DISCRETIZATION OF POSITIVE HARMONIC FUNCTIONS ON RIEMANNIAN MANIFOLDS AND MARTIN BOUNDARY

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Abstract. Let X be a separated subset in a connected Riemannian manifold M with bounded geometry such that the ε-neighbourhood of X is recurrent w.r.t. Brownian motion on M for some $\varepsilon > 0$. The main result of this paper says that the data in the discretization procedure of Lyons and Sullivan can be chosen such that the Green function of M and the resulting Markov chain on X coincide up to a constant on pairs (y, z), where $y \neq z$ are points in X.

Résumé. Soit X un sous-ensemble séparé d'une variété riemannienne M à géométrie bornée tel que le voisinage d'épaisseur ε de X est récurrent pour le mouvement brownien sur M pour au moins un ε positif. Le principal résultat de cet article dit que les données du procédé des discrétisations de Lyons et Sullivan peuvent être choisies de telle sorte que la fonction de Green de M et la chaîne de Markov sur X qui s'en déduit coïncident à une constante près sur les paires de points (y, z) avec $y \neq z$.

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INTRODUCTION

We are interested in the connection between potential theory of the Laplacian on Riemannian manifolds and the potential theory of Markov chains on discrete subsets. Such a connection has been established by Furstenberg [F] in the case of discrete subgroups of $Sl(2,\mathbb{R})$. We investigate the discretization procedure of Lyons and Sullivan [LS], which associates to a so-called $*$-*recurrent* (respectively *cocompact*) discrete subset X of a connected Riemannian manifold M a family of probability measures $\mu_y,\ y \in M$, on X such that

$$H(y) = \mu_y(H) := \sum_{x \in X} H(x)\mu_y(x)$$

for any bounded (respectively positive) harmonic function H on M. In particular, the restriction of H to X is a μ-harmonic function with respect to the Markov chain on X defined by the measures $\mu_x, \quad x \in X$ (that is, $\mu_x(H) = H(x)$ for all $x \in X$). Under some extra assumptions on the data involved in the construction, one obtains in this way all bounded (respectively positive) μ-harmonic functions on X (see [A], [K]) and, if X is cocompact, that Brownian motion on M is transient iff the Markov chain on X is transient [LS].

A more precise information about behaviour at infinity of harmonic functions is given by the Martin compactification $cl_\Delta M$ and the Martin boundary $\partial_\Delta M$ of M. By definition, $cl_\Delta M = M \dot{\cup} \partial_\Delta M$ is the closure of M in the space of positive superharmonic functions via the embedding $y \longmapsto K(.,y)$, where

$$K(.,y) = G(.,y)/G(x_0,y)$$

is the Martin kernel, G is the Green function of M and $x_0 \in M$ is a chosen origin. For convenience, we choose $x_0 \in X$. The Martin compactification $cl_\mu X$ and Martin boundary $\partial_\mu X$ of X with respect to a Markov chain on X are defined in the same way by using the Martin kernel k and the Green function g of the Markov chain. The definition of the Martin boundary requires that Brownian motion on M (respectively the Markov chain on X) has a Green function, i.e., that it is transient.

As a consequence of Theorems 1.11, 2.7, 2.8, 3.1 and Corollary 2.9 below we obtain the theorem

Main theorem. — *Assume that the geometry of M is bounded and that X is a discrete subset of M such that, for some $\varepsilon > 0$,*

(i) $dist(x,z) \geq 2\varepsilon$ for all $x \neq z$ in X; *(ii) $B_\varepsilon(X)$ is recurrent.*

Then, for some appropriate choice of data, the measures $\mu_y, y \in M$, of Lyons and Sullivan satisfy

(a) for some positive constant κ we have $g(x,z) = \kappa G(x,z)$ for all $x \neq z$ in X. In particular, the Markov chain on X is transient iff Brownian motion on M is.

