RICCI CURVATURE MODULO HOMOTOPY

Joachim LOHKAMP

D-MATH
ETH-Zentrum
CH-8092 Zürich (Switzerland)

Abstract. This article is a report summarizing recent progress in the geometry of negative Ricci and scalar curvature. It describes the range of general existence results of such metrics on manifolds of dimension ≥ 3. Moreover it explains flexibility and approximation theorems for these curvature conditions leading to unexpected effects. For instance, we find that "modulo homotopy" (in a specified sense) these curvatures do not have any of the typical geometric impacts.

Résumé. Cet article est un résumé des progrès récents dans la géométrie des variétés riemanniennes à courbure de Ricci ou scalaire négative. Il décrit le domaine de validité des résultats généraux d'existence pour de telles métriques sur les variétés de dimension ≥ 3. De plus, il explique les théorèmes de flexibilité et d'approximation pour ces conditions de courbure, ce qui conduit à des résultats inattendus. Par exemple, nous montrons que "modulo homotopie" (dans un sens précis), ces conditions de courbure n'impliquent aucune des conditions géométriques usuelles.

M.S.C. Subject Classification Index (1991) : 11F72, 11R39, 22E55.

TABLE OF CONTENTS

INTRODUCTION

This paper reports on recent progress in understanding negative Ricci and scalar curvature. We mainly intended to write a guide summarizing and tabulating the main results. We also alluded to some technical (or rather philosophical) background while this is just enough to give some orientation.

As will become clear, $\mathrm{Ric} < 0$-metrics can be met quite frequently in geometry, in a way unexpected before.

One of the insights is concerned with the contrast between positive and negative curvatures. In the case of sectional curvature the implied topological conditions exclude each others, while Ricci and scalar curvature behave quite differently. Here, one may think of a certain maximal amount of positive curvature which could be carried by a given manifold. Now, starting from any metric one can deform it into more and more strongly negatively curved ones. In other words, on each manifold there is an (individual) upper but definitely no lower bound for the spectrum of such an "amount" of Ricci or scalar curvature.

Beside other features there is an amazing resemblance to some existence theories in completely different contexts, for instance, Smale-Hirsch immersion theory. Namely, one may say that these geometric problems can be understood "modulo homotopy" from the algebraic structure of the differential relation which formalizes the geometric condition (e.g. $\mathrm{Ric} < 0$ as partial differential inequality of second order). We will discuss these things in more details in a later chapter.

Now, in order to start our $\mathrm{Ric} < 0$-story, we may notice that it was not even known whether each manifold could admit a $\mathrm{Ric} < 0$-metric. As this paper intends to lead beyond this first order question we start with a short sketch of how to prove that each closed manifold M^n of dimension $n \geq 3$ admits a metric with $\mathrm{Ric} < 0$.

First of all, we mention that it is an easier matter to get a $\mathrm{Ric} < 0$-metric on open manifolds, and thus it does not hurt to use this here. Secondly, we start only in dimension $n \geq 4$. The case $n = 3$, omitted here, can be handled similarly, but needs an extra argument.

Now, if $B \subset M^n$ is a ball, then B contains a closed submanifold N^{n-2} admitting a metric with $\mathrm{Ric} < 0$ and whose normal bundle is trivial. This is easily done in case $n = 4$ using the embedding of a hyperbolic surface in $\mathbb{R}^3 \subset \mathbb{R}^4$.

In higher dimensions we can use induction : S^{n-2}, $n \geq 5$, admits a metric with $\mathrm{Ric} < 0$ and we take the usual embedding $S^{n-2} \hookrightarrow \mathbb{R}^{n-1} \subset \mathbb{R}^n$. (Of course these metrics are not the induced metrics coming from the embedding.)

As mentioned above, we have a metric with $\mathrm{Ric} < 0$ on the open manifold $M \setminus N$, and, in addition, we can get a warped product metric on a tubular neighborhood U of N such that $U \setminus N$ may be identified with $]0, r[\times S^1 \times N$ equipped with $g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g_N$ for some strongly increasing $f \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^{>0})$. The manifold $(]0, r[\times S^1, g_{\mathbb{R}} + f^2 \cdot g_{S^1})$ looks like the spreading open end of the pseudosphere, and we would be done if it was possible to "close" this with a metric with Gaussian curvature $K < 0$. But this is impossible by the Gauß-Bonnet theorem.

