GEOMETRY OF TOTAL CURVATURE

Takashi SHIOYA

Graduate School of Mathematics
Kyushu University
Fukuoka 812-81 (Japan)

Abstract. This is a survey article on geometry of total curvature of complete open 2-dimensional Riemannian manifolds, which was first studied by Cohn-Vossen ([Col, Co2]) and on which after that much progress was made. The article consists of three topics : the ideal boundary, the mass of rays, and the behaviour of distant maximal geodesics.

Résumé. Cet article présente une synthèse sur la géométrie de la courbure totale des surfaces riemanniennes ouvertes, qui fut d'abord étudiée par Cohn-Vossen ([Co1, Co2]), et à propos de laquelle de grands progrès ont été faits ensuite. L'article couvre trois sujets : le bord idéal, la masse des rayons, et le comportement des géodésiques maximales à l'infini.

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TABLE OF CONTENTS

INTRODUCTION 563

1. THE IDEAL BOUNDARY WITH GENERALIZED TITS METRIC 567
   1. The construction and basic properties 567
   2. Relation between geodesic circles and the Tits metric 573
   3. Global and asymptotic behaviour of Busemann functions 575
   4. Angle metric and Tits metric 577
   5. The control of critical points of Busemann functions 579
   6. Generalized visibility surfaces 580

2. THE MASS OF RAYS 581
   1. Basics 581
   2. The asymptotic behaviour and the mean measure of rays 584

3. THE BEHAVIOUR OF DISTANT MAXIMAL GEODESICS 587
   1. Visual diameter of any compact set looked at from a distant point 587
   2. The shapes of plane curves 589
   3. Maximal geodesics in strict Riemannian planes 591
   4. Generalization to finitely connected surfaces 596

BIBLIOGRAPHY 597

INTRODUCTION

The total curvature of a closed Riemannian 2-manifold is determined only by the topology of the manifold. On the other hand, that of a complete open Riemannian 2-manifold is not a topological invariant but depends on the metric. The geometric meaning of the total curvature is an interesting subject. In this article, we survey some of our own results concerning the relations between the total curvature $c(M)$ of M and various geometric properties of M when M is a finitely connected, complete, open and oriented Riemannian 2-manifold.

Gromov [BGS] first defined the ideal boundary and its Tits metric for an n-dimensional Hadamard manifold as the set of equivalence classes of rays with respect to the asymptotic relation and investigated its geometric properties. This turns out to be useful in studying nonpositively curved n-manifolds. Here, the nonpositiveness of the sectional curvature implies that the asymptotic relation, which is originally due to Busemann [Bu], becomes an equivalence relation. However this is not true in general. The emphasis of the present article is that the ideal boundary together with the Tits metric can be constructed for M by a new equivalence relation between rays by using the total curvature. In particular, our construction is a natural generalization of that of Gromov, because both coincide on every Hadamard 2-manifold. It is natural to ask the influence of our Tits metric on the ideal boundary upon the geometric properties of M. The Tits metric defined here can be precisely described in terms of the total curvature of M, which plays an essential role throughout this article.

In Chapter 1, we construct the ideal boundary of M and its generalized Tits metric. For the Euclidean plane, the Tits distance between two points represented by two rays emanating from a common point is just the angle between the initial vectors of these rays. In the general case, we have various geometric properties on the analogy with the Euclidean case. All these properties are connected with the asymptotic behaviour. We apply these to the study of the detailed behaviour of

Busemann functions.

In Chapter 2, we investigate on the mass of rays in M. We view this as the Lebesgue measure $\mathfrak{M}(A_p)$ of the set A_p of all unit vectors which are initial vectors of rays emanating from a point p in M. A pioneering work of Maeda ([Md1], [Md2]) states that the infimum of $\mathfrak{M}(A_p)$ for all $p \in M$ is equal to $2\pi - c(M)$ provided M is a nonnegatively curved Riemannian plane (i.e., a complete nonnegatively curved manifold homeomorphic to $\mathbb{R}^2$). We investigate the asymptotic behaviour of the measure $\mathfrak{M}(A_p)$ for a general M with total curvature as p tends to infinity and the mean of $\mathfrak{M}(A_p)$ with respect to the volume of M.

In Chapter 3, we study the behaviour of maximal geodesics close enough to infinity (i.e., outside a large compact set) in a complete 2-manifold homeomorphic to $\mathbb{R}^2$ with total curvature less than 2π. Such manifolds will be called strict Riemannian planes. Any such maximal geodesic becomes proper as a map of $\mathbb{R}$ into M and has almost the same shape as that of a maximal geodesic in a flat cone. Moreover, we give an estimate for its rotation number and show that it is close to $\pi/(2\pi - c(M))$. Here, we have extended the notion of the rotation number of a closed curve due to Whitney [Wh] to that of a proper curve.

Basic concepts

The total curvature $c(M)$ of an oriented Riemannian 2-manifold M is defined to be the possibly improper integral $\int_M G\,dM$ of the Gaussian curvature G of M with respect to the volume element dM of M. We define the total positive curvature $c_+(M)$ and the total negative curvature $c_-(M)$ by $c_\pm(M) := \int_M G_\pm\,dM$, where $G_-+(p) := \max\{G(p), 0\}$ and $G_--(p) := \max\{-G(p), 0\}$ for $p \in M$. Then, the total curvature $c(M)$ exists if and only if at least one of $c_+(M)$ or $c_-(M)$ is finite. A well-known theorem due to Cohn-Vossen [Co1] states that if M is finitely connected and admits total curvature, then $c(M) \leq 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. When M is infinitely connected and admits total curvature, Huber's theorem [Hu] (cf. [Ba1]) states that $c(M) = -\infty$. Therefore, the total curvature exists if and only if the total positive curvature is finite.

Throughout this article, assume that M is a finitely connected, complete, open and oriented Riemannian 2-manifold admitting total curvature and that all geodesics of M are normal. The finite connectivity of M implies that there exists a homeomorphism $\varphi : M \to N - E$, where N is a closed and oriented 2-manifold and E is a finite subset of N. We call each point in E an *endpoint* of M. For instance, if M is a *Riemannian plane* (i.e., a complete Riemannian 2-manifold homeomorphic to $\mathbb{R}^2$), then N is homeomorphic to S^2 and E consists of a single point in N. A subset U of M is called a *neighbourhood of an endpoint* $e \in E$ if $\varphi(U) \cup \{e\}$ is a neighbourhood of e in N. For each endpoint e of M, we denote by $\mathfrak{U}(e)$ the set of all neighbourhoods of e which are diffeomorphic to closed half-cylinders with smooth boundary. Following Busemann [Bu], we call an element of $\mathfrak{U}(e)$ a *tube* of M.

For any region D of M with piecewise smooth boundary ∂D parameterized positively relative to D, we define the *total geodesic curvature* $\kappa(D)$ by the sum of the integrals of the geodesic curvature of ∂D together with the exterior angles of D at all vertices. Here, we allow $\kappa(D)$ to be infinite. When $\partial D = \phi$ (i.e., $D = M$), we set $\kappa(D) := 0$. The Gauss-Bonnet theorem states that if a region D has piecewise smooth boundary and is compact and finitely connected, then

$$\kappa(D) + c(D) = 2\pi\chi(D) \ .$$

For any region D of M admitting $\kappa(D) + c(D)$ (i.e., so that $\kappa(D)$ and $c(D)$ exist and if both $\kappa(D)$ and $c(D)$ are infinite, they have the same sign), we define

$$\kappa_\infty(D) := 2\pi\chi(D) - \kappa(D) - c(D) \ .$$

A slight generalization of Cohn-Vossen's theorem (cf. [Co2], [Sy5]) states that

$$\kappa_\infty(D) \geq \pi\chi(\partial D) \ ,$$

where $\chi(\partial D)$ is the Euler characteristic of ∂D, namely the number of connected components of ∂D which is homeomorphic to $\mathbb{R}$.

Geometrically, $\kappa_\infty(D)$ may be thought of as the total geodesic curvature of the boundary at infinity of D. This is seen as follows. Let $\{D_j\}$ be a monotone increasing sequence of compact regions with piecewise smooth boundary such that $\cup D_j = D$ and that the inclusion map from each D_j into D is a strong deformation retraction. Since $\chi(D_j) = \chi(D)$ for all j and $\lim_{j\to\infty} c(D_j) = c(D)$, the Gauss-Bonnet theorem implies that

$$\kappa_\infty(D) = 2\pi\chi(D) - \kappa(D) - \lim_{j\to\infty} c(D_j) = \lim_{j\to\infty} \kappa(D_j) - \kappa(D) \ .$$

Assume for convenience that D is closed and $\kappa_\infty(D) < +\infty$. Then, the total geodesic curvature of D supported by $\partial D - \partial D_j$ tends to zero. Thus, $\kappa_\infty(D)$ is equal to the limit of the total geodesic curvature of D_j supported by $\partial D_j - \partial D$.

We set

$$\kappa_\infty(e) := \kappa_\infty(U)$$

for each endpoint e of M and a tube $U \in \mathfrak{U}(e)$. $\kappa_\infty(e)$ is independent of the choice of U and satisfies

$$\sum_{e\in E} \kappa_\infty(e) = \kappa_\infty(M) \ .$$

After Bangert [Ba3], the quantity $\kappa_\infty(e)$ is called the *curvature deficit* for the endpoint e of M. We call $\kappa_\infty(M) = 2\pi\chi(M) - c(M)$ the *total deficit* of M. Considering an isometric embedding of a tube $U \in \mathfrak{U}(e)$ into a Riemannian plane M_U, we have $\kappa_\infty(e) = \kappa_\infty(M_U)$ by the Gauss-Bonnet theorem. Then, Cohn-Vossen's theorem implies that $0 \le \kappa_\infty(M) \le +\infty$ and $0 \le \kappa_\infty(e) \le +\infty$ for every endpoint e of M. The curvature deficits play an important role in the geometric characterization of M.

Let us now look at two typical examples.

Examples. (1) A complete open oriented Riemannian 2-manifold is said to be *conical* if it is flat outside some compact set. Every conical M is a finitely connected surface with finite total curvature, and for each endpoint e of M there is a flat tube $U \in \mathfrak{U}(e)$ which is embedded isometrically into the Euclidean 3-space $\mathbb{R}^3$. If moreover $0 < \kappa_\infty(M) < 2\pi$, then U is isometrically embedded in a standard cone in $\mathbb{R}^3$ with vertex angle $\kappa_\infty(M)$.