If Brownian motion on M is transient, then $\mu_x(z) = \mu_z(x)$ for all x, z in X and

(b) the inclusion $X \subset M$ extends to a homeomorphism of $cl_\mu X$ and $\overline{X}$, where $\overline{X}$ is the closure of X in $cl_\Delta M$;

(c) restriction defines an isomorphism between the simplex of positive harmonic functions on M spanned by $\overline{X} \cap \partial_\Delta M$ and the space of positive μ-harmonic functions on X which are 1 at x_0.

The Harnack inequality implies that $\overline{X} \cap \partial_\Delta M$ contains all extremal positive harmonic functions of M which are 1 at x_0 if X is a net, that is, if $B_R(X) = M$ for some $R > 0$. Thus (c) implies in this case that the space of positive harmonic functions on M and the space of positive μ-harmonic functions on X are isomorphic, a result due to Ancona [A].

If Γ is a discrete group of isometries of M and X is the orbit of a point x_0 on which Γ acts freely, then X satisfies (i). Property (ii) holds if $vol(M/\Gamma) < \infty$ or , more generally, if the Brownian motion on M/Γ is recurrent. If this is the case, then the Markov chain on X corresponds to a (left-invariant) symmetric random walk on Γ (via the natural identification of Γ and $X = \Gamma(x_0)$).

Corollary. — *There exists a symmetric random walk on the free group F_q with $q \geq 2$ generators with Martin boundary equal to a circle.*

As for the proof, recall that the Martin boundary of the hyperbolic plane H^2 is the circle (at infinity) and that F_q acts as a discrete group of isometries on H^2 with $vol(H^2/F_q) < \infty$.

It follows from Theorem 3.2 below that the measure defining the random walk on F_q has finite logarithmic moment with respect to the word norm on F_q and finite entropy. This has to be contrasted with the case of probabilities on F_q with finite support, for which the Martin boundary is known to be a Cantor set [D].

We would like to thank Martine Babillot to whom we owe the assertion and the proof of the symmetry of the measures μ_x in the above theorem. The second author gratefully acknowledges the support by the SFB 256 at the University of Bonn.

1. HARMONIC FUNCTIONS

Let M denote a connected Riemannian manifold. A *Brownian path* on M is a continuous curve

$$\omega : [0, \zeta(\omega)) \to M, \text{ where } \zeta(\omega) \in (0, \infty] \ .$$

For x in M, let P_x denote the measure on the space of Brownian paths on M with $\omega(0) = x$ defining the Brownian motion on M starting from x. For a Borel measure λ on M let P_λ be defined by $P_\lambda = \int P_x \lambda(dx)$. The measure P_λ describes the Brownian motion on M with initial distribution λ.

For a closed subset F of M and a Brownian path ω set

$$R^F(\omega) = inf\{t \geq 0 \quad | \quad \omega(t) \in F\} \ .$$

The *balayage* $\beta^F_\lambda = \beta(\lambda, F)$ of a measure λ onto F is the distribution of P_λ at the time R^F, i.e., for A a Borel subset of M,

$$\beta^F_\lambda(A) = P_\lambda\{\omega \quad | \quad R^F(\omega) < \zeta(\omega) \text{ and } \omega(R^F(\omega)) \in A\} \ .$$

For short we set $\beta^F_x = \beta(x, F) = \beta(\delta_x, F)$; then $\beta(\lambda, F) = \int \beta(x, F)\lambda(dx)$. For x in F, we have $\beta(x, F) = \delta_x$. We say that F is *recurrent* if $\beta^F_x(F) = 1$ for all x in M.

