CLASSICAL, EXCEPTIONAL, AND EXOTIC HOLONOMIES : A STATUS REPORT

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Abstract. I report on the status of the problem of determining the groups that can occur as the irreducible holonomy of a torsion-free affine connection on some manifold.

Résumé. Il s'agit d'un rapport sur le problème de la détermination des groupes qui peuvent être les groupes d'holonomie de connexions affines sans torsion.

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INTRODUCTION

0.1. Overview. The goal of this report is to present, in a unified way, what is known about the problem of prescribed holonomy of torsion-free affine connections smooth manifolds.

In §1, I give the fundamental definitions and develop the algebra needed to formulate Berger's criteria which a subgroup of $\mathrm{GL}(T_xM)$ must satisfy if it is to be the holonomy of a torsion-free affine connection on M which is not locally symmetric. I also develop the closely related notion of a torsion-free H-structure. The fundamental strategy is to 'classify' the torsion-free connections with a given holonomy H by first 'classifying' the torsion-free H-structures and then examining the problem of determining for any given torsion-free H-structure, its space of compatible torsion-free connections. In nearly all cases, there is a unique compatible torsion-free connection, but there are important exceptions that are closely related to the second-order homogeneous spaces.

I formulate the classification problem for general torsion-free H-structures as a problem treatable by the methods of Cartan-Kähler theory. Finally, I conclude this section with an appendix containing definitions of the various Spencer constructions that will be needed and a discussion of the history of the classification of the irreducible second-order homogeneous spaces. This classification turns out to be important in the classification of the affine torsion-free holonomies in §3.

In §2, I review Berger's list of the possible irreducible holonomies for pseudo-Riemannian metrics which are not locally symmetric. In the course of the review, I analyze each of the possibilities and determine the degree of generality of each one. Among the notable results are, first, that the group $\mathrm{SO}(n, \mathbb{H})$, which appeared on Berger's original list turns out not to be possible as the holonomy of a torsion-free connection, and, second, that there are two extra cases left off the usual lists (see §2.7-8). These can be viewed as alternate real forms of a group whose compact form is $\mathrm{Sp}(p)\cdot\mathrm{Sp}(1)$, the holonomy group of the so-called 'quaternionic-Kähler' metrics.

In §3, I turn to Berger's list of the possible irreducible holonomies for affine connections which are not locally symmetric and do not preserve any non-zero quadratic form. This list turns out to be quite interesting and the examples display a wide variety of phenomena. Actually, one has to remember that Berger's original list was only meant to cover all but a finite number of the possibilities, leaving open the possibility of a finite number of 'exotic' examples. Moreover, in Berger's original list, there was no attempt to deal with the different possibilities for the holonomy of the central part of the group; Berger's classification deals mainly with the classification of the semi-simple part of the irreducible holonomies. It turns out that the center of the group plays a very important role and gives rise to a wealth of examples that had heretofore not been anticipated.

Finally, in §4, I discuss what is known about the exotic examples so far (see Table 4). Perhaps the most interesting of these examples, aside from the examples in dimension 4 first discussed in [Br2], are the ones associated to the 'exceptional' second-order homogeneous spaces of dimension 16 and 27. For example, a consequence of this is that $E_6^{\mathbb{C}} \subset \mathrm{SL}(27,\mathbb{C})$ can occur as the holonomy of a torsion-free (but not locally symmetric) connection on a complex manifold of dimension 27! Unfortunately, as of this writing, the full classification of the possible exotic examples is far from complete.

0.2. Notation. In this report, I have adopted a slightly non-standard nomenclature for the various groups that are to be discussed. This subsection will serve to fix this notation, which is closely related to that used in [KoNa].

I will need to work with vector spaces over $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$. Conjugation has its standard meaning in $\mathbb{C}$ and $\mathbb{H}$; in each case, the fixed subalgebra is $\mathbb{R}$. The symbol $\mathbb{F}$ will be used to denote any one of these division algebras. The elements of the standard n-space $\mathbb{F}^n$ are to be thought of as columns of elements of $\mathbb{F}$ of height n. It is convenient to take all vector spaces over $\mathbb{H}$ to be *right* vector spaces.

For any vector space V over F, the group of invertible $\mathbb{F}$-linear endomorphisms of V will be denoted $\mathrm{GL}(V,\mathbb{F})$ or just $\mathrm{GL}(V)$ when there is no danger of confusion. The algebra of n-by-n matrices with entries in $\mathbb{F}$ will be denoted by $M_n(\mathbb{F})$. This algebra acts on the left of $\mathbb{F}^n$ by the obvious matrix multiplication, representing the algebra $\mathrm{End}_{\mathbb{F}}(\mathbb{F}^n)$. As usual, let $\mathrm{GL}(n,\mathbb{F}) \subset M_n(\mathbb{F})$ denote the Lie group consisting of the invertible matrices in $M_n(\mathbb{F})$, i.e., $\mathrm{GL}(n,\mathbb{F}) = \mathrm{GL}(\mathbb{F}^n)$. When $\mathbb{F} = \mathbb{R}$, the group $\mathrm{GL}(V)$ has two components and it is occasionally useful to use the notation $\mathrm{GL}^+(V)$ for the identity component. For any $A \in M_n(\mathbb{F})$, define $A^* \in M_n(\mathbb{F})$ to be the conjugate transpose of A, so that $(AB)^* = B^*A^*$ for all $A, B \in M_n(\mathbb{F})$.

For a vector space V over $\mathbb{R}$ or $\mathbb{C}$, the notation $\mathrm{SL}(V)$ has its standard meaning. There is no good notion of a quaternionic determinant; however, the obvious identification $\mathbb{H}^n \simeq \mathbb{R}^{4n}$ induces an embedding $\mathrm{GL}(n,\mathbb{H}) \hookrightarrow \mathrm{GL}(4n,\mathbb{R})$ and the subgroup $\mathrm{SL}(n,\mathbb{H}) \subset \mathrm{GL}(n,\mathbb{H})$ is then defined by $\mathrm{SL}(n,\mathbb{H}) = \mathrm{GL}(n,\mathbb{H}) \cap \mathrm{SL}(4n,\mathbb{R})$. Note that $\mathrm{SL}(n,\mathbb{H})$ has codimension 1 (not 4) in $\mathrm{GL}(n,\mathbb{H})$. In Chevalley's nomenclature, $\mathrm{SL}(n,\mathbb{H})$, which is a real form of $\mathrm{SL}(2n,\mathbb{C})$, is denoted $\mathrm{SU}^*(2n)$. My notation for the other real forms of $\mathrm{SL}(n,\mathbb{C})$ are the standard ones: $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SU}(p,q) = \{\, A \in \mathrm{SL}(n,\mathbb{C}) \,|\, A^*\, I_{p,q}\, A = I_{p,q} \,\}$. For simplicity, $\mathrm{SU}(n)$ denotes $\mathrm{SU}(n,0)$.

