A CONVERGENCE THEOREM IN THE GEOMETRY OF ALEXANDROV SPACES

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Abstract. The fibration theorems in Riemannian geometry play an important role in the theory of convergence of Riemannian manifolds. In the present paper, we extend them to the Lipschitz submersion theorem for Alexandrov spaces, and discuss some applications.

Résumé. Les théorèmes de fibration de la géométrie riemannienne jouent un rôle important dans la théorie de la convergence des variétés riemanniennes. Dans cet article, on les étend au cadre lipschitzien des espaces d'Alexandrov, et on donne quelques applications.

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0. INTRODUCTION

An Alexandrov space is a metric space with length structure and with a notion of curvature. In the present paper we study Alexandrov spaces whose curvatures are bounded below. Such a space occurs for instance as the Hausdorff limit of a sequence of Riemannian manifolds with curvature bounded below. Understanding such a limit space is significant in the study of structure of Riemannian manifolds themselves also, and it is a common sense nowadays that there is interplay between Riemannian geometry and the geometry of Alexandrov spaces through Hausdorff convergence.

Recently Burago, Gromov and Perelman [BGP] have made important progress in understanding the geometry of Alexandrov spaces whose curvatures are bounded below. Especially, they proved that the Hausdorff dimension of such a space X is an integer if it is finite and that X contains an open dense set which is a Lipschitz manifold. A recent result in the revised version [BGP2] and also Otsu and Shioya [OS] has extended the later result by showing that such a regular set actually has full measure. Since the notion of Alexandrov space is a generalization of Riemannian manifold, it seems natural to consider the problem : what extent can one extend results in Riemannian geometry to Alexandrov spaces ?

The notion of Hausdorff distance introduced by Gromov [GLP] has brought a number of fruitful results in Riemannian geometry. For instance, the convergence theorems and their extension, the fibration theorems, or other related methods have played important roles in the study of global structure of Riemannian manifolds. The main motivation of this paper is to extend the fibration theorem ([Y]) to Alexandrov spaces. In the Riemannian case we assumed that the limit space is a Riemannian manifold. Here, we employ an Alexandrov space as the limit whose singularities are quite nice in the following sense.

Let X be an n-dimensional complete Alexandrov space with curvature bounded below. In [BGP], it was proved that the space of directions Σ_p at any point $p \in X$

is an $(n-1)$-dimensional Alexandrov space with curvature ≥ 1, and that if Σ_p is Hausdorff close to the unit $(n-1)$-sphere S^{n-1}, then a neighborhood of p is bi-Lipschitz homeomorphic to an open set in $\mathbf{R}^n$. This fact is also characterized by the existence of (n,δ)-strainer. (For details, see Section 1). For $\delta > 0$, we now define the *δ-strain radius* at $p \in X$ as the supremum of $r > 0$ such that there exists an (n,δ)-strainer at p with length r, and the *δ-strain radius of X* by

$$\delta\text{-str. rad } (X) = \inf_{p \in X} \delta\text{-strain radius at } p\,.$$

For instance, X has a positive δ-strain radius if X is compact and if Σ_p is Hausdorff close to S^{n-1} for each $p \in X$.

For every two points x, y in X, a minimal geodesic joining x to y is denoted by xy, and the distance between them by $|xy|$. The angle between minimal geodesics xy and xz is denoted by $\angle yxz$. Under this notaton, we say that a surjective map $f : M \to X$ between Alexandrov spaces is an *ϵ-almost Lipschitz submersion* if

(0.1.1) — it is an ϵ-Hausdorff approximation.

(0.1.2) — For every $p, q \in M$ if θ is the infimum of $\angle qpx$ when x runs over $f^{-1}(f(p))$, then

$$\left| \frac{|f(p)f(q)|}{|pq|} - \sin\theta \right| < \epsilon\,.$$

Remark that the notion of ϵ-almost Lipschitz submersion is a generalization of ϵ-almost Riemannian submersion. Our main result in this paper is as follows.

Theorem 0.2. — *For a given positive integer n and $\mu_0 > 0$, there exist positive numbers $\delta = \delta_n$ and $\epsilon = \epsilon_n(\mu_0)$ satisfying the following. Let X be an n-dimensional complete Alexandrov space with curvature ≥ -1 and with δ-str.rad$(X) > \mu_0$. Then, if the Hausdorff distance between X and a complete Alexandrov space M with curvature ≥ -1 is less than ϵ, then there exists a $\tau(\delta,\epsilon)$-almost Lipschitz submersion $f : M \to X$. Here, $\tau(\delta,\sigma)$ denotes a positive constant depending on n, μ_0 and δ, ϵ and satisfying* $\lim_{\delta,\epsilon\to 0} \tau(\delta,\epsilon) = 0$.

Because of the lack of differentiability in X, it is unclear at present if the map f is actually a locally trivial fiber bundle. The author conjectures that this is true. In fact, in the case when both X and M have natural differentiable structures of class C^1, we can take a locally trivial fibre bundle as the map f. (See Remark 4.20).

Remark 0.3. — Under the same assumption as in Theorem 0.2, for any $x \in X$ let Δ_x denote the diameter of $f^{-1}(x)$. Then, there exists a compact nonnegatively curved Alexandrov space N such that the Hausdorff distance between N and $f^{-1}(x)$ having the metric multiplied by $1/\Delta_x$ is less than $\tau(\delta, \epsilon)$ for every $x \in X$. (See the proof of Theorem 5.1 in §5.)

In Theorem 0.2, if $\dim M = \dim X$ it turns out that

Corollary 0.4. — *Under the same assumptions as in Theorem 0.2, if* $\dim M = n$, *then the map f is $\tau(\delta, \sigma)$-almost isometric in the sense that for every $x, y \in M$*

$$\left| \frac{|f(x)f(y)|}{|xy|} - 1 \right| < \tau(\delta, \sigma) .$$

Remark 0.5. — In [BGP2], Burago, Gromov and Perelman have proved Corollary 0.4 independently. And Wilhelm [W] has obtained Theorem 0.2 under stronger assumptions. He assumed a positive lower bound on the injectivity radius of X and that M is an almost Riamannian space. His constant ϵ in the result depends on the particular choice of X. It should also be noted that Perelman [Pr1] has obtained a version of Corollary 0.4 in the general situation. He proved that any compact Alexandrov space X with curvature ≥ -1 has a small neighborhood with respect to the Hausdorff distance such that every Alexandrov space of the same dimension as X with curvature ≥ -1 which lies in the neighborhood is homeomorphic to X.

By using Corollary 0.4, one can prove the volume convergence.

Corollary 0.6. ([Pr2]) — *Suppose that a sequence (M_i) of n-dimensional compact Alexandrov spaces with curvature ≥ -1 converges to an n-dimensional one, say M, with respect to the Hausdorff distance. Then, the Hausdorff n-measure of M_i converges to that of M.*

As in the Riemannian case, Theorem 0.2 has a number of applications. The results in Riemannian geometry which essentially follow from the splitting theorem ([T],[CG],[GP1],[Y]) and the fibration theorem are still valid for Alexandrov spaces.

For instance, we have the following generalization of the main result in Fukaya and Yamaguchi [FY1].

Theorem 0.7. — *There exists a positive number ϵ_n such that if X is an n-dimensional compact Alexandrov space with curvature ≥ -1 and* $\mathrm{diam}(X) < \epsilon_n$*, then its fundamental group contains a nilpotent subgroup of finite index.*

The basic idea of the proof of Theorem 0.2 and the organization of the present paper is as follows. In section 1, after recalling some basic results in [BGP], we study a neighborhood of a point with singularities of small size. Such a neighborhood has nice properties similar to those of a small neighborhood in a Riemannian manifold. The proof of Theorem 0.2 starts from Section 2. We construct an embedding $f_X : X \to L^2(X)$ and a map $f_M : M \to L^2(X)$ by using distance functions, where $L^2(X)$ is the Hilbert space consisting of all L^2-functions on X. Similar constructions were made in [GLP],[K],[Fu1,2] and [Y] in the case where both X and M are smooth Riemannian manifolds. However, in our case, there appear some difficulties in proving the existence of a tubular neighborhood of $f_X(X)$ in $L^2(X)$ because $f_X(X)$ is just a Lipschitz manifold. Of course a tubular neighborhood of $f_X(X)$ does not exist in the exact sense because of singularities of X. To overcome this difficulty, we generalize the notion of tubular neighborhood. First, we show that the image of the directional derivative df_X of f_X at each point $p \in X$ can be approximated by an n-dimensional subspace Π_p in $L^2(X)$ because of the small size of singularities of X. Thus, a small neighborhood of $f_X(p)$ in $f_X(X)$ is approximated by the n-plane $f_X(p) + \Pi_p$. This fact is used in Section 3, a main part of the paper, to construct a smooth map ν of a neighborhood of $f_X(X)$ into the Grassmann manifold consisting of all subspaces in $L^2(X)$ of codimension n such that ν is almost perpendicular to $f_X(X)$. The point is to evaluate the norm of the gradient of ν in terms of apriori constants, which makes it possible to prove that ν actually provides a tubular neighborhood of $f_X(X)$ in the generalized sense, and to estimate the radius of the tubular neighborhood in terms of given constants. This idea is also effective in studying the projection $\pi : f_M(M) \to f_X(X)$ along ν. It turns out that π is locally Lipschitz continuous with Lipschitz constant close to one and that it is almost isometric in the directions almost parallel to $f_X(X)$. In Section 4, we show that the composed map $f = f_X^{-1} \circ \pi \circ f_M : M \to X$ is an almost Lipschitz submersion as required. The proof of Theorem 0.7 is given in section 5. Its machinery is the same as that in [FY1] except for the induction procedure, which is carried out after deriving the property of the "fibre" of f as described in Remark 0.3. In the Appendix, we discuss the relative volume comparison for Alexandrov spaces that is of Bishop and Gromov type.

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1. PROPERTIES OF A NEIGHBORHOOD OF A STRAINED POINT

First of all, we recall some basic facts on Alexandrov spaces. We refer the reader to [BGP] for details.

Let X be a locally compact complete Alexandrov space with curvature $\geq k$. For $x, y, z \in X$, let $\Delta(x, y, z)$ denote a geodesic triangle with sides xy, yz and zx. We also denote by $\widetilde{\Delta}(x, y, z)$ a geodesic triangle in the simply connected surface $M(k)$ with constant curvature k, with the same side lengths as $\Delta(x, y, z)$. The angle between xy and xz is denoted by $\angle yxz$, and the corresponding angle of $\widetilde{\Delta}(x, y, z)$ by $\widetilde{\angle} yxz$. Two minimal geodesics emanating from a point are by definition equivalent if one is a subarc of the other. For $p \in X$, let Σ'_p denote the set of all equivalence classes of minimal geodesics starting from p. The space of directions Σ_p at p is the completion of Σ'_p with respect to the angle distance. We denote by x' the set consisting of all directions represented by minimal geodesics joining p to x. If $\xi \in x'$, we use the familiar notation $\exp t\xi$ to denote the minimal geodesic px parametrized by arclength. From now on, all geodesics are assumed to have unit speed unless otherwise stated.

The following theorem, which corresponds to the Toponogov comparison theorem in Riemannian geometry, is of basic importance in the geometry of Alexandrov spaces.