For an open subset V of M and a Brownian path ω set

$$S^V(\omega) = inf\{t \geq 0 \quad | \quad \omega(t) \in M \backslash V\} \ .$$

We call $S^V(\omega)$ the *exit time* from V of the path ω. The distribution of P_λ at the time S^V will be denoted $\varepsilon_\lambda^V = \varepsilon(\lambda, V)$ and we set $\varepsilon_x^V = \varepsilon(x, V) = \varepsilon(\delta_x, V)$. For x in $M\backslash V$, we have $\varepsilon(x, V) = \delta_x$. For x in V, $\varepsilon(x, V)$ is supported on ∂V and is called the *harmonic measure* of x. By construction $\varepsilon(\lambda, V) = \beta(\lambda, M\backslash V)$.

Now let X be a discrete subset of M. A family of closed sets $(F_x)_{x\in X}$ and relatively compact open sets $(V_x)_{x\in X}$ will be called *Lyons-Sullivan data* or LS-data if

(D1) $x \in \overset{\circ}{F}_x$ and $F_x \subset V_x$ fo $x \in X$,

(D2) $F_x \cap V_z = \emptyset$ for all $x \neq z$ in X,

(D3) $F = \cup_{x\in X} F_x$ is recurrent;

(D4) there is a constant $C > 1$ such that

$$\frac{1}{C} < \frac{d\varepsilon(z, V_x)}{d\varepsilon(x, V_x)} < C$$

for all x in X and z in F_x.

We say that X is $*$-*recurrent* if X admits LS-data. Note that our notion is more restrictive than the one of Lyons and Sullivan.

Let X be $*$-recurrent and let $(F_x, V_x)_{x\in X}$ be a choice of LS-data. Consider the following modification, applied to a finite measure μ on M,

$$\mu' = \sum_{x\in X} \Big(\int_{F_x} (\varepsilon(z, V_x) - \frac{1}{C}\varepsilon(x, V_x)) \beta_\mu^F(dz) \Big) \text{ and } \mu'' = \frac{1}{C} \sum_{x\in X} \beta_\mu^F(F_x)\delta_x \ . \tag{1.1}$$

Now start with the measure

$$\mu_0 = \delta_y \text{ for } y \notin X, \quad \mu_0 = \varepsilon(y, V_y) \text{ for } y \in X, \tag{1.2}$$

and define recursively, for $n \geq 1$,

$$\mu_n = (\mu_{n-1})' \text{ and } \tau_n = (\mu_{n-1})''. \tag{1.3}$$

Then the LS-measure $\mu_y, y \in M$ is the probability measure on X given by

$$\mu_y = \sum_{n\geq 1} \tau_n \ . \tag{1.4}$$

Note that μ_y depend on the LS-data. The family of LS-measures has the following properties:

(1.5) $\mu_y(x) > 0$ for all x in X and y in M;

(1.6) for any isometry γ of M leaving X and the LS-data invariant we have $\mu_{\gamma y}(\gamma x) = \mu_y(x)$ for all y in M and x in X;

(1.7) for all x in X, $\quad \mu_x = \int_{\partial V_x} \mu_u \varepsilon(x, V_x)(du)$;

(1.8) for all x in X and y in F_x, $\quad y \neq x$,

$$\mu_y = \frac{1}{C}\delta_x + \int_{\partial V_x} (\frac{d\varepsilon(y, V_x)}{d\varepsilon(x, V_x)} - \frac{1}{C})\mu_u \, \varepsilon(x, V_x)(du) \ ;$$

(1.9) for any y in $M \backslash F$ and any stopping time $T \leq R^F$,

$$\mu_y = \int \mu_u \, \pi_y^T(du) \ ,$$

where π_y^T is the distribution of P_y at time T.

These properties readily follow from the definition. Use the strong Markov property for (1.9).

Let H be a positive harmonic function on M. Then $\beta_y^F(H) \leq H(y)$ for all y in M. We say that F *sweeps* H if $\beta_y^F(H) = H(y)$ for all y in M. Since F is recurrent, if H is bounded, then F sweeps H by the martingale convergence theorem. With these notations the discussion in [LS], page 317, gives the following.