When $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ and Q is a non-degenerate quadratic form on a vector space V over $\mathbb{F}$, the slightly non-standard usage $\mathrm{SO}(V,Q)$ (respectively, $\mathrm{CO}(V,Q)$) will refer to the identity component of the subgroup of $\mathrm{GL}(V)$ that fixes Q (respectively, that fixes Q up to a multiple). The notations $\mathrm{SO}(p,q)$ ($= \mathrm{SO}(p)$ when $q = 0$) and $\mathrm{CO}(p,q)$ ($= \mathrm{CO}(p)$ when $q = 0$) denote the identity components of the standard subgroups of $\mathrm{GL}(p{+}q,\mathbb{R})$, while $\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{CO}(n,\mathbb{C})$ denote the standard subgroups of $\mathrm{GL}(n,\mathbb{C})$. Finally, $\mathrm{SO}(n,\mathbb{H})$ stands for the subgroup consisting of those $A \in \mathrm{GL}(n,\mathbb{H})$ that satisfy $A^*\, iI_n\, A = iI_n$. In Chevalley's nomenclature, $\mathrm{SO}(n,\mathbb{H})$, which is a real form of $\mathrm{SO}(2n,\mathbb{C})$, is denoted $\mathrm{SO}^*(2n)$.

Finally, when $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ and Ω is a non-degenerate skew-symmetric bilinear form on a vector space V over $\mathbb{F}$, the notation $\mathrm{Sp}(V,\Omega)$ (respectively, $\mathrm{CSp}(V,\Omega)$) will stand for the subgroup of $\mathrm{GL}(V)$ that fixes Ω (respectively, that fixes Ω up to a multiple.) The notations $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{CSp}(n,\mathbb{R})$ denote the standard subgroups of $\mathrm{GL}(2n,\mathbb{R})$ while $\mathrm{Sp}(n,\mathbb{C})$ and $\mathrm{CSp}(n,\mathbb{C})$ denote the standard subgroups of $\mathrm{GL}(2n,\mathbb{C})$. (In Chevalley's notation, $\mathrm{Sp}(n,\mathbb{R})$ is denoted by $\mathrm{Sp}^*(n)$.) As for the other real forms of $\mathrm{Sp}(n,\mathbb{C})$, I use the usual $\mathrm{Sp}(p,q)$ to denote the subgroup of $\mathrm{GL}(p{+}q,\mathbb{H})$ consisting of those matrices $A \in M_{p+q}(\mathbb{H})$ that satisfy $A^*\, I_{p,q}\, A = I_{p,q}$, with $\mathrm{Sp}(n,0)$ abbreviated to $\mathrm{Sp}(n)$.

Now define the following subspaces

$$S_n(\mathbb{R}) = \{\ A \in M_n(\mathbb{R}) \,\big|\, A = {}^tA\ \}$$

$$S_n(\mathbb{C}) = \{\ A \in M_n(\mathbb{C}) \mid A = {}^tA\ \}$$
$$S_n(\mathbb{H}) = \{\ A \in M_n(\mathbb{H}) \mid A = -A^*\ \}$$

and

$$A_n(\mathbb{R}) = \{\ A \in M_n(\mathbb{R}) \mid A = -{}^tA\ \}$$
$$A_n(\mathbb{C}) = \{\ A \in M_n(\mathbb{C}) \mid A = -{}^tA\ \}$$
$$A_n(\mathbb{H}) = \{\ A \in M_n(\mathbb{H}) \mid A = A^*\ \}$$

and

$$H_n(\mathbb{C}) = \{\ A \in M_n(\mathbb{C}) \mid A = A^*\ \}\ .$$

The definitions of $S_n(\mathbb{H})$ and $A_n(\mathbb{H})$ may seem surprising at first glance, but these choices maintain a helpful consistency in the names of real forms of certain complex representations.

When $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, the matrix group $\mathrm{GL}(n,\mathbb{F})$ acts on the right on $M_n(\mathbb{F})$ by the rule $m \cdot A = {}^tA\, m\, A$, preserving the two subspaces $S_n(\mathbb{F})$ and $A_n(\mathbb{F})$, which are irreducible and inequivalent. On the other hand, when $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{H}$, the matrix group $\mathrm{GL}(n,\mathbb{F})$ acts on the right on $M_n(\mathbb{F})$ by the rule $m \cdot A = A^*\, m\, A$. When $\mathbb{F} = \mathbb{H}$, this action preserves the two subspaces $S_n(\mathbb{H})$ and $A_n(\mathbb{H})$, which are irreducible and inequivalent. When $\mathbb{F} = \mathbb{C}$, this action preserves the two complimentary subspaces $H_n(\mathbb{C})$ and $i\, H_n(\mathbb{C})$, which are irreducible and equivalent.

1. HOLONOMY AND G-STRUCTURES

1.1. Holonomy. Let M^n be a smooth, 1-connected n-manifold and let ∇ be a linear connection on its tangent bundle TM.

Let $\mathcal{P}(M)$ denote the set of piecewise smooth paths $\gamma : [0,1] \to M$. For each $\gamma \in \mathcal{P}(M)$, the connection ∇ defines a linear isomorphism $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$ known as *parallel translation* along γ. The *holonomy of* ∇ *at* x is defined to be the set

$$H_x = \{\, P_\gamma \mid \gamma \in \mathcal{P}(M) \text{ and } \gamma(0) = \gamma(1) = x \,\} \subset \mathrm{GL}(T_xM) \ .$$

1.1.1. Group properties. — Because M is assumed to be connected and simply connected, a theorem of Borel and Lichnerowicz [KoNo, Theorem 4.2, Chapter II] implies that H_x is a connected Lie subgroup of $\mathrm{GL}(T_xM)$.

The dependence on the basepoint x is well understood: For any $\gamma \in \mathcal{P}(M)$, the isomorphism $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$ identifies $H_{\gamma(0)}$ with $H_{\gamma(1)}$.

Let V be some vector space of dimension n over $\mathbb{R}$. Choose an $x \in M$ and an isomorphism $u : T_xM \to V$. Let $H_u \subset \mathrm{GL}(V)$ denote the subgroup that corresponds to $H_x \subset \mathrm{GL}(T_xM)$ under this isomorphism. Explicitly,

$$H_u = \left\{\, u \circ P_\gamma \circ u^{-1} \,\middle|\, \gamma \in \mathcal{P}(M) \text{ and } \gamma(0) = \gamma(1) = x \,\right\} \subset \mathrm{GL}(V) \ .$$

Because M is connected, the $\mathrm{GL}(V)$-conjugacy class of the subgroup H_u is independent of the choices of x and u. In fact, as u varies, the group H_u ranges over all of the subgroups in a fixed $\mathrm{GL}(V)$-conjugacy class. In discussions of holonomy groups, it is customary to fix a subgroup H in this conjugacy class and simply say that the holonomy of ∇ is H. I employ this abuse of language when it seems unlikely to be confusing.