Theorem 1.1. ([BGP, 4.2]) — *If X has curvature $\geq k$, then*

(1.1.1) *for any $x, y, z \in X$, there is a triangle $\widetilde{\Delta}(x, y, z)$ in $M(k)$ such that each angle of $\widetilde{\Delta}(x, y, z)$ is not less than the corresponding one of $\Delta(x, y, z)$.*

In the case where $k > 0$ and the perimeter of $\Delta(x, y, z)$ is less than $2\pi/\sqrt{k}$, such a triangle is uniquely determined up to isometry.

(1.1.2) — *Suppose that* $|xy| = |\tilde{x}\tilde{y}|$, $|xz| = |\tilde{x}\tilde{z}|$ *for* $x, y, z \in X$, $\tilde{x}, \tilde{y}, \tilde{z} \in M(k)$, *and that* $\angle yxz = \angle \tilde{y}\tilde{x}\tilde{z}$. *Then* $|yz| \le |\tilde{y}\tilde{z}|$.

In [BGP], (1.1.1) is proved in the case when the perimeter is less than $2\pi/\sqrt{k}$. Then, the rest follows along the same line as the Toponogov comparison theorem (cf. [CE]).

Next, we briefly discuss measure of metric balls. It is quite natural to expect that the curvature assumption should influence on it. From now on, we assume that X has finite Hausdorff dimension, denoted by n. For $r > 0$, $b_k^n(r)$ denotes the volume of a metric r-ball in the n-dimensional simply connected space $M^n(k)$ with constant curvature k. We fix $p \in M$ and $\bar{p} \in M^n(k)$, and put $B_p(r) = B_p(r, X) = \{x \in X \,|\, |px| < r\}$.

Lemma 1.2. — *There exists an expanding map* $\rho : B_p(r) \to B_{\bar{p}}(r)$.

Proof. We show by induction on n. Since Σ_p has curvature ≥ 1 and diameter $\le \pi$, we have an expanding map $I : \Sigma_p \to S^{n-1} = \Sigma_{\bar{p}}$. For every $x \in B_p(r)$, put $\rho(x) = \exp_{\bar{p}} |px| I(\xi)$, where ξ is any element in x'. Theorem 1.1.2 then shows that ρ is expanding. □

Let V_n denote the Hausdorff n-measure. Lemma 1.2 immediately implies

$$V_n(B_p(r)) \le b_k^n(r)\,. \tag{1.3}$$

In the Appendix, we shall discuss the equality case in (1.3) and relative volume comparison.

A system of pairs of points $(a_i, b_i)_{i=1}^m$ is called an (m, δ)-strainer at p if it satisfies the following conditions:

$$\begin{array}{ll} \tilde{\angle} a_i p b_i > \pi - \delta\ , & |\tilde{\angle} a_i p b_i - \pi/2| < \delta\ , \\ |\tilde{\angle} b_i p b_j - \pi/2| < \delta\ , & |\tilde{\angle} a_i p b_j - \pi/2| < \delta \quad (i \ne j)\ . \end{array}$$

The number $\min_{1 \le i \le m}\{|a_i p|, |b_i p|\}$ is called the *length* of the strainer (a_i, b_i). It should be remarked that one can make the length of (a_i, b_i) as small as one likes by retaking strainer on minimal geodesics from p to a_i, b_i.

From now on, we assume that X has curvature ≥ -1 for simplicity. For n and $\mu_0 > 0$ we use the symbol $\tau(\delta, \dots, \epsilon)$ to denote a positive function depending only on n, μ_0, $\delta, \dots, \epsilon$ satisfying $\lim_{\delta,\dots,\epsilon\to 0} \tau(\delta, \dots, \epsilon) = 0$.

A surjective map $f : X \to Y$ is called an ϵ-almost isometry if $||f(x)f(y)|/|xy| - 1| < \epsilon$ for all $x, y \in X$.

Theorem 1.4 ([BGP, 10.4]). — *There exists $\delta_n > 0$ satisfying the following. Let $(a_i, b_i)_{i=1}^n$ be an (n, δ)-strainer at p with length $\geq \mu_0$, $\delta \leq \delta_n$. Then, the map $f : X \to \mathbf{R}^n$ defined by $f(x) = (|a_1 x|, \dots, |a_n x|)$ provides a $\tau(\delta, \sigma)$-almost isometry of a metric ball $B_p(\sigma)$ onto an open subset of $\mathbf{R}^n$, where $\sigma \ll \mu_0$.*

A system $(A_i, B_i)_{i=1}^m$ of pairs of subsets in an Alexandrov space Σ with curvature ≥ 1 is called a global (m, δ)-strainer if it satisfies

$$\begin{array}{ll} |\xi_i \eta_i| > \pi - \delta\ , & ||\xi_i \xi_j| - \pi/2| < \delta\ , \\ ||\xi_i \eta_j| - \pi/2| < \delta\ , & ||\eta_i \eta_j| - \pi/2| < \delta \quad (i \neq j) \end{array}$$

for every $\xi_i \in A_i$ and $\eta_i \in B_i$. It should be remarked that if $(a_i, b_i)_{i=1}^m$ is an (m, δ)-strainer at $p \in X$, then $(a_i', b_i')_{i=1}^m$ is a global (m, δ)-strainer of Σ_p. The result for global strainers, corresponding to Theorem 1.4 is the following (compare [OSY]).

Theorem 1.5 ([BGP, 10.5]). — *There exists a positive number δ_n satisfying the following. Let Σ be an Alexandrov space with curvature ≥ 1 and with Hausdorff dimension $n - 1$. Suppose that Σ has a global (n, δ)-strainer $(A_i, B_i)_{i=1}^n$ for $\delta \leq \delta_n$. Then,*

(1.5.1) $|\sum_{i=1}^n \cos^2 |A_i \xi| - 1| < \tau(\delta)$ *for every $\xi \in \Sigma$,*

(1.5.2) *the map f of Σ to the unit $(n-1)$-sphere $S^{n-1} \subset \mathbf{R}^n$ defined by*

$$f(\xi) = \frac{(\cos |A_i \xi|)}{|(\cos |A_i \xi|)|}$$

is a $\tau(\delta)$-almost isometry.

As a result of Theorem 1.5, it turns out that the space of directions Σ_p at an (n, δ)-strained point p in X is $\tau(\delta)$-almost isometric to S^{n-1}.

Let $f : X \to \mathbf{R}$ be a Lipschitz function. The directional derivative of f in a direction $\xi \in \Sigma'_p$ is defined as

$$df(\xi) = \lim_{t \downarrow 0} \frac{f(\exp t\xi) - f(p)}{t} ,$$

if it exists. Then df extends to a Lipschitz function on Σ_p.

Proposition 1.6 ([BGP, 12.4]). — *If f is a distance function from a fixed point $p \in X$,*

$$df(\xi) = -\cos|\xi p'|$$

for every $x \in X$ and $\xi \in \Sigma_x$.

We now represent some basic properties of (n, δ)-strained points of X.

Lemma 1.7. — *Let X,p and δ, σ be as in Theorem* 1.4. *Then, for every $q, r, s \in B_p(\sigma)$ with $1/100 \le |qr|/|qs| \le 1$, we have $|\angle rqs - \tilde{\angle} rqs| < \tau(\delta, \sigma)$.*

Proof. This is an immediate consequence of Theorem 1.4.

Lemma 1.8. — *Let X,p and δ, σ be as in Theorem* 1.4. *Then for every $q \in B_p(\sigma/2)$ and $\xi \in \Sigma_q$, there exist points $r, s \in B_p(\sigma)$ such that*

(1.8.1) $|qr|, |qs| \ge \sigma/4$,

(1.8.2) $|\xi r'| < \tau(\delta, \sigma)$,

(1.8.3) $\tilde{\angle} rqs > \pi - \tau(\delta, \sigma)$.

Proof. For $\xi \in \Sigma_q$ and a fixed $\theta > 0$, let us consider the set $A = \{x = \exp t\eta \,|\, |\xi\eta| \le \theta, \sigma/4 \le t \le \sigma/2\}$. For $\bar{q} \in M^n(-1)$, let $I : \Sigma_q \to \Sigma_{\bar{q}}$ and $\rho : B_q(\sigma/2) \to B_{\bar{q}}(\sigma/2)$ be as in Lemma 1.2. Now suppose that A is empty. Then $\rho(B_q(\sigma/2)) \subset B_{\bar{q}}(\sigma/2) - \widetilde{A}$, where $\widetilde{A} = \{x = \exp\, t\eta \,|\, |I(\xi)\eta| \le \theta, \sigma/4 \le t \le \sigma/2\}$. It follows from (1.3) that

$$\frac{V_n(B_q(\sigma/2))}{b^n_{-1}(\sigma/2)} \le \frac{b_1^{n-1}(\pi) - b_1^{n-1}(\theta)}{b_1^{n-1}(\pi)} + \frac{b^n_{-1}(\sigma/4) b_1^{n-1}(\theta)}{b^n_{-1}(\sigma/2) b_1^{n-1}(\pi)} .$$

On the other hand since $B_q(\sigma/2)$ is $\tau(\delta, \sigma)$-almost isometric to $B(\sigma/2)$,

$$\frac{V_n(B_q(\sigma/2))}{b^n_{-1}(\sigma/2)} > 1 - \tau(\delta, \sigma) .$$

Therefore $\theta < \tau(\delta,\sigma)$. Thus we can find r satisfying (1.8.1) and (1.8.2). For (1.8.3) it suffices to take s such that $|f(q)f(s)| = \sigma/2$ and $\angle f(r)f(q)f(s) = \pi$.

Lemma 1.9. — *Let* X,p,δ,σ *be as in Theorem* 1.4. *Then for every* q *with* $\sigma/10 \le |pq| \le \sigma$ *and for every* x *with* $|px| \ll \sigma$, *we have*

$$|\angle xpq - \tilde{\angle} xpq| < \tau(\delta,\sigma,|px|/\sigma)\ .$$

Proof. By Lemma 1.8, we can take r such that $|pr| \ge \sigma/4$ and $\tilde{\angle} qpr > \pi - \tau(\delta,\sigma)$. Then the lemma follows from [BGP, Lemma 5.6].

We have just verified that the constant μ_0 or σ plays a role similar to the injectivity radius at p.

2. EMBEDDING X INTO $L^2(X)$

From now on, we assume that X is an n-dimensional complete Alexandrov space with curvature ≥ -1 satisfying

$$\delta\text{-str.rad}\ (X) > \mu_0 \tag{2.1}$$

for a fixed $\mu_0 > 0$ and a small $\delta > 0$. By definition, for every $p \in X$ there exists an (n,δ)-strainer (a_i, b_i) at p with length $> \mu_0$. Let σ be a positive number with $\sigma \ll \mu_0$. Then, by Lemmas 1.7 and 1.8, we may assume that for every $p \in X$

(2.2.1) — there exists an (n,δ)-strainer at every point in $B_p(\sigma)$,

(2.2.2) — for every $q \in B_p(\sigma)$ and for every $\xi \in \Sigma_q$, there exist points r, s such that $|qr| \geq \sigma$, $|qs| \geq \sigma$ and $|\xi r'| < \tau(\delta,\sigma)$, $\tilde{\angle} rqs > \pi - \tau(\delta,\sigma)$,

(2.2.3) — $|\angle rqs - \tilde{\angle} rqs| < \tau(\delta,\sigma)$, for any $q, r, s \in B_p(10\sigma)$ with $1/100 \leq |qr|/|qs| \leq 1$.

