CUT LOCI AND DISTANCE SPHERES ON ALEXANDROV SURFACES

Katsuhiro SHIOHAMA

Kyushu University
Graduate School of Mathematics
6-10-1 Hakozaki, Higashi-ku
Fukuoka 812 (Japan)

Minoru TANAKA

Tokai University
Faculty of Science
Department of Mathematics
Hiratsuka 259-12 (Japan)

Abstract. The purpose of the present paper is to investigate the structure of distance spheres and cut locus $C(K)$ to a compact set K of a complete Alexandrov surface X with curvature bounded below. The structure of distance spheres around K is almost the same as that of the smooth case. However $C(K)$ carries different structure from the smooth case. As is seen in examples of Alexandrov surfaces, it is proved that the set of all end points $C_e(K)$ of $C(K)$ is not necessarily countable and may possibly be a fractal set and have an infinite length. It is proved that all the critical values of the distance function to K is closed and of Lebesgue measure zero. This is obtained by proving a generalized Sard theorem for one-valuable continuous functions.

Our method applies to the cut locus to a point at infinity of a noncompact X and to Busemann functions on it. Here the structure of all co-points of asymptotic rays in the sense of Busemann is investigated. This has not been studied in the smooth case.

Résumé. L'objet de cet article est d'étudier la structure des sphères de distance et du cut locus $C(K)$ d'un ensemble compact.

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INTRODUCTION

The topological structure of the cut locus $C(p)$ to a point p on a complete, simply connected and real analytic Riemannian 2-manifold M was first investigated by Poincaré [P], Myers [M1], [M2] and Whitehead [W]. If such an M has positive Gaussian curvature, then (1) Poincaré proved that $C(p)$ is a union of arcs and does not contain any closed curve and its endpoints are at most finite which are conjugate to p, and (2) Myers proved that if M is compact (and hence homeomorphic to a 2-sphere), then $C(p)$ is a tree and if M is noncompact (and hence homeomorphic to $\mathbb{R}^2$), then it is a union of trees. Here, a topological set T is by definition a *tree* iff any two points on T is joined by a unique Jordan arc in T. A point x on a tree T is by definition an *endpoint* iff $T \setminus \{x\}$ is connected. Whitehead proved that if M is not simply connected, then $C(p)$ carries the structure of a local tree and the number of cycles in $C(p)$ coincides with the first Betti number of M. Here, a topological set C is by definition a *local tree* iff for every point $x \in C$ and for every neighborhood U around x, there exists a smaller neighborhood $T \subset U$ around x which is a tree.

The structure of geodesic parallel circles for a simple closed curve $\mathcal{C}$ in a real analytic Riemannian plane M was first investigated by Fiala [F] in connection with an isoperimetric inequality. Hartman extended Fiala's results (and also Myers' ones on $C(p)$) to a Riemannian plane with C^2-metric. Geodesic parallel coordinates for a given simply closed C^2-curve was employed in [H] to prove that there exists a closed set $\mathcal{E} \subset [0,\infty)$ of measure zero such that if $t \notin \mathcal{E}$, then

(1) the geodesic t-sphere $\mathcal{S}(\mathcal{C};t) := \{\, x \in M \,;\, d(x,\mathcal{C}) = t\}$ around $\mathcal{C}$ consists of a finite disjoint union of piecewise C^2-curves each component of which is homeomorphic to a circle,

(2) the length $L(t)$ of $\mathcal{S}(\mathcal{C};t)$ exists, and moreover $\frac{dL(t)}{dt}$ also exists and is continuous on $(0,\infty) \setminus \mathcal{E}$. Furthermore, the set $\mathcal{E}$ is determined by the topological structure of the cut locus and focal locus to $\mathcal{C}$.

These results were extended to complete, open and smooth Riemannian 2-manifolds (finitely connected or infinitely connected) in [S], [ST1], [ST2].

The purpose of the present article is to establish almost similar results on the structure of cut loci and geodesic spheres without assuming almost any differentiability. In fact, a simple closed curve in a C^2-Riemannian plane will be replaced in our results by a compact set in an Alexandrov surface. *From now on, let X be a connected and complete Alexandrov space without boundary of dimension 2 whose curvature is bounded below by a constant k.* Let $K \subset X$ be an arbitrary fixed compact set and $\rho : X \to \mathbb{R}$ the distance function to K. Let $\mathcal{S}(t) := \rho^{-1}(t)$ for $t > 0$ be the distance t-sphere of K. Let $C(K)$ be the cut locus to K and $C_e(K)$ the set of all endpoints of $C(K)$. With these notations our results are stated as follows.

Theorem A. — *For a connected component $C_0(K)$ of $C(K)$,*

(1) *$C_0(K)$ carries the structure of a local tree and any two points on it can be joined by a rectifiable Jordan arc in it ;*

(2) *the inner metric topology of $C_0(K)$ is equivalent to the induced topology from X ;*

(3) *there exists a class $\mathcal{M} := \{m_1, \cdots\}$ of countably many rectifiable Jordan arcs $m_i : I_i \to C_0(K)$, $i = 1, \cdots$, such that I_i is an open or closed interval and such that*

$$C_0(K) \setminus C_e(K) = \bigcup_{i=1}^{\infty} m_i(I_i) \ , \quad \textit{disjoint union} \ ;$$

(4) *each m_i has at most countably many branch points such that there are at most countably many members in $\mathcal{M}$ emanating from each of them.*

The above result is *optimal* in the sense that $C(K)$ in Example 4 cannot be covered by any countable union of Jordan arcs.

We see from (3) and (4) in Theorem A that $C(K)$ has, roughly speaking, a self similarity. The cut locus $C_0(K)$ is a *fractal set* iff the Hausdorff dimension of $C_0(K)$ in X is not an integer. Example 4 in §1 suggests that $C_0(K)$ will be a fractal set, where $C_e(K)$ is uncountable.

Theorem B. — *There exists a set $\mathcal{E} \subset (0, \infty)$ of measure zero with the following properties. For every $t \notin \mathcal{E}$,*

(1) *$\mathcal{S}(t)$ consists of a disjoint union of finitely many simply closed curves.*

(2) *$\mathcal{S}(t)$ is rectifiable.*

(3) *Every point $x \in \mathcal{S}(t) \cap C(K)$ is joined to K by at most two distinct geodesics of the same length t. Furthermore, if $x \in C(K) \cap \mathcal{S}(t)$ is joined to K by a unique geodesic, then $x \in C_e(K)$.*

(4) *There exists at most countably many points in $\mathcal{S}(t) \cap C(K)$ which are joined to K by two distinct geodesics.*

It should be noted that in contrast with the Riemannian case, the set $\mathcal{E}$ is not always closed. In fact, X admits a singular set $\mathrm{Sing}(X)$ and $\mathcal{E}$ contains $\rho(\mathrm{Sing(X)})$. Example 2 in §1 provides the case where $\rho(\mathrm{Sing}(X))$ is a dense set on $(0, \operatorname{diam} X)$.

In due course of the proof we obtain a generalized Sard theorem on the set of all critical values of a continuous (not necessarily of bounded variation) function, see Lemma 3.2, and prove the

Theorem C. — *The set of all critical values of the distance function to K is closed and of measure zero.*

The Basic Lemma applies to the cut locus of a point at infinity. Let $\gamma : [0, \infty) \to X$ be an arbitrary fixed ray. A *co-ray σ to γ* is by definition a ray obtained by the limit of a sequence of minimizing geodesics $\sigma_j : [0, \ell_j] \to X$ such that $\lim_{j\to\infty} \sigma_j(0) = \sigma(0)$ and such that $\{\sigma_j(\ell_j)\}$ is a monotone divergent sequence on $\gamma[0, \infty)$. Through every point on X there passes at least a co-ray to γ. A co-ray σ to γ is said to be *maximal* iff it is not properly contained in any co-ray to γ. Let $C(\gamma(\infty))$ be the set of all the starting points of all maximal co-rays to γ. In the Riemannian case the set $C(\gamma(\infty))$ is contained in the set of all non-differentiable points of the Busemann function F_γ with respect to γ. Here F_γ is defined by

$$F_\gamma(x) := \lim_{t\to\infty} [t - d(x, \gamma(t))], \qquad x \in X \ .$$

The set $C(\gamma(\infty))$ may be understood as the cut locus at a point $\gamma(\infty)$ of infinity, for it carries the same structure as cut locus. The structure of $C(\gamma(\infty))$ has not been discussed even in Riemannian case. Our proof method applies to investigate the structure of $C(\gamma(\infty))$ on X, and we obtain

Theorem D. — *Let $\gamma : [0, \infty) \to X$ be an arbitrary fixed ray.*

(1) *Theorem A is valid for each component of $C(\gamma(\infty))$.*

(2) *There exists a set $\mathcal{E}(\gamma(\infty)) \subset (0, \infty)$ of measure zero with the property that Theorem B is valid for the levels of F_γ .*

The proof of Theorem D is essentially contained in those of Theorems A and B and omitted here. The first two statements in Theorem A were proved by Hebda in [He] in the case where K is a point on a smooth Riemannian 2-manifold. In view of the proofs of these theorems, we recognize that the differentiability assumption in Riemannian case is not essential.