It is not hard to show that any connected Lie subgroup of $\mathrm{GL}(V)$ is (conjugate to) the holonomy of some linear connection on $\mathbb{R}^n$. Moreover, on a given manifold M, the problem of determining which subgroups of $\mathrm{GL}(V)$ can be the holonomy of some affine connection on TM is purely topological in nature; it reduces to the question of which structure reductions of the tangent bundle TM are possible topologically.

1.1.2. The torsion-free condition. — For a connection ∇ on the tangent bundle of a smooth manifold M, there is a tensor invariant called the *torsion* of ∇, defined by the rule $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ for any two vector fields X and Y on M. It is the lowest order invariant of a connection on TM. Most connections on the tangent bundle that arise in differential geometry are torsion-free, i.e., satisfy $T \equiv 0$, with the Levi-Civita connection of a Riemannian metric being the prime example.

This report is concerned with the following basic problem:

Problem. Which (conjugacy classes of) subgroups $H \subset \mathrm{GL}(V)$ can occur as the holonomy of some torsion-free connection ∇ on some n-manifold M?

Note that, by the remarks above, the torsion-free condition is the part of the problem which makes it interesting in a differential geometric sense.

In the pioneering work [Be1], M. Berger found conditions stemming from the Bianchi identities which must be satisfied by any subgroup $H \subset \mathrm{GL}(V)$ that occurs as the holonomy of some torsion-free connection. In §1.1.4, these conditions will be recalled, but first, it is convenient to review the structure equations of a torsion-free connection in a form that will be useful in later discussions.

1.1.3. The structure equations. — Again, let V be a fixed reference vector space of dimension n over the reals. Let $\pi : \mathcal{F} \to M$ denote the bundle of V- valued coframes. Thus, an element of the fiber $\mathcal{F}_x = \pi^{-1}(x)$ is an isomorphism $u : T_xM \to V$. The bundle $\mathcal{F}$ is naturally a smooth principal right $\mathrm{GL}(V)$-bundle over M where the right action is given by $R_A(u) = u\cdot A = A^{-1} \circ u$.

Any linear connection ∇ on TM has an associated connection form θ, i.e., a 1-form on $\mathcal{F}$ with values in $\mathfrak{gl}(V) = V \otimes V^*$ [KoNo]. The form θ is characterized by two conditions: First, it restricts to each fiber $\mathcal{F}_x$ to represent the canonical left-invariant

1-form on $\mathrm{GL}(V)$. Second, its kernel at each point u is the horizontal space of the connection ∇, i.e., a piecewise C^1 curve $\tilde{\gamma} : [0,1] \to \mathcal{F}$ satisfies $\tilde{\gamma}^*(\theta) = 0$ if and only if it is of the form $\tilde{\gamma}(t) = \tilde{\gamma}(0) \circ \big(P_{\gamma(t)}\big)^{-1}$ where $\gamma = \pi \circ \tilde{\gamma}$. These properties imply that θ satisfies $R_A^*(\theta) = A^{-1}\,\theta\,A$.

Because of the way $\mathcal{F}$ was defined, there is also a canonical V-valued 1-form ω on $\mathcal{F}$ defined by the rule $\omega(v) = u\big(\pi_*(v)\big)$ for $v \in T_u\mathcal{F}$. This 1-form obeys the equivariance condition $R_A^*(\omega) = A^{-1}\omega$. For each $u \in \mathcal{F}$, the linear map $\omega : T_u\mathcal{F} \to V$ is a surjection and its kernel is the tangent space at u to the fiber $\mathcal{F}_{\pi(u)}$.

The condition that ∇ be torsion-free is expressed in terms of θ and ω by the *first structure equation* of Élie Cartan:

$$d\omega = -\theta \wedge \omega \ . \tag{1}$$

The curvature of θ is the 2-form $\Theta = d\theta + \theta\wedge\theta$. (This latter equality is often called the *second structure equation* of Élie Cartan.) Taking the exterior derivative of the first structure equation yields the *first Bianchi identity*

$$\Theta \wedge \omega = 0 \ , \tag{2}$$

and the *second Bianchi identity* is simply $d\Theta = \Theta\wedge\theta - \theta\wedge\Theta$.

1.1.4. Berger's criteria. — The Bianchi identities can be used to derive information about the curvature of torsion-free connections with holonomy H. For any $u \in \mathcal{F}_x$, define the *holonomy bundle of* ∇ *through* u to be

$$\mathcal{B}_u = \left\{ \ u \circ P_{\gamma(1)} \,\middle|\, \gamma \in \mathcal{P}(M) \text{ and } \gamma(1) = x \ \right\} \ . \tag{3}$$

By the Reduction Theorem [KoNo, Theorem 7.1, Chapter II] the subset $\mathcal{B}_u \subset \mathcal{F}$ is a principal right H_u-subbundle of $\mathcal{F}$.

Suppose that ∇ has holonomy (conjugate to) H where $H \subset \mathrm{GL}(V)$ is a connected Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(V)$. Choose $u \in \mathcal{F}$ so that $H_u = H$. After pulling back the forms ω and θ to $\mathcal{B} = \mathcal{B}_u$, the form ω remains V-valued and surjective but, according to the Reduction Theorem, the form θ now takes values in the subalgebra $\mathfrak{h} \subset \mathfrak{gl}(V)$.

It follows that there exists a curvature function R on $\mathcal{B}$ with values in the subspace $\mathfrak{h}\otimes\Lambda^2(V^*)\subset\mathfrak{gl}(V)\otimes\Lambda^2(V^*)$ so that

$$\Theta = R\big(\omega\wedge\omega\big)\ .$$

The first Bianchi identity then becomes the relation

$$0=\Theta\wedge\omega = R\big(\omega\wedge\omega\big)\wedge\omega\ .$$

Thus, R takes values in the vector space $\mathcal{K}(\mathfrak{h})$, which is defined to be the kernel of the composition

$$\mathfrak{h}\otimes\Lambda^2(V^*)\hookrightarrow\mathfrak{gl}(V)\otimes\Lambda^2(V^*)=V\otimes V^*\otimes\Lambda^2(V^*)\longrightarrow V\otimes\Lambda^3(V^*)\ ,$$

where the final map is induced by exterior multiplication $V^*\otimes\Lambda^2(V^*)\to\Lambda^3(V^*)$. In terms of the Spencer complex described in the Appendix, $\mathcal{K}(\mathfrak{h})$ is the cycle group $Z^{1,2}(\mathfrak{h})=\mathfrak{h}\otimes\Lambda^2(V^*)\cap V\otimes S^2(V^*)\otimes V^*$.