Let $L^2(X)$ denote the Hilbert space consisting of all L^2 functions on X with respect to the Hausdorff n-measure. In this section we study the map $f_X : X \to L^2(X)$ defined by

$$f_X(p)(x) = h(|px|)\,,$$

where $h : \mathbf{R} \to [0,1]$ is a smooth monotone non-increasing function such that

(2.3.1) $\quad h = 1 \quad$ on $(-\infty, 0]$, $\qquad h = 0 \quad$ on $[\sigma, \infty)$.

(2.3.2) $\quad h' = 1/\sigma \quad$ on $[2\sigma/10, 8\sigma/10]$.

(2.3.3) $\quad -\sigma^2 < h' < 0 \quad$ on $(0, \sigma/10]$.

(2.3.4) $\quad |h''| < 100/\sigma^2$.

Remark that f_X is a Lipschitz map.

From now on, we use $c_1, c_2, \dots$ to express positive constants depending only on the dimension n. First we remark that by Theorem 1.4 there exist constants c_1 and c_2 such that for every $p \in X$,

$$c_1 < \frac{V_n(B_p(\sigma))}{b_0^n(\sigma)} < c_2 \,. \tag{2.4}$$

We next consider the directional derivatives of f_X. For $\xi \in \Sigma_p$, we put

$$df_X(\xi)(x) = -h'(|px|) \cos|\xi x'|, \quad (x \in X)\,. \tag{2.5}$$

Since $x \to |\xi x'|$ is upper semicontinuous, $df_X(\xi)$ is an element of $L^2(X)$, and by Lebesgue's convergence theorem and Proposition 1.6,

$$df_X(\xi) = \lim_{t \downarrow 0} \frac{f_X(\exp t\xi) - f_X(p)}{t} \qquad \text{in } L^2(X)\,.$$

From now on, we use the norm of $L^2(X)$ with normalization

$$|f|^2 = \frac{\sigma^2}{b(\sigma)} \int_X |f(x)|^2 d\mu(x)\,,$$

where $b(\sigma) = b_0^n(\sigma)$ and $d\mu$ denotes the Hausdorff n-measure.

Lemma 2.6. — *There exist positive numbers c_3 and c_4 such that for every $p \in X$ and $\xi \in \Sigma_p$,*

$$c_3 < |df_X(\xi)| < c_4\,.$$

Proof. By (2.2.2) take q such that $|pq| \geq \sigma/2$ and $|\xi q'| < \tau(\delta, \sigma)$. Then, it follows from (2.2.3) that for every $x \in B_q(\sigma/100)$, $\angle xpq < 1/20$ and hence $|\xi x'| < 1/10$. Then, the lemma follows from (2.3), (2.4) and (2.5). □

Lemma 2.7. — *There exist positive numbers c_5 and c_6 such that, for every $p, q \in X$ with $|pq| \leq \sigma$,*

$$c_5 < \frac{|f_X(p) - f_X(q)|}{|pq|} < c_6\,.$$

In particular f_X is injective.

Proof. By Lemma 2.6, we can take $c_6 = c_4$. Let $\ell = |pq|$. By (2.2.2) we can take a $(1, \tau(\delta, \sigma))$-strainer (p, r) at q with $|qr| = \sigma/2$. Let $c : [0, \ell] \to X$ be a minimal geodesic joining q to p. Then by (2.2.3), $\angle rc(t)x < 1/10$ for every x in $B_r(\sigma/100)$. It follows that

$$\begin{aligned} h(|px|) - h(|qx|) &= \int_0^\ell \frac{d}{dt} h(|c(t)x|)\, dt \\ &= \int_0^\ell h'(|c(t)x|) \cos \angle rc(t)x\, dt \\ &> \frac{\ell}{\sigma} \cos(1/10)\,, \end{aligned}$$

which implies

$$\frac{|f_X(p) - f_X(q)|}{|pq|} > \sqrt{c}_1 \cos(1/10) > 0\,.$$

□

Let $K_p = K(\Sigma_p)$ be the tangent cone at p. From definition, Σ_p can be considered as a subset of K_p. The map $df_X : \Sigma_p \to L^2(X)$ naturally extends to $df_X : K_p \to L^2(X)$. Next, we show that $df_X(K_p)$ can be approximated by an n-dimensional subspace of $L^2(X)$.

For a global (n, δ)-strainer (ξ_i, η_i) of Σ_p, let Π_p be the subspace of $L^2(X)$ generated by $df_X(\xi_i)$.

Lemma 2.8. — *For any $\xi \in \Sigma_p$,*

$$|df_X(\xi) - \sum_{i=1}^n c_i\, df_X(\xi)| < \tau(\delta)\,,$$

where $c_i = \cos |\xi_i \xi|$. In particular, $df_X(\xi_1), \dots, df_X(\xi_n)$ are linearly independent.

Proof. Let $\phi : \Sigma_p \to S^{n-1}$ be the $\tau(\delta)$ almost isometry defined by

$$\phi(\xi) = (\cos |\xi_i \xi|)/|(\cos |\xi_i \xi|)|\,.$$

(See Theorem 1.5). Using (1.5.1), one can verify

$$|\cos |\xi\eta| - \sum_{i=1}^n c_i \cos |\xi_i \eta|| < \tau(\delta)\,,$$

for every $\eta \in \Sigma_p$. It follows that

$$\begin{aligned}
&|df_X(\xi) - \sum_{i=1}^{n} c_i\, df_X(\xi_i)|^2 \\
&\qquad = \frac{\sigma^2}{b(\sigma)} \int_X (h'(|px|))^2 (\cos|\xi x'| - \sum_{i=1}^{n} c_i \cos|\xi_i x'|)^2\, d\mu(x) \\
&\qquad < \tau(\delta)\,.
\end{aligned}$$

Next, suppose that $\sum \alpha_i\, df_X(\xi_i) = 0$ for a nontrivial α_i. If we assume that $\sum \alpha_i^2 = 1$, then there exists a $\xi \in \Sigma_p$ such that $\phi(\xi) = (\alpha_1, \dots, \alpha_n)$. It turns out that

$$|df_X(\xi)| = |df_X(\xi) - \sum \alpha_i\, df_X(\xi_i)| < \tau(\delta)\,,$$

which contradicts Lemma 2.6 if δ is sufficiently small. □

Thus, $df_X(K_p)$ can be approximated by the n-dimensional subspace Π_p. In view of Lemma 2.8, one may say that df_X is almost linear.

3. CONSTRUCTION OF A TUBULAR NEIGHBORHOOD

In this section, we construct a tubular neighborhood of $f_X(X)$ in $L^2(X)$. In the case where X is a smooth Riemannian manifold with bounded curvature, Katsuda [K] studied a tubular neighborhood of a smooth embedding of X into a Euclidean space by using an estimate on the second fundamental form. However, in our case, $f_X(X)$ is a Lipschitz manifold. Hence, even the existence of a tubular neighborhood in a generalized sense is a priori nontrivial.

We begin with a lemma.

Lemma 3.1. — *For any $p, q \in X$, $d_H^{L^2}(df_X(\Sigma_p), df_X(\Sigma_q)) < \tau(\delta, \sigma, |pq|/\sigma)$, where $d_H^{L^2}$ denotes the Hausdorff distance in $L^2(X)$.*

Proof. By (2.2.2), for every $\xi \in \Sigma_q$ there exists r satisfying $|qr| \geq \sigma$ and $|\xi r'| < \tau(\delta, \sigma)$. We put $\xi_1 = r' \in \Sigma_p$. By using (2.2.3), we then have $||\xi x'| - |\xi_1 x'|| < \tau(\delta, \sigma, |pq|/\sigma)$ for all x with $\sigma/10 \leq |px| \leq \sigma$. It follows that $|df_X(\xi) - df_X(\xi_1)| < \tau(\delta, \sigma, |pq|/\sigma)$. □

We put $\widetilde{N}_p = f_X(p) + \Pi_p^{\perp}$, where $\perp$ denotes the orthogonal complement in $L^2(X)$.

Lemma 3.2. — *For any $p, q \in X$ and ξ in $q' \subset \Sigma_p$,*

$$\left| \frac{f_X(q) - f_X(p)}{|qp|} - df_X(\xi) \right| < \tau(\delta, \sigma, |pq|/\sigma)\,. \tag{3.2.1}$$

In particular, $f_X(B_p(\sigma_1)) \cap \widetilde{N}_p = \{f_X(p)\}$ if σ_1/σ is sufficiently small.

Proof. By Lemma 1.9, $|\angle xpq - \tilde{\angle} xpq| < \tau(\delta, \sigma, |pq|/\sigma)$ for all x with $\sigma/10 \leq |px| \leq \sigma$. We put $t = |pq|$. Since $||xq| - |xp| + t \cos \tilde{\angle} xpq| < t\tau(t/\sigma)$, it follows that

$$||xq| - |xp| + t \cos |\xi x'|| < t\,\tau(\delta, \sigma, t/\sigma)\,, \tag{3.3}$$

which yields (3.2.1). Since (3.2.1) shows that the vector $f_X(q) - f_X(p)$ is transversal to $\widetilde{N}_p$, we obtain $f_X(B_p(\sigma_1)) \cap \widetilde{N}_p = \{f(p)\}$ for sufficiently small σ_1/σ. $\square$

For $q \in B_p(\sigma_1)$ and $\sigma_1 \ll \sigma$, we put

$$\widehat{N}_q = f_X(q) + \Pi_p^{\perp} .$$

Then, Lemmas 2.8, 3.1 and 3.2 imply the following.

Lemma 3.4. — *We have $f_X(B_p(\sigma_1)) \cap \widehat{N}_q = \{f_X(q)\}$ for all $q \in B_p(\sigma_1)$.*

Let G_n be the infinite-dimensional Grassmann manifold consisting of all n-dimensional subspaces in $L^2(X)$. Let $\{p_i\}$ be a maximal set in X such that $|p_i p_j| \geq \sigma_1/10$, $(i \neq j)$, and $T_i : B_i \to G_n$ be the constant map, $T_i(x) = \Pi_{p_i}$, where $B_i = B_{f_X(p_i)}(c_6\sigma_1/10, L^2(X))$. Notice that $\{B_i\}$ covers $f_X(X)$ and that the multiplicity of the covering has a uniform bound depending only on n. (See Lemma 1.2, or Proposition A.4).

Our next step is to take an average of T_i in G_n to obtain a global map $T : \cup B_i \to G_n$. We need the notion of angle on G_n. The space G_n has a natural structure of Banach manifold. The local chart at an element $T_0 \in G_n$ is given as follows. Let N_0 be the orthogonal complement of T_0, and $L(T_0, N_0)$ the Banach space consisting of all homomorphisms of T_0 into N_0, where the norm of $L(T_0, N_0)$ is the usual one defined by

$$\|f\| = \sup_{0 \neq x \in T_0} \frac{|f(x)|}{|x|}, \quad (f \in L(T_0, N_0)) .$$

We put $V = \{T \in G_n \,|\, T \cap N_0 = \{0\}\}$. Then, $p(T) = T_0$ for every $T \in V$, where $p : L^2(X) \to T_0$ is the orthogonal projection. Hence, T is the graph of a homomorphism $\varphi_{T_0}(T) \in L(T_0, N_0)$. Thus, we have a bijective map $\varphi_{T_0} : V \to L(T_0, N_0)$, which imposes a Banach manifold structure on G_n.