On the other hand, we can use the additional factor (N, g_N). We can take a singular metric $g_{\text{sing.}}$ with $K < 0$ on the disk D such that the metric near the boundary looks like $(]0, r[\times S^1, g_{\mathbb{R}} + f^2\, g_{S^1})$ with $\{0\} \times S^1 = \partial D(!)$. Now, we can use $\mathrm{Ric}(g_N) < 0$ to smooth the singularities of $g_{\text{sing.}}$ getting a warped product metric with $\mathrm{Ric} < 0$ on $D \times N$ and glue it to $M \setminus U$. Thus, we have closed M again and it is equipped with a metric with $\mathrm{Ric} < 0$. Details and extensions are described in [L4].

We hope that including this rough existence argument already in the introduction motivates the search for refinements (in various directions) as treated in this paper. In the course of describing such results we will meet some important features of how $\mathrm{Ric} < 0$-metrics are "assembled" in general.

I. COLLECTION OF RESULTS

One of the main features of Ric $<$ 0-geometry is that many problems can be condensed into a local one and that, on the other hand, the local solution can be globalized.

In this chapter we start to describe the results available using this method of attack. It turns out that this particular interplay yields insights into the behaviour of Ric $<$ 0-metrics in a natural way.

I.1. General Existence Theorems.

I.1.1. Theorem. — *Each manifold $M^n, n \geq 3$, admits a complete metric g_M with*

$$-a(n) < r(g_M) < -b(n)\,,$$

for some constants $a(n) > b(n) > 0$ depending only on the dimension n.

We also have another result motivated partly by the existence of complete, finite area metrics with $K < -1$ on open surfaces, partly by S.T. Yau's theorem that each complete non-compact manifold with Ric >0 has infinite volume.

I.1.2. Theorem. — *Each manifold M^n, $n \geq 3$, admits a complete metric g'_M with $r(g'_M) < -1$ and $\mathrm{Vol}(M^n, g'_M) < +\infty$.*

I.1.1 - I.1.2 are proved in [L2].

I.2. Refined Results, Constrained Structures.

Riemannian embeddings and submersions are the two basic "morphisms" in Riemannian geometry. They appear to the same extent in $\mathrm{Ric} < 0$-geometry.

I.2.1. Theorem. — *Let (M^n, g_0), $n \geq 3$, be properly embedded into (N, g) and* $\operatorname{codim} M \geq c(n)$ *(for some $c(n) > 0$ depending only on n).*

Then, there is a metric g_1 on M^n with $\mathrm{Ric}(g_1) < 0$ and a proper embedding of (M, g_1) into (N, g) which is isotopic to the embedding of (M, g_0) by proper embeddings lying inside any prescribed neighborhood of (M, g_0).

The same conclusions hold for immersions instead of embeddings.

Note that I.2.1 could be proved combining Nash's isometric embeddings and the approximation result I.3.6 below. However, we can get the result modifying the proof of I.1.1. leading to a geometric reinterpretation of both those constructions involved and the value of $c(n)$. Secondly, we get

I.2.2. Theorem. — *Let $\pi : E \to N^n$, $n \geq 3$ be a fibre bundle with typical fibre F^m, $m \geq 3$. Then, there are metrics g_E on E, g_N on N and a continuous family of fibre metrics $g_{\pi^{-1}(x)}$ on $\pi^{-1}(x) \approx F$, $x \in N$, such that all metrics involved have* $\mathrm{Ric} < 0$ *and π is a Riemannian submersion.*

Note that this is not done from an argument of the sort : take g_F and g_N with $\mathrm{Ric} < 0$ and define (something like) $g_E = g_F + g_N$.

Actually, the proof uses results concerning the space of all metrics with $\mathrm{Ric} < 0$, cf. I.3.4 and I.3.6 below.

