Abelian varieties (due to Buser-Sarnak, see 2.D.6), a "typical" $W \in A_g$ has

$$\mathrm{syst}_1\, W \geq C_0 \sqrt{g},\ C_0 > 0\ .$$

On the other hand, each $W \in A_{d,\delta}$ contains a surface V of area δg which represents a homology class in $H_2(W)$ of rank $2g$. It follows by (3) that

$$\mathrm{syst}_1\, W \leq C\, g^{-\frac{1}{2}} \exp C' \sqrt{\log g}\ \sqrt{\delta}\ . \tag{$*$}$$

Therefore, $A_{g,\delta}$ in A_g is indeed thin if δ is significantly smaller than $g/\exp C'\sqrt{\log g}$ for a fixed large constant C', say $C' = 1000$.

Remarks. (a) Inequality $(*)$ above does not look sharp, and it is unclear what the (asymptotically) sharp inequality should be for large δ.

(b) A purely algebro-geometric consequence of $(*)$ is the positivity of codimension of $A_{g,\delta}$ in A_g for $\delta \ll g/\exp C'\sqrt{\log g}$. Probably, algebraic geometers know much more about $\operatorname{codim} A_{d,\delta}$ for various δ.

A related algebro-geometric question is an estimate of the minimal genus of a curve in W in terms of $\delta(W)$.

3.C.5. Inequalities (1) – (3) in 3.C.3 can be significantly strenghtened under additional geometric and topological assumptions on W and α. For example, the theorem of Buser-Sarnak gives estimates better than (1) - (3) for flat tori W, albeit the whole situation is far from clear even for the flat tori.

Another source of strengthening of these inequalities comes from symmetries of (W,α), as suggested by the result by Bavard on hyperelliptic curves cited in 2.C. For example, suppose W admits an isometric involution I which fixes α, and let $\operatorname{Vol}\alpha$ refer to the infimum of the volumes of I-invariant cycles in W. Then, one may suggest the same homological conditions on W, α and I mimicking those satisfied by hyperelliptic inductions, and *conjecture* for such W, α and I the inequality

$$s_1 \le C_n\, b^{-\frac{1}{n}} (\operatorname{Vol}\alpha)^{\frac{1}{n}} \ .$$

(A suitable homological condition in the case $n=2$ is $\operatorname{Trace} I_* \approx -b$ for the operator

$$I_* : H_1(W;\mathbb{R}) \to H_1\big(W;(R)\ .\big)$$

The effect of extra symmetries can also be expressed in the language of displacements where we have a discrete isometry group Γ acting on a contractible space X, and we want a bound

$$\inf_{\gamma\in\Gamma_\infty} \operatorname*{dist}_{x\in X} \big(x,\gamma(x)\big) \le C_\Gamma \big(\operatorname{Vol}(X/\Gamma)\big)^{\frac{1}{n}} \ , \qquad (*)$$

where Γ may have torsion and Γ_∞ denotes the torsionless part of Γ (compare Bavard's result on cristallographic groups cited in 2.C). It may happen that the presence of "strong torsion" makes C_Γ small for some groups Γ. (At the present stage of knowledge, one cannot rule out the possibility of Γ being pure torsion which would make $C_\Gamma = \infty$.)

Let us indicate several more specific questions.

Questions. Let V_k be homeomorphic to the connected sum of k copies of a fixed closed essential (e.g., aspherical) manifold V_1 of dimension $n \geq 3$. What is the asymptotics for $k \to \infty$ of the (best) constant $C_{n,k}$ in the inequality

$$\operatorname{syst} \pi_1(V_k) \leq C_{n,k} (\operatorname{Vol} V_k)^{\frac{1}{n}} \ ?$$

The most we can say about $C_{n,k}$ is where V_1 has *non-zero* simplicial volume $\|V_1\|_\Delta$ which ensures the bound $C_{n,k} \leq C_n (\log k) k^{-\frac{1}{n}}$ by the above (1) (since $\|V_k\|_\Delta = k\|V_1\|_\Delta$ according to [Gro]$_1$). Maybe this is the best possible estimate, and then, for every V_1, the manifolds $V_k,\ k = 1, 2, \ldots$, would admit metrics for which

$$(\operatorname{Vol} V_k)^{\frac{1}{n}} \big/ \operatorname{syst} \pi_1(V_k) \leq C\, k^{\frac{1}{n}} \big/ \log k$$

for some constant $C = C(V_1)$. On the other hand, we cannot rule out the possibility of the bound $C_{n,k} \leq C_{V_1} k^{-\frac{1}{n}}$. This would give us an ideal systolic inequality, namely

$$\operatorname{syst} \pi_1(V_k) \lesssim k^{-\frac{1}{n}} (\operatorname{Vol} V_k)^{\frac{1}{n}} \ ,$$

where the extremal metric may look something like in Figure 5.

(Notice that for $\dim V = 2$, we do have examples with $(\operatorname{Area} V_k)^{\frac{1}{2}} \big/ \operatorname{syst}(V_k) \approx k^{\frac{1}{2}} \big/ \log k$ obtained with congruence coverings (compare [Bu-Sa]) which look quite different from Figure 5, see below.)

The above question extends to many other natural sequences V_k, such as a sequence of cyclic k-sheeted coverings of a fixed V_1 corresponding to a given non-zero cohomology class in $H^1(V_1; \mathbb{Z})$. An interesting V_1 to start with is a closed manifold with a metric g of negative (e.g., constant negative) curvature. The covering manifolds V_k with induced metrics $\tilde{g}_k$ have $\operatorname{Vol} \big/ (\operatorname{syst}_1)^n \approx k$. The question is if one can find different metrics, say $\tilde{g}'_k$ on V_k, for which the ratio $\mathrm{Vol}\big/(\mathrm{syst}_1)^n$ becomes significantly smaller than k for large $k \to \infty$. In fact, this is not at all clear if we restrict to metrics $\tilde{g}'_k$ of negative (or even pinched negative) curvature. Here, as earlier, the case $\dim V_1 = 2$ is exceptional (one can significantly decrease $\mathrm{Area}\big/(\mathrm{syst})^2$ by a deformation with constant negative curvature), but one may expect certain "rigidity" of $\mathrm{Vol}\big/(\mathrm{syst}_1)^n$ for higher dimensions.

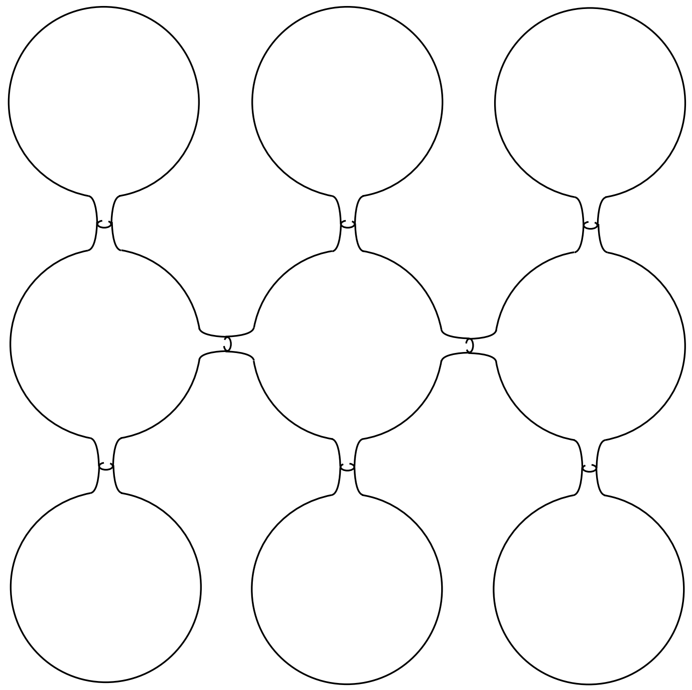

Figure 5

Provisional conjecture. Let (V, g) be a compact irreducible locally symmetric space with non-positive curvature and denote by $s_1^+(V)$ the *maximum* of the twice injectivity radius $2\,\mathrm{Rad}_v(V)$ over $v \in V$. (Notice that $s_1^+ > \mathrm{syst}_1$.) Then, the systole of an arbitrary metric g' on V normalized by the condition $\mathrm{Vol}(V, g') = \mathrm{Vol}(V, g)$ has

$$\mathrm{syst}\,\pi_1(V, g') \leq C s_1^+(V)$$

for some constant C depending only on $n = \dim V$.

Of course, this conjecture is violated for surfaces and so one has to assume that $n \geq 3$. Also, 3-dimensional manifolds of constant curvature may provide counterexamples. On the other hand, the conjecture stands a chance for more rigid locally symmetric spaces especially if one adds extra conditions on V and/or limits possible metrics g', e.g., as follows :

(i) the function $\mathrm{Rad}_v(V)$ is "nearly constant" in v. This means, $\mathrm{Rad}_v V \leq B \operatorname{syst}\pi_1(V)$, where B is a given constant which is kept fixed as the manifolds V vary (and where the above constant C may depend on B if one varies B). For example, if all our manifolds V appear as finite Galois covers of a single V_0, then Rad_v is "nearly constant" for these manifolds V. (It is, probably, not hard to find for each class of local isometry a sequence of compact manifolds V in this class for which the ratio $s_1^+(V)/\operatorname{syst}\pi_1(V)$ goes to infinity. In fact, finite non-Galois coverings of a fixed manifold V_1 seem to provide such examples. On the other hand, it is much harder to produce non-compact locally symmetric manifolds V with given behaviour of the function $v \mapsto \mathrm{Rad}_v V$ on V.)