Under the notation above, the angle $\angle(T_0, T_1)$ between T_0 and T_1 $(\in G_n)$ is given by

$$\angle(T_0, T_1) = \begin{cases} \operatorname{Arc\,tan} \|\varphi_{T_0}(T)\| & \text{if} \quad T_1 \cap T_0^{\perp} = \{0\} \\ \pi/2 & \text{if} \quad T_1 \cap T_0^{\perp} \neq \{0\} . \end{cases}$$

It is easy to check that the angle gives a distance on G_n and that the topology of G_n coincides with that induced from angle.

From now on, we use the simpler notation τ to denote a positive function of type $\tau(\delta, \sigma, \sigma_1/\sigma)$.

An estimate for the second fundamental form in case of X being a smooth Riemannian manifold can be replaced by the following more elementary lemma. We put $U = \cup B_i$.

Lemma 3.5. — *There exists a smooth map $T : U \to G_n$ such that*

(3.5.1) $\angle(T(x), T_i(x)) < \tau$ *if* $x \in B_i$,

(3.5.2) $\angle(T(x), T(y)) < C|x - y|$, *where* $C = \tau/\sigma_1$.

Proof. Let $\{\rho_i\}$ be a partition of unity associated with $\{B_i\}$ such that $|\nabla\rho_i| \leq 100/c_6\sigma_1$. First, put $T = T_1$ on B_1 and extend it on $B_1 \cup B_2$ as follows. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be orthonormal bases of T_1 and T_2 respectively such that $|v_i - w_i| < \tau$. Put $u_i(x) = \rho_1(x)v_i + (1 - \rho_1(x))w_i$, and let $T(x)$ be the n-plane generated by $u_1(x), \ldots, u_n(x)$, $(x \in B_1 \cup B_2)$. Then, $\{u_1(x), \ldots, u_n(x)\}$ is a τ-almost orthonormal basis of $T(x)$ in the sense that

$$| < u_i(x), u_j(x) > -\delta_{ij}| < \tau \, .$$

Notice that $\angle(T(x), T_i) < \tau$ if $x \in B_i$ (i=1,2), and $|\nabla u_i| < \tau/\sigma_1$.

Suppose that $T(x)$ and a τ-almost orthonormal basis $\{v_1(x), \ldots, v_n(x)\}$ of $T(x)$ are defined for $x \in U_j = \cup_{i=1}^{j} B_i$ in such a way that

(3.6.1) $\angle(T(x), T_i) < \tau$ if $x \in B_i$, $(1 \leq i \leq j)$,

(3.6.2) $|\nabla v_i| < \tau/\sigma_1$.

We extend them on U_{j+1} as follows. Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis of T_{j+1} such that $|v_i(x)w_i| < \tau$ on $U_j \cap B_{j+1}$. Now, we put

$$u_i(x) = \left(\sum_{\alpha=1}^{j} \rho_\alpha(x)\right) v_i(x) + \left(1 - \sum_{\alpha=1}^{j} \rho_\alpha(x)\right) w_i \, ,$$

and let $T(x)$ be the subspace genereted by $u_i(x)$. Then, it is easy to check that $T(x)$ and $u_i(x)$ satisfy the properties of (3.6). Thus, by induction, we have a smooth map $T : U \to G_n$ and a τ-almost orthonormal frame $u_i(x)$ for $T(x)$ satisfying (3.6). It follows from (3.6.2)

$$\begin{aligned}\angle(T(x), T(y)) &\leq \mathrm{const}_n \max_{1\leq i\leq n} |u_i(x) - u_i(y)| \\ &\leq \mathrm{const}_n \max_{1\leq i\leq n} |\nabla u_i||x - y| \\ &\leq (\tau/\sigma_1)|x - y|\,.\end{aligned}$$

□

Let G_n^* be the Grassmann manifold consisting of all subspaces of codimension n in $L^2(X)$, and $\nu : U \to G_n^*$ the dual of T, $\nu(x) = T(x)^\perp$. The angle $\angle(\nu(x), \nu(y))$ is also defined in a way similar to $\angle(T(x), T(y))$. Remark that the equality $\angle(\nu(x), \nu(y)) = \angle(T(x), T(y))$ holds. We put

$$N_x = x + \nu(x)\,.$$

By using (3.5.1), we have the following lemma in a way similar to Lemma 3.2.

Lemma 3.7. — *For every $p \in X$ and $q \in B_p(\sigma_1)$,*

$$f_X(B_p(\sigma_1)) \cap N_{f_X(q)} = \{f_X(q)\}\,.$$

For $c > 0$, we put

$$\mathcal{N}(c) = \{(x, v)|x \in f_X(X), v \in \nu(x), |v| < c\}\,.$$

Lemma 3.8. — *There exists a positive number $\kappa = \mathrm{const}_n\,\sigma_1$ such that $\mathcal{N}(\kappa)$ provides a tubular neighborhood of $f_X(X)$. Namely,*

(3.8.1) *$x_1 + v_1 \neq x_2 + v_2$ for every $(x_1, v_1) \neq (x_2, v_2) \in \mathcal{N}(\kappa)$;*

(3.8.2) *the set $U(\kappa) = \{x + v|(x, v) \in \mathcal{N}(\kappa)\}$ is open in $L^2(X)$.*

Proof. Suppose that $x_1 + v_1 = x_2 + v_2$ for $x_i = f_X(p_i)$ and $v_i \in \nu(x_i)$. If $|p_1p_2| > \sigma_1$ and $|v_i| \leq c_5\sigma_1/2$, a contradiction would immedately arise from Lemma 2.7. We consider the case $|p_1p_2| \le \sigma_1$. Put $K = N_{x_1} \cap N_{x_2}$, and let $y \in K$ and $z \in N_{x_2}$ be such that $|x_1y| = |x_1K|$,$|x_1y| = |yz|$ and that $\angle x_1yz = \angle(x_1 - y, N_{x_2}) \le \angle(N_{x_1}, N_{x_2})$. Then, Lemma 3.1 implies that $\angle x_1yz < \tau$. It follows from the choice of z that $|\angle(x_1 - z, N_{x_2}) - \pi/2| < \tau$. On the other hand, the fact $\angle(x_2 - x_1, T(x_1)) < \tau$ (Lemma 3.2) also implies that $|\angle(x_2 - x_1, N_{x_2}) - \pi/2| < \tau$. It follows that $|x_2z| < \tau|x_1x_2|$. Putting $\ell = |yx_1| = |yz|$ and using Lemma 3.5, we then have

$$\begin{aligned}|x_1z| &\le \ell\angle x_1yz \\ &\le \ell\angle(T(x_1), T(x_2)) \\ &\le \ell C|x_1x_2|, \quad C = \tau/\sigma_1\,.\end{aligned}$$

Thus, we obtain $\ell \ge (1-\tau)/C \ge \sigma_1/\tau$ as required. □

The proof of (3.8.2) follows from (3.8.1): For any $y \in U(\kappa)$ with $y \in N_{x_0}$, $x_0 \in f_X(X)$ and for any $z \in L^2(X)$ close to y, let T_0 be the n-plane through z and parallel to $T(x_0)$, and y_0 the intersection point of T_0 and N_{x_0}. If $x \in f_X(X)$ is near x_0, then N_x meets T_0 at a unique point, say $\alpha(x)$. Using (3.8.1), we can observe that α is a homeomorphism of a neighborhood of x_0 in $f_X(X)$ onto a neighborhood of y_0 in T_0. Hence $z \in U(\kappa)$ as required.

Remark 3.9. — The proof of Lemma 3.8 suggests the possibility that one can take the constant κ in the lemma such as $\kappa = \sigma_1/\tau$. In fact we can get the sharper estimate by a bit more refined argument. However, we omit the proof because we do not need the estimate in this paper.

Next, let us study the properties of the projection $\pi : \mathcal{N}(\kappa) \to f_X(X)$ along ν. By definition, $\pi(x) = y$ if $x \in N_y$ and $y \in f_X(X)$.

Lemma 3.10. — *The map $\pi : \mathcal{N}(\kappa) \to f_X(X)$ is locally Lipschitz continuous. More precisely, if $x, y \in \mathcal{N}(\kappa)$ are close each other and $t = |x\pi(x)|$, then*

(3.10.1) $|\pi(x)\pi(y)|/|xy| < 1 + \tau + \tau t/\sigma_1\,,$

(3.10.2) *if* $|\angle(y - x, N_{\pi(x)}) - \pi/2| < \tau$, *then*

$$|(y-x) - (\pi(y) - \pi(x))| < (\tau + \tau t/\sigma_1)|xy|\,.$$

Proof. First we prove (3.10.2). Let N be the affine space of codimension n parallel to $N_{\pi(x)}$ and through y. Let y_1 and y_2 be the intersections of $N_{\pi(y)}$ and N with $T_{\pi(x)}$ respectively. Let z be the point in $K = N \cap N_{\pi(y)}$ such that $|y_2 z| = |y_2 K|$, and $y_3 \in N_{\pi(y)}$ the point such that $|y_2 z| = |y_3 z|$ and $\angle y_2 z y_3 = \angle(y_2 - z, N_{\pi(y)}) \le \angle(N, N_{\pi(y)})$. An argument similar to that in Lemma 3.8 yields that

$$(3.11.1) \qquad |y_1 y_3| < \tau |y_1 y_2|\,,$$

$$(3.11.2) \qquad |y_2 y_3|/|z y_2| \le \angle(\nu(\pi(x)), \nu(\pi(y))) \le (\tau/\sigma_1)|\pi(x)\pi(y)|\,.$$

It follows that $|y_1 y_2| < (\tau/\sigma_1)t|\pi(x)\pi(y)|$. Furthermore the assumption implies $|(\pi(x) - y_2) - (x - y)| < \tau|xy|$. Therefore, we get

$$\begin{aligned} |(\pi(x) - y_1) - (x - y)| &\le |(\pi(x) - y_1) - (\pi(x) - y_2)| + |(\pi(x) - y_2) - (x - y)| \\ &\le |y_1 y_2| + \tau|xy| \\ &< (\tau/\sigma_1)t|\pi(x)\pi(y)| + \tau|xy|\,. \end{aligned}$$

On the other hand, since $\angle y_1 \pi(x)\pi(y) < \tau$,

$$|(\pi(x) - \pi(y)) - (\pi(x) - y_1)| < \tau|\pi(x)\pi(y)|\,.$$

Combining the two inequalities, we obtain that

$$|(\pi(x) - \pi(y)) - (x - y)| < (\tau + C't)|\pi(x)\pi(y)| + \tau|xy|\,,$$

from which (3.10.2) follows.

For (3.10.1), take $y_0 \in N_{\pi(y)}$ such that $|xy_0| = |xN_{\pi(y)}|$. Then, (3.10.2) implies

$$\begin{aligned} \frac{|\pi(x)\pi(y)|}{|xy|} &\le \frac{|\pi(x)\pi(y)|}{|xy_0|} \\ &\le 1 + \tau + \tau t/\sigma_1\,. \end{aligned}$$

□

4. f IS AN ALMOST LIPSCHITZ SUBMERSION

In this section, we shall prove Theorem 0.2.