Basic tools in Alexandrov spaces and length spaces are referred to [GLP] and [BGP]. The authors would like to express their thanks to H. Sato for valuable discussion on the treatment of the Sard theorem for continuous functions developed in Lemma 3.2, and also to J. Itoh for the discussion on the construction of Example 4. This work was achieved during the second author's visit to Kyushu University in 1992-93. He would like to express his thanks to Kyushu University for its hospitality while he was staying in the Department of Mathematics.

1. PRELIMINARIES

Let $M^2(k)$ be a complete simply connected surface with constant curvature k. An *Alexandrov space* X with curvature bounded below by a constant k is by definition a locally compact complete length space with the following properties :

(1) Any two points $x, y \in X$ are joined by a curve, denoted by xy and called a *geodesic*, whose length realizes the distance $d(x, y)$.

(2) Every point $x \in X$ admits a neighborhood U_x with the following property. There exists for every geodesic triangle $\Delta = \Delta(pqr)$ in U_x a corresponding geodesic

triangle $\tilde{\Delta} = \Delta(\tilde{p}\tilde{q}\tilde{r})$ with the same edge lengths sketched on $M^2(k)$ such that if s is a point on an edge qr of Δ and if $\tilde{s}$ is a point on the corresponding edge $\tilde{q}\tilde{r}$ of $\tilde{\Delta}$ with $d(q,s) = d(\tilde{q},\tilde{s})$, then $d(p,s) \geq d(\tilde{p},\tilde{s})$.

The above property makes it possible to define an angle $\angle yxz$ at $x \in X$ between two geodesics xy and xz, and to lead the Alexandrov convexity property as well as the Toponogov comparison theorem for geodesic triangles. Alexandrov spaces with curvature bounded below have the following properties which are used throughout this paper.

Fact 1 (see 2.8.2 Corollary in [BGP]) Every geodesic on X does not have branches. Namely, if a point $z \in X$ belongs to an interior of geodesics xy and xy_1, then these four points are on the same geodesic.

Fact 2 (see 2.8 in [BGP]) If $\{p_iq_i\}$ and $\{p_ir_i\}$ are sequences of geodesics such that $\lim_{i\to\infty} p_iq_i = pq$ and $\lim_{i\to\infty} p_ir_i = pr$, then

$$\liminf_{i\to\infty} \angle q_ip_ir_i \geq \angle qpr \ .$$

Fact 3 (The first variation formula ; see Theorem 3.5 in [OS]) For a geodesic xy and for a point $p \in X$ we have

$$d(p,y) - d(p,x) = -d(x,y) \cdot \cos \min{}_{px} \angle pxy + o(d(x,y)) \ ,$$

where the minimum is taken over all geodesics joining p to x.

From now on, let X be a 2-dimensional Alexandrov space with curvature bounded below by k. It was proved in § 11; [BGP] that if a 2-dimensional Alexandrov space X without boundary has curvature bounded below, then it is a topological 2-manifold. However it is not expected for such an X to admit a usual differentiable manifold structure. In fact, singular points may exist on X. It was proved in [OS] that X admits a full measure subset X_0 on which C^1-differentiable structure and $\mathrm{C}^{\frac{1}{2}}$-Riemannian structure is well defined. A point $p \in X$ is by definition a *singular point* iff the space of directions S_p at p is a circle of length less than 2π. It follows from the Toponogov comparison theorem that $\mathrm{Sing}(X)$ is a countable set, (see [G]).

Notice that the strict inequality in Fact 2 occurs only when $p \in X$ is a singular point. If $\dim X = 2$ in Fact 2, then $\lim_{i\to\infty} \angle q_i p_i r_i = \angle qpr$ holds for every $p \notin \mathrm{Sing}(X)$.

The following example (see [OS]) shows that $\mathrm{Sing}(X)$ forms a dense set in X.

Example 1. — Let $P_n \subset \mathbb{R}^3$ for $n \geq 4$ be a convex polyhedron contained in 2-ball around the origin with the following properties. All the vertices of P_n are those of P_{n+1} and the image under radial projection of all the vertices of P_n to the unit sphere $S^2(1)$ forms an δ_n-dense set on $S^2(1)$ with $\lim_{n\to\infty} \delta_n = 0$. If $X \subset \mathbb{R}^3$ is the Hausdorff limit of $\{P_n\}$, then X is an Alexandrov surface of curvature bounded below by 0 and its singular set is dense on it.

Let $K \subset X$ be an arbitrary fixed compact set. Let $\rho : X \to R$ be the distance function to K, i.e., $\rho(x) := d(x, K)$, $x \in X$. A geodesic joining x to a point $y \in K$ with length $\rho(x)$ is called *a geodesic from x to K*. Let $\Gamma(x)$ for $x \in X \setminus K$ be the set of all geodesics from x to K. A point $x \in X$ is by definition *a cut point to* K iff a geodesic in $\Gamma(x)$ is not properly contained in any geodesic to K. The cut locus $C(K)$ to K is by definition the set of all cut points to K. Notice that $X \setminus K$ has countably many components. Each bounded component of it contains a unique component of $C(K)$. Thus $C(K)$ has at most countably many components. Notice also that every singular point of X is a cut point to K because such a point cannot be an interior of any geodesic on X. In contrast to the Riemannian case, $C(K)$ is not necessarily closed in X, for instance see Examples 2 and 4. From the definition of cut locus to a compact set K we observe that $C(K) \cap K = \emptyset$, while we allow the existence of a sequence of cut points to K converging to a point on K.

Example 2. — Let D be a convex domain in $\mathbb{R}^2$. Then, its double $\mathcal{F}$ is an Alexandrov surface with curvature bounded below by 0. If a point p on the plane curve ∂D has positive curvature, then $C(p) = \partial D \setminus \{p\}$, and in particular $d(p, C(p)) = 0$.

A point $x \in X$ is by definition a *critical point of* ρ iff for every tangential direction $\xi \in S_x$ there exists a geodesic $xz \in \Gamma(x)$ whose tangential direction at x makes an angle with ξ not greater than $\pi/2$. The set of all critical points of ρ is denoted by

$\mathrm{Crit}(\rho)$. It is clear that $\mathrm{Crit}(\rho) \subset C(K)$. If $x \in \mathrm{Sing}(X)$ has the property that S_x has length not longer than π, then x is a critical point of ρ.

The following Example 3 of a flat cone shows that its vertex is a critical point of ρ and that the strict inequality in Fact 2 occurs. Thus, the behavior of geodesics on X is quite different from that on Riemannian manifolds.

Example 3. — Let X be a flat cone with its vertex x, at which S_x has length π. Let $K \subset X$ be a line segment which intersects a generating half line ℓ orthogonally at its midpoint. We develop X to a closed half plane H such that the double cover $\tilde{\ell}$ of ℓ is a line and forms the boundary of H. The developed image $\tilde{K} \subset H$ of K forms two parallel line segments orthogonal to $\tilde{\ell}$ and each of them has the same length. If a line segment $\tilde{p}\tilde{q}$ with $\tilde{p}, \tilde{q} \in \tilde{K}$ is parallel to $\tilde{\ell}$, then its midpoint $\tilde{r}$ has the preimage $r \in X$ as a critical point of ρ. The $\Gamma(r)$ consists of exactly two elements which are developed onto $\tilde{p}\tilde{q}$ making an angle π at $\tilde{r}$. If a sequence $\{r_n\}$ of such points converges to x, then $\{\Gamma(r_n)\}$ converges to a unique geodesic $\Gamma(x)$, and x is a critical point of ρ.

There are three types of cut points to K. A cut point p to K is an endpoint if the set of tangential directions to all elements of $\Gamma(p)$ forms either a point or a closed subarc of S_p. A cut point $q \in C(K)$ is by definition a *regular point* iff $\Gamma(q)$ consists of exactly two elements. A cut point q is by definition a *branch point* iff $\Gamma(q)$ contains at least three connected components.

The following Example 4 provides us with an Alexandrov surface $\mathcal{F}$ in $\mathbb{R}^3$ where the cardinality of the set of all end cut points to $p \in \mathcal{F}$ is uncountable.

Example 4. — A monotone increasing sequence $\{\mathcal{F}_n\}$ of convex polyhedra in $\mathbb{R}^3$ is successively constructed in such a way that if $\mathcal{F}$ is the Hausdorff limit of $\{\mathcal{F}_n\}$, then $\mathcal{F}$ admits a point p at which $C(p)$ has the following properties :

(1) the cardinality of the set of all endpoints of $C(p)$ is uncountable ;

(2) there exists a sequence of endpoints of $C(p)$ converging to an interior of some geodesic emanating from p.

Let $\Pi(a) \subset \mathbb{R}^3$ for $a \geq 0$ be the plane parallel to (x, y)-plane and given by $z = a$. For convex polygons $P, Q \subset \mathbb{R}^3$, not lying on the same plane, we denote by $\mathcal{C}(P; Q)$

the convex polyhedron generated by P and Q. A positive number q is identified with the point $(0,0,q) \in \mathbb{R}^3$ when there is no confusion.