Proposition. (Berger) — *Suppose that $\mathfrak{h}\subset\mathfrak{gl}(V)$ is the Lie algebra of the connected subgroup $H\subset\mathrm{GL}(V)$. Let $\mathfrak{h}'\subset\mathfrak{h}$ denote the smallest subspace that satisfies $\mathcal{K}(\mathfrak{h}')=\mathcal{K}(\mathfrak{h})$. Then $\mathfrak{h}'$ is the Lie algebra of a connected normal subgroup $H'\subset H$ which has the property that, if ∇ is a torsion-free connection on a 1-connected n-manifold M whose holonomy is (conjugate to) a subgroup of H, then its holonomy is (conjugate to) a subgroup of H'.*

Proof. First, note that if $\mathfrak{p}$ and $\mathfrak{q}$ are linear subspaces of $\mathfrak{gl}(V)$, then $\mathcal{K}\big(p\cap q\big)=\mathcal{K}(p)\cap\mathcal{K}(q)$. Thus, intersecting all of the subspaces $\mathfrak{p}\subset\mathfrak{h}$ that satisfy $\mathcal{K}(\mathfrak{p})=\mathcal{K}(\mathfrak{h})$ produces a unique minimal such subspace, say $\mathfrak{h}'$. Since the sequence defining $\mathcal{K}(\mathfrak{h})$ is H-equivariant, $\mathfrak{h}'\subset\mathfrak{h}$ must be invariant under the adjoint representation of H. In particular, $\mathfrak{h}'$ is an ideal of $\mathfrak{h}$, so that it is the Lie algebra of a connected normal subgroup $H'\subset H$.

Now, suppose that ∇ is a torsion-free connection on a 1-connected n-manifold M and that there exists a $u\in\mathcal{F}_x$ so that $H_u\subset H$. Let $\mathcal{B}=\mathcal{B}_u$. After restriction to $\mathcal{B}$, the connection form θ takes values in $\mathfrak{h}_u\subset\mathfrak{h}$ and hence the curvature function R

must take values in $\mathcal{K}(\mathfrak{h}) = \mathcal{K}(\mathfrak{h}')$. But now, for any piecewise C^1 path $\gamma : [0,1] \to M$ with $\gamma(1) = x$, parallel translation of any curvature endomorphism of ∇ on $T_{\gamma(0)}M$ to an endomorphism on $T_{\gamma(1)}M = T_xM$ yields an endomorphism which, relative to the coframe u, takes values in $\mathfrak{h}'$, possibly after being conjugated by an element of $H_u \subset H$. However, H' is a normal subgroup of H, so $\mathfrak{h}'$ is stable under conjugation by H_u. Thus, all parallel translations of curvature endomorphisms take values in $\mathfrak{h}'$. Now, by the holonomy theorem of Ambrose and Singer [KoNo, Theorem 8.1 of Chapter II] and the 1-connectedness of M, it follows that $H_u \subset H'$ as desired.

□

This yields the *first criterion of M. Berger:*

Criterion 1. — *If $H \subset \mathrm{GL}(V)$ can occur as the holonomy of a torsion-free connection, then $\mathcal{K}(\mathfrak{h}) \neq \mathcal{K}(\mathfrak{h}')$ for any proper ideal $\mathfrak{h}' \subset \mathfrak{h}$.*

Example. — Criterion 1 is very stringent. In [Br2], it is shown that, of all of the irreducible representations $\rho_n : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n{+}1,\mathbb{R})$ $(n \geq 1)$, only the groups $\rho_n\big(\mathrm{SL}(2,\mathbb{R})\big) \subset \mathrm{SL}(n{+}1,\mathbb{R})$ for $n = 1$, 2, 3, or 4 satisfy Criterion 1.

More generally, Berger showed that any reductive Lie group H has only a finite number of inequivalent representations $\rho : H \to \mathrm{GL}(V)$ so that the subgroup $\rho(H) \subset \mathrm{GL}(V)$ satisfies Criterion 1. In fact, making extensive use of representation theory, he compiled a list of almost all of the subgroups $H \subset \mathrm{GL}(V)$ that satisfy Criterion 1 and act irreducibly on V. This list was rather long. To reduce it, Berger formulated a second criterion that I will now describe.

A torsion-free connection ∇ is said to be *locally symmetric* if its curvature tensor is ∇-parallel. The problem of classifying the irreducible (affine) locally symmetric connections can be reduced to a (still formidable) algebra problem concerning Lie algebras [KoNo, Chapter XI]. Building on Cartan's work on the irreducible Riemannian symmetric spaces, Berger [Be2] solved this problem. Thus, the groups that can only occur as the holonomy of locally symmetric connections can be eliminated from further consideration.

In order to do this effectively, one needs a condition on subgroups $H \subset \mathrm{GL}(V)$ that is sufficient to force any torsion-free connection whose holonomy lies in H to be locally symmetric. Berger derived such a condition as follows: By the structure equations, the exterior derivative of the curvature function R can be written in the form

$$dR = \theta{\cdot}R + DR(\omega)$$

where the term $\theta{\cdot}R$ represents the 'fiber derivative' of R and and the term $DR(\omega)$ represents its 'covariant derivative'. Here, DR is a function on $\mathcal{B}$ with values in $\mathcal{K}(\mathfrak{h}) \otimes V^*$. The condition that ∇ be locally symmetric is just the condition $DR = 0$.

The second Bianchi identity now takes the form $DR(\omega)\big(\omega{\wedge}\omega\big) = 0$ and hence this represents a set of linear equations on DR. These equations express the condition that DR take values in the vector space $\mathcal{K}^1(\mathfrak{h})$, defined to be the kernel of the composition

$$\mathcal{K}(\mathfrak{h}) \otimes V^* \ \hookrightarrow\ \mathfrak{gl}(V) \otimes \Lambda^2(V^*) \otimes V^* \longrightarrow \mathfrak{gl}(V) \otimes \Lambda^3(V^*)\ ,$$

where the second map is just the identity on $\mathfrak{gl}(V)$ tensored with exterior multiplication $\Lambda^2(V^*) \otimes V^* \to \Lambda^3(V^*)$. This leads to the *second criterion of M. Berger:*

Criterion 2. — *If $H \subset \mathrm{GL}(V)$ can occur as the holonomy of a torsion-free connection which is not locally symmetric, then $\mathcal{K}^1(\mathfrak{h}) \neq 0$.*

Example. (CONTINUED) — Only $\rho_n\big(\mathrm{SL}(2,\mathbb{R})\big) \subset \mathrm{SL}(n{+}1,\mathbb{R})$ for $n = 1$, 2, or 3 satisfy Criterion 2 [Br3]. In fact, all three of these subgroups do occur as holonomy of non-symmetric torsion-free connections on manifolds of the appropriate dimension. When $n = 1$, such connections are the generic torsion-free connections on surfaces that preserve an area form, and when $n = 2$, since $\rho_2\big(\mathrm{SL}(2,\mathbb{R})\big) \simeq \mathrm{SO}(2,1) \subset \mathrm{SL}(3,\mathbb{R})$, such connections are the Levi-Civita connections of (generic) Lorentzian metrics on 3-manifolds. The case $n = 3$ is considerably more subtle. The reader may consult [Br2] or [Sc] for details.