Let M be an Alexandrov space with curvature ≥ -1. We suppose $d_H(M,X) < \epsilon$ and $\epsilon \ll \sigma_1$. Let $\varphi : X \to M$ and $\psi : M \to X$ be ϵ-Hausdorff approximations such that $|\psi\varphi(x), x| < \epsilon$, $|\varphi\psi(x), x| < \epsilon$, where we may assume that φ is measurable. Then, the map $f_M : M \to L^2(X)$ defined by, for $x \in X$

$$f_M(p)(x) = h(|p\varphi(x)|)\,,$$

should have the properties similar to those of f_X. We begin with a lemma.

Lemma 4.1. — *We have $f_M(M) \subset \mathcal{N}(c_7\epsilon)$.*

Proof. This follows immediately from

$$|f_M(p) - f_X(\psi(p))| < c_7\epsilon\,. \tag{4.2}$$

□

By Lemmas 3.8 and 4.1, the map $f = f_X^{-1} \circ \pi \circ f_M : M \to X$ is well defined.

Lemma 4.3. — *We have $d(f(p), \psi(p)) < c_8\epsilon$.*

Proof. It follows from (4.2) that $|f_X(f(p)) - f_X(\psi(p))| < 3c_7\epsilon$. Since we may assume that $|f(p)\psi(p)| < \sigma$, we have $|f(p)\psi(p)| < 3c_7\epsilon/c_5$ by Lemma 2.7. □

It follows from Lemmas 3.10 and 4.3 that f is a Lipschitz map.

Similarly to (2.5), $df_M(\xi) \in L^2(X)$, $\xi \in \Sigma_p$, is given by

$$df_M(\xi)(x) = -h'(|p\varphi(x)|)\cos|\xi\varphi(x)'|\,. \tag{4.4}$$

Lemma 4.5. — *For every $p, q \in M$ take ξ in $q' \subset \Sigma_p$. Then*

$$\left|\frac{f_M(q) - f_M(p)}{|qp|} - df_M(\xi)\right| < \tau(\delta, \sigma, \epsilon/\sigma, |pq|/\sigma)\,.$$

Proof. For every x with $\sigma/10 \le |px| \le \sigma$, take $y \in X$ such that $\tilde{\angle}\psi(x)\psi(p)y > \pi - \tau(\delta, \sigma)$. Since $\tilde{\angle}xp\varphi(y) > \pi - \tau(\delta, \sigma) - \tau(\epsilon/\sigma)$, it follows from an argument similar to Lemma 3.2 that $||qx| - |px| + |qp| \cos|\xi x'|| < |qp|\tau(\delta, \sigma, \epsilon/\sigma, |pq|/\sigma)$, which implies the required inequality. □

We now fix $p \in M$, and put $\bar{p} = f(p)$ and

$$H_p = \{\xi \,|\, \xi \in x' \subset \Sigma_p, |px| \ge \sigma/10\}\,,$$

which can be regarded as the set of "horizontal directions" at p.

Lemma 4.6. — *For every $\bar{\xi} \in \Sigma'_{\bar{p}}$, there exists $q \in M$ with $|pq| \ge \sigma$ such that*

$$|f(\exp t\xi), \exp t\bar{\xi}| < t\tau(\delta, \sigma, \sigma_1/\sigma, \epsilon/\sigma_1)\,,$$

for every ξ in $q' \subset \Sigma_p$ and sufficiently small $t > 0$.

Conversely, for every $\xi \in H_p$, there exists $\bar{\xi} \in \Sigma'_{\bar{p}}$ satisfying the above inequality.

In other words, the curve $f(\exp t\xi)$ is almost tangent to $\exp t\bar{\xi}$.

For the proof of Lemma 4.6, we need

Comparison Lemma 4.7. — *Let x, y, z be points in M, and $\bar{x}, \bar{y}, \bar{z}$ points in X such that $\sigma/10 \le |xy|, |yz| \le \sigma$. Suppose that $|\psi(x)\bar{x}| < \tau(\epsilon)$, $|\psi(y)\bar{y}| < \tau(\epsilon)$ and $|\psi(z)\bar{z}| < \tau(\epsilon)$. Then, for every minimal geodesics xy, yz, and $\bar{x}\bar{y}, \bar{y}\bar{z}$, we have*

$$|\angle xyz - \angle \bar{x}\bar{y}\bar{z}| < \tau(\delta, \sigma, \epsilon/\sigma)\,.$$

Proof. By (2.2.2), we take a point $\bar{w} \in X$ such that

$$\tilde{\angle}\bar{z}\bar{y}\bar{w} > \pi - \tau(\delta, \sigma) \tag{4.8}$$

and $|\bar{y}\bar{w}| \geq \sigma$. Put $w = \varphi(\bar{w})$. Then, Theorem 1.1 and (2.2.3) imply that

$$\angle xyz > \angle \bar{x}\bar{y}\bar{z} - \tau(\delta, \sigma) - \tau(\epsilon/\sigma)\,, \tag{4.9.1}$$

$$\angle xyw > \angle \bar{x}\bar{y}\bar{w} - \tau(\delta, \sigma) - \tau(\epsilon/\sigma)\,. \tag{4.9.2}$$

Since (4.8) implies

$$|\angle zyw - \pi| < \tau(\delta, \sigma) + \tau(\epsilon/\sigma)\,,$$

(4.9.1) and (4.9.2) yield the required inequality. □

Proof of Lemma 4.6. Take $\bar{q} \in X$ such that $|\bar{p}\bar{q}| \geq \sigma$ and $|\bar{\xi}\bar{q}'| < \tau(\delta, \sigma)$. Put $q = \varphi(\bar{q})$. For any ξ in $q' \subset \Sigma_p$ let $c(t) = \exp t\xi$, $\bar{c}(t) = \exp t\bar{\xi}$. By using (2.3),(2.5),(4.4) and Lemma 4.7 we get $|df_M(\xi) - df_X(\bar{\xi})| < \tau(\delta, \sigma, \epsilon/\sigma)$. Lemmas 3.2 and 4.5 then imply

$$\left| \frac{f_M(c(t)) - f_M(p)}{t} - \frac{f_X(\bar{c}(t)) - f_X(q)}{t} \right| < \tau(\delta, \sigma, \epsilon/\sigma)\,,$$

for sufficiently small $t > 0$. In particular, $f_M(c(t)) - f_M(p)$ is almost perpendicular to $N_{\pi(f_M(p))}$. It follows from (3.10.2) that

$$\left| \frac{f_M(c(t)) - f_M(p)}{t} - \frac{\pi \circ f_M(c(t)) - \pi \circ f_M(p)}{t} \right| < \tau(\delta, \sigma, \sigma_1/\sigma, \epsilon/\sigma_1)\,,$$

and hence $|\pi \circ f_M(c(t)) - f_X(\bar{c}(t))\,| < t\tau(\delta, \sigma, \sigma_1/\sigma, \epsilon/\sigma_1)$. Lemma 2.7 then implies the required inequality.

Similarly, we have the second half of the lemma. □

From now on, we use the simpler notation τ_ϵ to denote a positive function of type $\tau(\delta, \sigma, \sigma_1/\sigma, \epsilon/\sigma_1)$.

The following fact follows from Lemma 4.6. For all $\xi \in H_p$ and small $t > 0$,

$$\left| \frac{|f(\exp t\xi), \bar{p}|}{t} - 1 \right| < \tau_\epsilon\,. \tag{4.10}$$

Lemma 4.11. — *For every $p, q \in M$, we have*

$$\left| \frac{|f(p)f(q)|}{|pq|} - \cos\theta \right| < \tau_\epsilon\,,$$

where $\theta = |\xi H_p|$, $\xi = q' \in \Sigma_p$.

For the proof of Lemma 4.11, we need two sublemmas.

Sublemma 4.12. — $d_H(H_p, S^{n-1}) < \tau_\epsilon$.

Proof. For each $\xi \in H_p$ let $\bar{\xi}$ be an element of $\Sigma_{\bar{p}}$ as in the second half of Lemma 4.6, and let $\chi : H_p \to \Sigma_{\bar{p}}$ be the map defined by $\chi(\xi) = \bar{\xi}$. By Lemma 4.7, $||\chi(\xi_1)\chi(\xi_2)| - |\xi_1\xi_2|| < \tau_\epsilon$, and Lemma 4.6 shows that $\chi(H_p)$ is τ_ϵ-dense in Σ_q. Thus χ is a τ_ϵ-Hausdorff approximation as required. □

Sublemma 4.13. — *For* $\xi \in \Sigma_p'$, *let* $\theta = |\xi H_p|$ *and* $\xi_1 \in H_p$ *be such that* $|\theta - |\xi\xi_1|| < \tau_\epsilon$. *Then,*

$$|f(\exp t\xi), f(\exp t \cos\theta\xi_1)| < t\tau_\epsilon\,,$$

for every sufficiently small $t > 0$.

Proof. Since Σ_p has curvature ≥ 1, we have an expanding map $\rho : \Sigma_p \to S^{m-1}$ with $m = \dim M$. First, we show that $||\rho(v_1)\rho(v_2)| - |v_1v_2|| < \tau_\epsilon$ for every $v_1, v_2 \in H_p$. Let $v_1^* \in H_p$ be such that $|v_1v_1^*| > \pi - \tau_\epsilon$. Since ρ is expanding, we obtain that

$$||v_1v_2| - |\rho(v_1)\rho(v_2)|| < \tau_\epsilon, \quad ||v_1^*v_2| - |\rho(v_1^*)\rho(v_2)|| < \tau_\epsilon\,. \tag{4.14}$$

This argument also implies that $\rho(H_p)$ is Hausdorff τ_ϵ-close to a totally geodesic $(n-1)$-sphere S^{n-1} in S^{m-1}. Let $\zeta : H_p \to S^{n-1} \subset S^{m-1}$ be a τ_ϵ-Hausdorff approximation such that $d(\zeta(v), \rho(v)) < \tau_\epsilon$ for all $v \in H_p$. For a given $\xi \in \Sigma_p$, an argument similar to (4.13) implies that $||\xi v| - |\rho(\xi)\zeta(v)|| < \tau$ for all $v \in H_p$. Remark that, for any y with $\sigma/10 \leq |py| \leq \sigma$, an elementary geometry yields

$$\cos|\rho(\xi)\zeta(y')| = \cos|\rho(\xi)\eta| \cos|\eta\zeta(y')|\,,$$

where η is an element of S^{n-1} such that $|\rho(\xi)\eta| = |\rho(\xi)S^{n-1}|$. It follows that for sufficiently small $t > 0$

$$
\begin{aligned}
&|f_M(\exp t\xi) - f_M(\exp t \cos \xi_1)|^2/t^2 \\
&\quad = \frac{\sigma^2}{b(\sigma)} \int_X \Big(\frac{h(|\exp t\xi, \varphi(x)|) - h(|p\varphi(x)|)}{t} \\
&\qquad\qquad\qquad - \frac{h(\exp t \cos \theta \xi_1, \varphi(x)|) - h(|p\varphi(x)|)}{t} \Big)^2 \, d\mu(x) \\
&\quad \le \frac{\sigma^2}{b(\sigma)} \int_X (h'(|p\varphi(x)|))^2 (\cos |\xi\varphi(x)'| - \cos\theta \cos |\xi_1 \varphi(x)'|)^2 \, d\mu(x) + \tau_\epsilon \\
&\quad \le \frac{\sigma^2}{b(\sigma)} \int_X (h')^2 (\cos |\xi\varphi(x)'| - \cos |\rho(\xi)\zeta(\varphi(x)')| \\
&\qquad\qquad + \cos |\rho(\xi)\eta| \cos |\eta\zeta(\varphi(x)')| - \cos |\xi\xi_1| \cos |\xi_1\varphi(x)'|)^2 \, d\mu(x) + \tau_\epsilon \\
&\quad \le \tau_\epsilon \, .
\end{aligned}
$$