Let $\{r_n\}$ be a strictly decreasing sequence with $r_0 := 1$ such that $\lim r_n =: r > 0$, and $\{p_n\}$ a strictly increasing sequence with $p_0 := 0$ such that $\lim p_n =: p < \infty$. Let $\Delta_0 \subset \Pi(0)$ be a right triangle centered at the origin O of $\mathbb{R}^3$ whose inscribed circle has radius $r_0 = 1$. A sequence of right $3 \cdot 2^n$-gons $\Delta_n \subset \Pi(p_n)$ for $n = 1, \ldots$ with the inscribed circle $S(p_n; r_n)$ centered at a point p_n with radius r_n is successively constructed as follows. For given sequences $\{r_n\}$ and $\{p_n\}$, we choose a strictly increasing sequence $\{\theta_n\}$ such that $\theta_0 > 0$ and $\lim \theta_n < \pi/2$ and such that

$$\text{(1-1)} \quad (p_n - p_{n-1}) \tan \theta_{n-1} = r_{n-1} - r_n \ , \qquad \frac{r_{n-1}}{\cos \frac{\pi}{3 \cdot 2^n}} - r_n \leq (p_n - p_{n-1}) \tan \theta_n \ .$$

Let $\Delta_n \subset \Pi(p_n)$ be placed as follows. Every other edge of Δ_n is parallel to an edge of Δ_{n-1}. The plane containing these edges meets z-axis at a point q_n with $q_n = r_{n-1} \cot \theta_{n-1} + p_{n-1} = r_n \cot \theta_{n-1} + p_n$. If an edge of Δ_n is not parallel to any edge of Δ_{n-1}, then the plane containing q_n and this edge intersects $\Pi(p_i)$ and the line of intersection does not separate Δ_i for all $i = 0, \cdots n-1$. The relation (1-1) then implies that

$$\mathcal{F}_n := \partial\Big(\bigcup_{k=1}^{n} \mathcal{C}(\Delta_{k-1}; \Delta_k)\Big) \ ,$$

for all n is the boundary of a convex polyhedron. The polyhedron $\mathcal{F}_n$ has the property that the set of all its vertices coincides with the set of all endpoints of $C(p_n)$. If q is a vertex of Δ_k for some $k < n$, then the geodesic $p_n q$ on $\mathcal{F}_n$ intersects orthogonally an edge of every Δ_j for $j = k+1, \cdots, n$ at its midpoint. If $\mathcal{F}$ is the Hausdorff limit of $\{\mathcal{F}_n\}$, then $\mathcal{F}$ is an Alexandrov surface with curvature bounded below by zero. The point $p \in \mathcal{F}$ has the property that $\lim p_n = p$ and the set of all endpoints of $C(p)$ is the union of all vertices of all Δ_n's. The set of all accumulation points of those vertices lies on $S(p,r) \setminus \{$ geodesics joining p to all vertices of all Δ_n's $\}$. Therefore, the set of all endpoints of $C(p)$ is uncountable. Moreover, if $q \in \mathcal{F}$ is a vertex of Δ_n, then there exists a sequence of endpoints of $C(p)$ converging to an interior of pq. For a suitable choice of sequences $\{r_n\}$ and $\{p_n\}$ we see that $C(p)$ has an unbounded total length.

We now discuss a general property of cut locus of K. As discussed in [BGP], every point on X admits a disk neighborhood. For a point $x \in X$ and for an $r > 0$ we denote by $B(x; r)$ an open metric r-ball centered at x. For every compact set $A \subset X$ such that $d(A, K) > 0$, we find a positive number $r = r_A$ with the following properties:

(1) $d(A, K) \geq 4r$;

(2) there exist for every point $x \in A$ two disk neighborhoods $U_r(x)$ and $U_{2r}(x)$ such that $U_r(x) \supset B(x; r)$, $U_{2r}(x) \supset B(x; 2r)$; the boundaries $\partial U_r(x)$ and $\partial U_{2r}(x)$ are homeomorphic to a circle ;

(3) $\partial U_r(x) \subset \{z \in X; d(z, x) = r\}$ and $\partial U_{2r}(x) \subset \{z \in X; d(z, x) = 2r\}$. To each point $x \in C(K)$ we assign a sufficiently small positive number $\varepsilon(x)$ such that for $A := \overline{B(x; \frac{1}{2}d(x, K))}$ and for $r = r_A$ every point $x' \in B(x; \varepsilon(x))$ has the property that every member in $\Gamma(x')$ intersects $\partial U_r(x)$ (and also $\partial U_{2r}(x)$) at a unique point. If $\varepsilon(x)$ is taken sufficiently small then this property is justified by the Toponogov comparison theorem for a narrow triangle $\Delta(x\gamma'(r_1)\gamma'(2r))$, where $\gamma' \in \Gamma(x')$ and $\gamma'(r_1) \in \partial U_r(x)$. In fact the triangle has excess not greater than $\varepsilon(x)$.

It follows from the choice of $\varepsilon(x)$ that $U_{2r}(x) \setminus \Gamma(x')$ for every $x' \in B(x; \varepsilon(x))$ consists of a countable union of disk domains and each component of it is bounded by two subarcs of γ' and σ' for $\gamma', \sigma' \in \Gamma(x')$ and a subarc of $\partial U_{2r}(x)$ cut off by γ' and σ'.

The following notation of a *sector at a point* $x \in C(K)$ plays an important role in our investigation.

Definition. — *Each component of* $U_{2r}(x) \setminus \Gamma(x)$ *(respectively,* $U_r(x) \setminus \Gamma(x)$*) is by definition a* $2r$*-sector (respectively an* r*-sector) at* x*. The inner angle of a sector* $R_r(x)$ *is by definition the length of the subarc of* S_x *determined by* $R_r(x)$*.*

Let $\gamma, \sigma \in \Gamma(x)$ be the boundary of a sector $R_r(x)$ at $x \in C(K)$ such that $\gamma(0) = \sigma(0) = x$. Each sector $R_r(x)$ at $x \in C(K)$ has the following properties.

S0 If $z \in R_r(x)$, then every geodesic xz lies in $R_r(x)$. If $y, z \in R_r(x)$, then every geodesic yz is contained in $U_{2r}(x)$. If the inner angle at x of $R_r(x)$ is less than

$\frac{1}{2}L(S_x)$ and if $y, z \in \overline{R_r(x)}$ are sufficiently close to x, then every geodesic yz lies in $R_r(x)$.

S1 There is no element in $\Gamma(x)$ which passes through points in $R_r(x)$.

S2 There exists a sequence of cut points to K in $R_r(x)$ converging to x.

S3 If $\{q_j\}$ is a sequence of points in $R_r(x)$ converging to x, then every converging subsequence of geodesics in $\{\Gamma(q_j)\}$ has limit as either γ or else σ.

S4 If $x' \in C(K) \cap B(x, \varepsilon(x)) \cap R_r(x)$, then there exists a unique sector at x' which contains x.

S5 Let I, J be non-overlapping small subarcs of $\partial U_r(x)$ such that $\gamma(r) \in I$ and $\sigma(r) \in J$. Then, there exists a positive number $\delta(I, J) \leq \varepsilon(x)$ such that if $x' \in B(x; \delta(I, J)) \cap R_r(x)$ then every element in $\Gamma(x')$ meets $I \cup J$.

The property S0 is a direct consequence of the triangle inequality and also S1 follows directly from the definition of a sector. Suppose that S2 does not hold. Then, there exists an open set V around x such that $V \cap R_r(x)$ does not contain any cut point to K. For any point $y \in V \cap R_r(x)$, each geodesic from y to K is properly contained in some geodesic to K which can be extended so as to pass through x. Therefore, $R_r(x)$ is simply covered by geodesics to K passing through x, a contradiction. Clearly S3 follows from S1. Property S4 follows from the fact that every geodesic in $\Gamma(x')$ does not pass through x. Property S5 is a direct consequence of S3.

If there exists no sector at $x \in C(K)$, then $U_r(x)$ is simply covered by $\Gamma(x)$ and the component of $C(K)$ containing x is a single point x. Such a cut point is not discussed.

It is clear that $x \in C(K)$ is an endpoint of $C(K)$ if and only if there is a unique sector at x. A point $x \in C(K)$ is a regular (respectively, branch) cut point to K if and only if there exist exactly two (respectively, more than two) sectors at x.