(2) Consider a surface of revolution S embedded in $\mathbb{R}^3$ with rotation axis y with respect to the (x, y, z)-coordinates. Assume that S is a Riemannian plane and is generated by a unit speed smooth (x, y)-plane curve $\alpha : [0, +\infty) \to \mathbb{R}^2$. Then, the total curvature of S exists and is finite if and only if $\dot\alpha(t)$ converges as $t \to +\infty$. If the limit $\dot\alpha(+\infty)$ exists, we have $c(S) = 2\pi b$, where $(a, b) := \dot\alpha(+\infty)$. Here, $a^2 + b^2 = 1$ from the assumption that $\dot\alpha(t)$ is a unit vector for all t.

1. THE IDEAL BOUNDARY WITH GENERALIZED TITS METRIC

1.1. The construction and basic properties.

To construct the ideal boundary of M, we need some notations and definitions. For each *proper curve* $\alpha : [0, +\infty) \to M$ (i.e., for any monotone and divergent sequence $\{t_i\}$, $\{\alpha(t_i)\}$ has no accumulation points) an endpoint $e(\alpha)$ of M is uniquely determined by $\lim_{t \to +\infty} \varphi \circ \alpha(t) = e(\alpha)$. Then, for any tube $U \in \mathfrak{U}(e(a))$, there is a number t such that $\alpha|[t, +\infty)$ is contained in U. A ray is defined to be a half geodesic any subarc of which is a minimizing segment. Clearly, any ray is a proper curve. For any endpoint e of M and for any finitely many rays $\sigma_1, \dots, \sigma_m$ in M with $e(\sigma_i) = e$, we denote by $\mathfrak{U}(e; \sigma_1, \dots, \sigma_m)$ the set of all tubes $U \in \mathfrak{U}(e)$ having the following three properties :

(1) each σ_i intersects ∂U ;

(2) each $\dot{\sigma}_i(t_{\sigma_i})$ is perpendicular to ∂U, where $t_{\sigma_i} := \sup\{t \geq 0; \sigma_i(t) \in \partial U\}$;

(3) for $i \neq j$ in $1, \dots, m$, either $\sigma_i([t_{\sigma_i}, +\infty))$ does not intersect $\sigma_j([t_{\sigma_j}, +\infty))$, or else coincides with $\sigma_j([t_{\sigma_j}, +\infty))$.

Then, $\mathfrak{U}(e; \sigma_1, \dots, \sigma_m)$ is nonempty. Fix an endpoint e of M and take a tube $U \in \mathfrak{U}(e; \sigma, \gamma)$ for given rays σ and γ in M with $e(\sigma) = e(\gamma) = e$. Assume that the boundary ∂U of U, which is a simple closed smooth curve, is parameterized positively relative to U. Let κ be the geodesic curvature of ∂U relative to U. Let $I(\sigma, \gamma)$ be the closed subarc of ∂U from $\sigma(t_\sigma)$ to $\gamma(t_\gamma)$ and $D(\sigma, \gamma)$ the closed region in U bounded by $\sigma([t_\sigma, +\infty)) \cup I(\sigma, \gamma) \cup \gamma([t_\gamma, +\infty))$ (see Figure 1.1.f1). In the special case where $\sigma([t_\sigma, +\infty)) = \gamma([t_\gamma, +\infty))$, we set $I(\sigma, \gamma) := \{\sigma(t_\sigma)\} = \{\gamma(t_\gamma)\}$ and $D(\sigma, \gamma) := \sigma([t_\sigma, +\infty)) = \gamma([t_\gamma, +\infty))$. The arc $I(\sigma, \gamma)$ is often identified by the interval of $\mathbb{R}$ corresponding to the parameters of ∂U.

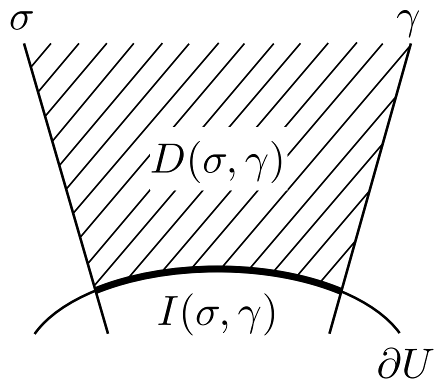

Figure 1.1.f1

We set

$$L(\sigma,\gamma) := \kappa_\infty(D(\sigma,\gamma)) - \pi = -c(D(\sigma,\gamma)) - \int_{I(\sigma,\gamma)} \kappa\, ds \ ,$$

which is independent of U by the Gauss-Bonnet theorem. Note that $L(\sigma,\gamma) = 0$ if $\sigma(t_\sigma) = \gamma(t_\gamma)$. $L(\sigma,\gamma)$ satisfies the following

1.1.1. Proposition ([Sy4, Proposition 1.1]). — *For any rays σ, τ and γ such that $e(\sigma) = e(\tau) = e(\gamma) = e$ and for any tube $U \in \mathfrak{U}(e;\sigma,\theta,\gamma)$, we have the following three properties :*

(1) $L(\sigma,\gamma) \geq 0$;

(2) *if* $\sigma(t_\sigma) \neq \gamma(t_\gamma)$, *then* $L(\sigma,\gamma) + L(\gamma,\sigma) = \kappa_\infty(e)$;

(3) *if* $\sigma(t_\sigma)$, $\tau(t_\tau)$ and $\gamma(t_\gamma)$ *lie on* ∂U *in that order, then* $L(\sigma,\tau) + L(\tau,\gamma) = L(\sigma,\gamma)$.

We define

$$d_\infty(\sigma,\gamma) := \begin{cases} \min\{L(\sigma,\gamma),\, L(\gamma,\sigma)\} & if\ e(\sigma) = e(\gamma) \\ +\infty & if\ e(\sigma) \neq e(\gamma)\ , \end{cases}$$

for two rays σ and γ in M. Then, by Proposition 1.1.1 this becomes a pseudo-distance on the set of rays in M (cf. [Sy2, §1]). The *ideal boundary* of M is defined to be the quotient metric space $(M(\infty), d_\infty)$ modulo the equivalence relation $d_\infty(\cdot,\cdot) = 0$. We denote by $\gamma(\infty)$ the equivalence class of a ray γ in M. Note that for an *Hadamard 2-manifold* (i.e., a nonpositively curved Riemannian plane) our ideal boundary coincides with that defined by Gromov in [BGS]. Setting

$$M_e(\infty) := \{\gamma(\infty) \in M(\infty); \gamma \text{ is a ray in } M \text{ with } e(\gamma) = e\}$$

for each endpoint e of M, we have that $d_\infty(M_e(\infty), M_{e'}(\infty)) = +\infty$ for any different endpoints e and e' and the decomposition

$$M(\infty) = \bigcup_{e \in E} M_e(\infty) \ .$$

For any point x in $M(\infty)$, we denote by $e(x)$ the endpoint of M so that $x \in M_{e(x)}(\infty)$.

A ray s in M is said to be *asymptotic* to a ray γ in M if there exist a monotone and divergent sequence $\{t_j\}$ and a sequence $\{\sigma_j : [0, l_j] \to M\}$ of minimizing segments in M converging to σ such that $\sigma_j(l_j) = \gamma(t_j)$ for all j. We have the following theorem.

1.1.2. Theorem ([Sy2, Theorem 5.1]). — *If a ray σ in M is asymptotic to a ray γ, then $\sigma(\infty) = \gamma(\infty)$.*

Let K be any compact subset of M. A ray γ is called a *ray from* K if $d(\gamma(t), K) = t$ for all $t \geq 0$, where d is the distance function of M induced from the Riemannian metric. By Theorem 1.1.2, for any $x \in M(\infty)$ there exists a ray from K such that $\gamma(\infty) = x$.

To describe the metric structure of the ideal boundary we need some more definitions. We define the *interior distance* $d_i : X \times X \to [0, +\infty]$ of a metric space (X, d) (cf. [G], [BGS]) as follows. For any two points p and q in X, if these points are contained in a common arcwise connected component of X, then

$$d_i(p, q) := \inf_c L(c) \ ,$$

otherwise

$$d_i(p, q) := +\infty \ ,$$

where $c : [a, b] \to X$ is any continuous curve joining p and q and the length $L(c)$ of c is defined by

$$L(c) := \sup_{a=s_0<\ldots<s_k=b} \sum_{i=0}^{k-1} d(c(s_i), c(s_{i+1})) .$$

Then, d_i is a new distance function for X and satisfies $d_i \geq d$ and $(d_i)_i = d_i$. Note that the identity map from (X, d) to (X, d_i) is not necessarily a homeomorphism. A metric space (X, d) is called a *length space* if $d = d_i$.

1.1.3. Theorem ([Sy2, Theorem 2.4], [Sy3, Theorem A]). — *The ideal boundary $(M(\infty), d_\infty)$ of M is a length space satisfying the following (1), (2) and (3) for each endpoint e of M :*

(1) *if $\kappa_\infty(e) = 0$, then $(M_e(\infty), d_\infty)$ consists of a single point ;*

(2) *if $0 < \kappa_\infty(e) < +\infty$, then $(M_e(\infty), d_\infty)$ is isometric to a circle with total length $\kappa_\infty(e)$;*

(3) *if $\kappa_\infty(e) = +\infty$, then each connected component of $(M_e(\infty), d_\infty)$ is isometric to a closed interval of $\mathbb{R}$, which may be a single point or an unbounded interval. There are at most a continuum of connected components in $M_e(\infty)$.*

Note that the case where $\kappa_\infty(e) = +\infty$ is quite different from the case where $\kappa_\infty(e) < +\infty$. Indeed, if a sequence $\{\sigma_i\}$ of rays in M tends to a ray σ, then $\{\sigma_i(\infty)\}$ tends to $\sigma(\infty)$ provided $\kappa_\infty(e(\sigma)) < +\infty$. However, when $\kappa_\infty(e(\sigma)) = +\infty$, there is always a sequence $\{\sigma_i\}$ of rays which tends to a ray σ and still $\{\sigma_i(\infty)\}$ does not tend to $\sigma(\infty)$. This phenomenon yields the noncompactness of the ideal boundary. Theorem 1.1.3 implies the completeness of the ideal boundary.