Therefore, by Lemmas 3.10 and 2.7 we conclude the proof of the sublemma. ☐

Proof of Lemma 4.11. Since f is a $\tau(\epsilon)$-Hausdorff approximation (Lemma 4.3), we may assume that $|pq| < \sigma^2 \ll \sigma$. Let $c : [0, \ell] \to M$ be a minimal geodesic joining p to q where $\ell = |pq|$. By using (2.2.2), one can show that

$$|\angle qc(t)x - \angle qpx| < \tau_\epsilon \, , \tag{4.15}$$

for every $t < \ell$ and for every $x \in M$ with $\sigma/10 \le |px| \le \sigma$. Let ξ be any element in $q' \subset \Sigma_{c(t)}$, and $\eta_0 \in H_p$ such that $|\xi_0 H_p| = |\xi_0 \eta_0|$. Take y such that $\eta_0 = y'$, $\sigma/10 \le |py| \le \sigma$ and η_t in $y' \subset \Sigma_{c(t)}$. Put $\theta_t = \angle qc(t)y$. It follows from Sublemma 4.13 and (4.15) that

$$|f \circ c(t+s), f(\exp s \cos \theta_0 \eta_t)| < \tau_\epsilon s \, . \tag{4.16}$$

Put $\bar{c}(t) = f \circ c(t)$, and take any $\bar{\eta}_t$ in $\psi(y)' \subset \Sigma_{\bar{c}(t)}$. Then, by Lemma 4.6,

$$|f(\exp s \cos \theta_0 \eta_t), \exp s \cos \theta_0 \bar{\eta}_t| < \tau s \, . \tag{4.17}$$

By (2.2.3), we see that for every $z \in X$ with $\sigma/10 \le |\bar{p}z| \le \sigma$,

$$|\angle \psi(y)\bar{c}(t)z - \angle \psi(y)\bar{p}z| < \tau_\epsilon \, . \tag{4.18}$$

Now, let (a_i, b_i) be an (n,δ)-strainer at $\bar p$ such that $|\bar p a_i| = \sigma$ and $\lambda : B_{\bar p}(\sigma^2) \to \mathbf{R}^n$ be the bi-Lipschitz map, $\lambda(x) = (|a_1 x|, \dots, |a_n x|)$. Put $u(t) = \lambda \circ \bar c(t)$. Combining (4.16),(4.17) and (4.18), we get

$$|\dot u(s) - \dot u(t)| < \tau, \qquad ||\dot u(s)| - \cos\theta_0| < \tau\,,$$

for almost all $s, t \in [0, \ell]$. Thus, we arrive at

$$\begin{aligned}|\ell \dot u(s) - (\lambda(f(y)) - \lambda(f(x)))|\\ &\le \int_0^\ell |\dot u(s) - \dot u(t)|\, dt \le \tau_\epsilon \ell\,.\end{aligned}$$

This completes the proof. □

We conclude the proof of Theorem 0.2 by showing

Lemma 4.19. — *For every $p \in M$ and $x \in X$, there exists $q \in M$ such that $f(q) = x$ and $|f(p), f(q)| \ge (1 - \tau_\epsilon)|p, q|$.*

Namely, f is $(1-\tau_\epsilon)$-open in the sense of [BGP1].

Proof. First we show that f is surjective. Since f is proper, $f(M)$ is closed in X. Suppose that there exists a point $x \in X - f(M)$, and take $\bar p \in f(M)$ such that $|x\bar p| = |xf(M)|$ and put $\bar p = f(p)$. By Lemma 4.6, for any $\bar\xi$ in $x' \subset \Sigma_{\bar p}$ we would find $\xi \in H_p$ satisfying $|f(\exp t\xi), \exp t\bar\xi| < \tau_\epsilon t$ for sufficiently small $t > 0$. Thus, it turns out that $|f(\exp t\xi), x| < |\bar p x|$, a contradiction.

By Lemma 4.3, we may assume that $|f(p), x| < \sigma^2$. By Lemma 4.6, there exists $p_1 \in M$ such that $p_1' \in H_p$ and $|f(p_1), x| < |f(p), x|\,\tau_\epsilon$. Inductively, we have a sequence $\{p_i\}$ such that $p_i' \in H_{p_{i-1}}$ and $|f(p_i), x| < |f(p_{i-1}), x|\,\tau_\epsilon$. Since $|p_i, p_{i+1}| < (1+\tau_\epsilon)|f(p_i), f(p_{i+1})|$ and

$$\begin{aligned}|f(p_i), f(p_{i+1})| &< |f(p_{i+1}), x| + |x, f(p_i)|\\ &< (1+\tau_\epsilon)|f(p_{i-1}), x|\,\tau_\epsilon\\ &< (1+\tau_\epsilon)\tau_\epsilon^i\,,\end{aligned}$$

we see that $\{p_i\}$ is a Cauchy sequence. It follows that $f(q) = x$ for the limit point q of $\{p_i\}$, and that

$$\begin{aligned}|f(p), f(q)| &\ge |f(p), f(p_1)| - \sum_{i=1}^{\infty} |f(p_i), f(p_{i+1})|\\ &> (1-\tau_\epsilon)|p, p_1| - C\tau_\epsilon |f(p), f(q)|\,.\end{aligned}$$

This implies that f is $(1-\tau_\epsilon)$-open. □

Lemmas 4.11 and 4.19 show in particular that the fibre is not a point if $\dim M > \dim X$. Thus, the proof of Theorem 0.2 is now complete.

Proof of Corollary 0.4. If $\dim M = n$, then 2δ-strain radius of M is greater than $\mu_0/2$ for sufficiently small $\epsilon > 0$. Lemma 1.8 then implies that H_p is $\tau(\delta,\sigma)$-dense in Σ_p for any $p \in M$. It follows from Lemma 4.11 that $|f(x)f(y)|/|xy| - \cos\tau(\delta,\sigma)| < \tau_\epsilon$. Thus, f is a τ_ϵ-almost isometry as required. □

Remark 4.20. — Suppose that both M and X have natural differentiable structures of class C^1 such that the distance functions are C^1-class. In this case, we can take a locally trivial fibre bundle of class C^1 in addition as the map f. It suffices only to replace the maps f_X and f_M by C^1-maps defined by

$$f_X(p)(x) = h\left(\frac{1}{V_n(B_x(\epsilon))}\int_{B_x(\epsilon)} |py|\, d\mu(y)\right),$$

$$f_M(p)(x) = h\left(\frac{1}{V_m(B_{\varphi(x)}(\epsilon))}\int_{B_{\varphi(x)}(\epsilon)} |py|\, d\mu(y)\right).$$

For instance, if every point in X is an $(n,0)$-strained point, then X has a natural C^1-structure ([OS]). Remark that the fibre of f is an "almost nonnegatively curved manifold" in the sense of [Y].

By the previous remark, one can modify the main result in [O] as follows. We denote by $e^d(M)$ the excess defined there.

Corollary 4.21. — *For given m and $D, d > 0$, $(D \geq d)$, there exists a positive number $\epsilon = \epsilon_m(D,d)$ such that if a compact Riemannian m-manifold M with sectional curvature ≥ -1 satisfies*

$$\mathrm{diameter}(M) \leq D, \quad \mathrm{radius}(M) \geq d, \quad e^d(M) < \epsilon,$$

then there exists an Alexandrov space X with curvature ≥ -1 having C^1 differentiable structure and a fibration $f : M \to X$ whose fiber is an "almost nonnegatively curved manifold".

In [O], Otsu constructed a smooth Riemannian manifold X' with a similar property as in Corollary 4.21. Unfortunately, the lower sectional curvature bound of X' goes to $-\infty$ when M changes such as $e^d(M) \to 0$.

Proof of Corollary 4.21. Suppose the corollary does not hold. Then we would have a sequence of compact m-dimensional Riemannian manifolds (M_i) with sectional curvature ≥ -1 such that $\mathrm{diam}(M_i) \leq D$, $\mathrm{rad}(M_i) \geq d$, $e^d(M_i) \to 0$ and that each M_i does not satisfies the conclusion. Passing to a subsequence, we may assume that (M_i) converges to an Alexandrov space X. Since $e^d(X) = 0$, we see that the injectivity radius of X is not less than d. Hence by [Pl], X admits a natural C^1-differentiable structure. Thus by Remark 4.20 we have a C^1-fibration of M_i over X for large i, a contradiction. □

Proof of Corollary 0.6. Let $A \subset M$ be the set of all (n, δ)-strained points for a small δ. By [BGP2] and [OS], $M \setminus A$ has measure zero. Thus, for any $\epsilon > 0$, we have a finite covering $\{B_j\}_{j=1,\dots,N}$ of $M \setminus A$ by metric balls of radii $\delta_j < \epsilon$ such that $\sum_j \delta_j^n < \epsilon$. By the construction of the map f in Theorem 0.2, we have $\tau(\epsilon_i)$-almost isometries $f_i : U_i \to A$, where $U_i \subset M_i$ and ϵ_i is the Hausdorff distance between M_i and M. Hence we see that $\liminf_{i\to\infty} V_n(M_i) \geq V_n(M)$. On the other hand, $M_i \setminus U_i$ have a finite covering $\{B_j^i\}_{j=1,\dots,N}$ such that $\mathrm{diam}(B_j^i) < \mathrm{diam}(B_j) + \tau(\epsilon_i)$. Therefore, we have $\lim V_n(M_i) = V_n(M)$. □

Remark 4.22. — In the construction of the map f, we used the embedding of X into $L^2(X)$. One can also employ an embedding of X into a Euclidean space by using the distance function from each point of a net in X. However, if one tries to extend our argument to a more general Alexandrov space Y, which may contain more serious singular points, $L^2(Y)$ is large enough to embed Y. This is the main reason why we employ $L^2(X)$ to embed X.

The remark above leads us to the following

Problem 4.23. — Find geometric conditions on an Alexandrov space X (other than the small size of singularities) that ensures the existence of a tubular neighborhood, in the generalized sense, of the embedding $f_X : X \to L^2(X)$.

An answer to the problem would provide, for instance, a geometric proof of Grove, Petersen and Wu's finiteness theorem [GPW]. (Compare [Pr1].)

5. PROOF OF THEOREM 0.7

The proof of Theorem 0.7 is based on the following

Theorem 5.1. — *For given positive integers m, n $(m \geq n)$ and $\mu_0 > 0$, there exist positive numbers δ, ϵ, σ and w depending only on a priori constants and satisfying the following. Let M and X be Alexandrov spaces with curvature ≥ -1 and with dimension m and n respectively. Suppose that δ-str. rad$(X) > \mu_0$. Then, if the Hausdorff distance between M and X is less than ϵ, then for any $p \in M$ the image Γ of the inclusion homomorphism $\pi_1(B_p(\sigma, M)) \to \pi_1(B_p(1, M))$ contains a solvable subgroup H satisfying*

(5.1.1) $[\Gamma : H] < w$,

(5.1.2) *the length of polycyclicity of H is not greater than $m - n$.*

For the defininition of the length of polycyclicity of a solvable group, see [FY1].