1.10. Theorem. — *For any positive harmonic function H on M, we have $\mu_y(H) \leq \beta_y^F(H)$ for all y in M; if $\beta_y^F(H) < H(y)$ for some y in M, then $\mu_y(H) < H(y)$ for all y in M; if F sweeps H, then $\mu_y(H) = H(y)$ for all y in M.*

We say that a function h on X is μ-harmonic if $\mu_x(h) = h(x)$ for all x in X. Theorem 1.10 implies that the restriction of a positive harmonic function H on M to X is μ-harmonic if and only if H is swept by F. Now denote by $\mathcal{H}_F^+(M)$ the space of positive harmonic functions swept by F and by $\mathcal{H}^+(X, \mu)$ the space of positive μ-harmonic functions on X.

1.11. Theorem. — *The restriction map $\mathcal{H}^+_F(M) \to \mathcal{H}^+(X,\mu)$ is an isomorphism.*

Proof. By Theorem 1.10 it remains to show that a μ-harmonic function on X is the restriction of a positive harmonic function H on M. We define $H(y) = \mu_y(h)$ and then need to show that H is harmonic. On $M \backslash F$ this is immediate since there

$$\mu_y(h) = \int \mu_u(h) \beta^F_y(du) \ .$$

Let x be in X. We shall establish that for y in V_x

$$(*) \qquad \mu_y(h) = \int \mu_u(h) \varepsilon(y, V_x)(du)$$

and this implies that H is harmonic on M. First for x itself $(*)$ is (1.7). Then from (1.8) we get for y in $F_x, y \neq x$.

$$\mu_y(h) = \frac{1}{C} h(x) + \int_{\partial V_x} \mu_u(h) \varepsilon(y, V_x)(du) - \frac{1}{C} \int_{\partial V_x} \mu_u(h) \varepsilon(x, \partial V_x)(du)$$

which is $(*)$ again by (1.7). Now let y be in $V_x \backslash F_x$ and let T be the exit time from $V_x \backslash F_x$. By (D2), $T \leq R^F$ for Brownian paths starting from y and hence by (1.9)

$$\mu_y(h) = \int \mu_u(h) \pi^T_y(du) \ .$$

Decompose $\pi^T_y = \varepsilon_1 + \varepsilon_2$, where ε_1 is supported on ∂V_x and ε_2 on F_x. Using $(*)$ on F_x we have

$$\mu_y(h) = \int_{\partial V_x} \mu_u(h) [\varepsilon_1 + \int_{F_x} \varepsilon(z, V_x) \varepsilon_2(dz)](du) \ .$$

Relation $(*)$ follows since by the strong Markov property of the Brownian motion

$$\varepsilon(y, V_x) = \varepsilon_1 + \int_{F_x} \varepsilon(z, V_x) \varepsilon_2(dz) \ . \qquad \square$$

1.12. Remark. — By analogous arguments we can prove Theorem 1.11 also under the more general uniform core condition of Kaimanovich [K].

2. MARTIN BOUNDARIES

Throughout this section, X is a $*$-recurrent subset of M and $(F_x, V_x)_{x\in X}$ is a fixed choice of LS-data. We now give a more detailed description of the construction of the measures μ_y, $y \in M$.

Let W be the space of all Brownian paths on M. For ω in W starting from a point y in F, define $S(\omega)$ to be the exit time from $V_{\varphi(y)}$, where $\varphi(y)$ is the unique point in X such that $y \in F_{\varphi(y)}$. Recursively we define the stopping times $R_n, n \geq 1$, and $S_n, n \geq 0$, by

$$S_0(\omega) = \begin{cases} 0 & \text{if } \omega(0) \notin X \\ S(\omega) & \text{if } \omega(0) \in X\,, \end{cases}$$

$$R_n(\omega) = \inf\{t \geq S_{n-1}(\omega) \mid \omega(t) \in F\}\,,$$

$$S_n(\omega) = \inf\{t \geq R_n(\omega) \mid \omega(t) \notin V_{X(n,\omega)}\}\,,$$