Examples. (1) The ideal boundary of the paraboloid of revolution consists of a single point.

(2) The ideal boundary of the Poincaré 2-disk has discrete topology.

(3) We take a nonpositively curved surface of revolution M with exactly two endpoints in such a way that there exists a closed geodesic γ in M dividing M into two open half-cylinders only one of which is flat (see Figure 1.1.f2). The ideal boundary

$\widetilde{M}(\infty)$ of the universal covering space $\widetilde{M}$ of M is isometric to the disjoint union of the interval $[0,\pi]$ of $\mathbb{R}$ and a discrete continuum.

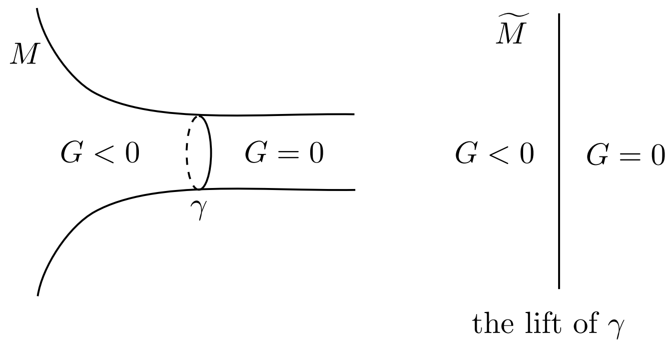

Figure 1.1.f2

A geodesic $\gamma : \mathbb{R} \to M$ is called a *straight line* if any subarc of γ is minimizing. We have

1.1.4. Proposition ([Sy4, Proposition 1.3]). — *Any straight line γ in M satisfies $d_\infty(\gamma(-\infty), \gamma(\infty)) \geq \pi$, where $\gamma(-\infty) \in M(\infty)$ is the class containing the ray opposite to γ, namely $t \mapsto \gamma(-t)$. In particular, if M contains a straight line γ with $e(\gamma(-\infty)) = e(\gamma(\infty))$, then $\kappa_\infty(e(\gamma)) \geq 2\pi$.*

By this proposition, if M has a unique endpoint, then the existence of a straight line in M restricts the curvature deficit of M. Conversely, Ohtsuka proved in [Ot1] that, if M has a unique endpoint and if $\kappa_\infty(M) > 2\pi$, then M contains at least one straight line. More generally, we have

1.1.5. Proposition. — *For any $x, y \in M(\infty)$ with $d_\infty(x,y) > \pi$, there exists a straight line γ such that $\gamma(\infty) = x$ and $\gamma(\infty) = y$. In particular, if $\kappa_\infty(M) > 2\pi$, then M contains infinitely many straight lines.*

We consider the case where M has a unique endpoint and $\kappa_\infty(M) = 2\pi$. Clearly, the Euclidean 2-space $\mathbb{R}^2$ contains straight lines and satisfies $\kappa_\infty(\mathbb{R}^2) = 2\pi$. On the other hand, as we shall show in the following example, there is a Riemannian plane M with $\kappa_\infty(M) = 2\pi$ and yet M contains no straight lines.

Example (cf. [Ot1]). For two fixed positive numbers y_0 and y_1 with $y_0 + \pi/2 < y_1$, let $f : (0, y_1) \to (0, +\infty)$ be a smooth function such that

$$f(0+) = +\infty \ ,$$
$$f > 1,\ f' < 0,\ f'' > 0 \text{ on } (0, y_0) \ ,$$
$$f = 1 \text{ on } [y_0, y_0 + \pi/2] \ ,$$
$$f(y_1-) = 0,\ f'(y_1-) = -\infty,\ f^{(n)}(y_1-) = 0 \text{ for any } n \geq 2 \ ,$$

where $a+$ (resp. $a-$) means $y < a$ (resp. $> a$) tending to a. Considering the (x, y, z)-coordinates of $\mathbb{R}^3$, the subset $\{(f(y), y, 0); y \in (0, y_1)\} \cup \{(0, y_1, 0)\}$ is the image of a smooth (x, y)-plane curve, which generates a surface of revolution M with rotation axis y (see Figure 1.1.f3). Then, M satisfies $\kappa_\infty(M) = 2\pi$. We divide M into the following three regions :

$M_1 := M \cap \{(x, y, z) \in \mathbb{R}^3; y_0 + \pi/2 \leq y \leq y_1\}$, which is an open disk domain of $G > 0$,

$M_2 := M \cap \{(x, y, z) \in \mathbb{R}^3; y_0 \leq y \leq y_0 + \pi/2\}$, which is a flat cylinder,

$M_3 := M \cap \{(x, y, z) \in \mathbb{R}^3; 0 < y < y_0\}$, which is a tube of $G < 0$.

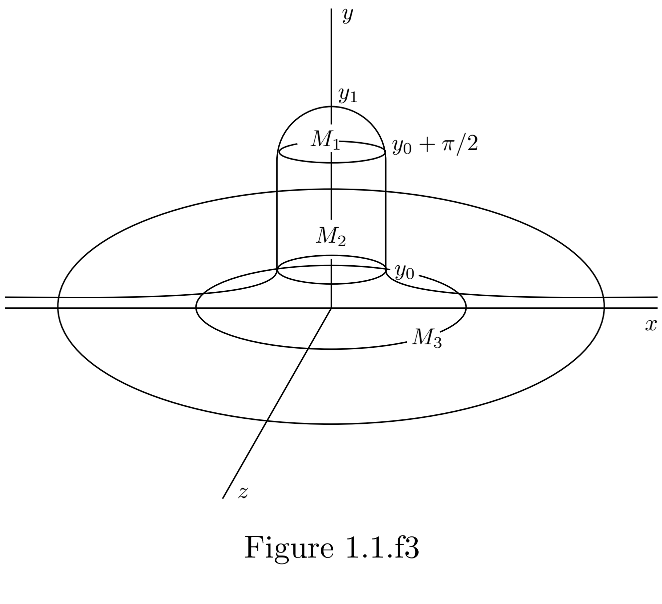

Figure 1.1.f3

Suppose that there is a straight line γ in M. If γ passes through a point in $M_1 \cup M_2$, then γ intersects M_1 and M_3, so that there are numbers $t_1 < t_2 < t_3$ such that

$\gamma(t_1)$, $\gamma(t_3) \in \partial M_3$ and $\gamma(t_2) \in M_1$. Hence $L(\gamma|[t_1, t_3]) > 2d(M_1, M_3) = \pi$ and $d(\sigma(t_1), \sigma(t_3)) \leq \pi$, which contradicts the minimizing properties of γ. Therefore, γ must be contained in M_3. Moreover, Proposition 1.1.4 and $\kappa_\infty(M) = 2\pi$ imply that $d_\infty(\gamma(-\infty), \gamma(\infty)) = \pi$ (see Theorem 1.1.3) and therefore, by the definition of d_∞, both of the two half planes bounded by γ have total curvature 0. This contradicts the fact that one of the two half planes is contained in M_3. Thus M contains no straight lines.

We outline the proof of Proposition 1.1.5 because it has never been published. The proof in the case where $e(x) \neq e(y)$ is obvious by Theorem 1.1.2. Assume that $e(x) = e(y) =: e$. We take a tube $U \in \mathfrak{U}(e)$ and rays σ, τ from ∂U such that $\sigma(\infty) = x$ and $\tau(\infty) = y$. If minimizing segments α_t joining $\sigma(t)$ and $\tau(t)$ for $t \geq 0$ have an accumulation straight line as $t \to +\infty$, Theorem 1.1.2 completes the proof. Otherwise, we may assume that there is a subsequence $\{\alpha_{t_i}\}$ of $\{\alpha_t\}$ such that each α_{t_i} is contained in $D(\sigma, \tau)$. Denote by D_i the disk domain bounded by $\sigma, \tau, I(\sigma, \tau)$ and α_{t_i}. The sequence $\{D_i\}$ is monotone increasing and covers $D(\sigma, \tau)$, which implies that $c(D_i)$ tends to $c(D(\sigma, \tau))$ as $i \to +\infty$. By applying the Gauss-Bonnet theorem to the domains D_i, we deduce that $L(\sigma, \tau) \leq \pi$ which is a contradiction.

1.2. Relation between geodesic circles and the Tits metric.

We consider the *geodesic parallel circles* $S_c(t) := \{x \in M; d(x, c) = t\}$ for a fixed simple closed curve c in M and for all $t > 0$. A minimizing segment α is called a *minimizing segment from* c if $d(\alpha(t), c) = t$ for all $t > 0$. A number $t > 0$ is said to be *exceptional* if there exists a cut point $p \in S_c(t)$ from c having one of the following three properties :

(1) p is the first focal point along some minimizing segment from c,

(2) there exist more than two minimizing segments from c to p, or

(3) there exist exactly two minimizing segments from c to p such that the angle between the two vectors at p tangent to these minimizing segments is equal to π.

Hartman [Ha] proved that the set of exceptional t-values is a closed subset of $\mathbb{R}$ of measure zero and that for any non-exceptional $t > 0$ $S_c(t)$ consists of simple closed piecewise smooth curves with only finitely many break points at the cut points from

c. Note that Hartman deals only with Riemannian planes. However, his argument is independent of the topology of M (cf. [ST]). Moreover, Shiohama [Sh4] proved that there exists an $R > 0$ such that, for any $t \geq R$, $S_c(t)$ is homeomorphic to a disjoint union of k circles, denoted by $S_{c,e}(t)$ for each endpoint e, where k is the number of endpoints, and $\varphi(S_{c,e}(t))$ tends to the endpoint e as $t \to +\infty$. Denote by d_t the interior distance of $S_c(t)$. Then, we have

1.2.1. Theorem ([Sy3, Theorem 5.3]). — *Any rays σ and γ from c satisfy*

$$\lim_{t\to+\infty} \frac{d_t(\sigma(t),\gamma(t))}{t} = d_\infty(\sigma(\infty),\gamma(\infty)) ,$$

where the limit is taken by evaluating the expression for t non-exceptional.

Kasue [Ks1] constructed the ideal boundary of an asymptotically nonnegatively curved manifold of any dimension. Any 2-dimensional asymptotically nonnegatively curved manifold has a total curvature and its ideal boundary coincides with ours by Theorem 1.2.1.

The following theorem is a generalization of [Sy3, Theorem A]. The basic idea of the proof is contained in the proof of [Sy3, Theorem A]. The precise proof of a more generalized version will be given in [SST2].