Basic Lemma. — *Let $R_r(x)$ be a sector at a point $x \in C(K)$. Then, for sufficiently small non-overlapping subarcs I and J of $\partial U_r(x)$ such that $\gamma(r) \in I$ and $\sigma(r) \in J$, there exists a point $x^* \in C(K)$ and a Jordan arc $m_R : [0,1] \to C(K) \cap \overline{R_r(x)} \cap B(x; \varepsilon(x))$ with the property that $m_R(0) = x$ and $m_R(1) = x^*$. Moreover, there exist* *at most countably many branch cut points on* $m_R[0,1]$. *If an interior point* $m_R(t)$ *is not a branch point, then there exist exactly two geodesics in* $\Gamma(m_R(t))$ *intersecting* I *and* J *respectively.*

Proof. We first note that if $\gamma = \sigma$, then I and J have a common endpoint at $\gamma(r) = \sigma(r)$. From S4 we find for every point $y \in B(x;\varepsilon(x)) \cap R_r(x) \cap C(K)$, a unique sector $R_{2r}(y;x)$ at y containing x. If

$$W_I := \{y \in B(x;\varepsilon(x)) \cap \overline{R_r(x)}; \text{there exists an element in } \Gamma(y) \text{ intersecting } I\}$$

and if

$$W_J := \{y \in B(x;\varepsilon(x)) \cap \overline{R_r(x)}; \text{there exists an element in } \Gamma(y) \text{ intersecting } J\},$$

then they are closed in X. Since $x \in C(K)$, every neighborhood $\mathcal{U}$ around x contains points in the interiors $\mathrm{Int}(W_I)$ of W_I and $\mathrm{Int}(W_J)$. Thus, $\mathcal{U}$ contains points on $W_I \cap W_J$, and hence $W_I \cap W_J$ is a nonempty closed set in X. Let $x^* \in W_I \cap W_J$ be chosen so as to satisfy that $R_r(x) \cap R_{2r}(x^*;x)$ is maximal in $W_I \cap W_J$. Namely, if $y \in W_I \cap W_J$, then $R_r(x) \cap R_{2r}(y;x) \subset R_r(x) \cap R_{2r}(x^*;x)$. It follows from Fact 1 that $\delta(I,J)$ tends to zero as I or J shrinks to a point. We may consider that I and J are taken to be $\partial U_r(x) \cap \overline{R_{2r}(x^*;x)} \cap \overline{R_r(x)} = I \cup J$. Setting for $y \in W_I \cap W_J$,

$$W(x;y) := R_r(x) \cap R_{2r}(y;x) \ ,$$

we observe that $W(x;y)$ for every $y \in W_I \cap W_J$ is divided by $W_I \cap W_J$, where $\mathrm{Int}(W_I)$, $\mathrm{Int}(W_J)$ and $W(x;x^*)$ are all disk domains.

We now prove that $W_I \cap W_J$ is a Jordan arc. To see this a continuous map $\bar{y} : J \to W_I \cap W_J$ joining x to x^* is constructed as follows. If $t \in J$ lies on a geodesic in $\Gamma(z)$ for some $z \in W_I \cap W_J$, then such a point is unique by Fact 1. We then define $\bar{y}(t) := z$. If $t_0 \in J$ is not on any geodesic in $\Gamma(z)$ for any $z \in W_I \cap W_J$, then there is a cut point z_0 to K with $z_0 \in W_J$ such that t_0 belongs to some geodesic in $\Gamma(z_0)$. Applying the discussion as developed in the last paragraph to the sector $R_{2r}(z_0;x)$ and two subarcs J_1, J_2 of J with $J_1 \cup J_2 = J \cap \overline{R_{2r}(z_0;x)}$, we find a point z_0^* in $W_I \cap W_J$ such that $R_{2r}(z_0;x) \cap R_{2r}(z_0^*;z_0)$ is maximal in $W_{J_1} \cap W_{J_2}$. Here $W_{J_i} :=$

$\{y \in R_{2r}(z_0; x);$ there exists an element of $\Gamma(y)$ intersecting $J_i\}$, $i = 1, 2$. Then, we define $\bar{y}(t_0) := z_0^*$. The continuity of $\bar{y} : J \to W_I \cap W_J$ is now clear. Choosing a suitable parameterization of J by a step function, we obtain a homeomorphic map $m_R : [0,1] \to W_I \cap W_J$. Similarly we obtain a continuous map $\hat{y} : I \to W_I \cap W_J$.

It follows by construction that a point $z \in W_I \cap W_J$ is a branch point if and only if $\bar{y}^{-1}(\{z\})$ or $\hat{y}^{-1}(\{z\})$ is a non-trivial subarc on $I \cup J$. If $z \neq w$ are branch points on m_R, then the corresponding open subarcs are disjoint on $I \cup J$. Therefore, m_R has at most countably many branch points. □

Corollary 1.1. — *Let $x \in C(K)$ and $x^* \in C(K) \cap B(x; \varepsilon(x))$ and $I, J \subset \partial U_r(x)$ be as in Basic Lemma. If $z \in C(K) \cap W(x; x^*)$, then there exists a unique Jordan arc joining z to some point on $m_R[0,1]$.*

2. CUT LOCUS AND SECTORS

As can be seen in the proof of the Basic Lemma, each sector at a cut point x to K contains a Jordan arc in $C(K)$. We shall assert that *every Jordan arc in $C(K)$ is obtained in the manner constructed in the Basic Lemma.* To see this, we fix an arbitrary given Jordan arc $c : [0,1] \to C(K)$. There exists for each $t \in [0,1]$ a small positive number $\delta = \delta(t)$ such that $c(t, t+\delta(t)]$ (respectively, $c[t-\delta(t), t))$ is contained entirely in a sector, say, $R_r^+(x)$ (respectively, $R_r^-(x)$) at $x := c(t)$ and such that $c[t-\delta, t+\delta] \subset B(x; \varepsilon(x))$. The first property follows from the fact that every geodesic in $\Gamma(x)$ does not meet $c([0,1])$ except at $x = c(t)$. Also, if $x^* \in C(K) \cap B(x; \varepsilon(x))$ is the point as obtained in the proof of Basic Lemma for $R_r(x) = R_r^+(x)$ and for non-overlapping subarcs on $\partial U_r(x) \cap R_r^+(x)$, then the resulting Jordan arc

$$m_{R_r^+(x)} : [0,1] \to C(K) \cap B(x; \varepsilon(x)) \cap \overline{R_r^+(x)}$$

joins x to x^* and has a small non-empty subarc around 0 which is contained entirely in $c([0,1])$. To see this, we recall that $C(K)$ does not contain a circle bounding a disk domain, see [P]. If otherwise supposed, then these two Jordan arcs meet only at their starting point x. Then, there is a geodesic from x to K lying in between these two arcs, a contradiction to S1. The same is true for $R_r^-(x)$ and $m_{R_r^-(x)}$. By iterating this procedure we prove the assertion. $\square$

Definition. — *A Jordan arc $m : [0,1] \to X$ is said to have left tangent (respectively right tangent) $v \in S_{m(c)}$ at $m(c)$ for $c \in (a,b]$ (respectively $c \in [a,b)$), iff the tangential direction $v_{m(c)m(t)}$ to any geodesic $m(c)m(t)$ converges to v in $S_{m(c)}$ as $t \to c-0$ (respectively $t \to c+0$).*

Lemma 2.1. — *Let $m : [0,1] \to C(K)$ be a Jordan arc. Then m has the right (respectively, left) tangent at $m(t)$ for all $t \in [0,1)$ (respectively, $t \in (0,1]$) and the right tangent (respectively, left tangent) bisects the sector $R_r^+(m(t))$ (respectively, $R_r^-(m(t))$).*

Proof. We only prove the statement for an arbitrary fixed point $x = m(t_0)$, $0 < t_0 < 1$. Let $\gamma^+, \sigma^+ \in \Gamma(x)$ bound the sector $R_r^+(x)$ and also $\gamma_t^-, \sigma_t^- \in \Gamma(m(t))$ for $t > t_0$ bound $R_r^-(m(t))$. Since x is an interior of m, $\gamma_t^\pm \neq \sigma_t^\pm$ for all $t \in (t_0, 1)$. Since $m[t_0, t_0+\delta_0]$ for a small $\delta_0 > 0$ coincides with $m_{R_r^+(x)}[0,1]$, for every $t \in (t_0, t_0+\delta_0]$ $\Gamma(m(t))$ contains two geodesics intersecting two subarcs of $R_r^+(x) \cap R_{2r}^-(m(t))$. The property S0 implies that both $\sigma_t^-(r)x$ and $\gamma_t^-(r)x$ are in $W(x; m(t))$. Clearly, $\lim_{t\to t_0+0}\sigma_t^-(r)x = \sigma^+[0,r]$ and $\lim_{t\to t_0+0}\gamma_t^-(r)x = \gamma^+[0,r]$. Assume that there is a sequence $\{m(t_i)\}$ with $\lim_{i\to\infty} t_i = t_0$ such that

$$\lim_{i\to\infty} \angle\sigma^+(r)xm(t_i) =: \theta\ , \qquad \lim_{i\to\infty} \angle\gamma^+(r)xm(t_i) =: \theta'\ .$$

From the triangle inequality we have

$$d(\sigma_{t_i}^-(r), m(t_i)) - d(\sigma_{t_i}^-(r), x) \le \rho\circ m(t_i) - \rho(x) \le d(\gamma^+(r), m(t_i)) - d(\gamma^+(r), x)\ .$$

Applying Fact 3 to both sides of the above relation,

$$-\cos\theta' \le \liminf_{i\to\infty} \frac{\rho\circ m(t_i) - \rho(x)}{d(m(t_i), x)} \le \limsup_{i\to\infty} \frac{\rho\circ m(t_i) - \rho(x)}{d(m(t_i), x)} \le -\cos\theta\ .$$

The above discussion being symmetric, we have $\theta = \theta'$.