where $X(n,\omega) = \varphi(\omega(R_n(\omega)))$. On $\tilde{W} = W \times [0,1]^{\mathbb{N}}$ we define recursively for $k \geq 0$

$$N_0(\omega,\alpha) = 0\,,$$

$$N_k(\omega,\alpha) = \inf\{n > N_{k-1}(\omega,\alpha) \mid \alpha_n < \kappa_n(\omega)\}\,,$$

where

$$\kappa_n(\omega) = \frac{1}{C}\frac{d\varepsilon(X(n,\omega), V_{X(n,\omega)})}{d\varepsilon(\omega(R_n(\omega)), V_{X(n,\omega)})}(\omega(S_n(\omega)))\,. \tag{2.1}$$

For y in M we denote by $\tilde{P}_y$ the product measure $P_y \otimes \lambda^{\mathbb{N}}$ on $\tilde{W}$, where λ is the Lebesgue measure on $[0,1]$. Since $F = \cup_{x\in X} F_x$ is recurrent, the stopping times R_n, S_n and N_k are finite almost surely. Now the LS-measures $\mu_y, y \in M$, are by definition given by

$$\mu_y(x) = \tilde{P}_y[X_{N_1} = x], \qquad x \in X\,. \tag{2.2}$$

The second main result of Lyons and Sullivan about the measures μ_y is as follows.

2.3. Theorem (see [LS], p 321). — *The process* $(X_{N_k})_{k\geq 1}$ *is a Markov process with time homogeneous transition probabilities* $p(x,z)=\mu_x(z)$ *for* x,z *in* X. *In fact, for* y *in* M *and* $x_1, x_2, \ldots x_k$ *in* X *we have*

$$\tilde{P}_y(X_{N_1}=x_1,\cdots,X_{N_k}=x_k)=\mu_y(x_1)\mu_{x_1}(x_2)\cdots\mu_{x_{k-1}}(x_k)\ .$$

Remark. — In [LS] this result is only stated in the so-called cocompact case. It is observed in [K] that it is also valid in this general set-up. Observe that here, by (D2), ∂V_x is assumed to be disjoint from X.

Fix y in M and define the Green function g of the Markov chain on X by

$$g(y,x)=\delta_y(x)+\sum_{k=1}^{\infty}\tilde{P}_y(X_{N_k}=x),\quad x\in X\ . \tag{2.4}$$

We want to compare the Green function G of the manifold M with g. We have

$$g(y,x)=\frac{1}{C}\sum_{n\geq 1}\nu_n(F_x)\qquad \text{for } y\neq x\ , \tag{2.5}$$

where ν_n denotes the distribution of $\omega(R_n)$ that is, for A a Borel subset of M,

$$\nu_n(A)=P_y(\omega(R_n(\omega))\in A)\ .$$

Proof of (2.5). Since $y\neq x$, we have

$$\begin{aligned}
g(y,x)&=\sum_{k\geq 1}\tilde{P}_y(X_{N_k}=x)\\
&=\sum_{k\geq 1}\sum_{n=k}^{\infty}\tilde{P}_y(\omega(R_n)\in F_x \text{ and } N_k(\omega,\alpha)=n)\\
&=\sum_{n\geq 1}\sum_{k=1}^{n}\tilde{P}_y(\omega(R_n)\in F_x \text{ and } N_k(\omega,\alpha)=n)\\
&=\sum_{n\geq 1}\tilde{P}_y(\omega(R_n)\in F_x \text{ and } \alpha_n<\kappa_n(\omega))\\
&=\frac{1}{C}\sum_{n\geq 1}\int_{F_x}(\int_{\partial V_x}\frac{d\varepsilon(x,V_x)}{d\varepsilon(z,V_x)}(\zeta)\varepsilon(z)(d\zeta))\nu_n(dz)\\
&=\frac{1}{C}\sum_{n\geq 1}\nu_n(F_x)\ ,
\end{aligned}$$

where we use the strong Markov property of the Brownian motion to express $\tilde{P}_y$ by an integral on M. $\square$