1.2.2. Theorem ([Sy3, Theorem A1]). — *For any $a, b > 0$ and rays σ and γ we have*

$$\lim_{t\to+\infty} \frac{d_t(\sigma(at),\gamma(bt))}{t} = \sqrt{a^2+b^2-2ab\min\{d_\infty(\sigma(\infty),\gamma(\infty)),\pi\}} .$$

Note that for any Hadamard manifold, Theorem 1.2.2 holds. On an Hadamard manifold, the function $f(t) := d(\sigma(t),\gamma(t))/t$ is monotone nondecreasing and hence (see [BGS, 4.4])

$$\lim_{t\to+\infty} f(t) \geq 2\sin\frac{\min\{d_\infty(\sigma(\infty),\gamma(\infty)),\pi\}}{2} .$$

However, f is not necessarily monotone in our case, and this makes the proof of Theorem 1.2.2 harder.

Theorem 1.2.2 leads us to the following

Corollary to Theorem 1.2.2. — *Assume that $\kappa_\infty(M) < +\infty$. Then, for any fixed point $p \in M$ we have that the pointed space $((M, d/t), p)$ tends to the cone over the ideal boundary $(M(\infty), d_\infty)$ as $t \to +\infty$ in the sense of the pointed Hausdorff distance.*

As for the definition of the pointed Hausdorff distance, see [Gr] and [BGS].

Denoting the diameter of a metric space by Diam, we have the following theorem as a consequence of Theorem 1.2.2.

1.2.3. Theorem ([Sy4, Theorem A2]). — *For each endpoint e of M, we have*

$$\lim_{t\to+\infty} \frac{\mathrm{Diam}(S_{c,e}(t), d)}{t} = 2\sin\frac{\min\{\mathrm{Diam}(M_e(\infty), d_\infty), \pi\}}{2} .$$

Note that $Diam(M_e(\infty), d_\infty) = \kappa_\infty(e)/2$ by Theorem 1.2.2.

1.3. Global and asymptotic behaviour of Busemann functions.

Busemann functions are first defined by Busemann in [Bu] and are very useful for the study of Riemannian manifolds (cf. [CG], [BGS]). In this section, we study the relation between Busemann functions and the distance d_∞.

The Busemann function $F_\gamma : M \to \mathbb{R}$ for a ray γ in M is defined by

$$F_\gamma(x) := \lim_{t\to+\infty} (t - d(x, \gamma(t))) .$$

First, we note

1.3.1. Theorem ([Sy3, Theorem 5.5]). — *Any rays σ and γ in M satisfy*

$$\lim_{t\to+\infty} \frac{F_\gamma \circ \sigma(t)}{t} = \cos\min\{d_\infty(\sigma(\infty), \gamma(\infty)), \pi\} .$$

If M is a Hadamard manifold, this theorem is proved as follows. Since any Busemann function is of class C^2 (see [HI]), we can apply L'Hospital's theorem. Then the left-hand side of the equality in Theorem 1.3.1 is equal to the limit of the angle between σ and the ray from $\sigma(t)$ asymptotic to γ, which tends to the right-hand side because of an easy discussion using Toponogov's triangle comparison theorem (see also Theorem 1.4.2). However, since a Busemann function is in general not differentiable, we need some delicate arguments. A function $f : M \to \mathbb{R}$ is called an *exhaustion* if $f^{-1}((-\infty, a])$ is compact for any $a \in f(M)$. We have Corollary 1.3.2, which was proved earlier by Shiohama [Sh2], as a consequence of Theorem 1.3.1.

1.3.2. Corollary ([Sh2]). — *Assume that M has a unique endpoint.*

(1) *If $\kappa_\infty(M) < \pi$, then all Busemann functions are exhaustions.*

(2) *If $\kappa_\infty(M) > \pi$, then all Busemann functions are nonexhaustions.*

Note that, if M has more than one endpoint, none of the Busemann functions is an exhaustion. Note also that there is a surface M with $\kappa_\infty(M) = \pi$ such that some of the Busemann functions are exhaustions while others are not (see [Sh2]). Nevertheless, when the Gaussian curvature of M is nonnegative outside a compact subset of M, the behaviour of the values of a Busemann function along a ray is described as follows.

1.3.3. Theorem ([Sy2, Theorem 4.9]). — *Assume that M has a unique endpoint, satisfies $\kappa_\infty(M) = \pi$, and has nonnegative Gaussian curvature outside some compact subset. If $d_\infty(\sigma(\infty), \gamma(\infty)) = \pi/2$ holds for the rays σ and γ in M, there exists a positive number t_0 such that $F_\gamma \circ \sigma|[t_0, +\infty)$ is monotone nonincreasing. In particular, all Busemann functions are nonexhaustions.*

Theorems 1.3.2 and 1.3.3 imply the following corollary, which was proved by Shiohama [Sh1] when the Gaussian curvature of M is nonnegative everywhere.

1.3.4. Corollary ([Sy2, Corollary 4.10]). — *Assume that M has a unique endpoint and the Gaussian curvature is nonnegative outside some compact subset of M. Then, we have :*

(1) *$\kappa_\infty(M) < \pi$ if and only if all Busemann functions are exhaustions ;*

(2) *$\kappa_\infty(M) \geq \pi$ if and only if all Busemann functions are nonexhaustions.*

1.4. Angle metric and Tits metric.

On a Hadamard manifold X, the Tits distance d_∞ on the ideal boundary is obtained as the interior distance of the angle distance $\sphericalangle$ defined by

$$\sphericalangle(x,y) = \sup_{p\in X} \sphericalangle_p(x,y)$$

for $x, y \in X(\infty)$, where $\sphericalangle_p(x,y)$ is the angle at p between the rays emanating from p to x and y. The angle distance satisfies $\sphericalangle(x,y) = \min\{d_\infty(x,y), \pi\}$ for all $x, y \in X(\infty)$. In our case, we observe that this does not necessarily hold. Nevertheless, we can see the asymptotic behaviour of the angles as follows.

1.4.1. Theorem ([Sy4, Theorem B1]). — *Assume that $\kappa_\infty(e) \geq 2\pi$ for all endpoints e of M. For any $x, y \in M(\infty)$ and for any sequence $\{p_j\}$ of points in M which has no accumulation points, let σ_j and γ_j be rays emanating from p_j such that $\sigma_j(\infty) = x$ and $\gamma_j(\infty) = y$ for every j. Then, we have*

$$\limsup_{j\to\infty} \sphericalangle(\dot\sigma_j(0), \dot\gamma_j(0)) \leq d_\infty(x,y)\ .$$

1.4.2. Remark. — The assumption that $\kappa_\infty(e) \geq 2\pi$ for all e is indispensable to Theorem 1.4.1. Indeed, consider a conical surface M such that $0 < \kappa_\infty(e) < 2\pi$ for some endpoint e of M. We can choose a pair of rays α and β in a flat tube $U \in \mathfrak{U}(e)$ such that for any $s, t \geq 0$ there are exactly two minimizing segment joining $\alpha(s)$ and $\beta(t)$ contained in U. For any $s \geq 0$, there are two different rays σ_s and γ_s emanating from $\alpha(s)$ which are asymptotic to β (see Figure 1.4.f).

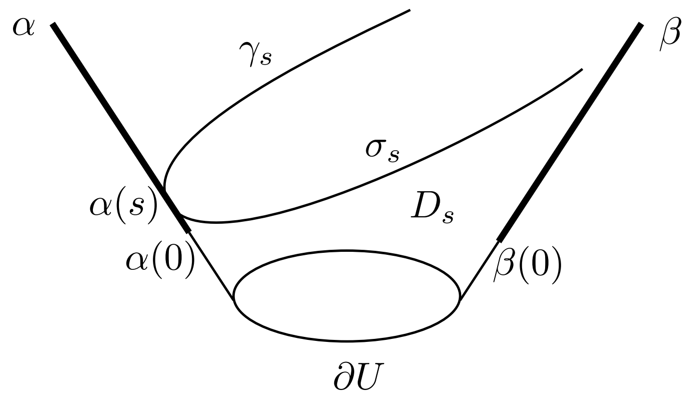

Figure 1.4.f

We have $\sigma_s(\infty) = \gamma_s(\infty) = \beta(\infty)$ for each $s \geq 0$ by Theorem 1.1.2. Let D_s for $s \geq 0$ be the region in U bounded by $\sigma_s \cup \gamma_s$ and containing β. Then, $\{D_s\}$ is a monotone increasing sequence with $\cup D_s = U$. Denoting the inner angle of D_s at $\alpha(s)$ by θ_s, we have

$$0 = d_\infty(\sigma_s(\infty), \gamma_s(\infty)) = -2\pi - \kappa(U) + \theta_s$$

for each $s \geq 0$, and hence

$$\theta_s = 2\pi - \kappa_\infty(e) ,$$

because $c(U) = 0$. Since $0 < \kappa_\infty(e) < 2\pi$, we have

$$\sphericalangle(\dot\sigma_s(0), \dot\gamma_s(0)) = \min\{\kappa_\infty(e), 2\pi - \kappa_\infty(e)\} > 0$$

for all $s \geq 0$, which is contrary to the inequality in Theorem 1.4.1.

1.4.3. Theorem ([Sy4, Theorem B2]). — *For any rays σ and γ in M, let γ_t be a ray emanating from $\sigma(t)$ which is asymptotic to γ. Then, we have*

$$\lim_{t\to+\infty} \sphericalangle(\dot\sigma(t), \dot\gamma_t(0)) = \min\{d_\infty(\sigma(\infty), \gamma(\infty)), \pi\} .$$

Theorem 1.4.3 for a Hadamard manifold is easily proved (see [BGS]) by Toponogov's comparison theorem. On the other hand, to prove Theorem 1.4.3 in our case we need the techniques of §1.3.

For any $x, y \in M(\infty)$ and for any subset B of M, we set

$$\begin{aligned}\sphericalangle(x, y; B) := \mathrm{Sup}\{&\sphericalangle(\dot\sigma(0), \dot\gamma(0)); \sigma \text{ and } \gamma \text{ are rays emanating from}\\ &\text{a common point in } M - B \text{ such that } \sigma(\infty) = x \text{ and } \gamma(\infty) = y\} .\end{aligned}$$

Then, Theorems 1.4.1 and 1.4.3 imply the following

1.4.4. Corollary ([Sy4, Corollary B3]). — *Assume that $\kappa_\infty(e) \geq 2\pi$ for all endpoints e of M. For any $x, y \in M(\infty)$ and for any $p \in M$, we have*

$$\lim_{t\to+\infty} \sphericalangle(x, y; B_p(t)) = \min\{d_\infty(x, y), \pi\} ,$$

where $B_p(t) := \{q \in M; d(p, q) < t\}$.