It is left to prove the case where $m(0)$ (or $m(1)$) is an endpoint of $C(K)$ and the boundary of $R_r^+(m(0))$, (or $R_r^-(m(1))$) consists of a single geodesic. The proof in this case is clear from the above discussion. Thus, the proof is complete. □

Lemma 2.2. — *Let $m : [0,1] \to C(K)$ be a Jordan arc in a sector $R_r(x)$ at $x := m(0)$. Let 2θ be the inner angle of $R_r(x)$ and $2\theta^+(t), 2\theta^-(t)$ for $t \in (0,1)$ the inner angles of $R_r^+(m(t))$, $R_r^-(m(t))$ respectively. If $t_0 \in (0,1)$, then*

$$\lim{}_{t\to t_0+0}\theta^+(t) = \theta^+(t_0), \qquad \lim{}_{t\to t_0+0}\theta^-(t) = \pi - \theta^+(t_0)\ ,$$

and

$$\lim{}_{t\to 0+}\theta^+(t) = \theta, \qquad \lim{}_{t\to 0+}\theta^-(t) = \pi - \theta\ .$$

Moreover, $\theta^+(t)$ and $\theta^-(t)$ are continuous on the set of all regular points on m and

$$\lim{}_{t\to 0+}\angle\sigma_t^-(r)m(t)x = \lim{}_{t\to 0+}\angle\gamma_t^-(r)m(t)x = \pi - \theta\ .$$

Proof. Take any positive number ε and a point $z_0 \in R_r(x)$ which is not a cut point to $\{x\}$ such that

$$\theta - \varepsilon < \angle\sigma(r)xz_0, \qquad \theta - \varepsilon < \angle\gamma(r)xz_0\ .$$

Since $\lim_{t\to 0+}\sigma_t^+ = \sigma$ and $\lim_{t\to 0+}\gamma_t^+ = \gamma$, we see that $R_r^+(m(t))$ for sufficiently small t contains z_0. From $2\theta^+(t) \geq \angle\sigma_t^+(r)m(t)z_0 + \angle\gamma_t^+(r)m(t)z_0$ and from Fact 3,

$$\liminf{}_{t\to 0+}\angle\sigma_t^+(r)m(t)z_0 \geq \angle\sigma(r)xz_0 \geq \theta - \varepsilon\ ,$$
$$\liminf{}_{t\to 0+}\angle\gamma_t^+(r)m(t)z_0 \geq \angle\gamma(r)xz_0 \geq \theta - \varepsilon\ .$$

Thus, we have

$$\liminf{}_{t\to 0+}2\theta^+(t) \geq 2\theta - 2\varepsilon\ .$$

Since ε is taken arbitrary small, $\liminf_{t\to 0+}\theta^+(t) \geq \theta$. From $2\theta^+(t) + 2\theta^-(t) \leq 2\pi$ for all $t \in (0,1)$, the first part is proved by showing that $\liminf_{t\to 0+}\theta^-(t) \geq \pi - \theta$.

To see this inequality we apply the Toponogov comparison theorem to geodesic triangles $\Delta(xm(t)\sigma_t^-(r))$ and $\Delta(xm(t)\gamma_t^-(r))$ to obtain

$$\liminf_{t\to 0+} \angle\sigma_t^-(r)m(t)x \geq \pi - \theta \ ,$$
$$\liminf_{t\to 0+} \angle\gamma_t^-(r)m(t)x \geq \pi - \theta \ .$$

From Lemma 2.1 m bisects $R_r(x)$ at x and that

$$\lim_{t\to 0+} \angle\sigma_t^-(r)x\sigma(r) = \lim_{t\to 0+} \angle\gamma_t^-(r)x\gamma(r) = 0 \ .$$

Therefore, the desired inequality is obtained by

$$\liminf_{t\to 0+} 2\theta^-(t) \geq \liminf_{t\to 0+}\{\angle\sigma_t^-(r)m(t)x + \angle\gamma_t^-(r)m(t)x\} \geq 2(\pi - \theta) \ .$$

The rest is now clear from Basic Lemma. □

Lemma 2.3. — *Every Jordan arc $m : [0,1] \to C(K)$ is rectifiable.*

Proof. As is asserted in the beginning of this section m is expressed by a finite union of Jordan arcs as obtained in Basic Lemma. We only need to prove the rectifiability of an $m := m_R : [0,1] \to C(K)$ for an arbitrary fixed sector $R_r(x)$ at a cut point $x \in C(K)$.

The Toponogov comparison theorem implies that $\partial U_r(x)$ and $\partial U_{2r}(x)$ are rectifiable and hence J has a length $L(J)$. Also there exists for a sufficiently small positive number h a constant $c(k,r,h) > 0$ depending continuously on h such that if $\Delta = \Delta(uvw)$ is a narrow triangle with $d(u,v), d(u,w) \gg d(v,w)$, and if $v_1 \in uv$ and $w_1 \in uw$ satisfy $r-h \leq d(u,v_1), d(u,w_1) \leq r+h$ and $2r-h \leq d(u,v), d(u,w) \leq 2r+h$ and $d(u,v)/d(u,w), d(u,v_1)/d(u,w_1) \in (1-h, 1+h)$, then $d(v,w) \leq c(k,r,h)d(u_1,v_1)$.

For an arbitrary fixed small positive number δ, we define $A(\delta) \subset m([0,1])$ by

$$A(\delta) := \{m(t); \theta^{\pm}(t) \in (\delta, \pi - \delta)\} \ .$$

In view of Lemma 2.2 we can choose $0 = t_0 \leq t_1 \leq \cdots \leq t_{2N} = 1$ such that

$$A(\delta) \subset \bigcup_{i=1}^{N} W(m(t_{2i}); m(t_{2i+1})) \ ,$$

and

$$\angle m(t_{i+1})m(t_i)\sigma_{t_i}^+(2r), \angle m(t_i)m(t_{i+1})\sigma_{t_i}^+(2r) \in (2\delta, \pi - 2\delta)$$

for every $i = 1, ..., 2N$, and such that if $u_i := \sigma_{t_i}^+(2r), v_i' := J \cap \sigma_{t_i}^+([0, 2r])$ and if $w_i' := J \cap \sigma_{t_i}^+(2r)m(t_{i+1})$, then $d(u_i, v_i')/d(u_i, w_i') \in (1-h, 1+h)$. If $d(u_i, m(t_i)) \geq d(u_i, m(t_{i+1}))$, we then set $w_i := m(t_{i+1})$ and v_i on $\sigma_{t_i}^+$ such that $d(u_i, v_i) = d(u_i, w_i)$. If $d(u_i, m(t_i)) < d(u_i, m(t_{i+1}))$, we then set $v_i := m(t_i)$ and w_i on $m(t_{i+1})u_i$ such that $d(u_i, v_i) = d(u_i, w_i)$. Then, Fact 3 implies that

$$\limsup{}_{N\to\infty} \sum_{i=1}^{N} d(m(t_{2i}), m(t_{2i+1})) \leq \frac{\sum_{i=1}^{N} d(v_{2i}, w_{2i})}{\sin 2\delta} \leq \frac{c(k, r, h)L(J)}{\sin 2\delta} .$$

Clearly, each interior point $m(t)$ of m belongs to $A(\delta)$ for some $\delta > 0$. The above discussion shows that every open subarc of m is rectifiable.

If the inner angle of $R_r(x)$ at x is 2π, then the proof is immediate from Fact 3.

□

Now the critical points of ρ are discussed.

Proposition 2.4. — *Assume that $x \in C(K)$ does not admit a sector with inner angle π. Then, there exists a positive number $\varepsilon_1(x) \leq \varepsilon(x)$ with the following properties.*

If $\Sigma_r(x)$ is a sector at x with inner angle less than π, then there is a point $x^* \in \Sigma_r(x) \cap C(K) \cap B(x; \varepsilon(x))$ and $m_\Sigma : [\rho(x^*), \rho(x)] \to C(K)$ such that

(a) *$\rho^{-1}(\rho(x) - \varepsilon_1(x), \rho(x)] \cap W(x; x^*) =: D_1(x)$ is a disk domain and contains no critical point of ρ ;*

(b) *if $y \in W(x; x^*) \cap C(K) \cap B(x; \varepsilon_1(x))$, then there exists a Jordan arc $m : [\rho(y), t_0] \to C(K) \cap W(x; x^*) \cap B(x; \varepsilon_1(x))$ joining y to a point $m_\Sigma(t_0)$ such that $\rho \circ m(t) = t$ holds for every $t \in [\rho(y), t_0]$;*

(c) *for every $t \in (\rho(y), t_0]$, the sector $R_r^-(m(t))$ has its inner angle less than π, while $R_r^+(m(t))$ has its inner angle greater than π.*

If $\Lambda_r(x)$ is a sector with inner angle greater than π, then there is a point $x^* \in C(K) \cap \Lambda_r(x) \cap B(x; \varepsilon(x))$ and $m_\Lambda : [\rho(x), \rho(x^*)] \to C(K)$ such that (a') $W(x;x^*) \cap \rho^{-1}(\rho(x) - \varepsilon_1(x), \rho(x) + \varepsilon_1(x)) \cap B(x;\varepsilon_1(x)) = D_2(x)$ *is a disk domain and contains no critical point of* ρ *,*