(ii) if the original metric g had negative curvature $K(g)$, one might restrict g' by $K(g') \leq 0$ or $-a \leq K(g') \leq -b$ for some $a, b \geq 0$. (One may try $|K(g')| \leq \mathrm{const}$, but this does not appear especially restrictive in the present context.) One also may restrict some global invariants of g', e.g., by requiring the diameter and/or the first eigenvalue of the Laplacian of g' to be close to those of g.

Next, we want to indicate some interesting (sequences of) manifolds which are far from being locally symmetric. We start with V_1 containing a submanifold $W \subset V_1$ of codimension 2 which is homologous to zero and take the cyclic k-sheeted ramified covers V_k of V_1 with ramification locus W (compare [Gr-Th]). About these V_k we ask the same questions as the ones for the connected sums $V_k = V_1 \# \dots \# V_1$ and for non-ramified coverings earlier. It seems plausible in view of an intersystolic inequality proven in [Gr-Th] that for certain V_1 and W such V_k may be systolically almost extremal, i.e., *every* metric g'_k on V_k normalized by $\mathrm{Vol}(V_k, g'_k) = k$ may have $\operatorname{syst}\pi_1(V_k, g'_k) \leq C$ for a constant $C = C(V, W)$.

Notice that the sequences of manifolds V_k we consider have the following common feature : each V_k can be triangulated into $\approx k$ simplices. This suggests two general questions indicating two opposite, mutually exclusive possibilities.

(A) Does every manifold V which can be divided into at most k simplices admit a metric for which

$$\operatorname{syst}\pi_1 \geq C_n\, k^{\frac{-1}{n}}(\log k)(\mathrm{Vol})^{\frac{1}{n}},\ n = \dim V\ ?$$

(B) Does there exist a sequence $V_k,\ k = 1, 2, \ldots,$ of Riemannian manifolds of a fixed dimension n where V_k can be triangulated into at most k simplices and such that

$$\operatorname{syst}\pi_1(V_k) \leq C'_n k^{-\frac{1}{n}}(\mathrm{Vol}\, V_k)^{\frac{1}{n}}\ ?$$

Of course, the true answer may lie somewhere *strictly* between (A) and (B).

3.C.6. Congruence coverings. Consider some coverings V_k of a fixed compact manifold V. Then,

$$\mathrm{Vol}\, V_k \approx k \underset{\text{def}}{=} \text{the number of sheets}\ ,$$

and $\operatorname{syst}\pi_1(V_k)$ can also be approximately expressed in terms of the subgroups $\Gamma_k = \pi_1(V_k) \subset \Gamma = \pi_1(V)$ as follows. Fix a finite generating set $C \subset \Gamma$ and denote $\operatorname{syst}_1(\Gamma_k, G)$ as the minimal G-word length of a non-identity element in $\Gamma_k \subset \Gamma$.

The general systolic problem for a finitely generated group Γ consists of finding the possible values of $\operatorname{syst}_1(\Gamma_k, G)$ for subgroups $\Gamma_k \subset \Gamma$ of index $k \to \infty$. Notice that $\operatorname{syst}_1(\Gamma_k, G)/\operatorname{syst}(\Gamma_k, G')$ is pinched between two constants independent of Γ_k, and so we may suppress G in the discussion of the rough asymptotics of $\operatorname{syst}_1(\Gamma_k, G)$ for $k \to \infty$.

Examples. (a) If Γ is the free Abelian group of rank n then, obviously, $\operatorname{syst}_1 \Gamma_k \lesssim k^{\frac{1}{n}}$ (i.e., $\leq C k^{\frac{1}{n}}$ for $C = C(\Gamma, G)$) for all Γ_k and (obviously), there are subgroups $\Gamma_k \subset \Gamma, k \to \infty$, where $\operatorname{syst}_1 \Gamma_k \approx k^{\frac{1}{n}}$.

(b) If Γ is a torsionless nilpotent group of polynomial growth of degree d then, clearly, $\operatorname{syst}_1 \Gamma_k \lesssim k^{\frac{1}{d}}$ for all subgroups Γ_k. This is sharp. In fact, Γ contains subgroups Γ_k of indices $k \to \infty$ with $\operatorname{syst}_1 \Gamma_k \approx k^{\frac{1}{d}}$ if and only if the Lie algebra corresponding to Γ is graded, as a simple argument shows.

(c) If Γ has exponential growth, then $\operatorname{syst}_1 \Gamma_k \lesssim \log k$, but this inequality may be non-sharp. Yet it is sharp for subgroups Γ of exponential growth in $SL_N\mathbb{Z}$ according to the following lemma.

Elementary Lemma. — *Let $\Gamma \subset SL_N\mathbb{Z}$ contain no unipotent elements and denote by $\Gamma'_k \subset \Gamma$ the subgroup of the matrices which are equal to the unit diagonal matrix modulo k.* Then,

$$\operatorname{syst}_1 \Gamma'_k \geq C \log k \ .$$

(Notice that the index of Γ'_k in Γ does not equal k but it is $\leq k^N$ which is as good for our purpose.)

Now, if we take a compact locally symmetric manifold V with a fundamental group Γ embeddable into $SL_N\mathbb{Z}$, then we shall have k-sheeted coverings V_k of V with $\operatorname{syst} \pi_1(V_k) \approx \log k$. (If V is of non-compact type with no flat factor, then the simplicial volume of V_k is $\approx k$ (see [Sav]), and so *every* metric on V_k with volume k has $\operatorname{syst} \pi_1 \lesssim \log k$.) Notice that arithmetic groups embed into $SL_N\mathbb{Z}$ and also that there are non-arithmetic examples. Also, the congruence construction of Γ_k extends to S-arithmetic groups (and to more general groups of matrices with entries m/s for $m \in \mathbb{Z}$ and $s \in S$, where S a finitely generated multiplicative semigroup in $\mathbb{Z}_+$) where the situation is similar to the above.

Remarks on λ_1 and diameter. (a) If Γ is arithmetic, then one knows (this is rather deep) that the first eigenvalue of the Laplace operator on V_k for *prime numbers* k is bounded away from zero,

$$\lambda_1(V_k) \geq \varepsilon > 0 \quad \text{for} \ \ k \to \infty \ ,$$

(and this property can be expressed combinatorially in terms of the Cayley graphs of Γ/Γ_k). Then, it easily follows that the ratio $\operatorname{Diam} V_k / \operatorname{syst} \pi_1(V_k)$ remains bounded as $k \to \infty$. Notice that the diameter of V_k approximately equals the minimal number D, such that $\Gamma_k \subset \Gamma$ can be generated by some elements $\gamma \in \Gamma_k$ of G-length $\leq D$. This number D can be called $\operatorname{Diam} \Gamma/\Gamma_k$, and there are many (how many ?) examples of non-arithmetic groups $\Gamma \subset SL_N\mathbb{Z}$ for which $(\operatorname{Diam} \Gamma/\Gamma_k)/ \operatorname{syst}_1 \Gamma_k$ remains bounded for $k \to \infty$.

Counter example. The Heisenberg subgroup $\Gamma \subset SL_3\mathbb{Z}$ of triangular matrices, $\Gamma = \left\{ \begin{matrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{matrix} \right\}$, has $\operatorname{Diam} \Gamma/\Gamma_k \approx k$ and $\operatorname{syst}_1 \Gamma_k \approx \sqrt{k}$. (Yet, there is a sequence of "round" subgroups Γ'_k where Diam $\approx \mathrm{syst}_1 \approx k$. These are defined by the congruences $a, b \equiv 0 (\mathrm{mod}\, k)$ and $c \equiv 0(\mathrm{mod}\, k^2)$.)

(b) It was suggested by Sakai (during my lecture in Tokyo) that a complicated topology of a manifold V may force a non-trivial bound on $\mathrm{syst}\, \pi_1(V)/\operatorname{Diam} V$ (which is trivially bounded by 2, unless $\pi_1(V) = 0$). The above examples indicate that such a bound cannot be too strong for locally symmetric spaces. Furthermore, for every non-trivial finitely presented group Γ, one can construct a (piecewise Riemannian metric of curvature $+1$) on a 2-polyhedron P corresponding to the presentation of Γ, such that $\mathrm{syst}\, \pi_1(P) = 2\pi$ and $\operatorname{Diam} P = \pi$. It follows that every manifold of dimension ≥ 5 admits a smooth Riemannian metric g for which $\mathrm{syst}\, \pi_1(V)/\operatorname{Diam} V \geq 2 - \varepsilon$ for an arbitrarily chosen $\varepsilon > 0$. But, if $\mathrm{syst}\, \pi_1(V) = 2 \operatorname{Diam} V$ and g is *smooth* Riemannian, then (V, g) is, probably, isometric to the projective space (compare [Ber-Ka]).

3.C.7. Systolic finiteness problem. Choose some number $n = 2, 3, \ldots$, take $\varepsilon > 0$, and consider all closed aspherical Riemannian manifolds V of dimension n, such that

$$\mathrm{syst}\, \pi_1(V) \geq \varepsilon (\operatorname{Vol} V)^{\frac{1}{n}} \ .$$

Then, we ask whether the number of isomorphism classes of the fundamental groups $\Gamma = \pi_1(V)$ is *finite* (and thus bounded by $N = N(n, \varepsilon)$). The known regularization techniques (see §6 in [Gro2]) show that there is a finite number (bounded by $N = N(n, \varepsilon)$) of finitely presentable groups Δ, such that each $\Gamma = \pi_1(V)$ is *dominated* by some Δ which means there exists a (split) epimorphism $\alpha : \Delta \to \Gamma$ and an embedding $\beta : \Gamma \to \Delta$, such that $\alpha \circ \beta = \mathrm{Id}$. This gives a bound on the torsion of $H_1(\Gamma)$ (and also on the torsion τ of all of $H_*(\Gamma)$ by $\log \tau \leq C_n \operatorname{Vol}/(\mathrm{syst}\, \pi_1)^n$ as follows from §6 in [Gro2]). This solves the finiteness problem for certain essential (rather than aspherical) manifolds V with Abelian $\Gamma = \pi_1(V)$. For example, if V is covered by a homotopy n-sphere, then the order of the fundamental group $\Gamma = \pi_1(V)$ (which is *cyclic* since it is *assumed* Abelian) is bounded by $N = N(n, \varepsilon)$ (in fact, by $\exp C_n \left(\operatorname{Vol}(V)/\left(\mathrm{syst}\, \pi_1(V)\right)^n\right)$. Also notice that, if Γ is nilpotent of nilpotency degree d, then the group Δ may also be assumed nilpotent of degree d which strongly (how strongly?) restricts Γ.