On the other hand, the subgroup $\rho_4\big(\mathrm{SL}(2,\mathbb{R})\big) \subset \mathrm{SL}(5,\mathbb{R})$ (which satisfies Criterion 1 but not Criterion 2) occurs only as the holonomy of a locally symmetric torsion-free connection on a 5-manifold. In fact, up to diffeomorphism, such a connection must be locally equivalent to the canonical symmetric connection on either the symmetric space $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(2,1)$ or the symmetric space $\mathrm{SU}(2,1)/\,\mathrm{SO}(2,1)$.

The list of (conjugacy classes of) subgroups $H \subset \mathrm{GL}(V)$ that act irreducibly on V and satisfy Berger's two criteria is manageably short. Berger himself compiled this list except for a few small modifications that will be explained in later sections. In this report, this list is essentially Tables 1 and 3. (The division into two parts is original with Berger. The first part consists of the groups satisfying both Criteria that also preserve a non-degenerate quadratic form and the second part contains all the rest.)

1.1.5. Sufficient conditions. — There remains the task of determining which of the groups on these lists can actually occur as holonomy, in other words determining *sufficient* conditions for a subgroup to be holonomy of a torsion-free connection.

The most direct method of proving sufficiency would be to explicitly construct a torsion-free connection with holonomy H for each of the groups H satisfying Berger's criteria. However, this approach has difficulties and limitations.

The main difficulty is that the condition on a connection that it have holonomy in a certain subgroup is an integro-differential condition, difficult to capture or test locally. Except for the trivial cases where H is one of $\mathbf{1}_V$, $\mathbb{R}^+ \cdot \mathbf{1}_V$, or $\mathrm{SL}(V)$ (the only connected normal subgroups of $\mathrm{GL}(V)$), there is certainly no set of differential equations on connections whose solutions are precisely the connections with holonomy (conjugate to) H.

The main limitation of explicit construction is that it may not address the problem of *moduli*, that is, construction of one example may not give any clue as to 'how many' torsion-free connections there are with a given holonomy.

There is, however, a general strategy for resolving these problems for any specific subgroup H. After reviewing the relevant details from the theory of G-structures, this strategy will be outlined in §1.3.

1.2. ***H*-structures.** If $H \subset \mathrm{GL}(V)$ is any subgroup, an H-structure on M is, by definition, a smooth H-subbundle $\mathcal{B} \subset \mathcal{F}$.

When H is a closed subgroup of $\mathrm{GL}(V)$, the space of H-structures on M is simply the space of sections of the quotient bundle $\mathcal{S}_H = \mathcal{F}/H \to M$ whose typical fiber is isomorphic to $\mathrm{GL}(V)/H$.[1]

In this report, I am going to concentrate on the local geometry of H-structures, avoiding global topological questions about whether or not there exists a section of the bundle $\mathcal{S}_H$ over a given manifold M^n. For the problems that I will discuss, taking $M = \mathbb{R}^n$ would suffice, but, for the most part, I will continue to discuss the general n-manifold case as a way of emphasizing the diffeomorphism invariance of the problem being treated. Note that, since the bundles $\mathcal{F}$ and $\mathcal{S}_H$ are (non-canonically) trivial over $\mathbb{R}^n$, one may think of the space of H-structures on $\mathbb{R}^n$ as the space of mappings of $\mathbb{R}^n$ into $\mathrm{GL}(V)/H$. This is what I mean by the statement "Local H-structures depend on $\dim\big(\mathrm{GL}(V)/H\big) = n^2 - \dim H$ functions of n variables."

1.2.1. Torsion-free structures. — Since an H-structure $\mathcal{B}$ on M is a subbundle of $\mathcal{F}$, it follows that every connection on $\mathcal{B}$ extends canonically to a connection on $\mathcal{F}$. Naturally, a connection on $\mathcal{B}$ is said to be torsion-free if its extension to $\mathcal{F}$ is torsion-free.

I will say that an H-structure $\mathcal{B}$ itself is torsion-free if it admits at least one torsion-free connection and that it is *locally flat*[2] if every point $x \in M$ has a neighborhood U over which $\mathcal{B}$ has a closed section, i.e., a section η which satisfies $d\eta = 0$. From a partition of unity argument it follows that a locally flat H-structure is torsion-free and that the condition of being torsion-free is itself a local condition on an H-structure. The converse is not true; for most subgroups H of interest, *torsion-free* does not generally imply *locally flat*.

For simplicity of notation, I will use the symbol ω to denote the pull-back to $\mathcal{B}$ of the canonical V-valued 1-form ω on $\mathcal{F}$. The condition that $\mathcal{B}$ be torsion-free is then equivalent to the condition that there exist a 1-form ϕ on $\mathcal{B}$ with values in $\mathfrak{h} \subset \mathfrak{gl}(V)$ satisfying the structure equation $d\omega = -\phi{\wedge}\omega$. For many (in fact, most) subgroups H of $\mathrm{GL}(V)$, being torsion-free is a non-trivial condition on $\mathcal{B}$, as will be seen in the next section. First, here are three examples:

[1] Even when H is not closed, it is possible to regard the space of H-structures as the space of sections of a (non-Hausdorff) bundle, but the closed case will suffice here.

[2] Some authors, such as [KuSp], say *integrable* although this terminology conflicts with that of earlier authors, cf. [Ch].

Example. — If $H = \mathbf{1}_V$, then an H-structure on M is simply a section of $\mathcal{F}$, i.e., a V-valued 1-form η on M with the property that $\eta_x : T_xM \to V$ is an isomorphism for all $x \in M$. This $\mathbf{1}_V$-structure is torsion-free if and only if $d\eta = 0$. Clearly, in this case, torsion-free is the same as locally flat.

Example. — If $J : V \to V$ is a complex structure on V and $H \subset \mathrm{GL}(V)$ is the commuting group of J, then an H-structure on M is simply an almost complex structure on M. It is not hard to show that an H-structure is torsion-free if and only if the Nijnhuis tensor of the almost complex structure vanishes. The Newlander-Nirenberg theorem [Ni] implies that this is equivalent to the condition that the almost complex structure be integrable to a complex structure. Thus, for this subgroup also, torsion-free is the same as locally flat.