(b') *if* $y \in W(x;x^*) \cap C(K) \cap B(x;\varepsilon_1(x))$*, then there exists a Jordan arc* $m : [\rho(y), t_0] \to C(K) \cap W(x;x^*)$ *joining* y *to a point* $m_\Lambda(t_0)$ *such that* $\rho \circ m(t) = t$ *for all* $t \in [\rho(y), t_0]$ *,*

(c') *for every* $t \in [\rho(y), t_0]$ *the sector* $R_r^+(m(t))$ *has inner angle greater than* π*, while* $R_r^-(m(t))$ *has inner angle less than* π *.*

Proof. Suppose that (a) does not hold for any $\varepsilon \in (0, \varepsilon(x)]$. Then, there is a sequence $\{q_j\}$ of critical points of ρ in $W(x;x^*) \cap B(x;\varepsilon)$ converging to x. There exists a positive number δ such that if $\Sigma_r(x)$ is any sector at x with inner angle less than π, then its inner angle is not greater than $\pi - \delta$. Let $\gamma_j, \sigma_j \in \Gamma(q_j)$ bound the sector $R_r(q_j;x)$. Since q_j is a critical point of ρ, we may consider that γ_j satisfies $\angle x q_j \gamma_j(r) \le \pi/2$. By applying Fact 3 to a triangle $\Delta(x q_j \gamma_j(r))$,

$$\begin{aligned}\rho(q_j) - \rho(x) &\le d(q_j, \gamma_j(r)) - d(x, \gamma_j(r)) \\ &= -d(x, q_j) \cos\, \min{}_{xq_j} \angle q_j x \gamma_j(r) + o(d(x, q_j)) \ ,\end{aligned}$$

and similarly (by using $\Delta(x q_j \gamma_j(r))$),

$$\begin{aligned}\rho(x) - \rho(q_j) &\le d(x, \gamma_j(r)) - d(q_j, \gamma_j(r)) \\ &= -d(x, q_j) \cos\, \min{}_{xq_j} \angle x q_j \gamma_j(r) + o(d(x, q_j)) \ .\end{aligned}$$

Thus, a contradiction is derived from $\min{}_{xq_j} \angle q_j x \gamma_j(r) \le (\pi - \delta)/2$. This proves (a).

Notice that the constant $\varepsilon_1(x)$ as obtained above does not depend on x but on the number δ bounding the inner angles less than π.

For the proof of (c) we assert that there is an open set U around x such that every point in U does not admit any sector with inner angle π. Suppose this is false. Then, there is a sequence $\{q_j\}$ of cut points converging to x such that q_j for every j admits a sector Π_j with inner angle π. Since q_j is a critical point of ρ, the above argument shows that there exists a sector $R_r(x)$ at x with inner angle greater than π such that almost all q_j's are contained in it. Suppose x is not a singular point of X. Then, the equality in Fact 2 holds at x, and S3 implies that the inner angle of $R_r(x)$ is the limit of those of Π_j, a contradiction.

Let $x \in \mathrm{Sing}(X)$. By means of the Basic Lemma, all of q_j's but a finite number are on $m_R[0,1]$. In fact, if an infinite subsequence $\{q_k\}$ of $\{q_j\}$ are contained in W_J (or in W_I), then there is a sequence $t_k \in (0,1)$ with $\lim t_k = 0$ such that $m_R(t_k)$ admits a sector Σ_k with inner angle less than π such that $q_k \in \Sigma_k$ is a critical point of ρ and $\lim d(m_R(t_k), q_k) = 0$. This contradicts to (a).

We therefore have either $R_r^+(m(t_j)) = \Pi_j$ or else $R_r^-(m(t_j)) = \Pi_j$. If there is an infinite sequence with $R_r^+(m(t_j)) = \Pi_j$, then Lemma 2.2 implies that the limit of their inner angles is π, a contradiction. If there is an infinite sequence with $R_r^-(m(t_j)) = \Pi_j$, then Lemma 2.2 derives a contradiction that $R_r(x)$ has its inner angle less than π.

We find an $\varepsilon_1(x) \in (0, \varepsilon(x)]$ such that $B(x; \varepsilon_1(x))$ contains no critical point of ρ and there is no point on $B(x; \varepsilon_1(x))$ admitting a sector with inner angle π. This proves (c).

For the proof of (b), the Jordan arc is obtained as in the Basic Lemma. We see from (a) that there is no critical point on this arc, and the derivative $(\rho \circ m)'(t)$ does not vanish. Therefore, m intersects each level of ρ at a unique point, and is parameterized by ρ.

Because for every $t \in [0,1)$ $R_r^-(m_\Lambda(t))$ has inner angle less than π, the proof of the rest part follows from (a), (b) and (c). □

The following proposition is analogously proved. The proof is omitted.

Proposition 2.5. — *Assume that x admits a sector $\Pi_r(x)$ with inner angle π. Then, there exists a rectifiable Jordan arc $m_\Pi : [0,1] \to C(K)$ emanating from x in $\Pi_r(x) \cap B(x; \varepsilon(x))$ and a positive number $\varepsilon_2(x)$ satisfying the following properties*

(1) *$\rho(m_\Pi(t)) \in [\rho(x) - \varepsilon_2(x), \rho(x) + \varepsilon_2(x)]$ holds for each $t \in [0,1]$,*

(2) *there is no critical point of ρ in $D_3(x) := \Pi_r(x) \cap R_r^-(m_\Pi(t)) \cap \rho^{-1}[\rho(x) - \varepsilon_2(x), \rho(x) + \varepsilon_2(x)]$ except possibly the points on m_Π,*

(3) *if $y \in D_3(x) \cap C(K)$, then there is a Jordan arc $m : [\rho(y), t_0] \to C(K) \cap W(x; x^*)$ such that $m(t_0) \in m_\Pi[0,1]$ and $\rho \circ m(s) = s$ for all $s \in [\rho(y), t_0]$.*

Proof of Theorem A(1). Let x be a cut point to K and V any neighborhood around x. We shall construct a tree $T(x)$ such that $T(x) \subset V$ and such that $T(x)$ is a neighborhood of x in $C(K)$, i.e., $T(x)$ is a tree neighborhood around x in $C(K)$ such that $T(x) \subset V$.

If every sector at x has its inner angle less than π, we then take a positive number $\varepsilon_1(x)$ so as to satisfy $\varepsilon_1 \in (0, \varepsilon_1(x))$ and $\rho^{-1}(\rho(x) - \varepsilon_1, \rho(x)] \subset V$. Since $\rho(x)$ is a local maximum, $\rho^{-1}(\rho(x) - \varepsilon_1, \rho(x)]$ is open. Let $T(x)$ be the set of all cut points in $\rho^{-1}(\rho(x) - \varepsilon_1, \rho(x)]$. It follows from Proposition 2.4 that any cut point in $T(x)$ is connected to x by a unique rectifiable arc in $\rho^{-1}(\rho(x) - \varepsilon_1, \rho(x)] \cap C(K)$, hence in $T(x)$. Thus, $T(x)$ is a tree neighborhood around x with $T(x) \subset V$.

Suppose that x admits a sector $\Lambda_r(x)$ with inner angle greater than π. Take a rectifiable Jordan arc m_Λ and a positive number $\varepsilon_1(x)$ as in Proposition 2.4. Choose a positive number $\varepsilon_2 \leq \varepsilon_1(x)$ so as to satisfy

$$R_r^-(m_\Lambda(\rho(x) + \varepsilon_2)) \cap \rho^{-1}[\rho(x) - \varepsilon_2, \rho(x) + \varepsilon_2] \subset V \ .$$

We then define a tree neighborhood $T(x)$ around x by the set of all cut points in $R_r^-(m_\Lambda(\rho(x) + \varepsilon_2)) \cap \rho^{-1}[\rho(x) - \varepsilon_2, \rho(x) + \varepsilon_2]$. If $x' \in T(x)$ lies in a sector at x with inner angle less than π, then Proposition 2.4 implies that x' is joined to x by a rectifiable Jordan arc in $T(x)$. If $x' \in T(x)$ lies in $\Lambda_r(x)$, then x' lies in the arc m_Λ or in a sector at a point on m_Λ with inner angle less than π. Hence, by Proposition 2.4, x' can be joined to a point on m_Λ by a rectifiable Jordan arc in $T(x)$. Therefore any $x' \in T(x)$ is joined to x by a rectifiable Jordan arc in $T(x)$.