3.C.8. Compactness problem and the thick-thin decomposition of general manifolds. One would like to have a (pre)compactness theorem for closed n-dimensional manifolds V with $\operatorname{Vol} V \leq \text{const}$ (and, possibly, with $\operatorname{syst} \pi_1(V) \geq \varepsilon > 0$) similar to that known for n-cycles in a fixed compact manifold. Such (pre)compactness is immediate if all ρ-balls in V have $\operatorname{Vol} B(\rho) \geq \delta = \delta(\rho) > 0$, e.g., $\delta(\rho) \geq \delta\rho^n$ for a fixed positive $\delta = \delta_n$. In general, in order to achieve (pre)compactness, we should restrict to the "thick part" of V where the volumes of balls *are* bounded from below. Of course, the thick-thin decomposition of an individual manifold V is quite ambiguous, but, if we have a sequence of manifolds V_i, then a sequence of points $v_i \in V$ can be called *thick* if $\operatorname{Vol} B(v_i, \rho) \geq \delta = \delta(\delta) > 0$ for all $\rho > 0$ (independently of $i = 1, 2 \ldots$). One may try to produce some (sub)limit space (in the Hausdorff sense) out of thick sequences and one wants to know how much is lost with the thin parts of V_i.

Questions. Let V be δ-*thin*, which may be understood in the following two different ways.

(a) Every unit ball in V has volume $\leq \delta$.

(b) Every ρ-ball in V for $\rho \leq 1$ has volume $\leq \delta\rho^n$.

Does it follow that

$$\operatorname{syst} \pi_1 V \leq C_n \delta^{\frac{1}{n}} ,$$

provided V is essential and $\delta \leq \delta_n$ for some sufficiently small $\delta_n > 0$ for $n = \dim V$? (This is so for $n = 2$, by 5.2.A in [Gro2]).

More generally, without assuming that V is essential, one may ask whether the *filling radius* of V (as defined in [Gro2]) is bounded by $C_n \delta^{\frac{1}{n}}$ for sufficiently small $\delta \leq \delta_n$.

Finally, the most optimistic (and least realistic) conjecture would be a bound of the $(n-1)^{\text{th}}$ Uryson width of V (see [Gro6]) in terms of δ. Notice that this width is essentially the same as $\operatorname{Diam}_{n-1}$ and Rad_{n-1} defined in [Gro2] and that even the bound of this width by $\operatorname{Vol} V$ remains an open problem for $n \geq 3$ (see p. 127 in [Gro2]). Notice that this width may be used to define the thick-thin decomposition of an arbitrary V. Namely, a point $v \in V$ is called (ρ, δ)-thin if the ρ-balls around v have $\text{width}_{n-1} \leq \delta\rho$ and this notion seems reasonably behaved as we vary ρ and δ.

(A similar definition can be made up with width_k for each k as well as with Fill Rad.) The main problem is to relate this thinness to the one defined with volumes of balls. In fact, the ideal (and improbable) result would be a bound of width_{n-1} by the *hyper-Euclidean size* of $B(\rho)$ (instead of $\mathrm{Vol}\, B(\rho)$) that is the maximal radius ρ_0 of the Euclidean ball $B(\rho_0) \subset \mathbb{R}^n$ for which there exists a proper distance decreasing map $B(\rho) \to B(\rho_0)$ of positive degree (compare [Gro5], [C-G-M], [Katz2]).

Systolic area of groups. Let Γ be a finitely presented group, consider all 2-polyhedra P with piecewise linear metrics such that $\pi_1(P) = \Gamma$, and set

$$\mathrm{syst\,area}\,\Gamma \underset{\mathrm{def}}{=} \inf_P \mathrm{Area}\, P \big/ \big(\mathrm{syst}\, \pi_1(P)\big)^2 .$$

It is not hard to show that $\mathrm{syst\,area}\,\Gamma \geq \varepsilon > 0$ unless Γ is free (see 6.7.A in [Gro2]), but one knows little else about the function $\Gamma \mapsto \mathrm{syst\,area}\,\Gamma$. For example, one does not know how large is the set of isomorphism classes of groups Γ having $\mathrm{syst\,area}\,\Gamma \leq C$ for a given (large) constant C. Another specific question is that of the evaluation of the systolic area of the free product of k non-free groups.

In fact, many of the systolic inequalities for manifolds extend to *essential* n-dimensional polyhedra P where P is called essential if the classifying map $P \to K(\Gamma, 1)$, for $\Gamma = \pi_1(P)$, does not contract to an $(n-1)$-dimensional subset in $K(\Gamma, 1)$. For example, every metric on an essential polyhedron P has

$$\mathrm{syst}\, \pi_1(P) \leq C_n (\mathrm{Vol}\, P)^{\frac{1}{n}}$$

(see Appendix 2 in [Gro2]). In fact, this inequality can be given the intersystolic shape by defining the absolute systole $\mathrm{absyst}_n(W)$ of a Riemannian manifold W as the infimum of n-volumes of those subsets in W which cannot be homotoped to $(n-1)$-dimensional subsets in W. It is clear that $\mathrm{absyst}_n \leq \mathrm{syst}\, H_n$ and yet this absolute systole bounds $\mathrm{syst}\, \pi_1 (= \mathrm{absyst}_1)$ if $\pi_k(W) = 0$ for $k = 2, \ldots, n$. Probably, such a bound can be strengthened in the spirit of (1) - (3) in 3.C.2, using some regularization based on the filling indicated in Appendix 2 in [Gro2].

3.C.9. Absolute systoles of congruence coverings. Consider a sequence of k-sheeted coverings V_k of a fixed manifold V and try to evaluate $\mathrm{absyst}_m V_k$ for a given m and $k \to \infty$. Here are several observations in this regard.

1. If V is a closed aspherical manifold, then the m-skeleton is essential in V for every $m = 1, 2, \dots n = \dim V$, and thus

$$\text{absyst}_m V_k \le C\, k$$

for some constant $C = C(V, m)$.

2. If the universal covering $\widetilde{V}$ of V satisfies the m-dimensional isoperimetric inequality of exponent $\alpha = \beta/\beta - 1$, i.e., if every $(m-1)$-dimensional polyhedron $Q \subset \widetilde{V}$ bounds a cone such that

$$\text{Vol}_m \text{ cone } \le \text{const}(\text{Vol}_{m-1} Q)^\alpha \ ,$$

then,

$$\text{absyst}_m V_k \ge \text{const}'(\text{absyst}_1 V_k)^\beta \ ,$$

provided $\alpha > 1$. If $\alpha = 1$, then

$$\text{absyst}_m V_k \ge \text{const}' \exp(\lambda \text{ absyst}_1 V_k)$$

for some positive const$'$ and λ.

Remark. — Notice, that the asymptotic behaviour of absyst V_k for $k \to \infty$ for aspherical manifolds V depends solely on $\Gamma = \pi_1(V)$ and $\Gamma_k = \pi_1(V_k) \subset \Gamma$. Similarly, the isoperimetric exponents of $\widetilde{V}$ are determined by Γ for contractible $\widetilde{V}$.

Example. If V has non-positive sectional curvature, then $\widetilde{V}$ satisfies the isoperimetric inequality of exponent $m/m-1$ in dimension m for all $m = 2, 3, \dots, n = \dim V$. Furthermore, if V is compact and $\widetilde{V}$ contains no m-dimensional flat (i.e., an isometric copy of $\mathbb{R}^m$), then, there is the linear inequality (with exponent $\alpha = 1$) in dimensions $m,\ m+1, \dots, n$. This applies, in particular, to locally symmetric spaces of non-compact type of rank $\le m$ (and their congruence coverings, see 3.C.6).

Questions. Let V be a compact (locally) irreducible, locally symmetric space of non-compact type and V_k be a sequence of congruence coverings. What are the asymptotic relations between the absolute systoles of V_k in different dimensions as $k \to \infty$. In particular, are there meaningful upper bounds on absyst_m ? For example, is absyst_m for $m \leq \operatorname{rank} V$ bounded by $\mathrm{const}(\mathrm{absyst}_1)^\beta$ for some $\beta \in [m, \infty)$? Is there a bound

$$\mathrm{absyst}_m \leq \mathrm{const}(\mathrm{Vol})^\gamma$$

for given $m < n = \dim V$ and $\gamma < 1$? Notice that in the arithmetic case V may contain arithmetic totally geodesic submanifolds $U \subset V$ of dimension m and then, the bound $\mathrm{absyst}_m \leq \mathrm{const}(\mathrm{Vol})^\gamma$ does hold for certain $\gamma < 1$ as $k \to \infty$.

3.C.10. Generalized intersystolic bounds on syst_1. The inequalities $\mathrm{syst}_1 \lesssim (\mathrm{Vol})^{\frac{1}{n}}$ and/or $\mathrm{syst}_1 \lesssim (\mathrm{syst}_n)^{\frac{1}{n}}$ can be generalized in two ways.