It is also interesting to see the effect of openness of the manifold.

I.2.3. Theorem. — *Let (M^n, g_0) be an open manifold. Then, there is a complete metric $g = e^{2f}\, g_0$ in the conformal class of g_0 with* $\mathrm{Ric}(g) < 0$.

Thus, $\mathrm{Ric} < 0$-geometry is compatible with a lot of topological structures. But there are geometric restrictions for the case (M, g) is closed, since $\mathrm{Ric}(g) < 0$ implies by a theorem due to Bochner that the isometry group of this metric $\mathrm{Isom}(M, g)$ is finite. In this context we have

I.2.4. Theorem. — *Let M^n, $n \geq 3$, be closed, $G \subset \mathrm{Diff}(M)$ a subgroup. Then, $G = \mathrm{Isom}(M, g)$ for some metric g with $\mathrm{Ric}(g) < 0 \Leftrightarrow G$ is finite.*

It is quite easy to prove the same for surfaces M^2 with $\chi(M) < 0$, while the philosophy is quite different, as will be explained later on.

I.3. Flexibility Results.

I.3.1. Theorem. — *For (M^n, g_0), $n \geq 3$, let $S \subset M$ be a closed subset and $U \supset S$ an open neighborhood, and let $\mathrm{Ric}(g_0) \leq 0$ on U.*

Then there is a metric g_1 on M with

(i) $g_1 \equiv g_0$ on S

(ii) $\mathrm{Ric}(g_1) < 0$ on $M \setminus S$.

The most important special case is described in the following corollary. Actually, we will see that, in turn, it implies the theorem.

I.3.2. Corollary. — *On $\mathbb{R}^n$, $n \geq 3$, there is a metric g_n with $\mathrm{Ric}(g_n) < 0$ on $B_1(0)$ and $g_n \equiv g_{\mathrm{Eucl.}}$ outside.*

Perhaps, it is interesting to note that for each $\varepsilon > 0$, we can find a concrete metric g_n as in I.3.2 with $\mathrm{Vol}(B_1(0), g_n) < \varepsilon$. We also note another consequence.

I.3.3. Corollary. — *Let M^n, $n \geq 3$, be compact with boundary $B \neq \emptyset$ and g_0 any fixed metric on B. Then, there is a metric g on M with $g \equiv g_0$ on B, $\mathrm{Ric}(g) < 0$ on M and such that each component of B is totally geodesic.*

Up to now we considered single metrics. But it is also interesting (and sometimes necessary) to understand the space of all such metrics.

As a motivation, recall from [LM] that the space of metrics with positive scalar curvature on a closed manifold M $S^+(M)$ can be quite complicated :

$$S^+(M) \text{ can be empty and/or } \pi_i(S^+(M)) \neq 0.$$

Things are very different in the negative case : Denote by $\mathrm{Ric}^{<\alpha}(M)$ the space of metrics g with $r(g) < \alpha$ on M, $\alpha \in \mathbb{R}$ ($S^{<\alpha}(M)$ is defined analogously).

I.3.4. Theorem. — *The spaces* $\mathrm{Ric}^{<\alpha}(M)$ *and* $S^{<\alpha}(M)$ *are highly non-convex but contractible Fréchet-manifolds.*

As already mentioned above, the fibration result I.2.2 is one of the applications of I.3.4 (and I.3.6). Another one can be derived using some elliptic theory.

I.3.5. Corollary. — *The space of metrics with constant negative scalar curvature is contractible.*

Next recall, for instance, using Bishop's comparison theorem that metrics in $\mathrm{Ric}^{>\alpha}(M)$ cannot "mimick" the fine geometry of negative curvature. For instance, the C^0-closure of $\mathrm{Ric}^{>\alpha}(M)$ in $\mathcal{M}(M)$ is contained in $\mathrm{Ric}^{\geq\alpha}(M)$. On the other hand we have

I.3.6. Theorem. — *The spaces* $\mathrm{Ric}^{<\alpha}(M)$ *and* $S^{<\alpha}(M)$ *are dense in* $\mathcal{M}(M)$ *for each* $\alpha \in \mathbb{R}$, *with respect to* C^0*- and Hausdorff-topology.*

Furthermore, there is the following finer approximation result

I.3.7. Theorem. — *Let* (M^n, g_0), $n \geq 3$, *be flat, then* g_0 *can be approximated by metrics in* $\mathrm{Ric}^{<0}(M)$ *even in* C^∞*-topology.*

For proofs we refer to [L1], [L3] and [L5].