For a cut point x admitting a sector with inner angle π, the construction of $T(x)$ is left to the reader, since it is similar by making use of Propositions 2.4 and 2.5. This proves Theorem A(1). □

By means of Theorem A(1), we can introduce an interior metric δ on $C(K)$ as follows. If $p, q \in C_0(K)$ are in a component $C_0(K)$ of $C(K)$, we then define

$$\delta(p, q) := \inf\{L(c); c \text{ is a rectifiable arc in } C_0(K) \text{ joining } p \text{ and } q\} \ ,$$

and also, if p, q are not in the same component of $C(K)$,

$$\delta(p, q) := +\infty \ .$$

Proof of Theorem A(2). Let $C_0(K)$ be a component of $C(K)$ and $x \in C_0(K)$. Since $d \leq \delta$ on $C(K)$, we only need to prove that $\lim_{n\to\infty}\delta(x_n, x) = 0$ for any sequence $\{x_n\}$ of points in $C_0(K)$ with $\lim_{n\to\infty}d(x_n, x) = 0$. Suppose that there exists a sequence $\{x_n\}$ of points in $C_0(K)$ and a positive number η such that $\delta(x_n, x) \geq \eta$ for any n and $\lim_{n\to\infty}d(x_n, x) = 0$. If $x_n \in B(x; \varepsilon_1(x))$ lies in a sector at x with inner angle less than π, then Lemma 2.1 and Fact 3 imply that $(\rho \circ c)'(s) \geq \sin\frac{\delta}{2}$ for almost all $s \in [0, \delta(x_n, x)]$. Here, $\pi - \delta$ denotes the maximal inner angle of all sectors at x with inner angles less than π, and $c : [0, \delta(x_n, x)] \to C_0(K)$ the minimizing curve joining x_n to x parameterized by arclength. By integrating the inequality, we get

$$\delta(x_n, x) \leq \frac{\rho(x) - \rho(x_n)}{\sin\frac{\delta}{2}} \leq \frac{d(x_n, x)}{\sin\frac{\delta}{2}} \ .$$

From what we have supposed, $\delta(x_n, x) \geq \eta$ for any n we see that the sequence $\{x_n\}$ contains an infinite subsequence all of whose members lie in a sector $R_r(x)$ whose inner angle is not less than π. Without loss of generality, we may assume that for all x_n,

$$\delta(x_n, x) \geq \eta, \qquad x_n \in R_r(x) \cap B(x; \varepsilon_2(x)) \ .$$

Take a rectifiable Jordan arc m_R emanating from x in $R_r(x)$ as obtained in the Basic Lemma. Hence x_n is connected to a point x'_n on m_R and x_n lies in a sector at x'_n with inner angle less than π. Since $C_0(K)$ is a local tree, $\lim_{n\to\infty}d(x'_n, x) = 0$, and hence $\lim_{n\to\infty}d(x_n, x'_n) = 0$ by the triangle inequality. Thus

$$\lim{}_{n\to\infty}\delta(x_n, x'_n) \leq \lim{}_{n\to\infty}\frac{d(x_n, x'_n)}{\sin\frac{\delta}{2}} = 0$$

for some positive $\delta < \pi$.

On the other hand $\lim_{n\to\infty}\delta(x, x'_n) = 0$, since m is rectifiable and $C_0(K)$ is a local tree. Therefore we have

$$\lim{}_{n\to\infty}\delta(x_n, x) \leq \lim{}_{n\to\infty}\delta(x, x'_n) + \lim{}_{n\to\infty}\delta(x'_n, x_n) = 0 \ .$$

This contradicts $\delta(x, x_n) \geq \eta$ for all n. □

Definition. — *A rectifiable Jordan arc in $C(K)$ is called a path iff it is parameterized by arclength. A path $m : [a, b] \to C(K)$ is called a main path iff for each $t \in (a, b)$ the* *inner angle of $R_r^+(m(t))$ at $m(t)$ is maximal of the inner angles of all sectors at $m(t)$ except $R_r^-(m(t))$.*

Proof of Theorem A(3),(4). Let $x \in C_0(K)$. Let $D(x)$ be an open set of x as constructed in the proof of Theorem A(1) such that $D(x) \cap C_0(K)$ is a tree neighborhood in $C_0(K)$ around x. Since X satisfies the second countability axiom, $C_0(K)$ is covered by a countable union $\bigcup_{i=1}^{\infty} D(x_i)$ for $x_i \in C_0(K)$. Thus, $C(K)$ is the union of a countable tree neighborhoods. From this fact we only need to prove that $T(x) \setminus C_e(K)$ for a tree neighborhood $T(x)$ of a cut point x is covered by a countable union of paths in $C_0(K)$. In each sector at x we choose a main path which is maximal in $T(x)$. Let A_1 be the set of all such paths in $T(x)$. Notice that the existence of such a main path is clear from Lemma 2.2. Because there exists at most countable sectors at x, A_1 is a countable set. Each sector at $y \in \bigcup_{I \in A_1} I =: |A_1|$, which does not contain any point on $|A_1|$, corresponds to an open subarc $J(y) \subset \partial U_r(x)$. The $J(y)$ is cut off by two elements in $\Gamma(y)$ and has the property that if y' is another such point not lying on the same main path, then $J(y') \cap J(y) = \emptyset$. Hence, there exist at most countably many points $y \in |A_1|$ admitting a sector which does not contain any point of $|A_1|$. In each sector at each point $y \in |A_1|$, which does not contain any point of $|A_1|$, choose a main path emanating from y which is maximal in $T(x) \setminus |A_1|$. If A_2 denotes the set of all such paths, then A_2 is also countable. Define a sequence of countable sets $A_1, A_2, \ldots$ inductively. If $T(x) = |A_1| \cup \ldots \cup |A_k|$ for some finite integer k, then $T(x)$ is clearly covered by countable paths. If the sequence $\{A_i\}$ is infinite, we shall prove that $T(x) \setminus C_e(K)$ is covered by $\cup_{i=1}^{\infty} |A_i|$. Suppose that there exists a point $q \in T(x) \setminus C_e(K)$ such that $q \notin |A_i|$ for any i. Let $c : [0, \delta(x,q)] \to T(x)$ be the unique path joining x to q. Let $R(x)$ be the sector at x containing $c(0, \delta(x,q)]$ and $I_1 \in A_1$ the curve in $R(x)$. Since c and I_1 lie in the same sector, there exists a positive number a_1 such that $c[0, a_1]$ is the intersection of $c[0, \delta(x,q)]$ and I_1. Let $R(c(a_1))$ be the sector at $c(a_1)$ containing $c(a_1, \delta(x,q))$ and $I_2 \in A_2$ the curve in $R(c(a_1))$ emanating from $c(a_1)$. Then, there exists a positive number a_2 such that $c[a_1, a_1 + a_2]$ is the intersection of $c[a_1, \delta(x,q)]$ and I_2. Since $q \notin |A_i|$ for any i, we get an infinite sequence $\{a_i\}$ of positive numbers with $\sum_{i=1}^{\infty} a_i \leq \delta(x,q)$. If $\tilde{R}(x_k)$, where $x_k = c(\sum_{i=1}^{k} a_i)$, denotes the sector at x_k containing $I_k \setminus \{x_k\}$, then it follows from Lemma 2.2 that the inner angles of $\tilde{R}(x_k)$ and $R(x_k)$ at x_k tend to zero and a positive number respectively as $k \to \infty$. But this contradicts the assumption that each I_k is a main path emanating from x_k lying in $\tilde{R}(x_k)$. $\square$

3. GEODESIC SPHERES ABOUT K

Let $C_{ea}(K) := \{x \in C_e(K); \sharp\Gamma(x) > 1\}$. We observe from the proof of Theorem A(3) that $C_{ea}(K)$ is countable. As is seen in Example 4, the set $C_e(K) \setminus C_{ea}(K)$ has the special property that there is no reason to distinguish it from points which are not on $C(K)$. This is caused by the lack of differentiability. A sufficient condition for $C_e(K) \setminus C_{ea}(K)$ to be uncountable is stated as Proposition 3.1.

Proposition 3.1. — *Let X be a simply connected Alexandrov surface with curvature bounded below and $\{p_i\}$ a sequence of cut points to a point p such that each $\Gamma(p_i)$ consists of a single element. If any subsequence of the sequence does not converge to p and if the set of all tangential directions v_{pp_i} of geodesics pp_i is a dense subset in an open subarc J of S_p, then there exist uncountably many cut points to p which are endpoints.*

Proof. Notice that X is homeomorphic to a 2-sphere or Euclidean plane, since X is simply connected. Hence, for each cut point x of p, each connected component of $X \setminus \Gamma(x)$ is bounded by two geodesics joining x to p. Each component of $X \setminus \Gamma(x)$ shall be called a (global) sector at x. Let A be the set of all monotone decreasing sequences $\{\Sigma(x_i)\}$ of bounded sectors at cut points x_i of the point p such that the inner angle of $\Sigma(x_i)$ at x_i tends to zero as i goes to infinity. It is trivial that the sequence $\{x_i\}$ converges to a unique cut point of p, which is an endpoint of $C(p)$. If for two elements $\{\Sigma(x_i)\}$, $\{\Sigma(y_i)\}$ of A, there exists an integer N such that $\Sigma(x_i)$ and $\Sigma(y_j)$ are disjoint for any $i, j \geq N$, then we shall say that $\{\Sigma(x_i)\}$ and $\{\Sigma(y_i)\}$ are *nonequivalent*. Notice that $\lim_{i\to\infty} x_i$ and $\lim_{i\to\infty} y_i$ are distinct if $\{\Sigma(x_i)\}$ and $\{\Sigma(y_i)\}$ are nonequivalent. Let $\{\epsilon_i\}$ be any sequence of positive numbers with $\lim_{i\to\infty} \epsilon_i = 0$. For the cut point p_1 of p, there exists a cut point x_1 of p and a sector $\Sigma(x_1)$ at x_1 containing p_1 such that the subarc of S_p determined by $\Sigma(x_1)$ is a subarc of J and that the inner angle of $\Sigma(x_1)$ at x_1 is less than π. Since the set $\{v_{pp_i}; i = 1, 2, 3...\}$ is dense in J, there exist at least two distinct cut points p_i,p_j in $\Sigma(x_1)$. Hence, there exist two cut points $q(1)$,$q(2)$ of p and two disjoint sectors $\Sigma(q(1))$,$\Sigma(q(2))$ lying in $\Sigma(x_1)$ such that the inner angle of $\Sigma(q(i))$ at $q(i)$ (i=1,2) is less than ϵ_1. One can inductively define cut points $q(i_1, ..., i_n)$ of p and sectors $\Sigma(q(i_1, ..., i_n))$ so as to satisfy that $\Sigma(q(i_1, ..., i_n))$ is contained in $\Sigma(q(i_1, ..., i_{n-1}))$, the two sectors $\Sigma(q(i_1, ..., i_{n-1}, 1))$ and $\Sigma(q(i_1, ..., i_{n-1}, 2))$ are disjoint and the inner angle of $\Sigma(q(i_1, ..., i_n))$ at $q(i_1, ..., i_n)$ is less than ε_n. This implies that the set A contains uncountably many nonequivalent elements. In particular, there exist uncountably many cut points of p which are endpoints. $\square$