1. Instead of locating a single short non-contractible curve one may look for certain systems of such curves. Namely, one may take a (possibly disconnected) 1-complex (graph) S with a prescribed subset of subgraphs $S_i \subset S$, $i = 1, \ldots, p$ and a certain set Φ of continuous maps $\varphi = S \to P$. Typically, Φ consists of homotopy classes of maps, e.g., of all non-contractible maps $S \to P$ or of all maps for which the induced homomorphism $\pi_1(S) \to \pi_1(P)$ is injective or surjective. Then, one tries to find a "short" map $\varphi \in \Phi$ with a certain bound on the lengths of $\varphi(S_i) \subset P$ or on combinations of these lengths, where such a bound should be linked to some n-dimensional volume characteristic of P, e.g., to $\mathrm{syst}_n P$. (Here, P is a polyhedron with a certain metric.)

The prototypical example of such a bound is the following Minkowski theorem extending the bound on the 1-systole of a flat torus T^n cited at the beginning of this article.

Second Minkowski theorem. — *There exist closed curves $S_1, \ldots, S_n$ in T^n which generate $H_1(T^n)$ and such that*

$$\prod_{i=1}^{n} \operatorname{length} S_i \leq C_n \operatorname{Vol} T^n \ .$$

It is unknown if such an inequality remains valid for non-flat tori, but some results in this direction can be obtained with the regularization techniques mentioned earlier (see 5.4, 6.5, 6.6 and 7.5 in [Gro2]).

2. The bound on syst_1 and other 1-dimensional characteristics of P can sometimes be improved if one allows a use of the volumes of more elaborate higher dimensional configurations than those incorporated into $\mathrm{Vol}_n P$ or $\mathrm{syst}_n P$. For example, one may consider the volumes V_i of the i-skeletons of P for a given set I of dimensions $i \geq 3$ and then try to bound syst_1 (and the lengths of more complicated graphs in P) in terms of these V_i. More generally, one may choose some subpolyhedra $P_i \subset P$ of dimensions n_i and look for length bounds in P in terms of $\mathrm{Vol}_{n_i} P_i$. (Here, one can restrict a relevant set Φ of maps $\varphi : S \to P$ by requiring certain subgraphs of S to go to some chosen subpolyhedra in P.)

Example. (See [Gr-Th].) — *Let P be covered by m subpolyhedra $P_1, \ldots, P_m$ homeomorphic to a fixed compact manifold of negative curvature with totally geodesic boundary, such that all P_i's intersect inside P across this boundary, called $P_0 \subset P$. Then, there exists a point $p_0 \in P_0$ and m loops $S_i \subset P_i$, $i = 1, \ldots, m$ based at p_0, such that these loops are freely independent in $\pi_1(P, p_0)$ and their lengths are bounded by ℓ for*

$$\ell = C_n \max\Big((\mathrm{Vol}_{n-1} P_0)^{\frac{1}{n-1}},\ (\mathrm{Vol}_n P_1)^{\frac{1}{n}},\ (\mathrm{Vol}_n P_2)^{\frac{1}{n}}, \ldots, (\mathrm{Vol}_n P_m)^{\frac{1}{n}}\Big)\ ,$$

where $n = \dim P$ is assumed to be ≥ 3.

One does not know what is the true general inequality for which the above serves as an example.

3.C.11. Filling and embolic inequalities. If $\pi_1(P) = 0$ (or P is non-essential), then there is no meaningful bound on $\mathrm{syst}_1 P$, but there are other 1-dimensional geometric invariants which can sometimes be bounded by $\mathrm{Vol}\, P$. Here is a remarkable instance of such a bound.

Embolic inequality. (See [Ber3,5,7].) — *The injectivity radius of a Riemannian manifold V is (sharply!) bounded by*

$$\mathrm{Inj\,Rad}\, V \leq \pi(\mathrm{Vol}\, V / \mathrm{Vol}\, S^n)^{\frac{1}{n}}\ ,$$

where S^n is the unit n-sphere for $n = \dim V$.

There are more general (but non-sharp) inequalities of this kind where the injectivity radius is replaced by the minimal radius R of a ball in V which cannot be contracted within the concentric ball of radius $\rho(R)$ for a given function $\rho(R)$. These follow from bounds on the *filling radius* of V in [Gro2]. (See [Ber8], [Gro5], [Katz1,4], [Gre-Pe].)

4. EVALUATION OF k-DIMENSIONAL SYSTOLES FOR $k \geq 2$

4.A. The first results concerning higher homological k-systoles are due to Berger (see [Ber1]) who computed syst_k for the projective spaces

– $\mathrm{syst}_k \mathbb{R}P^n$. The homology of the real projective space mod 2 is generated in each dimension $k = 1, 2, \ldots, n$ by a k-dimensional subspace $\mathbb{R}P^k \subset \mathbb{R}$ and Berger shows for the standard metric in $\mathbb{R}P^n$ that

$$\mathrm{syst}_k \big(= \mathrm{syst}\, H_k(\mathbb{R}, P^n; \mathbb{Z}_2)\big) = \mathrm{Vol}\, \mathbb{R}P^n \ .$$

Proof. The inequality $\mathrm{syst}_k \leq \mathrm{Vol}\, \mathbb{R}P^n$ follows from the fact that $\mathbb{R}P^k \subset \mathbb{R}P^n$ is not homologous to zero. To prove the opposite inequality, we must show that every k-cycle C in $\mathbb{R}P^n$ non-homologous to zero has $\mathrm{Vol}\, C \geq \mathrm{Vol}\, \mathbb{R}P^n$. This is obtained with integral geometry by observing that $\mathrm{Vol}\, C$ equals the integral of the number of the intersection points of C with the $(n-k)$-dimensional projective subspaces in $\mathbb{R}P^n$. If $\mathrm{Vol}\, C < \mathrm{Vol}\, \mathbb{R}P^n$, this integral is less than that for $\mathbb{R}P^k$ in $\mathbb{R}P^n$, and so, there is an $(n-k)$-subspace missing C. Hence, C is homologous to zero. $\square$

— $\mathrm{syst}_{2k} \mathbb{C}P^n$. The (integral) homology in every even dimension is generated by $\mathbb{C}P^k$ and

$$\mathrm{syst}_{2k} \big(= \mathrm{syst}\, H_{2k}(\mathbb{C}P^n)\big) = \mathrm{Vol}\, \mathbb{C}P^k \ .$$

Proof. Let us show that every $2k$-cycle $C \subset \mathbb{C}P^n$ with $\mathrm{Vol}\, C < \mathrm{Vol}\, \mathbb{C}P^k$ misses some projective subspace in $\mathbb{C}P^n$ of (complex) dimension $n-k$, and thus C is homologous to zero. The averaged (oriented) intersection number of an arbitrary $2k$-chain C with

$(n-k)$-subspaces equals, by integral geometry, the integral $\int\limits_C \Omega$, where Ω is the $2k$-form on $\mathbb{C}P^n$ obtained by averaging the $2k$-current corresponding to $\mathbb{C}P^{n-k}$ over the isometry group G of $\mathbb{C}P^n$. Plainly speaking, Ω is *defined* by

$$\int_C \Omega = \int_G \#\big(C \cap g(\mathbb{C}P^{n-k})\big)\, dg$$

for all $2k$-chains C (where dg refers to the normalized Haar measure). Since Ω is G-invariant, it necessarily equals a scalar multiple of ω^k for the Kähler form ω of $\mathbb{C}P^n$. We agree to normalize the metric in $\mathbb{C}P^n$ to have $\int_{\mathbb{C}P^n} \omega^n = 1 = n!\operatorname{Vol}\mathbb{C}P^n$ which makes $\Omega = \omega^k$. Then, we recall the *Wirtinger inequality*

$$\operatorname{comass}\omega^k \leq k!$$

which means, by definition,

$$\int_C \omega^k \leq k!\operatorname{Vol} C$$

for all $2k$-chains C. Therefore, if $\operatorname{Vol} C < \operatorname{Vol}\mathbb{C}P^k = (k!)^{-1}$, then some $(n-k)$-subspace misses C, and so C is homologous to zero. $\square$

— $\operatorname{syst}_{4k}\mathbb{H}P^n$. Here, in the quaternionic case, we have again

$$\operatorname{syst}_{4k} = \operatorname{Vol}\mathbb{H}P^k \ ,$$

and the proof boils down to the inequality $\operatorname{comass}\Omega \leq 1$, where Ω is the $4k$-form obtained by the G-averaging of the current corresponding to $\mathbb{H}P^{n-k}$. The form Ω, being G-invariant, is unique (up to a scalar constant) and can be written down explicitly. It is proven in [Ber1] that, indeed, $\operatorname{comass}\Omega \leq 1$.

– $\operatorname{syst}_8\mathbb{C}a\,P^2$. In this case (of the Cayley plane) the proof is as above, though the inequality $\operatorname{comass}\Omega \leq 1$ is rather complicated (see [Ber1]). The conclusion is the same as earlier

$$\operatorname{syst}_8 = \operatorname{Vol}\mathbb{C}aP^1 \ .$$

4.A.1. Remark. — The above argument (due to Berger) is called, nowadays, the method of *calibrations*. A calibration for us is a closed k-form Ω with a controlled *comass*,

$$\operatorname{comass}\Omega \leq a \ ,$$

where comass is defined as the supremum of the values of Ω on all orthonormal k-frames in the Riemannian manifold V where Ω lives. The above inequality is equivalent to

$$\int_C \Omega \leq a \operatorname{Vol} C$$

for all k-chains. Therefore, if Ω is *integral*, i.e., the class $[\Omega]$ is contained in $H^k(V;\mathbb{Z}) \subset H^k(V;\mathbb{R})$, then the volume of every homology class $h \in H_k(V)$ satisfying $\langle h, [\Omega]\rangle \neq 0$ is bounded from below by

$$\operatorname{Vol} h \geq a^{-1} \ , \tag{$*$}$$

where $\operatorname{Vol} h$ is defined as the infimum of the volumes of cycles C representing h.