The essential idea of the proof of Theorem 5.1 is the same as that in [FY1,7.1]. However, in our case we do not know yet if the map in Theorem 0.2 is a fibre bundle. This is the point for which we have to be careful.

Proof. The proof is done by downward induction on n and by contradiction. By Corollary 0.4, the theorem holds for $n = m$. Suppose that it holds for $\dim X \geq n+1$, but not for n. Then, we would have sequences M_i, X_i of Alexandrov spaces satisfying

(5.2.1) $\dim M_i = m, \quad \dim X_i = n.$

(5.2.2) δ_i-str. rad $(X_i) > \mu_0$, where $\lim_{i\to\infty} \delta_i = 0$.

(5.2.3) $d_H(M_i, X_i) < \epsilon_i$, where $\lim_{i\to\infty} \epsilon_i = 0$.

(5.3) For some $p_i \in M_i$ and for sequences $\sigma_i \to 0, w_i \to \infty$, the image of the inclusion homomorphism $\pi_1(B_{p_i}(\sigma_i, M_i)) \to \pi_1(B_{p_i}(1, M_i))$ does not contain a solvable subgroup satisfying (5.1) for $w = w_i$. □

Let $f_i : M_i \to X_i$ be the $\tau(\delta_i, \epsilon_i)$-almost Lipschitz submersion constructed in Theorem 0.2, and Δ_i the diameter of $f_i^{-1}(x_i)$, $x_i = f_i(p_i)$. For $\sigma_0 \ll \mu_0$, we put $\bar{B}_i = B_{x_i}(\sigma_0, X)$, $B_i = f_i^{-1}(\bar{B}_i)$. Remark that $B_{p_i}(\sigma_0/2, M_i) \subset B_i \subset B_{p_i}(2\sigma_0, M_i)$. Let $\pi_i : \widetilde{B}_i \to B_i$ be the universal cover, and Γ_i the deck transformation group. Let d_i and $\bar{d}_i$ be the distances of M_i and X_i respectively. From now on, we consider the scaled distances d_i/Δ_i and $\bar{d}_i/\Delta_i$ implicitly. Passing to a subsequence, we may assume that (B_i, p_i) (resp. $(\bar{B}_i, x_i)$) converges to a pointed space (Y, y_0) (resp. to $(\mathbf{R}^n, 0)$) with respect to the pointed Hausdorff distance. We may also assume that the Lipschitz map $f_i : B_i \to \bar{B}_i$ converges to a Lipschitz map $f : Y \to \mathbf{R}^n$ with Lipschitz constant 1. Since one can lift n-independent lines in $\mathbf{R}^n$ to those in Y, the splitting theorem ([GP],[Y]) implies that Y is isometric to a product $\mathbf{R}^n \times N$, where N is compact with diameter 1. Furthermore, since the property of f_i in Lemma 4.11 is invariant under scaling of metrics, one can check that $f : \mathbf{R}^n \times N \to \mathbf{R}^n$ is actually the projection.

In particular, it turns out that the fiber $f_i^{-1}(x_i)$ with the distance d_i/Δ_i converges to the nonnegatively curved Alexandrov space N. This implies the properties of fiber stated in Remark 0.3.

For $\tilde{p}_i \in \pi_i^{-1}(p_i)$, by using [FY1,3.6], we may assume that $(\widetilde{B}_i, \Gamma_i, \tilde{p}_i)$ converges to $(Z, G, \tilde{p}_\infty)$ with respect to the pointed equivariant Hausdorff distance, where G is a closed sugbroup of the group of isometries of Z. As before, one can prove that Z is isometric to $\mathbf{R}^{n+\ell} \times Z'$, where Z' is compact, and that π_i converges to the projection $\pi_\infty : \mathbf{R}^{n+\ell} \times Z' \to \mathbf{R}^n \times N$ by the action of G. Remark that G acts on $\mathbf{R}^\ell \times Z'$. Let C be the diameter of $N = (\mathbf{R}^\ell \times Z')/G$.

For a triple (X, Γ, x_0), we use the notation in [FY1,§3] such as

$$\Gamma(R) = \{\gamma \in \Gamma \mid |\gamma x_0 x_0| < R\}.$$

Then we have easily.

Lemma 5.4. — *The group G is generated by $G(2C)$.*

To apply [FY1,3.10], we need to restrict ourselves to a compact set of $\mathbf{R}^n$. Let $\bar{U}_i = B_{x_i}(10C+1, \bar{d}_i/\Delta_i)$, $U_i = f^{-1}(\bar{U}_i)$. Remark that U_i has a uniform bound D on its diameter.

Since f_i is not known to be a fibre bundle, we need the following lemma.

Lemma 5.5. — *There exists a positive integer I such that Γ_i is generated by $\Gamma_i(8C+1)$ for each $i > I$. In particular, the inclusion homomorphism $\pi_1(U_i) \to \Gamma_i$ is surjective.*

Proof. First, we prove that $\pi_i^{-1}(U_i)$ is connected. Suppose that it has two connected components V_i and W_i. Since the diameter of U_i is uniformely bounded, we can take $y_i \in V_i$ and $z_i \in W_i$ such that $|y_i z_i| = |V_i W_i|$ and that $|\tilde{p}_i y_i|$ is uniformly bounded. Let $\tilde{c}_i = \exp t\tilde{\xi}_i$ be a minimal geodesic joining y_i to z_i, and ℓ_i the length of $\tilde{c}_i$. Since the action of G on $\mathbf{R}^n$-factor is trivial, ℓ_i must go to infinity as $i \to \infty$. For $x \in \widetilde{B}_i$ let $\widetilde{H}_x \subset \Sigma_x$ be the set that project down to $H_{\pi_i(x)}$. (See §4). From the convergence $(\widetilde{B}_i, \tilde{p}_i) \to (\mathbf{R}^{n+\ell} \times Z')$ and from the choice of y_i and z_i, it follows that $|\tilde{\xi}_i \widetilde{H}_{y_i}| \to 0$ as $i \to \infty$. Now let $c_i = \pi_i \circ \tilde{c}_i = \exp t\xi_i$. Take w_i such that $|\pi_i(y_i) w_i| \geq \sigma_0/\Delta_i$ and $|\xi_i w_i'| < \tau(\delta_i, \epsilon_i)$, and put $\eta_i(t) = \exp tw_i'$. A generalized version of Theorem 1.1 (see [CE]) implies that $|\pi_i(z_i)\eta_i(\ell_i)| < \ell_i \tau(\delta_i, \epsilon_i)$. Take $\gamma_1, \gamma_2 \in \Gamma_i$ such that $|\gamma_1 \tilde{p}_i, y_i| < 2D$, $|\gamma_2 \tilde{p}_i, z_i| < 2D$. It turns out

$$\begin{aligned}
0 &= |\pi_i(\gamma_1 \tilde{p}_i), \pi_i(\gamma_2 \tilde{p}_i)| \\
&\geq |\pi_i(y_i)\pi_i(z_i)| - |\pi(\gamma_1 \tilde{p}_i)\pi_i(y_i)| - |\pi_i(\gamma_2 \tilde{p}_i)\pi_i(z_i)| \\
&\geq \ell_i - \ell_i \tau(\delta_i, \epsilon_i) - 4D > 0\,,
\end{aligned}$$

for each sufficiently large i, a contradiction.

Now, for any $\gamma \in \Gamma_i$, let $c_1(t)$ be a curve in $\pi_i^{-1}(U_i)$ joining $\tilde{p}_i$ to $\gamma\tilde{p}_i$ with length say, R. For each j, $1 \leq j \leq R$ and for sufficiently large i, one can take $\gamma_j \in \Gamma_i$ such that $|c_1(j)\gamma_j \tilde{p}_i| < 4C$. Thus, γ is written as the product

$$\gamma = (\gamma\gamma_{[L]}^{-1})(\gamma_{[L]}\gamma_{[L]-1}^{-1}) \cdots (\gamma_2\gamma_1^{-1})\gamma_1\,,$$

each of whose factor has length less than $8C+1$. This completes the proof of the lemma. $\square$

Let $\widetilde{U}_i$ be the universal cover of U_i, and Λ_i the deck transformation group. As before, we may assume that $(\widetilde{U}_i, \Lambda_i, \tilde{p}_i)$ converges to a triple $(\mathbf{R}^k \times W, H, 0)$, where both W and $(\mathbf{R}^k \times W)/H$ are compact. The main theorem in [FY2] implies that H/H_0 is discrete, where H_0 is the identity component of H.

We next show that H/H_0 is almost abelian. Since H preserves the splitting $\mathbf{R}^k \times W$, we have a homomorphism $p : H \to \mathrm{Isom}(\mathbf{R}^k)$. Let K and L denote the kernel and the image of p respectively. The compactness of K implies the closedness of L. It follows from [FY, 4.1] that L/L_0 is almost abelian. Since KH_0/H_0 is finite, the exact sequence

$$1 \longrightarrow \frac{KH_0}{H_0} \longrightarrow \frac{H}{H_0} \longrightarrow \frac{L}{L_0} \longrightarrow 1$$

implies that H/H_0 is almost abelian as required. (See [FY1, 4.4]).

Now, by [FY1, 3.10], we can take the "collapsing part" Λ_i' of Λ_i in the following sense:

(5.6.1) $(\widetilde{U}_i, \Lambda_i', \tilde{p}_i)$ converges to $(\mathbf{R}^k \times W, H_0, 0)$ with respect to the pointed equivariant Hausdorff distance ;

(5.6.2) Λ_i/Λ_i' is isomorphic to H/H_0 for large i ;

(5.6.3) for any $\epsilon > 0$ there exists I_ϵ such that Λ_i' is generated by $\Lambda_i'(\epsilon)$ for every $i > I_\epsilon$.

The final step is to show that Λ_i' is almost solvable. We go back to the Hausdorff convergence of U_i to $B^n(C') \times N$, where $C' = 10C+1$ and $B^n(C') = B_0(C', \mathbf{R}^n)$. By [BGP], we can take a good point x_0 in $B^n(C') \times N$. This means that $((B^n(C') \times N, d/\epsilon), x_0)$ converges to $(\mathbf{R}^{n+s}, 0)$ as $\epsilon \to 0$, where d is the original distance of $B^n(C') \times N$ and s is the Hausdorff dimension of N ($s \geq 1$). Let $\epsilon_{m,n+s}(1)$ and $\sigma_{m,n+s}(1)$ be the constants ϵ, σ given by the inductive assumption for $m, n+s$ and $\mu_0 = 1$. Now, fix a small ϵ and take a large i so that the pointed Hausdorff distance between $((U_i, d_i/\Delta_i\epsilon), q_i)$ and $(\mathbf{R}^{n+s}, 0)$ is less than $\epsilon_{m,n+s}(1)$, where q_i is a point in U_i Hausdorff close to x_0. By induction we can conclude that the image $\widetilde{\Gamma}_i$ under the inclusion homomorphism of $\pi_1(B_{q_i}(\sigma_{m,n+s}(1), d_i/\Delta_i\epsilon))$ to $\pi_1(B_{q_i}(1, d_i/\Delta_i\epsilon))$ contains a solvable subgroup H_i such that

(5.7.1) $[\widetilde{\Gamma}_i : H_i]$ has a uniform bound independent of i ;

(5.7.2) the length of polycyclicity of H_i is not greater than $m - n - s$.