II. LOCALIZATION AND DISTRIBUTION OF CURVATURE

The central point in the proof of those results above is the fact that one can solve the problem in a localized version allowing us to circumvent any global inhibition using (additionally) a distributing-curvature-technique.

Rephrasing this in more technical terms, there are two main steps in the argument : the existence of a metric g_n in $\mathbb{R}^n$, $n \geq 3$, with $\mathrm{Ric}(g_n) < 0$ on $B_1(0)$ and $g_n \equiv g_{\mathrm{Eucl.}}$

outside and a covering argument for arbitrary manifolds giving a "compatible" covering by negatively Ricci curved balls like $(B_1(0), g_n)$, which yields metrics with $\mathrm{Ric} < 0$ on each manifold of dimension ≥ 3.

Thus, the first step consists in constructing local deformation in the flat case.

II.1.1. Proposition. — *On $\mathbb{R}^n$, $n \geq 3$, there is a metric g_n with $r(g_n) < 0$ on $B_1(0)$ and $g_n \equiv g_{\mathrm{Eucl.}}$ outside.*

We will outline a transparent (while coarse) construction easy to survey, cf. [L2] and [L3] for refined deformations needed to understand the spaces of such metrics.

We start in dimension $n = 3$. It is simple to find a positive C^∞-function f of $\mathbb{R}$ with $f \equiv \mathrm{id}$ on $\mathbb{R}^{\geq 1}$ which is symmetric in $\delta \in]0,1[$, i.e. $f(r) = f(2\delta - r)$ and satisfies $\mathrm{Ric}(g_{\mathbb{R}} + f^2 \cdot g_{S^2}) < 0$ on $]2\delta - 1, 1[\times S^2$.

Now, consider instead of the Euclidean metric, the metric $g_{\mathbb{R}} + f^2 \cdot g_{S^2}$ on $\mathbb{R}^3 \setminus B_\delta(0)$. It has two symmetries: a first one under reflections R_E along planes $E \subset \mathbb{R}^3$ with $0 \in E$, and a second "imaginary" one along $\partial B_\delta(0)$ coming from the symmetry of f in δ, in particular $\partial B_\delta(0)$ is totally geodesic. Now, choose one such plane E and consider the quotient space of $\mathbb{R}^3 \setminus B_\delta(0)$ under identification along $\partial B_\delta(0)$ via R_E.

This can be "canonically" attached with the differentiable structure of $\mathbb{R}^3$ (according to Milnor's "smoothing of corners") and the metric on this $\mathbb{R}^3$ is smooth outside the geodesic curve γ corresponding to $\partial B_\delta(0) \cap E$ has $\mathrm{Ric} < 0$ on $B_1(0)$ and is Euclidean outside.

The singularity along γ can be smoothed (with $\mathrm{Ric} < 0$) providing us with a regular metric g_3 as claimed.

The case $n \geq 4$ can be handled in the same way as described in the introduction. We choose a codim 2-submanifold $N \subset \mathbb{R}^n$ with trivial normal bundle and which admits a metric with $\mathrm{Ric} < 0$. Next, we bend $\mathbb{R}^n \setminus N$ "outwards" giving $\mathrm{Ric} < 0$ on $B \setminus N$ for some ball $B \subset \mathbb{R}^n$ and subsequently we use the same method as indicated in the introduction in order to close $\mathbb{R}^n$ again (preserving $\mathrm{Ric} < 0$) and obtain the desired metric g_n. □

Now, we will give some ideas of how to derive the following result whose proof is typical for many results.