One can place $(*)$ in a more conceptual framework by defining the norm "mass" on k-currents (in particular on k-cyles) as the dual to the comass norm on forms. In other words, mass is the minimal norm for which

$$\int_C \Omega \leq (\operatorname{mass} C)(\operatorname{comass} \Omega)$$

for all C and Ω. Then, $(*)$ reads

$$\operatorname{Vol} h \geq \operatorname{mass} h$$

where $\operatorname{mass} h$ is the infimum of mass of (real) closed k-currents representing h. The following theorem by Federer (see [Fed]) renders a geometric meaning to mass.

Federer's formula.

$$\operatorname{mass} h = \lim_{i\to\infty} i^{-1} \operatorname{Vol} ih \ .$$

4.A.2. Berger asks in [Ber1] what happens to syst_k of a projective space when the standard metric g is deformed keeping the volume unchanged. We shall see presently that

1. *there are many non-trivial (i.e., non-Kähler) deformations g' of the standard metric g of $\mathbb{C}P^n$, such that the systoles* syst_2 *do not change* ;

2. *every small deformation g'' of g on $\mathbb{C}P^2$ with* $\operatorname{Vol} g'' = \operatorname{Vol} g$ *has*

$$\operatorname{syst}_2 g'' \leq \operatorname{syst}_2 g \ ;$$

3. *there are arbitrarily small deformations g' of g on $\mathbb{C}P^n$ for every $n \geq 3$, keeping the total volume unchanged and such that*

$$\mathrm{syst}_{2k}(g') > \mathrm{syst}_{2k}(g)$$

for $k = 2, 3, \ldots, n-1$.

Quasi-Kähler deformations of g. The Kähler form ω on $\mathbb{C}P^n$ defines a (linear) symplectic structure in each tangent space $T_v(\mathbb{C}P^n)$, and we consider the (gauge) group $\mathcal{S}p\ell$ of fiberwise (linear) symplectic transformations of $T(\mathbb{C}P^n)$. The metrics g on $\mathbb{C}P^n$ of the form $g' = \sigma(g)$ for $\sigma \in \mathcal{S}p\ell$ are called *quasi-Kähler deformations* of g. (If we use the full gauge group of all fiberwise linear transformations, then the orbit of g equals the set of *all* Riemannian metrics on $\mathbb{C}P^n$. If we use the transformations preserving ω^n in the tangent spaces we obtain all metrics having the same volume element as g.)

Theorem. — *Every quasi Kähler deformation g' of g satisfies*

$$\mathrm{syst}_2\, g' = \mathrm{syst}_2\, g\ .$$

Proof. Let J' denote the almost complex structure obtained by σ from the original complex structure J on $\mathbb{C}P^n$, i.e., $J' = \sigma(J)$. Then, according to [Gro4], there exists a *J'-holomorphic curve* $C \subset (\mathbb{C}P^n, J^n)$ representing the generator in $H_2(\mathbb{C}P^n)$. It is easy to see that $\mathrm{area}\, C = \mathrm{area}(\mathbb{C}P^1, g)$, and so $\mathrm{syst}_2\, g' \leq \mathrm{syst}_2\, g$. This implies that the theorem as the opposite inequality $\mathrm{syst}_2\, g' \geq \mathrm{syst}_2\, g$ follows from the Wirtinger inequality for ω with respect to g',

$$\mathrm{comass}_{g'}\, \omega (= \mathrm{comass}_g\, \omega) \leq 1$$

(as ω is gauge invariant).

Conformal changes of g. — *Let $g = \varphi g$ for some positive function φ on $\mathbb{C}P^n$ such that*

$$\big(\mathrm{Vol}(\mathbb{C}P^n, g_\varphi) = \big) \int_{\mathbb{C}P^n} \varphi^n dg = \mathrm{Vol}(\mathbb{C}P^n, g)\ .$$

Then,

$$\operatorname{syst}_{2k} g_\varphi \le \operatorname{syst}_{2k} g,\ k = 1, \dots, n-1 .$$

Proof. The required "small" $2k$-cycle for g_φ comes from some k-dimensional complex projective subspace in $\mathbb{C}P^n$. Namely, there is such a subspace $S(= \mathbb{C}P^k)$ in $\mathbb{C}P^n$ whose g_φ-volume, i.e., $\int_S \varphi^k$ is less than or equal to $\operatorname{Vol}\mathbb{C}P^k$.

In fact, the average of $\big(\operatorname{Vol}(S,g)\big)^{\frac{n}{k}}$ over all subspaces $S \subset \mathbb{C}P^n$ is bounded by $\big(\operatorname{Vol}(S,g)\big)^{\frac{n}{k}} = (\operatorname{Vol}\mathbb{C}P^k)^{\frac{n}{k}}$. To simplify the computation, we normalize the metric g on $\mathbb{C}P^n$ such that $\operatorname{Vol}(S,g)$ becomes 1. We denote by Σ the (Grassmann) manifold of all k-dimensional subspaces S in $\mathbb{C}P^n$ and observe the following property of Σ.

($\star$) *There exists a smooth positive measure μ on Σ, such that, for every function ψ on $\mathbb{C}P^n$, one has*

$$\int_{\mathbb{C}P^n} \psi\, dg = \int_\Sigma d\mu \int_S \psi\, ds\ , \tag{$\star$}$$

where dg denotes the Riemannian volume element on $(\mathbb{C}P^n, g)$ and ds is the volume element on $S \subset \mathbb{C}P^n$ with the induced metric.

Proof. The properly normalized (Haar) measure on Σ invariant under isometries of $\mathbb{C}P^n$ satisfies the above, as everybody knows.

Now, we apply the above formula to $\psi = \varphi^n$ and use the Schwarz inequality $\int_S \varphi^n \ge \big(\int_S \varphi^k\big)^{\frac{n}{k}}$ (issuing from $\operatorname{Vol} S = 1$). We recall that $\int_{\mathbb{C}P^n} \varphi^n = 1$ and conclude to the inequality

$$\int_\Sigma d\mu (\operatorname{Vol} S, g_\varphi)^{\frac{n}{k}} \le \operatorname{Vol}(\mathbb{C}P^n, g_\varphi) = \operatorname{Vol}(\mathbb{C}P^n, g)\ .$$

This implies our assertion, since

$$\int_\Sigma d\mu \big(\operatorname{Vol}(S,g)\big)^{\frac{n}{k}} = \operatorname{Vol}(\mathbb{C}P^n, g)\ ,$$

according to ($\star$) applied to $\psi = 1$. ☐

Stability of ($\star$). Let us slightly C^∞-perturb the family Σ and denote the perturbed submanifolds $S' \subset \mathbb{C}P^n$. Here, each S' is C^∞-close (and diffeomorphic) to some S and the variety Σ' of all S' is diffeomorphic to Σ. (The simplest perturbations are those obtained by small diffeomorphisms of $\mathbb{C}P^n$.) Then, we slightly perturb the metric g

to some g' on $\mathbb{C}P^n$, such that the g'-volumes of all S' become ≤ 1 (where we assume that the original metric is normalized to have g-volumes of S equal one).

($\star'$) *There exists a positive measure μ' on Σ' such that every function ψ on $\mathbb{C}P^n$ satisfies*

$$\int_{\mathbb{C}P^n} \psi\, dg' = \int_{\Sigma'} d\mu' \int_{S'} \psi\, ds' \ . \tag{$\star$}'$$

Proof. Denote by R' the Radon operator (transform) from functions on $\mathbb{C}P^n$ to those on Σ' defined by $R'\psi(S') = \int\limits_{S'} \psi\, ds'$ for all $S' \in \Sigma'$ and let R'_* denote the adjoint operator from measures on Σ' to those on $\mathbb{C}P^n$. Our claim can be stated in this language by saying that the Riemannian measure dg' on $\mathbb{C}P^n$ lies in the image of R'_*. Moreover, there exists a *positive* measure μ' on Σ' solving the equation $R'_*(\mu') = dg'$. (We use interchangingly μ' and $d\mu'$ in the hope that no confusion follows.) One knows in this regard that the Radon transform R for the original Σ and g is injective. Moreover, it is bijective (for appropriate function spaces) for $\operatorname{codim}_{\mathbb{R}} S = 2$ (see [Hel]). Furthermore, R and R_* are *elliptic Fourier integral operators* for $\operatorname{codim}_{\mathbb{R}} S = 2$ (see [Gu-St]) and therefore, bijectivity remains intact for small perturbations R' and R'_* corresponding to Σ' and g'. Moreover, the solution of $R'_*(\mu') = dg'$ remains C^0-close to the Haar measure μ on Σ which implies positivity of μ'. This proves ($\star'$) for $\operatorname{codim} S' = 2$ and the general case (which we do not use in the sequel) is left to the reader.

Corollary. — *Every conformal metric $g'_\varphi = \varphi g'$ with $\operatorname{Vol} g'_\varphi \leq \operatorname{Vol} g'$ has*

$$\int_{\Sigma'} \big(\operatorname{Vol}(S', g'_\ell)\big)^{\frac{n}{k}} d\,\mu' \leq \int_{\Sigma'} d\,\mu'$$

and, consequently, there exists $S' \subset \Sigma$ for which $\operatorname{Vol}(S', g'_\ell) \leq 1$.