Example. — If H is the group of isometries of some positive definite quadratic form on V, then an H-structure on M is simply a Riemannian metric on M. By the Fundamental Lemma of Riemannian geometry, such a structure always has a (unique) torsion-free connection. Thus, in this case, all H-structures are torsion-free. However, as this example illustrates, 'torsion-free' need not imply 'flat'. The generic Riemannian metric is certainly not flat when $n > 1$.

1.2.2. Differential equations. — It is not hard to show that 'torsion-free' is the same as 'locally flat to first order' [Br1]. In fact, it is worthwhile to look at the differential equations which an H-structure must satisfy in order to be torsion-free. Since this is a local condition, I can simply assume that $M = \mathbb{R}^n = V$ with standard coordinate $x : V \to \mathbb{R}^n$. Then any map $g : M \to \mathrm{GL}(V)$ determines an H-structure by the formula

$$\mathcal{B}_g = \left\{ \, h^{-1} g^{-1} \, dx_p \, \middle| \, h \in H, \ p \in M \, \right\} \ .$$

Clearly, $\mathcal{B}_g$ depends only on the reduced mapping $[g] : M \to \mathrm{GL}(V)/H$. This H-structure will be torsion-free if and only if there exists a 1-form ϕ on M with values in $\mathfrak{h}$ which satisfies the equation

$$g^{-1} dg \wedge g^{-1} dx = \phi \wedge g^{-1} dx \ .$$

In other words, writing $g^{-1}dg{\wedge}g^{-1}dx = T(g)\big(g^{-1}dx{\wedge}\, g^{-1}dx\big)$, where $T(g) : M \to V \otimes \Lambda^2(V^*)$ is a non-linear first-order operator on maps $g : M \to \mathrm{GL}(V)$, the H-structure $\mathcal{B}_g$ is torsion-free if and only if $T(g) = \delta(F)$ where $F : M \to \mathfrak{h} \otimes V^*$ is some mapping and $\delta : \mathfrak{h} \otimes V^* \to V \otimes \Lambda^2(V^*)$ is the Spencer map defined in the Appendix. The cokernel of this mapping is the Spencer cohomology group $H^{0,2}(\mathfrak{h})$, whose dimension is $h^{0,2}(\mathfrak{h})$. It follows that the condition of being torsion-free is a set of at most $h^{0,2}(\mathfrak{h})$ independent first-order partial differential equations on an H-structure.

It is not difficult to show that these equations are all independent at a point and that these equations are actually equations on the mapping $[g] : M \to \mathrm{GL}(V)/H$. The principal difficulties of dealing with these equations can be enumerated as follows:

(1) They are invariant under the diffeomorphism group. This precludes them from being elliptic or hyperbolic in the usual senses, so that it can be difficult to apply analytic techniques. On the other hand, this can sometimes be turned into an advantage, as in Malgrange's proof of the Newlander-Nirenberg Theorem [Ni].

(2) They are generally overdetermined. It is almost always true that $h^{0,2}(\mathfrak{h}) > \dim\big(\mathrm{GL}(V)/H\big)$. For example, subgroups H which satisfy $h^{0,2}(\mathfrak{h}) = 0$, so that all H-structures are torsion-free, are very restricted. For a discussion of this, see the Appendix.

(3) For most subgroups $H \subset \mathrm{GL}(V)$, they are neither involutive or formally integrable, so that delicate methods from Cartan-Kähler theory must be brought to bear in their analysis. For example, for the subgroups $H = \mathrm{SO}(p) \times \mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(2p,\mathbb{R})$ studied in Example 1, the corresponding equations are not formally integrable.

Nevertheless, the methods of Cartan-Kähler theory can be used to study these problems and this is what I will be doing in a large part of this report.

1.2.3. An exterior differential system. — I will now describe an exterior differential system on $\mathcal{S}_H$ whose integral manifolds are the (local) torsion-free H-structures. The Appendix contains the information on the Spencer complex needed in this description. First, here are some basic facts about the geometry of $\mathcal{F}$.

For each $x \in \mathfrak{gl}(V)$, the right action of the 1-parameter subgroup $e^{tx} \in \mathrm{GL}(V)$ on $\mathcal{F}$ is the flow of a vertical vector field on $\mathcal{F}$ which I denote by V_x. The assignment $x \mapsto V_x$ is a linear map which satisfies the identity $V_{[x,y]} = \big[V_x, V_y\big]$ and the identities $\omega(V_x) = 0$ and $V_x \lrcorner\, d\omega = -x\,\omega$ are easily verified.

The double complex $A^{*,*}(V) = S^*(V^*)\otimes\Lambda^*(V^*)$ is defined in the Appendix. Now, define a map $\xi : A^{*,*}(V) \to \mathcal{A}^*(\mathcal{F})$ by setting $\xi(1 \otimes \alpha) = \alpha \circ \omega$ and $\xi(\alpha \otimes 1) = \alpha \circ d\omega = d\big(\alpha \circ \omega\big)$ and then extending ξ to all of $A^{*,*}(V)$ as an algebra map. Thus, for example, $\xi\big(A^{0,*}(V)\big)$ consists of the differential forms on $\mathcal{F}$ which are polynomial in the components of ω with constant coefficients. If $e_1, \ldots, e_n$ is a basis of V and one sets $\omega = e_i\,\omega^i$, then a typical element $\varphi \in \xi\big(A^{1,q}(V)\big)$ has the form

$$\varphi = \frac{1}{q!} \sum_{i,j_1,\ldots,j_q} c_{ij_1\cdots j_q}\, d\omega^i \wedge \omega^{j_1} \wedge \cdots \wedge \omega^{j_q}$$

where the constants $c_{ij_1\cdots j_q}$ are skew-symmetric in the last q indices. The n-form $\Omega = \omega^1\wedge\omega^2\wedge\cdots\wedge\omega^n$ generates $\xi\big(A^{0,n}(V)\big)$ and is non-vanishing on any submanifold of $\mathcal{F}$ which is transverse to the fibers of $\mathcal{F} \to M$.

Define

$$I^q(\mathfrak{h}) = \big\{\varphi \in \xi\big(A^{1,q}(V)\big) \mid V_x \lrcorner\, \varphi = 0 \text{ for all } x \in \mathfrak{h}\,\big\}\ .$$

It is obvious that $I^*(\mathfrak{h})$ is invariant under right action by H. By construction, it is semi-basic for the projection $\mathcal{F} \to \mathcal{F}/H = \mathcal{S}_H$. It follows that there is a differential ideal $\mathcal{I}_H$, well-defined on $\mathcal{S}_H$, which has the property that its pullback to $\mathcal{F}$ is generated by $I^*(\mathfrak{h})$. The n-form Ω is only well-defined on $\mathcal{S}_H$ when H is a subgroup of $\mathrm{SL}(V)$, however, it is well-defined up to a multiple on $\mathcal{S}_H$ and so, by abuse of language, I will usually refer to it as providing an independence condition for the ideal $\mathcal{I}_H$.