For an open subset V of M denote by G_V the Green function of V. For y not in V_x we have

$$G(y,x) = \sum_{n\geq 1} \int_{F_x} G_{V_x}(z,x)\nu_n(dz) \ . \tag{2.6}$$

Proof of (2.6). Let $B \subset F_x$ be a neighbourhood of x. Then

$$\int_B G(y,u)du = E_y(\int_0^\infty \chi_B(\omega(t))dt) \ .$$

Since $\omega(t)$ is not in F for $S_n(\omega) < t < R_{n+1}(\omega)$ and since $B \subset F$, the right hand side is equal to

$$\sum_{n=1}^{\infty} E_y(\int_{R_{n(\omega)}}^{S_{n(\omega)}} \chi_B(\omega(t))dt) \ .$$

Now $S_n(\omega) = R_n(\omega)+S(\omega(R_n(\omega)))$ and hence we get from the strong Markov property of Brownian motion that the above expression is equal to

$$\sum_{n=1}^{\infty} \int_{F_x} E_z(\int_0^{S(\omega)} \chi_B(\omega(t))dt)\nu_n(dz) \ .$$

Since S is the exit time from V_x we get

$$\int_B G(y,u)du = \sum_{n\geq 1} \int_{F_x} (\int_B G_{V_x}(z,u)du)\nu_n(dz) \ .$$

The measures ν_n are supported on ∂F (and y if $y \in X$), and $G(y,.)$ and $G_{Vx}(z,.)$, $z \in \partial F_x$, are uniformly bounded and continuous on a small neighbourhood $\overline{B}(x,\delta) \subset \overset{\circ}{F}_x$ of x. Taking $B = B(x,\varepsilon)$ in the above formula, dividing by $vol(B)$ and letting ε tend to 0, we obtain formula (2.6) as the limit. $\square$

Say that LS-data $(F_x, V_x)_{x\in X}$ are *balanced* if

(D5) there is a constant D such that $G_{V_x}(z,x) = D$ for all $x \in X$ and $z \in \partial F_x$.

From (2.5) and (2.6) we get the first part of our main theorem.

2.7. Theorem. — *If $(F_x, V_x)_{x\in X}$ are balanced LS-data for X, then*

$$G(y,x) = CDg(y,x)$$

for all x in X and all y not in V_x. In particular, the Brownian motion on M is transient if and only if the Markov process on X is transient. In the transient case we have $\mu_x(z) = \mu_z(x)$ for all x, z in X.

Proof. Except for the last assertion, all claims follow immediately from what is said above. As for the last claim, recall that

$$g(y,x) = \sum_{k\geq 0} \mu_y^{(n)}(x) \ .$$

For a positive function f on X we set

$$Pf(x) = \sum_z \mu_x(z) f(z), \quad Uf(x) = \sum_z g(x,z) f(z) \ .$$

If f has finite support we obtain

$$U(I-P)f = f \ .$$

Now U is symmetric with respect to

$$< f, h >= \sum_{x\in X} < f(x), h(x) >$$

and hence

$$\begin{aligned} < (I-P)f, h > &=< (I-P)f, U(I-P)h > \\ &=< U(I-P)f, (I-P)h >=< f, (I-P)h > \end{aligned}$$

for all positive functions f, h on X with finite support. The assertion follows. □

2.8. Theorem. — *Assume the Brownian motion on M is transient. If $(F_x, V_x)_{x\in X}$ are balanced LS-data for a $*$-recurrent set X, then the inclusion $X \hookrightarrow M$ extends to a convex homeomorphism between $\partial_\mu X$ and $\partial_\Delta M \cap \overline{X}$, where $\overline{X}$ is the closure of X in the Martin compactification $cl_\Delta M$ of M.*