We shall prove Theorems B and C. The set $\mathcal{E} \subset (0, \infty)$ of all exceptional values for geodesic spheres is defined by

$$\mathcal{E} := \rho(\mathrm{Sing}(\rho)) \cup \rho(\mathrm{Crit}(\rho)) \cup \rho\{\text{ the set of all branch cut points to } K\} \cup \rho(C_{ea}(K)) \,.$$

First of all we prove that $\mathcal{E}$ is of Lebesgue measure zero. As stated in Section 1, $\mathrm{Sing}(X)$ is countable, and so is $\rho(\mathrm{Sing}(X))$. Also we observe from Theorem A(3) that the set of all branch cut points is countable.

In connection with the critical points of ρ and sectors at branch and regular cut points to K, we classify cut points into six sets as follows

$$C_{BIC}(K) := \{x \in C(K) \setminus \mathrm{Sing}(X); \sharp\Gamma(x) > 2, \text{every sector at } x \text{ has inner angle less than } \pi\} \quad \text{(branch-isolated critical)},$$

$$C_{RNC}(K) := \{x \in C(K) \setminus \mathrm{Sing}(X); \sharp\Gamma(x) = 2,\ x \text{ admits a sector with inner angle greater than } \pi\} \quad \text{(regular-noncritical)},$$

$$C_{BNC}(K) := \{x \in C(K) \setminus \mathrm{Sing}(X); \sharp\Gamma(x) > 2,\ x \text{ admits a sector with inner angle greater than } \pi\} \quad \text{(branch-noncritical)},$$

$$C_{RC}(K) := \{x \in C(K) \setminus \mathrm{Sing}(X); \sharp\Gamma(x) = 2, \text{ and the two sectors have the same inner angle } \pi\} \quad \text{(regular-critical)},$$

$$C_{BC}(K) := \{x \in C(K) \setminus \mathrm{Sing}(X); \sharp\Gamma(x) > 2, \ x \text{ admits a sector with inner angle } \pi\} \quad \text{(branch-critical)},$$

$$C_{IC}(K) := \{x \in C(K) \setminus \mathrm{Sing}(X); \text{ there exists no sector at that point}\} \quad \text{(isolated critical)}.$$

In order to prove that the set $\rho(\mathrm{Crit}(\rho))$ is of Lebesgue measure zero, we need a generalized Sard theorem for continuous functions.

Lemma 3.2. — *Let $f : [0,1] \to R$ be a continuous function. If*

$$A := \{t \in [0,1] \,;\, f'(t) \text{ exists and equals } 0\} \ ,$$

then $f(A)$ is of Lebesgue measure zero.

Proof. For an arbitrary fixed $\varepsilon > 0$ and for each positive integer n, we set

$$A_n^\varepsilon := \{\, t \in [0,1] \,;\, \sup{}_{|h| \le \frac{1}{n}} |\frac{f(t+h) - f(t)}{h}| \le \varepsilon\} \ .$$

The continuity of f implies that A_n^ε is closed, hence measurable. Clearly, A_n^ε is increasing in n. Setting $A^\varepsilon := \bigcup_{n \ge 1} A_n^\varepsilon$, we observe $A \subset A^\varepsilon$ for any $\varepsilon > 0$. We now fix an $\varepsilon > 0$ and an n and set

$$\Gamma_k := A_n^\varepsilon \cap (\frac{k-1}{n}, \frac{k}{n})$$

for $1 \le k \le n$. Then, $f(\Gamma_k) \subset [\inf_{t \in \Gamma_k} f(t), \sup_{t \in \Gamma_k} f(t)]$, and hence

$$\mu(f(\Gamma_k)) \le \sup{}_{t,t' \in \Gamma_k} |f(t) - f(t')| \le \frac{\varepsilon}{n} \ ,$$

where μ is the Lebesgue measure on $\mathbb{R}$. Thus we have

$$\mu(f(A_n^\varepsilon)) \le \sum_{k=1}^{n} \mu(f(\Gamma_k)) \le \varepsilon \ ,$$

and in particular

$$\mu(f(A)) \leq \mu(f(A^{\varepsilon})) = \lim{}_{n\to\infty}\mu(f(A_n^{\varepsilon})) \leq \varepsilon \ .$$

This proves Lemma 3.2. □

Proof of Theorems B and C. Clearly the set $\mathrm{Crit}(\rho)$ is closed, and hence so is $\rho(\mathrm{Crit}(\rho))$. Using above notations we observe that

$$\mathrm{Crit}(\rho) \subset \mathrm{Sing}(X) \cup C_{BIC}(K) \cup C_{RC}(K) \cup C_{IC}(K) \ .$$

In order to prove that $\mathcal{E}$ is of Lebesgue measure zero, we only need to check that $\rho(\mathrm{Crit}(\rho))$ is of measure zero. If $x \in C_{RC}(K)$, then there exists a path $m : [-a, a] \to C(K)$ for a sufficiently small $a > 0$ with $m(0) = x$ such that $(\rho \circ c)'(0) = 0$. By means of Theorem A(3) there are at most countably many paths in the interior of which $C_{RC}(K)$ is contained. Then, Lemma 3.2 implies that $\mathcal{E}$ is of measure zero. This also proves Theorem C.

For the proof of Theorem B(1) we suppose that $\mathcal{S}(t)$ has infinitely many components, say $\mathcal{S}_1(t), \mathcal{S}_2(t), \mathcal{S}_3(t), \ldots$ for some $t \in (0,\infty) \setminus \mathcal{E}$. Take any point x_i on the component $\mathcal{S}_i(t)$ for each i. Since $\mathcal{S}(t)$ is compact, by choosing a subsequence if necessary, we may assume that the sequence $\{x_i\}$ converges to a point x of a component $\mathcal{S}_\infty(t)$ of $\mathcal{S}(t)$ and that any x_i does not lie on $\mathcal{S}_\infty(t)$. Choose a disk domain $U(x)$ containing x so as to satisfy that some proper subarc I_0 of $\mathcal{S}_\infty(t)$ divides $U(x)$ into two components U_+, U_-. Let I_1 be a proper subarc of I_0 containing x and let I_+(respectively I_-) denote the set of all points y in I_1 such that there exists a geodesic from y to K passing through U_+ (respectively U_-). Since x is not a critical point of ρ, the intersection of I_+ and I_- is empty. Furthermore, I_+ and I_- are both closed in I_1 and $I_+ \cup I_- = I_1$. Thus either I_+ or I_- is empty. Therefore, we may assume I_+ is empty. Take any sufficiently small positive ϵ and any points y, z in I_0 sufficiently close to x such that the subarc I_{yz} of I_0 with endpoints y, z contains x in its interior. Then, a disk domain D in U_- is bounded by I_{yz}, a subarc of $\mathcal{S}(t-\epsilon)$ and subarcs of geodesics in $\Gamma(y), \Gamma(z)$. Since I_+ is empty, x_i for all sufficiently large i is contained in D. Hence, $\mathcal{S}_i(t)$ is contained in D for all sufficiently large i. Therefore, there exists a sequence of local maximal points with respect to ρ which converges to x. Since a local maximal point is a critical point of ρ, x is a critical point of ρ. This contradicts the assumption that $t = \rho(x)$ is not a critical value of ρ.

The proofs of Theorem B(3) and (4) are clear from the definition of $\mathcal{E}$.

Let t be a positive number in $(0,\infty) \setminus \mathcal{E}$. It suffices for the proof of (2) to show that every $x \in \mathcal{S}(t)$ has a sufficiently small open subarc of $\mathcal{S}(t)$ which is rectifiable. If x is such a point that has a unique geodesic in $\Gamma(x)$, then it is orthogonal to $\mathcal{S}(t)$. We conclude the proof in the same manner as in the proof of Lemma 2.3. $\square$

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