The importance of this ideal is explained by the following proposition. The proof consists of unwinding the definitions and will be omitted.

Proposition. — *The n-dimensional integral manifolds of $\mathcal{I}_H$ that are transverse to the fibers of $\mathcal{S}_H \to M$ are locally graphs of torsion-free H-structures. Conversely, the projection of each torsion-free H-structure $\mathcal{B}$ over M to $\mathcal{S}_H$ is an n-dimensional integral manifold of $\mathcal{I}_H$ that is transverse to the fibers of $\mathcal{S}_H \to M$.*

□

This proposition makes the study of the generality of torsion-free H-structures amenable to the techniques of Cartan-Kähler theory. If, for example, the ideal-plus-independence condition $(\mathcal{I}_H, \Omega)$ is involutive on $\mathcal{S}_H$, then this allows one to determine the 'generality' of torsion-free H-structures and to make claims about the behavior of the higher order jets of torsion-free H-structures. This information can be quite useful in the analysis of H-structures, cf. [Br1] and below.

Before leaving this construction, I should remark that there is also a relative version of this construction and this will be called upon at various times during the report. In the H-structures to be studied below, it sometimes happens that there is a group G satisfying $H \subset G \subset \mathrm{GL}(V)$ which has the property that torsion-free G-structures are all (locally) flat. The most common cases of this are when $G = \mathrm{Sp}(V, \Omega)$ or is the group of complex linear transformations of V endowed with a fixed complex structure. In such cases, it is frequently advantageous to take advantage of Darboux' theorem or the Newlander-Nirenberg theorem to reduce the original problem to the study a differential system whose integral manifolds correspond to the (local) reductions of a torsion-free G-structure to a torsion-free H-structure. I will not go through the construction of the corresponding differential system in full generality here. Instead, I will content myself with examples. The reader can see these examples in application in §2.5, which deals with torsion-free $\mathrm{Sp}(p, q)$-structures and in §3.1.3, which deals with $\mathrm{G}_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{H})$-structures.

1.3. Torsion-free H-structures and connections. Here is how the problem of describing the local torsion-free H-structures is related to the problem of describing the torsion-free connections with holonomy H: When the holonomy of a torsion-free connection ∇ is conjugate to a subgroup $H \subset \mathrm{GL}(V)$, say, with $H_u = H$, then the bundle $\mathcal{B}_u$ is a torsion-free H-structure. In fact, the connection ∇ determines a family of torsion-free H-structures which, in some sense, ought to be considered equivalent: Let $N_H \subset \mathrm{GL}(V)$ denote the normalizer of H in $\mathrm{GL}(V)$. Then for every $g \in N_H$, the bundle $\mathcal{B}_u \cdot g = \mathcal{B}_{u \cdot g}$ is also a torsion-free H-structure on M. This construction gives rise to a family of H-structures parametrized by the homogeneous space N_H/H.

Conversely, starting with any torsion-free H-structure $\mathcal{B} \subset \mathcal{F}$, one may choose a compatible connection ∇ which will be torsion-free and whose holonomy will be a (possibly proper) subgroup of H.

If being able to describe all torsion-free H-structures is to be useful in determining whether or not H can be the holonomy of a torsion-free connection, one needs a verifiable condition on an H-structure $\mathcal{B}$ which will guarantee that it supports a torsion-free connection with holonomy equal to the full group H. The remainder of this section develops a sufficient condition which will be used frequently in the rest of the report.

1.3.1. A sufficiency condition. — Now, for any fixed torsion-free H-structure $\mathcal{B}$, the space of $\mathcal{B}$-compatible torsion-free connections is an affine space modeled on the space of sections of the vector bundle $\mathcal{B}\times_H \mathfrak{h}^{(1)}$ over M whose typical fiber is isomorphic to $\mathfrak{h}^{(1)}$. In fact, supposing that ϕ is an $\mathfrak{h}$-valued 1-form on $\mathcal{B}$ that represents the restriction to $\mathcal{B}$ of a torsion-free connection on $\mathcal{F}$, then $d\omega = -\phi{\wedge}\omega$ and any other $\mathcal{B}$-compatible torsion-free connection is represented by a 1-form $\tilde{\phi} = \phi + a\,\omega$ where a is a function on $\mathcal{B}$ with values in $\mathfrak{h}^{(1)} \subset \mathfrak{h}\otimes V^*$ which satisfies the equivariance condition $R_A^*(a\,\omega) = A^{-1}\,(a\,\omega)\,A$. Since $\Phi = d\phi + \phi{\wedge}\phi = R(\omega{\wedge}\omega)$, where R takes values in $\mathcal{K}(\mathfrak{h})$, it is easy to compute that

$$\begin{aligned}\tilde{\Phi} &= d\tilde{\phi} + \tilde{\phi}\wedge\tilde{\phi} = \Phi + \big(da + \phi{\cdot}a + Q'(a)(\omega)\big)(\omega)\\ &= \Phi + \big(Da(\omega)\big)\wedge\omega = \big(R + \delta(Da)\big)(\omega\wedge\omega)\\ &= \tilde{R}\,\big(\omega\wedge\omega\big)\end{aligned}$$

where $Q' : \mathfrak{h}^{(1)} \to \mathfrak{h}^{(1)}\otimes V^*$ is an appropriate quadratic mapping and Da is a function on $\mathcal{B}$ with values in $\mathfrak{h}^{(1)}\otimes V^*$ which may be thought of as representing the covariant differential of the section of $\mathcal{B}\times_H \mathfrak{h}^{(1)}$ represented by a. In particular, note that the quotient mapping

$$[R] : \mathcal{B} \to H^{1,2}(\mathfrak{h}) = Z^{1,2}(\mathfrak{h})/B^{1,2}(\mathfrak{h}) = \mathcal{K}(\mathfrak{h})/B^{1,2}(\mathfrak{h})$$

is well-defined independent of choice of connection. Thus, $[R]$ represents a section of $\mathcal{B}\times_H H^{1,2}(\mathfrak{h})$ that can be regarded as the intrinsic curvature of the torsion-free H-structure $\mathcal{B}$.

Now, in the case where the differential system $(\mathcal{I}_H, \Omega)$ is involutive, the fact that there is a (local) integral manifold tangent to every integral element coupled with the description of the integral elements just given shows that for every element $[R_0] \in H^{1,2}(\mathfrak{h})$, there is a local torsion-free H-structure whose curvature function assumes the value $[R_0]$. Moreover the formula given for the effect on curvature of variation of connection then shows that for each element of $R_0 \in Z^{1,2}(\mathfrak{h}) = \mathcal{K}(\mathfrak{h})$, there is a connection on some torsion-free H-structure whose curvature assumes the value R_0.