II.1.2. Proposition. — *Each manifold M^n, $n \geq 3$, admits a complete metric g_M with $-a(n) < r(g_M) < -b(n)$ for constants $a(n) > b(n) > 0$ depending only on n.*

One easily gets a metric on M such that $\exp_p : B_{100}(0) \to \exp_p(B_{100}(0))$ is a diffeomorphism which is arbitrarily near to an isometry independent of $p \in M$ in C^k-topology. (In case M is compact, just scale any given metric.)

Indeed, we may presently assume $M = (\mathbb{R}^n, g_{\text{Eucl.}})$.

Consider a covering of $\mathbb{R}^n$ by closed balls $\overline{B_5(p_i)}$, $p_i \in A \subset \mathbb{R}^n$ satisfying the following conditions:

(i) $d(p,q) > 5$ for $p \neq q \in A$,

(ii) $\#\{p \in A \mid z \in B_{10}(p)\} \leq c(n)$, $c(n)$ independent of $z \in \mathbb{R}^n$,

and define $g(A,d,s) := \prod_{p \in A} \exp(2\, F_{d,s}\, h(10 - d(p,id)))\, g_A$ with $g_A = g_{\text{Eucl.}}$ on $\mathbb{R}^n \setminus \bigcup\limits_{p \in A} B_1(p), g_A = f_p^*(g_n)$ on $B_1(p)$ for $f_p(x) = x - p$.

Furthermore, $F_{d,s} := s \cdot \exp(-d/id_{\mathbb{R}})$, $h \in \mathrm{C}^\infty(\mathbb{R}, [0,1]), h \equiv 0$ on $\mathbb{R}^{\geq 9{,}6}, h \equiv 1$ on $\mathbb{R}^{\leq 9{,}4}$.

While the rigorous proof is not quite immediate, it should be conceivable that one can find $d, s > 0$ such that $-a < r(g(A,d,s)) < -b$ holds at each point of $\mathbb{R}^n$ and each direction for constants $a > b > 0$.

As noted above, we can find a nearly flat metric $g(M)$ on each manifold. Furthermore we can construct a covering satisfying the same conditions on each of these manifolds (a "Besicovitch covering").

It is not hard to visualize that (almost) the same $d, s > 0$ and pinching constants $a > b > 0$ can be obtained for the Ricci curvature of an analogously defined metric $g(A,d,s)$ on an arbitrary manifold starting from $g(M)$. $\square$

The covering argument above can be used to produce as much negative curvature as is necessary to "hide" each metric of some compact family of metrics behind a "veil" of negative Ricci curvature.

This observation leads to a suggestive argument for the contractibility of $\mathrm{Ric}^{<\alpha}(M)$ and $S^{<\alpha}(M)$. We just explain the idea of how to prove that $\mathrm{Ric}^{<0}(M)$ is connected.

Thus, start with two metrics $g_0, g_1 \in \mathrm{Ric}^{<0}(M)$, and take any path between g_0, g_1, for instance, $\gamma_t := t \cdot g_0 + (1-t) \cdot g_1,\ t \in [0,1]$. As will be shown below, γ_t does not stay in $\mathrm{Ric}^{<0}(M)$ in general.

However, we can "produce Ric < 0" in such a way that γ_t continuously shifts into $\mathrm{Ric}^{<0}(M)$. Thus, the edge-paths fit together to a path in $\mathrm{Ric}^{<0}(M)$ joining g_0 and g_1.

The reader might have noticed an analogy between the contractibility of $\mathrm{Ric}^{<0}(M^n)$ for $n \geq 3$ and the well-known result that the space of metrics with $K < 0$ on surfaces with $\chi(F) < 0$ is also contractible. But this latter fact is based on a different philosophy. Namely, on surfaces we have an unambigious and fixed "amount" of curvature, the *integral curvature*, which is determined uniquely from the topology via the Gauß-Bonnet formula.