Proof. Choose an origin x_0 in X and define for $x \neq x_0$ in X, y in M

$$k(y,x) = \frac{g(y,x)}{g(x_0,x)} \text{ and } K(y,x) = \frac{G(y,x)}{G(x_0,x)} \ .$$

From (2.7) we have $k(y,x) = K(y,x)$ for all $x \neq x_0$ in X and y in M not in V_x. Consider a convergent sequence $(x_n)_{n\geq 1}$ in the Martin compactification of (X,μ). Then for any fixed y, $k(y,x_n) = K(y,x_n)$ for n large enough and any Martin limit point H of the sequence $(K(\cdot,x_n))_{n\geq 1}$ satisfies $H|_X = h$. By Theorem 1.11 we have $H(y) = \mu_y(h)$ and H is unique. This shows that the sequence $(x_n)_{n\geq 1}$ converges in $cl_\Delta M$ and that the correspondence is convex and continuous. The converse is clear. □

It follows from Theorem 2.8 and its proof that the restriction map defines an isomorphism between the linear cone generated by $\overline{X}$ in $\mathcal{H}^+(M)$ and $\mathcal{H}^+(X,\mu)$. Comparing with Theorem 1.11 we get the following

2.9. Corollary. — *Let X be a discrete subset of M admitting balanced LS-data $(F_x, V_x)_{x\in X}$. Then a positive harmonic function H is swept by $F = \cup_{x\in X} F_x$ if and only if it can be written as an average of minimal harmonic functions in $\overline{X}$.*

Proof. We identified the cone generated by $\overline{X}$ with $\mathcal{H}^+_F(M)$. But by definition extremal directions in $\mathcal{H}^+_F(M)$ correspond to minimal harmonic functions. The same is therefore true for the cone generated by $\overline{X}$ in $\mathcal{H}^+(M)$. □

Corollary 2.9 can also be read the other way around : a family of neighbourhoods $(F_x)_{x\in X}$ has the same potential theory as X if $F = \cup_{x\in X} F_x$ is recurrent and if one can find open relatively compact $(V_x)_{x\in X}$, $V_x \supset F_x$, satisfying (D2), (D4) and (D5).

3. EXAMPLES

We say that the *geometry* of M is *bounded* in the ε-neighbourhood $B_\varepsilon(X)$ of a subset X of M if the injectivity radius in $B_\varepsilon(X)$ is positive and if the sectional curvature is bounded in $B_\varepsilon(X)$. For example, if X is the orbit of a point x_0 under a group of isometries, then the geometry of M is bounded in the ε-neighbourhood of X for any $\varepsilon > 0$ such that $B_\varepsilon(x_0)$ is relatively compact.

3.1. Theorem. — *If $X \subset M$ satisfies for some $\varepsilon > 0$*

(C1) *the geometry of M is bounded in $B_\varepsilon(X)$;*

(C2) *$dist(x, z) \geq 2\varepsilon$ for all $x \neq z$ in X ;*

(C3) *$\overline{B_\varepsilon(X)} = \cup_{x\in X}\overline{B}_\varepsilon(x)$ is recurrent,*

then X admits a choice of balanced LS-data $(F_x, V_x)_{x\in X}$ such that any isometry of M, which leaves X invariant, permutes the sets $(F_x, V_x)_{x\in X}$.

Remark. — If N is a recurrent Riemannian manifold, $M \to N$ a Riemannian covering and X the preimage in M of a point in N, then X satisfies the assumptions of Theorem 3.1. Note that N is recurrent if N is complete, of finite volume and with Ricci curvature bounded from below.