1.5. The control of critical points of Busemann functions.

In this section we consider the distribution of the critical points of Busemann functions. For a Lipschitz function $f : M \to \mathbb{R}$ with Lipschitz constant 1, we denote by $V(f)$ the closure in the tangent bundle TM of the set of gradient vectors of f at all points of differentiability. Note that for any ray γ in M and for any unit tangent vector $\nu \in T_pM$ we have that $\nu \in V(F_\gamma)$ if and only if the geodesic $t \mapsto \exp_p t\nu$ is a ray asymptotic to γ (see for instance [Sh6]). A point p in M is said to be a *critical point* of f if, for any unit tangent vector $u \in T_pM$, there exists a vector $\nu \in V(f)$ at p such that $< u, \nu >\geq 0$. We set

$$\mathrm{Crit}(M) := \{p \in M; p \text{ is a critical point of some Busemann function on } M\} .$$

Shiohama proved that, if M has a unique endpoint and if $\kappa_\infty(M) < \pi$, then $\mathrm{Crit}(M)$ is bounded. We have extended this to the following results as an application of the arguments in the proof of Theorem 1.4.1.

1.5.1. Theorem ([Sy4, Theorem C1]). — *If $\kappa_\infty(e) \neq \pi$ for all endpoints e, then* $\mathrm{Crit}(M)$ *is bounded.*

Remark. — When $\kappa_\infty(e) = \pi$ for some endpoint e, $\mathrm{Crit}(M)$ is not necessarily bounded. Indeed, we consider a conical surface M such that $\kappa_\infty(e) = \pi$ for some endpoint e. Let α, β, σ_s and γ_s be rays in M as in Remark 1.4.2. Since $\sphericalangle(\dot\sigma_s(0), \dot\gamma_s(0)) = \pi$, the points $\alpha(s)$ for all $s \geq 0$ are critical points of F_β. This means that $\mathrm{Crit}(M)$ is unbounded. Moreover, we observe that $\mathrm{Crit}(M)$ becomes a neighbourhood of e.

Nevertheless, we have the following

1.5.2. Theorem ([Sy4, Theorem C2]). — *If the set $\{p \in M; G(p) = 0\}$ is compact, then* $\mathrm{Crit}(M)$ *is bounded.*

1.6. Generalized visibility surfaces.

Once we have established the notion of the ideal boundary $M(\infty)$ of a complete open surface M with total curvature, we can define the notion of a visibility surface in a way analogous to [BGS]. A finitely connected oriented complete open 2-manifold M admitting total curvature is called a *visibility surface* if, for any two different points $x, y \in M(\infty)$, there exists a straight line $\gamma : \mathbb{R} \to M$ with $\gamma(-\infty) = x$ and $\gamma(\infty) = y$. Note that the total curvature of any visibility surface is $-\infty$. We have the following result.

1.6.1. Theorem ([Sy3]). — *Assume that M is finitely connected and admits total curvature. Then, the following statements are equivalent :*

(1) *M is a visibility surface ;*

(2) *there exists a positive ϵ such that $d_\infty(x, y) \geq \epsilon$ for any different points $x, y \in M(\infty)$;*

(3) *if $x, y \in M(\infty)$ are different points, then $d_\infty(x, y) = +\infty$;*

(4) *if σ and γ are rays with $\sigma(\infty) \neq \gamma(\infty)$, then $\lim_{t \to +\infty} F_y \circ \sigma(t) = -\infty$;*

(5) *for any rays σ and γ with $\sigma(\infty) \neq \gamma(\infty)$, $F_\sigma^{-1}([a, +\infty)) \cap F_\gamma^{-1}([b, +\infty))$ is bounded for all $a, b \in \mathbb{R}$;*

(6) *for any rays σ and γ with $\sigma(\infty) \neq \gamma(\infty)$, $F_\sigma^{-1}([a, +\infty)) \cap F_g^{-1}([b, +\infty)) = \phi$ for some $a, b \in \mathbb{R}$.*

Note that for any Hadamard 2-manifold, the conclusion of Theorem 1.6.1 holds (see [BGS, 4.14]). By definition, a Hadamard manifold X satisfies the *visibility axiom* if and only if, for any $p \in X$ and for any $\epsilon > 0$, there exists a number $r(p, \epsilon)$ with the property that if $\sigma : [a, b] \to X$ is a geodesic segment with $d(p, \sigma) \geq r(p, \epsilon)$, then $\sphericalangle_p(\sigma(a), \sigma(b)) \leq \epsilon$, where $\sphericalangle_p(x, y)$ is an angle at p between geodesics joining p to x, y (see [EO]). Note also that a Hadamard 2-manifold satisfies the visibility axiom if and only if it is a visibility surface. However, a visibility surface does not necessarily satisfy the visibility axiom as follows. Suppose that a visibility surface M is a Riemannian plane containing a point p such that M contains a cut point to p. Then, there are a sequence $\{q_i\}$ of cut points to p tending to the endpoint of M and two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ of minimizing segments tending to distinct rays α and β such that both α_i and β_i join p and q_i for any i. Since the angle between α_i and β_i at p tends to the angle between α and β at p, such an M does not satisfy the visibility axiom.

2. THE MASS OF RAYS

2.1. Basics.

For any point p in M, let A_p be the set of unit vectors at p which are initial vectors of rays emanating from p. To measure the mass of rays emanating from a point p in M, we consider the Lebesgue measure $\mathfrak{M}$ on the unit tangent sphere S_pM induced from the Riemannian metric of M, which satisfies $\mathfrak{M}(S_pM) = 2\pi$. Since the limit of rays in M is a ray, A_p is a closed subset of S_pM and hence it is measurable with respect to $\mathfrak{M}$. Moreover, the function $\mathfrak{M}(A_.) : p \mapsto \mathfrak{M}(A_p)$ is upper-semicontinuous and so locally integrable in the sense of Lebesgue. We call $\mathfrak{M}(A_p)$ the measure of rays at p.

As an example, let us consider the case where M is a conical Riemannian plane. If $\kappa_\infty(M) = 0$, since M contains a flat half-cylinder, for any point p in M close enough to the endpoint of M (i.e. $\varphi(p)$ close enough to the endpoint), A_p consists of only one vector and hence $\mathfrak{M}(A_p) = 0$. If $0 < \kappa_\infty(M) < 2\pi$, then with the notations of Remark 1.4.2 we have for $p = \alpha(s)$

$$A_p = \{\nu \in S_pM; \exp_p t\nu \in M - D_s \text{ for any } t > 0\} ,$$

which is a subarc of S_pM with length $2\pi - \theta_s$, and thus

$$\mathfrak{M}(A_p) = \kappa_\infty(M) .$$

Moreover, the above holds for any point p in some neighbourhood of the endpoint of M.

The first instance in which an estimate for the measure of rays was given is due to Maeda and is the following

2.1.1. Theorem ([Md2], [Md3]). — *If M is a nonnegatively curved Riemannian plane, then*

$$\inf_{p\in M} \mathfrak{M}(A_p) = \kappa_\infty(M)\ .$$

Shiga extended this to the case where the sign of the Gaussian curvature changes.

2.1.2. Theorem ([Sg2]). — *If M is a Riemannian plane with a total curvature, then*

$$2\pi - c_+(M) \leq \inf_{p\in M} \mathfrak{M}(A_p) \leq \kappa_\infty(M)\ .$$

Note that Theorem 2.1.2 implies Theorem 2.1.1. For the proof of Theorem 2.1.2, the following lemma is essential.

Let M be a finitely connected surface with a total curvature and p a point in M. Let α be a subarc of S_pM the endpoints u and ν of which are contained in A_p. Denote by γ_u and γ_v the two rays from p whose initial vectors are u and v respectively, and assume that γ_u and γ_v together bound a closed region D_α in M of side α, i.e.,

$$\alpha = \{w \in S_pM; \exp_p tw \in D_\alpha \text{ for any small } t > 0\}\ .$$

Obviously, the length $\mathfrak{M}(\alpha)$ of α is equal to the inner angle of D_α. Cohn-Vossen's theorem implies that $\kappa_\infty(D_\alpha) \geq \pi$. Moreover, we have

2.1.3. Lemma (cf. [Md3], [Sg2]). — *If $\alpha \cap A_p = \{u, v\}$, then*

$$\kappa_\infty(D_\alpha) = \pi\ .$$

In particular, if D_α is homeomorphic to the half plane, then

$$c(D_\alpha) = \mathfrak{M}(\alpha)\ .$$

Maeda and Shiga treated only the case of Riemannian planes. However, the proof is essentially independent of the topology of M.

The proof of the right-hand side of the inequality in Theorem 2.1.2 is sketched as follows. Under the assumption of Lemma 2.1.3 for a Riemannian plane M, since $S_pM - \mathrm{Int}(\alpha) \supset A_p$ (where Int denotes the interior), we have

$$2\pi - c(D_\alpha) = 2\pi - \mathfrak{M}(\alpha) \geq \mathfrak{M}(A_p) \ .$$

Now, we may consider only the case where $\kappa_\infty(M) < 2\pi$. In this case, by Proposition 1.1.4, M contains no straight lines, which shows that for any compact subset K of M and for any point $p \in M$ close enough to the endpoint of M, any ray from p does not intersect K. This proves that, for any point $p \in M$ close enough to the endpoint of M, there exists a subarc α_p satisfying the assumption of Lemma 2.1.3 such that

$$\lim_{p \to e} c(D_{\alpha_p}) = c(M) \ ,$$

so that

$$\kappa_\infty(M) = 2\pi - \lim_{p \to e} c(D_{\alpha_p}) \geq \limsup_{p \to e} \mathfrak{M}(A_p) \ . \tag{$*$}$$

Shiga also gave the following estimate from above.

2.1.4. Theorem ([Sg1]). — *If M is a nonpositively curved and finitely connected surface, then*

$$\mathfrak{M}(A_p) \leq \kappa_\infty(M)$$

for any point p in M.