This follows by the same argument as used in the case of (Σ, g) with ($\star'$) in place of ($\star$).

Small perturbations of $\mathbb{C}P^2$. Let g'' be an arbitrary Riemannian metric on $\mathbb{C}P^2$ and let ω'' be the harmonic form generating $H^2(\mathbb{C}P^2;\mathbb{Z}) \subset H^2(\mathbb{C}P^2;\mathbb{R})$. This form is self-dual, i.e., at each point $v \in \mathbb{C}P^2$ there exists a g''-orthonormal frame where $\omega'' = \varphi\, dx_1 \wedge dy_1 + \varphi\, dx_2 \wedge dy_2$. Therefore, if $\varphi = \varphi(v)$ does not vanish on $\mathbb{C}P^2$, the metric g'' is conformal to a quasi-Kähler metric g' with respect to which $\omega'' = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. (This was pointed out to me by Berger.) If g'' is close to the standard metric g on $\mathbb{C}P^2$, then so is g', and in fact, g' is isometric to some small quasi-Kähler deformation of g (since ω'' is simplectomorphic to ω by a transformation of $\mathbb{C}P$ close to the identity). We assume that g' itself is such a deformation and invoke the corresponding J'-holomorphic curves in $\mathbb{C}P^2$. One knows (see [Gro4]) that there is a family Σ' of such "curves"(which are topologically 2-spheres) in $\mathbb{C}P^2$ which is close to the family Σ of the projective lines and where each $S' \in \Sigma'$ has $\operatorname{area}_{g'} = 1 = \operatorname{area}_g S$. Hence, our metric g'', conformal to g', admits some $S' \in \Sigma'$ with g''-area ≤ 1. Thus, every metric g'' close to the standard metric g and having $\operatorname{Vol} g'' \leq \operatorname{Vol} g = 2$ satisfies the systolic bound

$$\operatorname{syst}_2 g'' \leq \operatorname{syst}_2 g = 1 \ ,$$

as was claimed in 2 earlier.

Finally, we prove the claim 3 by observing that a *generic* quasi-Kähler deformation g' of g on $\mathbb{C}P^n$ has

$$\operatorname{syst}_{2k} g' \geq \operatorname{syst}_{2k} g$$

for $2 \leq k < n$. In fact, by the Wirtinger inequality, every non-trivial k-cycle C with

$$\operatorname{Vol}_{g'} C = \operatorname{Vol}_g C = (k!)^{-1} \int_C \omega$$

must be J'-holomorphic (since $\omega^k(\tau_1, \ldots, \tau_{2k}) < k!$ for g'-orthonormal frames with non-complex spans). On the other hand, a generic J' (being quite non-integrable) admits no germs of J'-holomorphic submanifolds for $2 \leq k < n$, and our claim follows by applying all that to the minimal cycle C realizing a relevant homology class in $H_2(\mathbb{C}P^2)$. $\square$

Remarks and open questions. (a) One recaptures the sharp systolic inequality for all metrics on $\mathbb{C}P^n$ if one uses mass instead of the volume. Namely, the mass of the generator $h \in H_{2k}(\mathbb{C}P^n)$ is bounded by

$$\operatorname{mass} h \leq \frac{k!}{n!} (\operatorname{Vol})^{\frac{k}{n}}$$

for all Riemannian metrics on $\mathbb{C}P^n$, provided n is divisible by k. More generally, generators $h_i \in H_{2k_i}(\mathbb{C}P^n)$, $i = 1, \ldots, j$, with $\sum_{i=1}^{j} k_j = m \leq n$, satisfy

$$\prod_{i=1}^{j} \operatorname{mass} h_i \leq C \operatorname{mass} h_m$$

where h_m is the generator in $H_{2m}(\mathbb{C}P^n)$ and $C = \big(\prod_i (k!)\big)/m!$. This follows from the discussion in §7.4 of [Gro2] (which also yields similar results for $\mathbb{H}P^n$ and $\mathbb{C}aP^2$).

Example. If g' is a quasi-Kähler deformation of the standard metric g on $\mathbb{C}P^n$, then the mass of the $(n-k)$-th power of the corresponding symplectic form $\omega' = \omega$ viewed as a $2k$-current equals $k!$ (where, as earlier, g and g' are normalized by $\int_{\mathbb{C}P^n} \omega^n = 1$). It follows by Federer's formula cited earlier that the minimizing $2k$-cycles $C_i \subset (\mathbb{C}P^n, g')$ representing $i\,h$ for the generator $h \in H_{2k}(\mathbb{C}P^n)$ satisfy

$$\lim_{i \to \infty} i^{-1} \operatorname{Vol} C_i = (k!)^{-1} .$$

It follows by the Wirtinger inequality that the cycles C_i must be "almost J'-holomorphic" most of the time. This means, there are subsets $C_i' \subset C_i$ which contain no singular points of C_i, which have $\operatorname{Vol}_{2k} C_i' \big/ \operatorname{Vol}_{2k} C_i \to 1$ for $i \to \infty$, such that the tangent space T to C_i at each point $c \in C_i'$ is ε_i-*complex* with respect to J' for $\varepsilon_i \to 0$ as $i \to \infty$ (where "ε-complex" means $\operatorname{dist}(T, J'T) \leq \varepsilon$). On the other hand, for generic g' and J' and $2 \leq k < n$, no C^1-smooth $2k$-dimensional submanifold can be J'-complex. It follows that there is no geometric limit of the cycles C_i, although they may weakly converge to the $2k$-current corresponding to ω^{n-k}. (I must admit that I am unable to visualize the actual geometry of C_i for $i \to \infty$.)

(b) The statements 1, 2 and 3 provide partial answers to the local (in the neighbourhood of the standard metric) systolic problem for $\mathbb{C}P^n$ raised by Berger in [Ber2]. This problem remains open for non-complex projective spaces and for $\operatorname{syst}_2 g'$ for small non-quasi-Kähler deformations of g on $\mathbb{C}P^n$, $n \geq 3$. On the other hand, there probably exist metrics g' far away from the standard one where the ratio $\operatorname{syst}_{2k} g' / (\operatorname{Vol} g')^{\frac{k}{n}}$ becomes arbitrarily large. (In fact, such examples must be rather easy for k not dividing n. Compare 4.A.3 and 4.A.5 below.)

4.A.3. Systolic problem for manifolds of general topological type. This problem consists of finding inequalities between $\mathrm{syst}_k V$, $k = 1, \dots, n = \dim V$. For example, if $H_k(V) \neq 0$, one looks for an inequality of the form $\mathrm{syst}_k \mathrm{syst}_{n-k} \leq \mathrm{const}\,\mathrm{Vol}$. More generally, if there are cohomology classes $h_i, i = 1, \dots, j$, of degrees k_i, such that $\sum_{i=1}^{j} k_i = m \leq n = \dim V$ and the cup product $h_1 \smile h_2 \smile, \dots \smile h_j$ does not vanish, then the desired inequality reads

$$\prod_{i=1}^{j} \mathrm{syst}_{k_i} \leq \mathrm{const}\,\mathrm{syst}_m \ . \tag{$*$}$$

The inequalities of the shape of $(*)$ are known for the systoles defined with the mass instead of the volume and these can be most conveniently expressed in terms of the geometry of the *mass-Jacobians*

$$J_k(V) = H_k(V; \mathbb{R}) / H_k(V; \mathbb{Z})$$

with the (flat Minkowski) metrics coming from the mass on $H_k(V; \mathbb{R})$ (see §7.4 in [Gro2], [Heb2] and [Bab1,2,3]). On the other hand, there is no single known upper bound on syst_k for $2 \leq k < n$ in terms of other systoles and/or more sophisticated metric characteristics of V pertaining to dimensions not equal to k.

In fact, there are obvious counterexamples to the inequality $\mathrm{syst}_1 \mathrm{syst}_{n-1} \leq \mathrm{const}\,\mathrm{Vol}$. For instance, if a simply connected manifold V_0 admits a free isometric S^1-action, then $V = (V_0 \times \mathbb{R})/\mathbb{Z}$ may have arbitrarily large ratio $(\mathrm{syst}_1\ \mathrm{syst}_{n-1})/\mathrm{Vol}$ for a suitable action of $\mathbb{Z}$ on $V_0 \times \mathbb{R}$. Namely, if we rotate V_0 by a small angle $\alpha = 2\pi/i \in S^1$ and translate $\mathbb{R}$ by $\varepsilon = \alpha/i \approx \alpha^2$, then the corresponding isometry of $V_0 \times \mathbb{R}$ and *all* its multiples have displacement $\gtrsim \alpha$ and so, the corresponding manifold $V = V_{\alpha,\varepsilon}$ has

$$\mathrm{syst}_1 V \geq \mathrm{const}\,\alpha \ .$$

On the other hand, the masses of the $(n-1)$-cycles in V for $n - 1 = \dim V_0$ equal those in V_0 (by an easy argument) and so

$$\mathrm{syst}_{n-1} V = \mathrm{Vol}\, V_0$$

(while evaluation of syst_k for $2 \leq k \leq n-2$ is more interesting in this example). As $\mathrm{Vol}\, V = \varepsilon \,\mathrm{Vol}\, V_0$, we have

$$\mathrm{syst}_1\ \mathrm{syst}_{n-1} \,/\, \mathrm{Vol} \approx \alpha/\varepsilon = i$$

which can be made as large as we wish. Next, suppose we can modify the metric g_0 on V_0 in order to make $\mathrm{Vol}\, V_0$ small without diminishing the induced distance function on the S^1-orbits. (This can easily be achieved, for instance, if $V_0 = (V_1 \times V_2,\ g_0 = g_1 \oplus g_2)$ with the S^1-action free on V_1, as we may take $g_0' = g_1 \oplus \delta g_2$ with small $\delta > 0$. But such modification is impossible if V_0 is a homotopy sphere as follows from the isosystolic inequality for $V_0/\{0, \alpha, 2\alpha, \ldots\},\ \alpha = 2\pi/i$). Then, we adjust

$$\mathrm{syst}_1 \approx \alpha,\ \mathrm{syst}_{n-1} \approx \alpha^{n-1},\ \mathrm{Vol} \approx \varepsilon\alpha^{n-1} \approx \alpha^{n+1}$$

which scales to a metric on V with syst_1 and syst_{n-1} bounded from below by one and $\mathrm{Vol} = \alpha \to 0$.