Thus, in our terminology, we could say we can neither produce nor lose curvature. Given any path γ_t between two such $K < 0$-metrics, we observe (as above) that γ_t need not stay in the space of $K < 0$-metrics. But the integral curvature remains negative and, as an additional extra structure on surfaces, each such metric γ_t can be deformed into a $K < 0$-metric (i.e., one may distribute the integral curvature uniformly) leading to the result for surfaces.

Finally, we will justify the claim that these spaces of metrics are "highly" non-convex. For notational simplicity, we restrict to $S^{<0}(M)$.

II.1.3. Lemma. — *For any $g \in S^{<0}(M)$ and each ball $B \subset M$ there is a diffeomorphism φ with $\varphi \equiv id$ on $M \setminus B$ and $t \cdot g + (1-t) \cdot \varphi^*(g) \notin S^{<0}(M)$ for some $t \in]0,1[$.*

Using scaling arguments we can reduce to the case $\mathbb{R}^n \supset B_1(0) \equiv B$. Here, we can take φ with $\varphi(t,x) = (f(t),x),\ (t,x) \in \mathbb{R}^+ \times S^{n-1} \equiv \mathbb{R}^n \setminus \{0\}$ for some diffeomorphism $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f \equiv id$ on $]0, \frac{1}{10}[\cup]\frac{9}{10}, +\infty[$ and $f \equiv id + \frac{1}{10}$ on $]\frac{3}{10}, \frac{7}{10}[$.

Now, it is not hard to check using warped product formulas that

$$\frac{1}{2}\, g_{\mathrm{Eucl.}} + \frac{1}{2}\varphi^*(g_{\mathrm{Eucl.}}) \notin S^{\leq 0}(\mathbb{R}^n)\,.$$

III. RELATIONS MODULO HOMOTOPY

We resume listing results (or rather interpretations of results) on Ric < 0 and $S < 0$. In particular, we describe some relations of the results presented above and "homotopy principles" (abb. h-principles), a conceptual language introduced in a broad context by M. Gromov, cf. his monograph on this subject [G].

We start with some definitions. Let $\pi : X \to M$ be a smooth fibration f over some manifold M, and denote by X^κ the space of κ-jets of germs of smooth sections of π and the induced fibration of M by $\pi_\kappa, \pi_\kappa : X^\kappa \to M$.

A section φ of π_κ is called *holonomic* if there is a section f of π whose κ-jet is φ.

A *differential relation* $\mathcal{R}$ of order κ imposed on sections of π is just a subset $\mathcal{R} \subset X^\kappa$, and a section f of π is called a *solution* of $\mathcal{R}$ if its κ-jet lies in $\mathcal{R}$.

Finally, let $\pi_{\kappa,m}$ denote the canonical projection $\pi_{\kappa,m} : X^\kappa \to X^m$ for $0 \leq m \leq \kappa$. Hence a holonomic section φ lying in $\mathcal{R}$ projects to a solution $\pi_{\kappa,0}(\varphi)$ of $\mathcal{R}$.

The concept of h-principle relies on the following (idea of) solving strategy : first, construct a (possibly non holonomic) section of X^κ lying in $\mathcal{R}$. This is basically a problem in Algebraic Topology. Then, (try to) pass inside of $\mathcal{R}$ to a holonomic one. This allows to make sure that a resulting holonomic section is really a solution and that we do not lose information we already had.

Now, denote by Sol $\mathcal{R}$ the set of all solutions of $\mathcal{R}, C(\mathcal{R})$ the space of all sections of X^κ lying in $\mathcal{R}$ and by J_κ : Sol $\mathcal{R} \to C(\mathcal{R})$ the map $J_\kappa(\varphi) = \kappa$-jet of φ.

III.1.1. Definition. — *The relation $\mathcal{R}$ fulfils the h-principle if J_κ is a weak homotopy equivalence.*

(Recall that a map $f : X \to Y$ is called a *weak homotopy equivalence* if all the induced maps between homotopy groups $f_n : \pi_n(X) \to \pi_n(Y)$ are isomorphisms.)