2.2. The asymptotic behaviour and the mean measure of rays.

In this section, let M be a finitely connected surface with a total curvature. We consider the asymptotic behaviour of the measure of rays at p when p tends to an endpoint e of M (i.e. $\varphi(p)$ tends to e).

2.2.1. Theorem ([Sh5], [Sy1]). — *For each endpoint e of M, we have*

$$\lim_{p\to e} \mathfrak{M}(A_p) = \min\{\kappa_\infty(e), 2\pi\} .$$

This theorem was first proved by Shiohama [Sh5] in the case where M contains no straight lines, and later by the author [Sy1] in the case where M contains a straight line. The methods of proofs for the two cases are quite different.

The proof of Theorem 2.1.2 (sketched in the previous section) means a part of Theorem 2.2.1 (see (*)). For the complete proof of Theorem 2.2.1, we need more delicate arguments.

Shiohama proved the following

2.2.2. Theorem ([Sh5]). — *Let e be an endpoint of M. If the volume of a tube $U \in \mathfrak{U}(e)$ is finite, then there exists a subset Z of M of measure zero such that emanating from each point in $M - Z$, there is a unique ray γ with $e(\gamma) = e$.*

If a tube $U \in \mathfrak{U}(e)$ has finite volume, all rays relative to the endpoint e are asymptotic to each other and determine the same Busemann function F of M. The set Z in Theorem 2.2.2 is defined to be the set of all non-differentiable points of F.

The mean of an integrable function $f : K \to \mathbb{R}$ for a compact subset K of M is defined as

$$\operatorname{mean}(f, K) := \frac{\int_K f\, dM}{\operatorname{vol}(K)} ,$$

where $\operatorname{vol}(K)$ denotes the volume of K. As a consequence of Theorems 2.2.1 and 2.2.2, we have the asymptotic behaviour of the mean of the measure of rays.

2.2.3. Theorem ([Sh5], [Sy1]). — *If $\{K_j\}$ is a monotone increasing sequence of compact subset of M such that $\cup K_j$ is a neighbourhood of an endpoint e, then*

$$\lim_{j\to+\infty} \operatorname{mean}(\mathfrak{M}(A_\cdot), K_j) = \min\{\kappa_\infty(e), \pi\} .$$

2.2.4. Theorem ([SST], [Sy1]). — *For any monotone increasing sequence* $\{K_j\}$ *of compact subsets of* M *with* $\cup K_j = M$, *we have*

$$\min_{e\in E}\{\kappa_\infty(e), 2\pi\} \le \liminf_{j\to+\infty} \operatorname{mean}(\mathfrak{M}(A_\cdot), K_j)$$
$$\le \limsup_{j\to+\infty} \operatorname{mean}(\mathfrak{M}(A_\cdot), K_j) \le \max_{e\in E} \min\{\kappa_\infty(e), 2\pi\}\ .$$

Note that this theorem is best possible, i.e., we have the following

2.2.5. Theorem (cf. [SST, Remark 2]). — *For any* $\lambda \in \mathbb{R}$ *such that*

$$\min_{e\in E}\{\kappa_\infty(e), 2\pi\} \le \lambda \le \max_{e\in E} \min\{\kappa_\infty(e), 2\pi\}\ ,$$

there exists a monotone increasing sequence $\{K_j\}$ *of compact subsets of* M *such that* $\cup K_j = M$ *and*

$$\lim_{j\to+\infty} \operatorname{mean}(\mathfrak{M}(A_\cdot), K_j) = \lambda\ .$$

Proof. For simplicity, we set $a(e) := \min\{\kappa_\infty(e), 2\pi\}$. For any endpoint e of M there exists $\lambda_e \in [0, 1]$ such that

$$\lambda = \sum_{e\in E} \lambda_e a(e) \quad \text{and} \quad \sum_{e\in E} \lambda_e = 1\ . \tag{2.2.5.1}$$

We may assume that $\lambda_e := 0$ for each endpoint e with $\kappa_\infty(e) = 0$. Fix any number $\epsilon > 0$. By Theorem 2.2.1, there exist disjoint tubes $U_e \in \mathfrak{U}(e)$ for all endpoints e of M such that

$$|\mathfrak{M}(A_p) - a(e)| < \epsilon \tag{2.2.5.2}$$

for all $p \in \mathfrak{U}_e$. Note that, for any number $\mu \ge 0$, there exists a tube $V_e \in \mathfrak{U}(e)$ such that $V_e \subset U_e$ and $\operatorname{vol}(U_e - V_e) = \mu$ provided $\operatorname{vol}(U_e) = +\infty$, and that if $\operatorname{vol}(U_e) < +\infty$, then $\kappa_\infty(e) = 0$ (see [Sh5]) and hence $\lambda_e = 0$. Therefore, for each endpoint e of M, there exists a monotone decreasing sequence $\{V_{e,j}\}_{j\ge1} \subset \mathfrak{U}(e)$ such that

(1) $\bigcap_{j=1}^\infty V_{e,j} = \phi$,

(2) $V_{e,j} \subset U_e$ for all $j \ge 1$,

(3) if $\mathrm{vol}(U_e) = +\infty$, then $v_{e,j} = j\lambda_e$ for all $j \geq 1$,

where $v_{e,j} := \mathrm{vol}(U_e - V_{e,j})$. Then, for any endpoint e of M, we have

$$\lim_{j\to\infty} \frac{\nu_{e,j}}{j} = \lambda_e \ . \tag{2.2.5.3}$$

Let $K_j := M - \bigcup_{e\in E} V_{e,j}$ for $j \geq 1$. Clearly, $\{K_j\}$ is a monotone increasing sequence of compact sets with $\cup K_j = M$. Since each K_j is the union of the disjoint sets K and $U_e - V_{e,j}$ for all $e \in E$, by (2.2.5.2), we have

$$\begin{aligned}
\mathrm{mean}(\mathfrak{M}(A_.), K_j) &= \frac{\int_K \mathfrak{m}(A_.)\, dM + \sum\limits_{e\in E} \int\limits_{U_e - V_{ej}} \mathfrak{m}(A_.)\, dM}{\mathrm{vol}(K) + \sum\limits_{e\in E} \nu_{e,j}} \\
&\leq \frac{\int\limits_K \mathfrak{m}(A_.)\, dM + \sum\limits_{e\in E} \nu_{e,j}(a(e)+\epsilon)}{\mathrm{vol}(K) + \sum\limits_{e\in E} \nu_{e,j}} \ ,
\end{aligned}$$

which tends to $\lambda + \epsilon$ as $j \to +\infty$ by (2.2.5.3) and (2.2.5.1). Therefore we obtain

$$\limsup_{j\to+\infty} \mathrm{mean}(\mathfrak{M}(A_.), K_j) \leq \lambda + \epsilon$$

as well as

$$\liminf_{j\to+\infty} \mathrm{mean}(\mathfrak{M}(A_.), K_j) \geq \lambda + \epsilon$$

This completes the proof. □

Nevertheless, if the exhaustion is by geodesic balls $\{B_p(t)\}_{t>0}$, then we have the following stronger result.

2.2.6. Theorem ([SST], [Sy1]). — *For any point p in M, we have*

$$\lim_{t\to+\infty} \mathrm{mean}(\mathfrak{M}(A_.), B_p(t)) = \begin{cases} \dfrac{\sum\limits_{e\in E} \kappa_\infty(e) \min\{\kappa_\infty(e), 2\pi\}}{\kappa_\infty(M)} & \text{if } \kappa_\infty(M) > 0 \ , \\ 0 & \text{if } \kappa_\infty(M) = 0 \ . \end{cases}$$

For the proof of this theorem, we need Theorem 2.2.3 as well as the isoperimetric theorem due to [Sh4] (cf. [Fi], [Ha], [Sh3]).

2.2.7. Remark. — We can generalize Theorems 2.2.1 and 2.2.6 to the case where M is any nonnegatively curved open manifold of dimension greater than 2, which will be written in [Sy6]. In [Sy6], we generalized Theorems 2.2.1 and 2.2.6 to the case where M is any nonnegatively curved complete open manifold of dimension greater than 2, or more generally any nonnegatively curved noncompact Alexandrov space. The situation in that case is more delicate because the asymptotic behaviour of the mass of rays at point p depends on the limit point of p in the ideal boundary. In the two-dimensional case, the ideal boundary is isometric to a circle and then homogeneous (when the total curvature is finite), which is the reason why the two-dimensional case is simpler than the higher-dimensional case.

3. THE BEHAVIOUR OF DISTANT MAXIMAL GEODESICS

The aim of this chapter is to describe our work in [Sy5] and [Sy7] about the behaviour of maximal geodesics close enough to infinity in a finitely connected surface M. For the results in this chapter, we need to introduce a new condition. A finitely connected, complete, open and oriented 2-manifold is said to be *strict* if it has a total curvature and the curvature deficit $\kappa_\infty(e)$ for every endpoint e is positive.

3.1. Visual diameter of any compact set looked at from a distant point.

Let us define

$$\Gamma_p(K) := \{v \in S_pM; \exp_p t\nu \in K \text{ for some } t \geq 0\}$$

for a subset K of M and a point p in M. Trivially if $p \in K$, then $\Gamma_p(K) = S_pM$. The *visual diameter* of a subset K of M at a point p in M is defined to be the diameter $\operatorname{Diam}\Gamma_p(K)$ of $\Gamma_p(K)$ with respect to the angle distance $\sphericalangle$ on S_pM. Then, the function $M \ni p \mapsto \operatorname{Diam}\Gamma_p(K) \in [0, \pi]$ for any fixed subset K is upper-semicontinuous. We have

3.1.1. Theorem [Sy5]). — *For any compact subset K of a strict surface M and for any endpoint e of M, we have*

$$\lim_{p\to e} \operatorname{Diam}\Gamma_p(K) = 0 \ .$$

Note that we proved earlier in [SST] that

$$\lim_{p\to e} \operatorname{Diam}\Gamma_p(K) \cap A_p = 0 \ ,$$

in the proof of which the minimizing properties of geodesics tangent to vectors in $\Gamma_p(K) \cap A_p$ simplifies the proving process. On the other hand, the proof of Theorem 3.1.1 needs to study the behaviour of non necessarily minimizing geodesics.