Proof. For x in X let $V_x = B(x, \varepsilon)$. Since the geometry of V_x is uniformly bounded, $\cup_{x\in X}\overline{B}_\delta(x)$ is recurrent for any $\delta > 0$ and the Green functions G_{V_x} admit uniform estimates. In particular, if $D > 0$ is any given constant, there is a $\delta \in (0, \varepsilon)$ such that $G_{V_x}(., x) \geq D$ on $\overline{B}_\delta(x)$. Hence

$$F_x = \{z \in V_x \quad | \quad G_{V_x}(z, x) \geq D\}$$

is a closed neighbourhood of x such that $G_{V_x}(z, x) = D$ on ∂F_x. Moreover, $F = \cup_{x\in X}F_x$ is recurrent since $\overline{B}_\delta(x) \subset F_x$ for all x in X. There is also a positive $\varepsilon' < \varepsilon$ such that $F_x \subset B(x, \varepsilon')$ for all x in X, hence (D4) is satisfied. □

3.2. Theorem. — *If M is simply connected, complete and with sectional curvature satisfying $-b^2 \leq K \leq -a^2 < 0$, and if Γ is a discrete group of isometries such that $vol(M/\Gamma) < \infty$, then Γ admits a symmetric probability μ such that*

(a) the Martin boundary of the random walk directed by μ is equal to the geometric boundary of M;

(b) μ has a finite moment with respect to the geometric norm on Γ and finite entropy.

Proof. The Martin compactification $cl_\Delta M$ of M is equal to the geometric compactification, see [AS]. Now choose $x_0 \in M$ such that Γ acts freely on x_0 and identify Γ with $\Gamma(x_0)$. Then Γ is $*$-recurrent in M since $vol(M/\Gamma) < \infty$. Hence $X = \Gamma(x_0)$ satisfies the assumptions of Theorem 3.1. Choose balanced LS-data $(F_x, V_x)_{x\in X}$ and let

$$\mu(\gamma) = \mu_{x_0}(\gamma x_0) \ .$$

Now Assertion (a) follows from Theorem 2.8 since the limit set of Γ is equal to the geometric boundary of M.

As for the proof of (b), we follow the construction of Lyons and Sullivan as described in section 2. We need that the functions

$$A_1(z) = E_z[S(\omega)],\ z \in F_x$$
$$A_2(y) = E_y[R_1(\omega)],\ y \in \partial V_x$$

are uniformly bounded. We will show this for A_2, the proof for A_1 is similar. If $\pi : M \to M/\Gamma$ is the projection, then $\pi(F) = \pi(F_x) =: C$ for any $x \in X$ and $\pi|_{F_x}$ is a homeomorphism. We have for y in ∂V_x

$$A_2(y) = T(\pi(y)) \ ,$$

where $T(z)$ is the average of the hitting time of C for Brownian motion starting in z. Since T is either identically $+\infty$ on $(M/\Gamma)\backslash C$ or smooth and solving $\Delta T = -1$, it suffices to show that T is finite on $(M/\Gamma)\backslash C$. Observe that

$$T(z) \le R(z)$$

where $R(z)$ is the average of the first time in $\mathbb{N}$ when Brownian motion starting in z hits C. By Kač formula [Ka] we have

$$\frac{|M/\Gamma|}{|C|} = \int_C R(z)dz \ge \int_C \int_{(M/\Gamma)\backslash C} p_1(x,y)T(y)\,dy\,dx \ .$$

Hence T is finite and A_2 is uniformly bounded on ∂V_x. Let A be a common bound for A_1 and A_2. We have for all x in M

$$E_x(R_n(\omega)) \le 2nA$$
$$\tilde{E}_x(R_{N_1}(\omega)) \le 2AE(N_1) \le 2AC^2 \ .$$

Since the average distance of the Brownian path to x_0 grows at most linearly with speed $(\dim M-1)b$, cf. for example [P], we get that the first moment is finite,

$$\sum_{\Gamma} dist(x_0, \gamma x_0)\mu(\gamma) = \tilde{E}_{x_0}(dist(x_0, X_{N_1}(\omega))) < +\infty \ .$$

The estimate on the entropy follows (see e.g. [BL], Lemma 2.1). □

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