Now, in our context of curvature conditions, we specify X = the bundle of pointwise positive definite symmetric (2,0)-tensors (i.e., whose sections are metrics) and we consider differential relations $\mathcal{R} \subset X^2$ which simply restricts the curvature of a section $\pi : X \to M$. For instance,

$$\mathcal{R} = \{\varphi \in X^2 \mid \mathrm{Ric}(\varphi) < 0\} \equiv \mathrm{Ric} < 0\,.$$

Next, we want to see that some of the flexibility results of I.3 can be reinterpreted using the h-principle language. Therefore we must have a look at $C(\mathcal{R})$ and check the following result (cf. also [G]) :

III.1.2. Lemma. — *The fibers of the fibrations* Sec $< \alpha$, Ric $< \alpha$, *and* $S < \alpha$ *are non-empty and contractible. The same holds in case "*$> \alpha$*".*

We have to show contractibility for the space of 2-jets of germs of metrics near $0 \in \mathbb{R}^n$ with Sec $< \alpha$ etc. These curvature relations contain the first two derivatives of the metric. Now, there are two easily verified features:

- for each 1-jet φ_1 of metric, there is 2-jet φ_2 with $\pi_{2,1}(\varphi_2) = \varphi_1$ and $\mathrm{Sec}(\varphi_2) < \alpha$ etc.,

- secondly the curvature depends linearly on the second derivatives.

This implies the fiber over each 1-jet φ_1 is non-empty and convex. Furthermore the space of all 1-jets is contractible, hence the whole space is contractible $\square$

It is a well-known result from elementary obstruction theory that fibrations with contractible fibers always have a section and the space of sections is also (weakly) contractible.

III.1.3. Corollary. — *The spaces* $C(\mathrm{Sec} < \alpha)$ *etc. are (weakly) contractible.*

Hence, we can reformulate I.3.4 as follows:

III.1.4. Theorem. — *On each manifold* M^n, $n \geq 3$, *the differential relations* Ric $< \alpha$ *and* $S < \alpha$ *fulfil the* h*-principle.*

In contrast to III.1.4 we have an at first sight "converse" approach due to Gromov [G] which starts from Topology and arrives at Geometry.

III.1.5. Theorem. — *Each open, diffeomorphism invariant differential relation* $\mathcal{R}$ *on an open manifold satisfies the* h*-principle.*

It is obvious that $\mathrm{Sec} < \alpha$, etc. are open and diffeomorphism invariant. Hence, we get

III.1.6. Corollary. — *The spaces* $\mathrm{Sec} < \alpha$ *(resp.* $> \alpha$*) etc. satisfy the* h*-principle on each open manifold and, in particular, each open manifold carries a (non-complete) metric with* $\mathrm{Sec} < \alpha$ *as well as one with* $\mathrm{Sec} > \alpha$*.*

In order to explain this apparent interplay, we note that the h-principle for open manifolds III.1.5 originated from Smale-Hirsch theory for topological immersions of open manifolds. In other words, III.1.5 is the abstracted version of an h-principle derived in a very concrete setting and leads, in this general form and language, to many new conclusions. Actually, many other h-principles for abstract problems had been obtained similarly.

Thus, besides its purely philosophical meaning the $\mathrm{Ric} < 0$-h-principles might be used in the same fashion exploiting those concrete methods to get new applications from abstraction.

BIBLIOGRAPHY

[G] M. Gromov, *Partial differential relations,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag Berlin Heidelberg, 3. Folge **9** (1986).

[LM] B. Lawson, M.-L. Michelsohn, *Spin geometry,* Princeton Univ. Press, Princeton (1989).

[L1] J. Lohkamp, *The space of negative scalar curvature metrics,* Inventiones Math. **110** (1992), 403–407.

[L2] J. Lohkamp, *Metrics of negative Ricci curvature*, Ann. Math. **140** (1994), 655–683.

[L3] J. Lohkamp, *Curvature h-principles*, Ann. Math. **142** (1995), 457–498.

[L4] J. Lohkamp, *Negative bending of open manifolds*, J. Differential Geom. **40** (1994), 461–474.

[L5] J. Lohkamp, *Notes on Localized Curvatures*, preprint.