Remark. — The strictness of M is indispensable in Theorem 3.1.1. Indeed, a conical Riemannian plane M with $c(M) = 2\pi$ does not satisfy the conclusion of Theorem 3.1.1. Since a flat half-cylinder is embedded isometrically in the complement of a compact subset K of M, for a point p of M close enough to infinity, there is a simple closed geodesic passing through p which bounds a disk D_p containing K (see Figure 3.1.f). All half-geodesics emanating from p and facing to the interior of D_p intersect K. Thus we have $\operatorname{Diam}\Gamma_p(K) = \pi$.

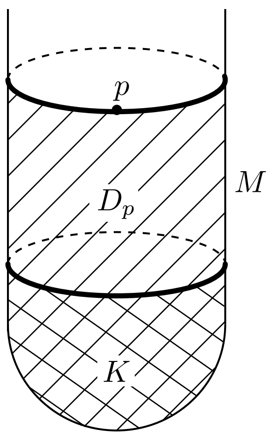

Figure 3.1.f

As a direct consequence of Theorem 3.1.1, we have the following corollary, which means the existence of many maximal geodesics close enough to infinity.

3.1.2. Corollary ([Sy5]). — *For any compact subset K of a strict surface M and for any number $\epsilon > 0$ there exists a number $r(K,\epsilon) > 0$ such that for any point p with $d(p,K) > r(K,\epsilon)$ there exists a unit tangent vector u at p such that if a unit vector ν satisfies $\sphericalangle(u,v) < \pi - \epsilon$, then γ_v does not intersect K, where γ_v is the maximal geodesic tangent to v at p.*

3.2. The shapes of plane curves.

In this section, we establish some definitions for the shapes of curves in order to state the main results. Let V be a surface diffeomorphic to $\mathbb{R}^2$. A proper (differentiable) immersion $\alpha : \mathbb{R} \to V$ is said to be *transversal* if the inverse image of every point in $\alpha(\mathbb{R})$ contains at most two points, and for a and b two distinct real numbers with $\alpha(a) = \alpha(b)$, $\dot\alpha(a)$ and $\dot\alpha(b)$ are linearly independent. Note that the set of double points of such an α is a discrete subset of V.

3.2.1. Definition. — *Let $\alpha : \mathbb{R} \to V$ be a proper transversal immersion. A double point $\alpha(a) = \alpha(b)$ such that $a < b$ is said to be positive if $(\dot\alpha(a), \dot\alpha(b))$ is compatible with a fixed orientation of V and negative otherwise. Denote by $n_+(s,t)$ (resp. $n_-(s,t)$) the numbers of positive (resp. negative) double points of the open arc $\alpha|(s,t)$. The rotation number* $\mathrm{rot}(\alpha)$ *of α is defined by*

$$\mathrm{rot}(\alpha) := \limsup_{\substack{s\to-\infty \\ t\to+\infty}} |n_+(s,t) - n_-(s,t)| \ .$$

Note that $\mathrm{rot}(\alpha)$ does not depend on the parameterization of α and is an invariant of the compactly supported regular homotopy class of α in the following sense. Let α and β be two proper transversal immersions. We say α is *compactly supported regular homotopic* to β if $\alpha(t) = \beta(t)$ for all t outside some bounded open interval (a,b) and if there exists a regular homotopy between $\alpha|[a,b]$ and $\beta|[a,b]$ fixing $\dot\alpha(a)$ and $\dot\alpha(b)$.

3.2.2. Definition. — *A proper transversal immersion α is called a semi-regular curve if there exists a nested sequence of intervals $[a_1,b_1] \subset [a_2,b_2] \subset \ldots \subset [a_n,b_n]$ with finite or infinite length $0 \le n \le +\infty$ such that $\{\alpha(a_i) = \alpha(b_i)\}_{i=1,\ldots,n}$ is the set of double points of α.*

Let α be a semi-regular curve. For each $i \ge 1$, let B_i be the compact disk bounded by $\alpha([a_i,a_{i-1}] \cup [b_{i-1},b_i])$, where $a_0 = b_0$ is some number in (a_1,b_1). Then,

B_1 is bounded by a simple loop and is called the *teardrop* of α. For each $i \geq 2$, one of the following two cases occurs (see Figure 3.2.f).

(1) The signs of the double points $\alpha(a_{i-1})$ and $\alpha(a_i)$ are distinct and

$$\operatorname{Int} B_i \cap \bigcup_{j=1}^{i-1} \operatorname{Int} B_j = \phi ,$$

in which case B_i is called a *lemon*.

(2) The signs of the double points $\alpha(a_{i-1})$ and $\alpha(a_i)$ are equal and

$$B_i \supset \bigcup_{j=1}^{i-1} \operatorname{Int} B_j ,$$

in which case B_i is called a heart.

Figure 3.2.f

3.2.3. Definition. — *The index* $\operatorname{ind}(\alpha)$ *of a semi-regular curve* α *is defined to be the number of all* B_i *which are not lemons.*

Note that a semi-regular curve α is simple if and only if $\operatorname{ind}(\alpha) = 0$. For any semi-regular curve α, we have $\operatorname{rot}(\alpha) \leq \operatorname{ind}(\alpha) \leq n$, where n is the number of double points of α.

3.2.4. Definition. — *A semi-regular curve* α *is said to be regular if* α *has no lemons. A semi-regular curve* α *is said to be almost regular if the largest heart contains no lemons.*

For any semi-regular curve α we have the following properties. If α is regular, then $\operatorname{rot}(\alpha) = \operatorname{ind}(\alpha) = n$. Conversely, if $\operatorname{ind}(\alpha) = n$, then α is regular. However, $\operatorname{rot}(\alpha) =$

$\operatorname{ind}(\alpha)$ does not imply α to be regular or almost regular. An almost regular curve may have infinitely many double points even when its index is finite. Nevertheless, for any almost regular curve α, we have $\operatorname{rot}(\alpha) = \operatorname{ind}(\alpha)$ or $\operatorname{ind}(\alpha) - 1$.

3.3. Maximal geodesics in strict Riemannian planes.

The aim of this section is to investigate the behaviour of maximal geodesics in strict Riemannian planes. First, we consider the following easy case.

In a flat cone F in $\mathbb{R}^3$ any geodesic γ (not passing through the vertex) is regular. We will show that $\operatorname{ind}(\gamma) = n(\theta)$, where $n(t)$ for $t > 0$ denotes the maximal integer k such that $k < \pi/t$ and θ the vertex angle of F. Let $\widetilde{F}$ be the universal covering space of $F - \{\text{the vertex}\}$ and $\widetilde{\gamma}$ a lift in $\widetilde{F}$ of γ. There is a ray σ in F from the vertex such that a lift in $\widetilde{F}$ of σ is parallel to $\widetilde{\gamma}$ (see Figure 3.3.f). The index of γ coincides with the number of all lifts of σ intersecting γ, which is equal to $n(\theta)$.

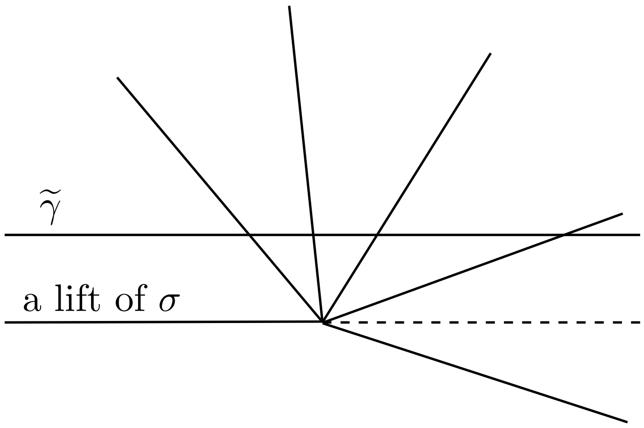

Figure 3.3.f

Let M be a conical Riemannian plane with $0 < \kappa_\infty(M) < 2\pi$ and e the endpoint of M. Since there is a tube $U \in \mathfrak{U}(e)$ isometrically embedded into a flat cone in $\mathbb{R}^3$ with vertex angle $\kappa_\infty(M)$, any geodesic γ in M close enough to infinity is regular and

$$\operatorname{ind}(\gamma) = n(\kappa_\infty(M)) \ .$$

Cohn-Vossen proved in [Co2] that if M is a positively curved Riemannian plane, then any maximal geodesic in M has at most one simple loop $\gamma|[a, b]$, and moreover both

$\gamma|(-\infty, a]$ and $\gamma|[b, +\infty)$ are simple and do not intersect $\gamma|[a, b]$. This follows from the non-existence of locally concave subsets of M. Moreover, this property shows the semi-regularity of any maximal geodesic. In the more general case where M is a strict Riemannian plane, there exists a compact contractible subset K of M such that the boundary of any concave subset of M intersect K. Therefore, in the same way as above, we have the semi-regularity of any maximal geodesic outside K. Recall that such a maximal geodesic exists by Corollary 3.1.2. Using much more delicate arguments we have the following

3.3.1. Theorem ([Sy5]). — *Let M be a strict Riemannian plane such that either $\kappa_\infty(M) = +\infty$, or $\pi/\kappa_\infty(M)$ is not an integer. Then, there exists a compact subset K of M such that any maximal geodesic outside K is regular of index $[\pi/\kappa_\infty(M)]$, where $[.]$ denotes the integral part of a real number.*

In the case where $c(M) < \pi$, Theorems 3.3.1 means that any geodesic close enough to infinity is proper and simple.