By [FY1,7.11], (5.6.3) can be strengthened as :

(5.6.3)$'$ for any $\epsilon > 0$, there exists a positive integer I_ϵ such that Λ'_i is generated by the set $\{\gamma \in \Lambda'_i \mid |\gamma x x| < \epsilon\}$ for every $x \in \widetilde{U}_i$.

It follows that Λ'_i is included in the image of $\pi_1(B_{q_i}(\sigma_{m,n+s}, d_i/\Delta_i\epsilon)) \to \Lambda_i$. Therefore, also Λ'_i contains a solvable subgroup satisfying (5.7). Thus, it follows from (5.6.2) that Λ_i is almost solvable. Therefore, Lemma 5.4 yields the almost solvability of Γ_i. This is a contradiction to (5.3). The proof of Theorem 5.1 is now complete. □

By using Theorem 5.1, we can prove the following theorem, a generalized Margulis lemma along the same line as [FY1, 10.1, A2]. The details are omitted.

Theorem 5.8. — *For given m, there exists a positive number σ_m satisfying the following. Let M be an m-dimensional Alexandrov space with curvature ≥ -1. Then, for any $p \in M$ the image of the inclusion homomorphism $\pi_1(B_p(\sigma_m, M)) \to \pi_1(B_p(1, M))$ contains a nilipotent subgroup of finite index.*

Our Theorem 0.7 is a special case of Theorem 5.8.

6. APPENDIX : RELATIVE VOLUME COMPARISON

Let X be an n-dimensional Alexandrov space with curvature $\geq k$. We fix $p \in M$ and $\bar{p} \in M^n(k)$, and put $D_p(r) = \{x \in X \,|\, |px| \leq r\}$. First, we study the equality case in (1.3).

Proposition A.1. — *Suppose $V_n(B_p(r)) = b_k^n(r)$. Then, $B_p(r)$ with the length structure induced from the inclusion $B_p(r) \subset X$ is isometric to $B_{\bar{p}}(r)$ with the induced length structure.*

Furthermore one of the following occurs:

(A.2.1) $D_p(r)$ with the induced length structure is isometric to $D_{\bar{p}}(r)$ with the induced length structure ;

(A.2.2) $X = D_p(r)$ and there exists an isometric $\mathbf{Z}_2$-action on the boundary of $D_{\bar{p}}(r)$ such that X is isometric to the quotient space $B_{\bar{p}}(r) \cup_{\mathbf{Z}_2} \partial D_{\bar{p}}(r)$.

In the case $k > 0$, $\pi/2\sqrt{k} < r < \pi/\sqrt{k}$, (A.2.2) does not occur.

Proof. By Lemma 1.2, the map $\rho : B_p(r) \to B_{\bar{p}}(r)$ there does not decrease measure, and hence preserves measure in the equality case. To show that $B_p(r)$ is isometric to $B_{\bar{p}}(r)$, it suffices to show that ρ is a local isometry. For any $x \in B_p(r)$, take an $\epsilon > 0$ such that $B_x(\epsilon) \subset B_p(r)$, and suppose that $|\rho(y_1)\rho(y_2)| > |y_1y_2|$ for some $y_1, y_2 \in B_x(\epsilon/2)$. Put $2s = |y_1y_2|$, $2t = |\rho(y_1)\rho(y_2)|$, $\widetilde{B}_i = B_{\rho(y_i)}(t)$ and $B_i = B_{y_i}(t)$. Let z be the midpoint of a minimal geodesic y_1y_2, and $B = B_z(t-s)$. Then, from $V_n((B_1 \cup B_2)^c) \leq V_n((\widetilde{B}_1 \cup \widetilde{B}_2)^c)$ and $V_n(B_i) \leq V_n(\widetilde{B}_i)$, we would have

$$
\begin{aligned}
V_n(B_p(r)) &< V_n(B_1) + V_n(B_2) + V_n((B_1 \cup B_2)^c) - V_n(B) \\
&\leq V_n(\widetilde{B}_1) + V_n(\widetilde{B}_2) + V_n((\widetilde{B}_1 \cup \widetilde{B}_2)^c) - V_n(B) \qquad (A.3) \\
&= b_k^n(r) - V_n(B)\,,
\end{aligned}
$$

which is a contradiction. □

The proof of (A.2.2) is essentially due to [GP2]. Suppose that ρ is not continuous on the boundary $\partial D_{\bar{p}}(r)$. Let $\mu : D_{\bar{p}}(r) \to D_p(r)$ be the continuous map such that $\mu = \rho^{-1}$ on $B_{\bar{p}}(r)$. We show that $\sharp\mu^{-1}(x) \leq 2$ for all $x \in \partial D_{\bar{p}}(r)$. Suppose that there are three points x_1, x_2, x_3 in $\mu^{-1}(x)$. Now, we have three minimal geodesics $\gamma_i : [0,\ell] \to X$ joining p to x, where $\ell = |px|$. For a sufficiently small $\epsilon > 0$, put $y_i = \gamma_i(\ell - \epsilon)$. Then, it follows from an argument similar to $(A.3)$ measuring volume loss that for every $1 \leq i \neq j \leq 3$, the ball $B_{y_i}(\epsilon)$ does not intersects with $B_{y_j}(\epsilon)$. Thus, it turns out that the segments $y_i x$ and $x y_j$ form a minimal geodesic. This contradicts the non-branching property of geodesic.

Now we have an involutive homeomorphism Φ on $\partial D_{\bar{p}}(r)$ such that $\mu(\Phi(x)) = \mu(x)$. Since a curve in $\partial D_{\bar{p}}(r)$ can be approximated by curves in $B_{\bar{p}}(r)$, we can see that Φ preserves the length of curves and hence is an isometry. Thus, $D_p(r)$ is isometric to the quotient $B_{\bar{p}}(r) \cup_{\mathbf{Z}_2} \partial D_{\bar{p}}(r)$. If Φ is nontrivial, then again the non-branching property of geodesic implies $X = D_p(r)$. However, in case of $k > 0$ and $\pi/2\sqrt{k} < r < \pi/\sqrt{k}$, the nontrivial quotient $B_{\bar{p}}(r) \cup_{\mathbf{Z}_2} \partial D_{\bar{p}}(r)$ does not have curvature $\geq k$. Hence, ρ must be continuous in this case. It follows that $\rho = \mu^{-1}$ is an isometry with respect to the induced length structure because it preserves the length of curves.

Next, we prove a relative version of (1.3), which corresponds to the Bishop and Gromov volume comparison theorem ([GLP]) in Riemannian geometry.

Proposition A.4. — *For $r < R$, we have*

$$\frac{V_n(B_p(R))}{V_n(B_p(r))} \leq \frac{b_k^n(R)}{b_k^n(r)}\,.$$

Proof. Put $S_p(t) = \{x \in X \mid |px| = t\}$. By the recent result in [BGP2] and [OS], the set of all (n,δ)-strained points in X has full measure for any $\delta > 0$. Hence, in view of Theorem 1.4, we can apply the coarea formula ([Fe]) to obtain

$$(A.5) \qquad V_n(B_p(R)) = \int_0^R V_{n-1}(S_p(t))\,dt\,.$$

Now we show that

$$(A.6)\qquad \frac{V_{n-1}(S_p(R))}{V_{n-1}(S_p(r))} \le \frac{V_{n-1}(S_{\bar p}(R))}{V_{n-1}(S_{\bar p}(r))}\,.$$

Let us suppose that $k<0$. The other cases can be treated similary. For $x \in S_p(R)$ (resp. $\bar x \in S_{\bar p}(R)$), let $\rho(x)$ (resp. $\bar\rho(\bar x)$) denote the intersection of a minimal geodesic px with $S_p(r)$ (resp. $\bar p\bar x$ with $S_{\bar p}(r)$). We know that for any $\epsilon>0$ there exists $\delta>0$ such that if $|\bar x\bar y|<\delta$, then

$$\left|\frac{|\bar\rho(\bar x)\bar\rho(\bar y)|}{|\bar x\bar y|} - \frac{\sinh\sqrt{-k}R}{\sinh\sqrt{-k}r}\right| < \epsilon\,,$$

which implies

$$(A.7)\qquad \frac{V_{n-1}(S_{\bar p}(R))}{V_{n-1}(S_{\bar p}(r))} = \left(\frac{\sinh\sqrt{-k}R}{\sinh\sqrt{-k}r}\right)^{n-1}\,.$$

Theorem 1.1 yields that $|\rho(x)\rho(y)| \ge |\bar\rho(\bar x)\bar\rho(\bar y)|$ for every $x,y\in S_p(R)$ and $\bar x,\bar y \in S_{\bar p}(R)$ with $|xy|=|\bar x\bar y|$. Hence, if $|xy|<\delta$, then

$$(A.8)\qquad \frac{|\rho(x)\rho(y)|}{|xy|} > \frac{\sinh\sqrt{-k}R}{\sinh\sqrt{-k}r} - \epsilon\,.$$

Now, (A.6) immediately follows from (A.7) and (A.8).

We put $A(t)=V_{n-1}(S_p(t))$, $\bar A(t)=V_{n-1}(S_{\bar p}(t))$ and

$$f(t) = \frac{V_n(B_{\bar p}(t))}{V_n(B_p(t))} = \frac{\int_0^t \bar A(t)dt}{\int_0^t A(t)dt}\,.$$

Since

$$\begin{aligned} f'(t) &= \frac{\bar A(t)\int_0^t A(t) - A(t)\int_0^t \bar A(t)}{(\int_0^t A(t))^2} \\ &= \left(\frac{\bar A(t)}{A(t)}\int_0^t A(t) - \int_0^t \bar A(t)\right)\frac{A(t)}{\left(\int_0^t A(t)\right)^2}\,, \end{aligned}$$

it follows from (A.6) that

$$\left(\frac{\bar A(t)}{A(t)}\int_0^t A(t) - \int_0^t \bar A(t)\right)' \ge 0\,.$$

This completes the proof. □

By using Proposition A.4, one can obtain the volume sphere theorem extending one in [OSY].

Proposition A.9. — *There exists a positive number $\epsilon = \epsilon_n$ such that if an n-dimensional Alexandrov space X with curvature ≥ 1 satisfies $V_n(X) > b_1^n(\pi) - \epsilon$, then X is $\tau(\epsilon)$-almost isometric to S^n.*

Proof. Let $\rho : X \to S^n$ be an expanding map as in Lemma 1.2. For some $y_1, y_2 \in X$ suppose that $2s = |y_1y_2| < |\rho(y_1)\rho(y_2)| = 2t$. Then, by the argument in (A.3),

$$V_n(X) < b_1^n(\pi) - V_n(B_z(t-s)), \tag{A.10}$$

where z is the midpoint of a minimal geodesic y_1y_2. On the other hand, from Proposition A.4 and the assumption on $V_n(X)$, we have $V_n(B_z(t-s)) > (1-\epsilon/b_1^n(\pi))b_1^n(t-s)$. Together with (A.10), this implies $|t-s| < \tau(\epsilon)$. Thus $d_H(X, S^n) < \tau(\epsilon)$ because $\rho(X)$ is $\tau(\epsilon)$-dense in S^n. Therefore, by Theorem 1.5 we obtain a $\tau(\epsilon)$-almost isometry between X and S^n. □

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