The simplest example where this happens is $V = S^3 \times S^2 \times S^1$ with a suitable family of homogeneous (!) metrics.

I realized somewhat belatedly that the above examples can be topologically varied and simplified by a suitable geometrically controlled surgery. For example, it is not hard to exhibit (highly non-homogeneous) metrics on $S^5 \times S^5$ with $\mathrm{syst}_5 \geq 1$ and $\mathrm{Vol} \to 0$ (compare [Ber9] and see [Be-Ka] and [Pit] for a more recent development; also see [Gro10] for a similar surgical construction of large manifolds with large eigenvalues of the Laplacian).

The above examples, as well as those indicated in 4.A.5 and 4.A.6 make quite interesting the evaluation of systoles for particular classes of manifolds and finding non-trivial inequalities between metric invariants (including the systoles) in these classes. Here are some possibilities.

(a) *Manifolds with a bound on the absolute values of the sectional curvature.* These are especially interesting when they *collapse* to lower dimensional manifolds (which, for $|K| \leq \mathrm{const}$, amounts to $\mathrm{inj\,rad} \to 0$). The simplest example of a collapsed manifold V is a circle bundle over some V_0. This is determined, besides the metric

g_0 on V_0, by the (closed integral) curvature 2-form ω on V_0 and the length ε of the implied circle. If ε is small compared to $\|\omega\|$, the curvature of V is approximately the same as that of V_0 and, in particular, it remains bounded for $\varepsilon \to 0$. Yet the geometry of k-dimensional cycles in V for $k = 2, \ldots, n-1$, is not quite easy to see. (Actually, this example is quite interesting for large $\|\omega\|$ where the curvature of V may blow up.)

(b) *Manifolds with* $K \geq -\mathrm{const}$. These are in many respects similar to the above but the proofs are harder. For example, according to Perelman, their local geometry is roughly conical which, probably, allows an upper bound on the systoles in the non-collapsed case. In the collapsed case, the k-dimensional volume looks harder to understand but the Uryson widths $w_0, \ldots, w_{k-1}$ behave as expected. Namely, Perelman proved the conjecture from [Gro6] claiming a double-sided bound on the product of the widths by the volume, the diameter and the lower bound on the sectional curvatures.

(c) *Manifolds with* Ricci $\geq -\mathrm{const}$. These have been vigorously investigated in the last couple of years by Anderson, Cheeger, Colding and Perelman. Yet, we do not know how to construct k-dimensional submanifolds in such manifolds with controlled volumes. For example, we do not know if every V with Ricci ≥ 1 admits a generic smooth map to $\mathbb{R}^{n-k}$ where the pull-backs of all points have $\mathrm{Vol}_k \leq \mathrm{const}_n$ (compare with 4.A.7). Notice that this is quite easy for $k = n-1$ (use the distance function) and for all k assuming $|K| \leq \mathrm{const}$. Also, the case $K \geq -\mathrm{const}$ looks within reach.

(d) *Manifolds with positive scalar curvature.* The condition $Sc \geq 0$ is incomparably weaker than Ricci ≥ 0 but yet it has non-trivial metric consequences obtained with the minimal surface technique of Schoen-Yau and with the Dirac operator. For example, 3-manifolds with $Sc \geq 1$ have Uryson's width w_1 universally bounded (see [Gro2], [Katz3] and [Gro10].)

(e) *Random manifolds.* If we are given a probability measure on the space of Riemannian metrics on V, we may expect the values of geometric invariants to be concentrated near their respective expectation values, and then we may speak, say, of the systoles of random manifolds. For instance, we may have a sequence μ_i of such measures, each supported on some finite dimensional subspace, where μ_i-random systoles have nice asymptotics for $i \to \infty$. For example, let V be a circle bundle over the flat torus T^n governed by the curvature form ω on T^n. Then, we may look at Gaussian measures μ_i on the spaces of Fourier polynomials of degree i (decomposing 2-forms) and try to evaluate the average and the typical values of the systoles

$$\mathrm{syst}_k,\ k = 1, 2 \ldots n+1 \text{ for } i \to \infty \ .$$

Further classes of manifolds are indicated in 4.A.4 and 4.A.6 below.

4.A.4. Systolic invariants of symplectic manifolds. Let V be a closed symplectic manifold with the structure from ω and let us look at an *adapted* Riemannian metric g on V for which there exists a (necessarily unique) almost complex structure $J : T(V) \to T(V)$, such that g is quasi-Kähler with respect to J. This means g and Ω are J-invariant and $\omega(\tau_1, \tau_2) = g(\tau_1, J\tau_2)$. Then, define

$$\mathrm{syst}_k(V, \omega) = \sup_g \mathrm{syst}_k(V, g) \ ,$$

where g runs over all metrics adapted to ω.

One knows for some manifolds (V, ω) that $\mathrm{syst}_2(V, \omega) < \infty$ as these V contain J-holomorphic curves, but this is unknown in general. Here some test questions.

(a) Let V be the $2n$-torus with a standard (translation invariant) symplectic structure ω. Then, we ask whether $\mathrm{syst}_2(V, \omega) < \infty$.

(b) Does every ω on V diffeomorphic to $\mathbb{C}P^n$ have

$$\mathrm{syst}_2(V, \omega) = \left(\int_V \omega^n \right)^{\frac{1}{n}} \ ?$$

One would especially like to know if, for every ω on $\mathbb{C}P^2$, there is an adapted metric for which some non-zero multiple $i\,h$ of the generator $h \in H_2(\mathbb{C}P^2)$ has $\mathrm{Vol}(i\,h) \leq i \left(\int_{\mathbb{C}P^2} \omega^2 \right)^{\frac{1}{2}}$. (Such $i\,h$ is necessarily realized by a J-holomorphic curve in $\mathbb{C}P^2$ and the existence of such a curve implies that ω is symplectomorphic to the standard symplectic structure on $\mathbb{C}P^2$, as follows by the techniques of [Gro4]. Recently, two remarkable new methods of constructing symplectic submanifolds came to light, one is due to Donaldson and the other to Taubes. Both methods apply to $\mathbb{C}P^2$ and rule out exotic symplectic structures there.)

4.A.5. Further examples of metrics with bounded volume and $\mathrm{syst}_k \to \infty$**.** If $2k > n = \dim V$, then there is no universal relation between $\mathrm{Vol}\, V$ and syst_k. This is shown by constructing metrics on V with $\mathrm{Vol} \leq \mathrm{const}$ and $\mathrm{syst}_k \to \infty$ as follows. Take a submanifold $M \subset V$ of codimension k with trivial normal bundle whose connected components generate $H_{n-k}(V;\mathbb{R})$ and take the family of metrics f_ε obtained by blowing up a fixed metric g in the directions normal to M by ε^{-a} and contracting g along M by ε^b. The blow-up (with $a > 0$) makes the mass norm on $H_k(V)$ go to infinity for $\varepsilon \to 0$, while the contraction in the M-direction with $b \geq \frac{a(n-k)}{k}$ keeps the volume of (V, g^ε) bounded. (The actual expansion-contraction takes place in a fixed trivialized tubular neighbourhood of $M \subset V$, see §2 in [Gro8].)

In general, we conjecture that all non-trivial intersystolic inequalities for simply connected manifolds are associated to multiplicative relations in the cohomology in the corresponding dimensions. This conjecture applies to the mass as well as to the volume, but for the volume we actually expect no inequalities at all as every closed, simply connected manifold probably admits a metric with arbitrarily given systoles $\mathrm{syst}_2,\ \mathrm{syst}_3, \dots, \mathrm{syst}_n = \mathrm{Vol}$ (and for the non-simply connected case the *only* intersystolic inequalities are probably tied up with the π_1-essentiality). On the other hand, our conjecture should be refined in the case of mass by describing the range of the geometries of the Jacobians $J_*(V, g)$ as g runs over all metrics on V. More specifically, we expect that the variation of systoles (and Jacobians) required by conjecture is achieved by blowing and contracting a fixed metric in V along some stratification in V. Here is an example.

Let V be a closed manifold and k be an *odd* number which *does not divide* $n = \dim V$. Then, there exists a family of metrics g_ε on V for which the mass-norm on $H_k(V;\mathbb{R})$ goes to infinity for $\varepsilon \to 0$ while $\mathrm{Vol}\, V$ remains bounded. To see that, take some k-codimensional submanifolds in V with trivial normal bundles whose fundamental classes span $H_{n-k}(V;\mathbb{R})$. To simplify the matter, assume that there are only two M's, say M_1 and M_2, transversally intersecting along $M = M_1 \cap M_2$. Then, we blow up a fixed metric g in V normally to M_1 and M_2 and simultaneously contract it along M and along M_1 and M_2 away from M. The normal expansion (blow-up) of g makes the mass-norm on $H_k(V)$ go to infinity while the contraction forces the volume to stay bounded. (We suggest the reader would make up the details by him-/herself).