As is seen in Theorem 3.3.1, when $\pi/\kappa_\infty(M)$ is not an integer, the topological shapes of all maximal geodesics close enough to infinity are completely controlled. On the other hand, when $\pi/\kappa_\infty(M)$ is an integer, the topological shapes of such geodesics are almost completely controlled as follows. Denoting by M_+ (resp. M_-) the positive (resp. negative) curvature locus of M, we have

3.3.2. Theorem ([Sy5]). — *Let M be a strict Riemannian plane such that $\pi/\kappa_\infty(M)$ is an integer. For a geodesic γ in M, consider the following three conditions :*

(i) *γ is regular and*

$$\mathrm{ind}(\gamma) = \pi/\kappa_\infty(M) - 1 \ ;$$

(ii) *γ is regular and*

$$\mathrm{ind}(\gamma) = \begin{cases} \pi/\kappa_\infty(M) & \text{if } c(M) \neq \pi \\ 0 \text{ or } 1 & \text{if } c(M) = \pi \ ; \end{cases}$$

(iii) *γ is not regular but almost regular (with possibly infinitely many double points) and*

$$\mathrm{ind}(\gamma) = \pi/\kappa_\infty(M) \ .$$

Then, there exists a compact subset K of M such that for a maximal geodesic γ outside K, we have :

(a) *if M_- is bounded, all γ satisfy Condition (i) ;*

(b) *if M_- is unbounded and M_+ is bounded, then all γ satisfy Condition (ii) ;*

(c) *if both M_+ and M_- are unbounded, then all γ may satisfy either (i), (ii) or (iii).*

Example. Concerning Theorem 3.3.2 (b), we can construct an example of a Riemannian plane M with $c(M) = \pi$ containing two geodesics γ_0 and γ_1 such that $\mathrm{ind}(\gamma_i) = i$ for $i = 0, 1$. Take a smooth bijective function $f : [0, 1) \to [0, +\infty)$ such that $f(0) = 0, f'' > 0$ and set

$$F := \{(x, y, 0) \in [0, +\infty) \times [0, +\infty) \times \mathbb{R}; y \leq f(x) \text{ if } x < 1\} .$$

Then, for a fixed small $0 < \epsilon << 1$, the boundary of the ϵ-neighbourhood of F in $\mathbb{R}^3$ is a Riemannian plane with C^1-metric which we denote by M'. Let S be the set of points at which the metric of M' is not C^∞. Then S is contained in the boundary of the ϵ-neighbourhood of $\{(x, f(x)), 0) \in \mathbb{R}^3; 0 \leq x < 1\}$. By smoothing M' in the δ-neighbourhood of S in $\mathbb{R}^3$ for some small $0 < \delta << \epsilon$, we obtain a smooth Riemannian plane M in $\mathbb{R}^3$ with $c(M) = \pi$ such that M_+ is bounded and M_- is unbounded. Since the plane $\{(x, y, z) \in \mathbb{R}^3; x = t\}$ for any $t > 1$ does not intersect S, the intersection of this plane for $t > 1 + \delta$ and M consists of a simple maximal geodesic γ_0 of M. Moreover, we will show that any geodesic γ_1 intersecting M_- and not intersecting M_+ is not simple. Suppose that such a γ_1 is simple. Then, γ_1 divides M into two half planes, one of which contains M_+. If we denote this by H, then

$$c(H) = c(M) - c(M - H) < \pi ,$$

which contradicts Cohn-Vossen's theorem for H.

Considering the attachment of $M(\infty)$ to M, we obtain the compactification of $\overline{M}$ of M equipped with a natural topology (cf. [Ks1], [Sy5]). For the Riemannian plane M as in the previous example, the closure of M_- in $\overline{M}$ intersects $M(\infty)$ at only one point. In general, under the assumption of Theorem 3.3.2 (b) with $c(M) = \pi$, if the closure of M_- in $\overline{M}$ contains more than one point in $M(\infty)$, then we have $\mathrm{ind}(\gamma) = 1$.

3.3.3. Remark. — *To obtain the conclusions of Theorems 3.3.1 and 3.3.2, the compact set K is taken to be contractible and to satisfy the following conditions. If $-\infty \le c(M) < \pi$, then*

$$(*) \qquad c_+(M-K) < \pi \text{ and } c_+(M-K) + c(K) < \pi \ .$$

In this case, we have that any geodesic outside K is proper and simple, whereby the theorems follow. If $c(M) \ge \pi$, then

$$(**) \qquad c_+(M-K) < \epsilon_+(\kappa_\infty(M)) \text{ and } c_-(M-K) < \epsilon_-(\kappa_\infty(M)) \ ,$$

where we set, for $t > 0$,

$$n(t) := \max\{k \in \mathbb{Z}; kt < \pi\}, \quad n'(t) := [\pi/t] \ ,$$
$$\epsilon_+(t) := \frac{\pi - n(t)t}{n'(t)+1} \text{ and } \epsilon_-(t) := \frac{(n'(t)+1)t - \pi}{n'(t)+1} \ .$$

Note that

$$\pi - n(t)\kappa_\infty(t) = d_{\mathbb{R}}(\pi, t\mathbb{Z} \cap [0,\pi))$$

and

$$(n'(t)+1)t - \pi = d_{\mathbb{R}}(\pi, t\mathbb{Z} \cap (\pi, +\infty)) \ ,$$

where $d_{\mathbb{R}}$ is the canonical distance of $\mathbb{R}$, and in particular $\epsilon_\pm(t)$ are positive for all $t > 0$. Note also that when $-\infty < c(M) < \pi$, since $\epsilon_+(\kappa_\infty(M)) = \pi$ and $\epsilon_-(\kappa_\infty(M)) = \pi - c(M)$, the two conditions (*) and (**) are equivalent.

Next, we consider all geodesics in a Riemannian plane M under the stronger assumption that $c_+(M) < 2\pi$.

3.3.4. Theorem ([Sy5]). — *In a Riemannian plane M with $c_+(M) < 2\pi$, any maximal geodesic γ is semi-regular and satisfies*

$$\operatorname{ind}(\gamma) < \frac{\pi}{2\pi - c_+(M)} \ .$$

Assume that M contains a pole p, i.e., $\exp_p : T_pM \to M$ is a local diffeomorphism. Now, by the simply connectivity of M, $\exp_p : T_pM \to M$ is a diffeomorphism. Let (θ, r) be the geodesic polar coordinate centered at p, where $\theta : M \to \mathbb{R}$ is the angle at p and $r : M \to [0, +\infty)$ the distance from p. For any maximal geodesic γ not passing through p, $\theta \circ \gamma$ is a strictly monotone function. Therefore, we have the following

3.3.5. Corollary ([Sy5]). — *If a Riemannian plane M with $c_+(M) < 2\pi$ contains a pole, any maximal geodesic in M is regular.*

Corollary 3.1.2 and Theorems 3.3.1, 3.3.2 and 3.3.4 prove the following

3.3.6. Corollary ([Sy5]). — *In a nonnegatively curved strict Riemannian plane M, any maximal geodesic γ is semi-regular and*

$$\mathrm{rot}(\gamma) \leq n(\kappa_\infty(M)) \ ,$$

where $n(\cdot)$ is as above. Moreover, if γ is close enough to infinity, then it is regular of index $n(\kappa_\infty(M))$. In particular, we have

$$\max_\gamma \mathrm{rot}(\gamma) = n(\kappa_\infty(M)) \ .$$

For behaviour of geodesics in non-strict Riemannian planes, we have the following

3.3.7. Theorem ([Sy6]). — *Any Riemannian plane M with $c(M) = 2\pi$ satisfies at least one of the following (1) and (2) :*

(1) *M contains a sequence of closed geodesics tending to the endpoint of M :*

(2) *for any $n \geq 1$ there exists a compact subset K_n of M such that any maximal geodesic γ outsdide K_n is semi-regular and contains a subarc whose rotation number is not less than n ; in particular,* $\mathrm{ind}(\gamma) \geq n$.

Remark. — In the above theorem, we cannot say whether a maximal geodesic outside K_n exists or not. More precisely, all Riemannian planes M with $c(M) = 2\pi$ are classified into the following three cases :

(1) M contains a sequence of closed geodesics tending to the endpoint of M, in which case M is called a **building Riemannian plane** ;

(2) M does not satisfy (1) and contains a compact subset which any maximal geodesic intersects, in which case M is called a **contracting Riemannian plane** ;

(3) M does not satisfy (1) and for any compact subset K of M there exists a maximal geodesic outside K, in which case M is called an **expanding Riemannian plane**.

For instance, if a Riemannian plane M with $c(M) = 2\pi$ satisfies that $G = 0$ (resp. < 0, > 0) outside a compact set, then M is a building (resp. contracting, expanding) Riemannian plane. Note that, if M is a strict Riemannian plane, then M satisfies (3).

3.4. Generalization to finitely connected surfaces.

Let M be a finitely connected complete open surface. A compact region C of M is called a *core* of M if $M - C$ is the union of disjoint tubes $U_e \in \mathfrak{U}(e)$ for all endpoints e of M. Note that, for any compact subset C' of M, there exists a core C of M containing C'. Hence, there exists a core C of M such that all the associated tubes U_e satisfy the following two conditions :

(1) if $\kappa_\infty(e) > \pi$, then

$$(\#) \qquad c_+(U_e) < \pi \ \text{ and } \ c_+(U_e) + \kappa(U_e) + \pi < 0 \ ,$$

where we recall that $\kappa(U_e)$ can be close enough to $\kappa_\infty(e)$ by taking U_e to be small enough ;

(2) if $0 < \kappa_\infty(e) \leq \pi$, then

$$(\#\#) \qquad c_+(U_e) < \epsilon_+(\kappa_\infty(e)) \ \text{ and } \ c_-(U_e) < \epsilon_-(\kappa_\infty(e)) \ .$$

For any such tube U_e, there exists an isometric embedding $\iota_e : U_e \to M_e$ of U_e into a Riemannian plane M_e. Then, for each endpoint e of M, the compact disk $K := M_e - \iota_e(U_e)$ has the properties made in Remark 3.3.3. In fact, (#) and (##) are respectively corresponding to (*) and (**) in Remark 3.3.3 independently of the choice of ι_e and M_e. Thus, with these notations we have the following

3.4.1. Theorem. — *Let M be a finitely connected complete open surface for which the total curvature exists and assume that M contains no sequence of closed geodesics tending to infinity. Then, there exists a core C of M such that any maximal geodesic γ outside C satisfies the following (1), (2), and (3) :*

(1) *the geodesic $\iota_e \circ \gamma$ in M_e is semi-regular, where e is the endpoint of M such that $\gamma \subset U_e$;*

(2) *if* $\kappa_\infty(e) > 0$, *then* $\iota_e \circ \gamma$ *satisfies the conclusions of Theorems 3.3.1 and 3.3.2 with* $\kappa_\infty(M_e) = \kappa_\infty(e)$ *;*

(3) *if* $\kappa_\infty(e) = 0$ *and if the core* C *is taken to be large enough against a given number* $n \geq 1$, *then a subarc of* $\iota_e \circ \gamma$ *has rotation number not less than* n.

Note that any strict surface satisfies the condition of Theorem 3.4.1. For more study on behaviour of geodesics, see [Sy5].

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