Let us indicate what should be done for k even. In this case, the topology of V may not allow submanifolds M with trivial normal bundles as the cohomology dual to [M] may have non-vanishing cup-squares. Yet, the above expansion-contraction procedure can be adapted to this case as follows. Assume that there is a single M whose intersection with a small generic perturbation M' of M has trivial normal bundle. Then, we blow up the metric normally to this intersection $M_0 = M \cap M'$ and we also blow up normally to M away from M_0, where we assume (in order for the blow-up to have the desirable effect on mass_k) that the triviality of the normal bundle of $M - M_0$ in V.

Remarks. (a) Suppose we take a submanifold M in a Riemannian manifold (V, g), such that the normal neighbourhood of M does not split. We still may perform the expansion-contraction of g along M in an "infinitely small" normal neighbourhood of M in V, but now the overall geometric effect of that will heavily depend on the geometry of M, first of all on the curvature of the normal bundle of M. Some idea of what happens near M may be gotten from the discussion in the following section 4.A.6.

(b) Suppose V is endowed with a symplectic structure ω. We want to construct a family of quasi-Kähler metrics g_ε which are *all adapted to* ω (thus having a fixed volume independent of ε) and which blow up transversally to a given k-codimensional submanifold M. (This makes, for a suitable M, the systole $\text{syst}_k V$ to go to infinity.) To achieve this one needs, technically speaking, a closed k-form μ with the support in an ε-neighbourhood U of M which is cohomologically dual to M, such that the norm $\|\mu\|_{g_\varepsilon} \to 0$ for $\varepsilon \to 0$. Such μ can be constructed on split neighbourhoods $U = M \times \mathbb{R}^k$ by pulling back a standard form from $\mathbb{R}^k$ to U by the projection $U \to \mathbb{R}^k$. This works very well for general g_ε unrestricted by any ω (see [Gro8]) but as we want g_ε adapted to ω, we need the splitting $U = M \times \mathbb{R}^k$ to be also adapted to ω as follows. The restrictions of ω to the fibers $M \times x,\ x \in \mathbb{R}^k$ must be *singular*, i.e., of rank $< n-k =$ the dimension of the fibers. The simplest splittings of this kind live on neighbourhoods U of *Lagrangian tori* $M \subset V$ where the corresponding fibers are Lagrangian. Using these, one can easily construct, for instance, a family of quasi-Kähler metrics g_ε adapted to ω on every even dimensional torus $V = T^{2m}$ with a translation invariant ω, such that $\operatorname{syst}_k g_\varepsilon \to \infty$ for all $k > m$. But for general symplectic manifolds V constructing such families g_ε seems more difficult.

4.A.6. Behaviour of systolic invariants for degenerate metrics. There are some particularly nice ways in which a Riemannian metric g on V may degenerate (or go to infinity), similar to geodesic rays in the space of flat tori. For example, let $A^t : T(v) \to T(V),\ t \in\,]0,\infty[$, be a group of linear automorphisms of the tangent bundle of V. Then, the family $g_t = A^t_g$ for a fixed g constitutes such an interesting path of (possibly degenerating) metrics for $t \to \infty$. (One can generalize by using different t on different parts of V, e.g., by restricting the effect of A^t to a certain stratification in V as in the previous section.) For example, one takes two mutually normal subbundles T' and T'' in $T(M)$ and blows up g in the T'-direction. This means, g splits as $g = g' + g''$ where g' vanishes on T'' and g'' on T' and then, g_t is defined by $g_t = tg' + g''$. This is equivalent, up to scaling g_t, to contracting along T'' which means taking $g_t = g' + t^{-1}g''$.

General problem (or program). Determine the asymptotic behaviour of metric invariants of (V, g_t) for $t \to \infty$.

We shall make below a few comments on $\operatorname{syst}_k g_t$ in the special case of $g_t = tg' + g''$ for $t \to \infty$.

Contact case. Suppose T'' is a contact structure on V, i.e., the kernel of a differential 1-form η, such that the restriction of $d\eta$ to T'' is a *non-singular* 2-form. Then, $\operatorname{syst}_k g_t$ remains bounded for $t \to \infty$, provided $2k < \dim V$ and $H_k(V) \neq 0$. (Here syst_k refers to the homological systole $\operatorname{syst} H_k(V)$.) In fact, the k-dimensional homology of V (with arbitrary coefficients) for $2k < \dim V$ (notice that $\dim V$ is odd in the contact case) can be realized by k-cycles in V tangent to T'' (see [Thom], and p.p. 109 and 339 in [Gro3]).

Now, we take k above the middle dimension, $2k > \dim V$, and claim that mass-norm on $H_k(V)$ grows as fast as t for $t \to \infty$ (provided $H_k(V; \mathbb{R}) \neq 0$). In fact, it is obvious that $\mathrm{Vol}(V, g_t) \approx t$, and one can show that every non-trivial k-cycle also has $\mathrm{Vol} \approx t$ as it is "uniformly non-tangent" to T'' (see [Gro7] and [Gro9]).

Remark. — The family $g_t,\ t \to \infty$, converges to a limit (non-Riemannian Carnot-Caratheodory) metric g_∞ for which the Hausdorff dimension of V equals $n+1$ for $n = \dim_{\mathrm{top}} V$ (see [G-L-P]). Every k-dimensional homology class for $2k < n$ can be realized by a k-cycle of Hausdorff dimension k but if $2k > n$, then every k-cycle non-homologous to zero necessarily has g_∞-Hausdorff dimension $\geq k+1$ (see [Gro9]).

Generic subbundles of middle dimension. Let $n = \dim V$ be even and $\mathrm{rank}\, T'' = m = \frac{1}{2} n$. We assume that T'' is generic and $m \geq 3$ and then, the limit (Carnot-Caratheodory) metric g_∞ gives to V Hausdorff dimension $n+m$. Next we try to make a k-cycle C in V *ℓ-tangent* to T'' which means $\dim T_c(C) \cap T''_c \geq \ell$ for the regular points $c \in C$. We think locally of our C at a regular point c as a graph of a map $\mathbb{R}^k \to \mathbb{R}^{n-k}$ and then, the ℓ-tangency condition is expressed in terms of the homomorphism $\delta : T(C) \to T' = T(V)/T''$ by $\mathrm{rank}\, \delta \leq k - \ell$ which amounts to $(m-i+\ell)(k-i+\ell)$ equations for $i = \min(k, m)$. Therefore, if

$$n - k > (m-i+\ell)(k-i+\ell) \tag{$*$}$$

the system of P.D.E. expressing the relation $\mathrm{rank}\, \delta \leq k-\ell$ is *undetermined* and it is, probably, not hard to prove that every homology class can be realized by a k-cycle ℓ-tangent to T'' (compare 2.3.8 in [Gro3]). In such a case one bounds the growth $\mathrm{syst}_k\, g_t$ by $\mathrm{const}\, t^{k-\ell}$. For example, if $k = m$, then $(*)$ reduces to $k > \ell^2$ and the expected growth of $\mathrm{syst}_k\, g_t$ is at most $t^{k-\sqrt{k}}$. (The expected Hausdorff dimension of minimal k-cycles is $\leq 2k - \sqrt{k}$.) On the other hand, if $n - k > (m-i+\ell)(k-i+\ell)$, then our P.D.E. system is *overdetermined*. We may think that the k-cycles are "uniformly non-ℓ-tangent" to T'' in the overdetermined case and, consequently, $\mathrm{syst}_k \gtrsim t^{k-\ell}$. In particular, for $k = m$, the systole should grow at least as $t^{k-\sqrt{k}}$ (and every non-trivial k-cycle would have Hausdorff dimension $\gtrsim 2k - \sqrt{k}$). This would make $(\mathrm{syst}_k)^2 / \mathrm{Vol} \to \infty$ for $t \to \infty$, and would settle (in the negative) the basic systolic problem. Unfortunately, present day techniques give no better than

$\mathrm{syst}_k\, g_t \gtrsim t^{\sqrt{k}}$ as these techniques apply via mass estimates. A related open problem is that of finding the minimal g_∞-Hausdorff dimension of non-trivial k-cycles in V and/or of general topologically k-dimensional subsets. In fact, one does not know if every (Carnot-Caratheodory) metric space of Hausdorff dimension N contains a subset of middle topological dimension and of Hausdorff dimension $\leq N/2$. Similarly, one asks about the minimal Hausdorff dimensions of non-trivial k-cycles in C_k in V with $\dim_{\mathrm{top}} V = n$ and $\dim_{\mathrm{Hau}} V = N$ if these (ever) satisfy inequalities of the form $\dim_{\mathrm{Hau}} C_k + \dim_{\mathrm{Hau}} C_\ell \leq N$ for $k + \ell = n$.

Finally, we indicate another interesting class of families of metrics where the above problems are also essentially open. These come with a dynamical system on V, say, the iterates f^i of a single diffeomorphism f, and g_j are defined as the pullbacks of a fixed metric g by

$$g_j = \sum_{i=1}^{j} (f^i)^* g$$

or by

$$g_j = \max_{0 \leq i \leq j} \, (f^i)^*(g) \ .$$

Notice that such g_i are similar to the above g_t (defined with an operator A^t on $T(V)$) for hyperbolic (Anosov) diffeomorphisms f.

4.A.7. Families of cycles and isosystolic manifolds. One may generalize the notion of syst_k by considering families of k-cycles with prescribed topological properties and minimizing the maximum of the volumes of cycles in such families. In fact, cycles come in families for many natural manifolds, such as the family of projective subspaces in a projective space, or the family of algebraic subvarieties of given dimension and degree in $\mathbb{C}P^n$. Unfortunately, the known results are limited to examples (e.g., see [Fra-Ka]) and 2-cycles in some quasi-Kähler manifolds (see [Gro4], [Ruan]).

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