Let $\mathcal{K}^\bullet(\mathfrak{h}) \subset \mathcal{K}(\mathfrak{h})$ denote the (possibly empty) subset consisting of those elements that do not lie in any subspace of the form $\mathfrak{p}\otimes\Lambda^2(V^*)$ for any proper subalgebra $\mathfrak{p} \subset \mathfrak{h}$. For groups H that satisfy Criterion 1, it frequently happens that $\mathcal{K}^\bullet(\mathfrak{h})$ is dense in $\mathcal{K}(\mathfrak{h})$. When $\mathfrak{h}^{(1)} \neq 0$, it can even happen that $\mathcal{K}^\bullet(\mathfrak{h}) \cap B^{1,2}(\mathfrak{h})$ is dense in $B^{1,2}(\mathfrak{h})$.

Definition. — *A connection on an H-structure $\mathcal{B}$ will be said to have $\mathfrak{h}$-full curvature if its curvature at some point assumes a value in $\mathcal{K}^\bullet(\mathfrak{h})$.*

By the Ambrose-Singer Holonomy theorem, any torsion-free connection on an H-structure $\mathcal{B}$ with $\mathfrak{h}$-full curvature will necessarily have its holonomy be all of H. Moreover, up to local diffeomorphism there is at most a finite dimensional space of torsion-free connections with holonomy H that are locally symmetric. Thus, a Cartan-Kähler analysis of the system $\big(\mathcal{I}_H, \Omega\big)$ together with an understanding of the set $\mathcal{K}^\bullet(\mathfrak{h})$ can suffice to prove that a torsion-free, not-locally-symmetric connection with holonomy H actually does exist. It is this general approach to sufficiency that will be used in this report.

I summarize this discussion in the enunciation of the following sufficient criterion.

Criterion 3. — *If $H \subset \mathrm{GL}(V)$ is a connected Lie subgroup for which the ideal $(\mathcal{I}_H, \Omega)$ is involutive and for which $\mathcal{K}^\bullet(\mathfrak{h})$ is non-empty, then there exist torsion-free connections with holonomy H.*

□

Example. — When $\mathfrak{h}^{(1)} = 0$, a torsion-free H-structure possesses a unique compatible, torsion-free connection. For example, $\mathfrak{so}(p,q)^{(1)} = H^{0,2}\big(\mathfrak{so}(p,q)\big) = 0$. (In fact, this is a restatement of the Fundamental Lemma of Riemannian geometry: Every (pseudo-)Riemannian metric (i.e., H-structure) possesses a unique compatible torsion-free connection.) In this case, there is no freedom in the choice of a compatible torsion-free connection and, since every $\mathrm{SO}(p,q)$-structure is torsion-free, it is not surprising that the system $\big(\mathcal{I}_{\mathrm{SO}(p,q)},\Omega\big)$ is involutive. Since $\mathcal{K}^\bullet\big(\mathfrak{so}(p,q)\big)$ is easily seen to be dense in $\mathcal{K}\big(\mathfrak{so}(p,q)\big)$, it follows that the generic $\mathrm{SO}(p,q)$-structure has holonomy equal to the identity component of $\mathrm{SO}(p,q)$.

Example. — At the other extreme, for groups like $\mathrm{GL}(V)$ or $\mathrm{SL}(V)$ when $n \geq 2$, which also have $H^{0,2}(\mathfrak{h}) = 0$, all H-structures are locally flat, $\mathcal{K}(\mathfrak{h}) = B^{1,2}(\mathfrak{h})$, and $\mathcal{K}^\bullet(\mathfrak{h})$ is dense in $\mathcal{K}(\mathfrak{h})$. Thus, in these cases also, the generic compatible connection has holonomy equal to H.

For most cases, however, the Cartan-Kähler analysis is non-trivial and the results are more subtle. In this report, I will concentrate exclusively on the case where H acts irreducibly on V, even though the general method does not need this restriction.

A. Appendix: Spencer cohomology

In this appendix, I collect definitions and facts about Spencer cohomology which will be needed in this report. Let V be a vector space of dimension n over a ground field $\mathbb{F}$ of characteristic zero. I use the standard notations $S^p(V^*)$ and $\Lambda^p(V^*)$ to denote, respectively, the symmetric and alternating p-linear functions on V.

A.1. The Spencer complex. The space $A^{p,q}(V) = S^p(V^*) \otimes \Lambda^q(V^*)$ can be thought of as the space of q-forms on V whose coefficients are homogeneous polynomial functions on V of degree p. Exterior differentiation then defines a linear map $\delta : A^{p,q}(V) \to A^{p-1,q+1}(V)$ which makes $A^{*,*}(V) = \bigoplus_{p,q\geq 0} A^{p,q}(V)$ into a bigraded complex satisfying

$$H^{*,*}\big(A^{*,*}(V),\delta\big) = H^{0,0}\big(A^{*,*}(V),\delta\big) \simeq \mathbb{F} \ .$$

Let W be another vector space over $\mathbb{F}$, define $\delta_W : W\otimes A^{*,*}(V) \to W\otimes A^{*,*}(V)$ to be $\delta_W = 1_W \otimes \delta$, and let $L \subset W \otimes V^*$ be any linear subspace. Define subspaces $L^{(k)} \subset W \otimes S^{k+1}(V^*)$ by the rules $L^{(-1)} = W$, $L^{(0)} = L$, and, for $k \geq 1$, the inductive formula

$$L^{(k)} = \delta_W^{-1}\big(L^{(k-1)} \otimes V^*\big) \ .$$

(The space $L^{(k)}$ is known as the *k-th prolongation* of L.) The natural inclusion $S^{k+1}(V^*) \subset V^* \otimes S^k(V^*)$ allows one to write

$$L^{(k)} = \big(L \otimes S^k(V^*)\big) \cap W \otimes S^{k+1}(V^*)\ .$$

The Spencer complex $\big(C^{*,*}(L), \delta\big)$ is then defined by setting

$$C^{p,q}(L) = L^{(p-1)} \otimes \Lambda^q(V^*) \subset W \otimes S^p(V^*) \otimes \Lambda^q(V^*) = W \otimes A^{p,q}(V)\ .$$

It is not hard to see that $\delta_W\big(C^{p,q}(L)\big) \subset C^{p-1,q+1}(L)$, so $\big(C^{*,*}(L), \delta_W\big)$ is indeed a subcomplex of $\big(W \otimes A^{*,*}(V), \delta_W\big)$. For simplicity, I shall write δ on this subcomplex instead of δ_W. The cohomology groups of this complex are denoted $H^{p,q}(L)$ and, as usual, $h^{p,q}(L)$ is defined to be $\dim_{\mathbb{F}} H^{p,q}(L)$. It is not hard to see directly from the definitions that $H^{p,0}(L) = H^{p,1}(L) = 0$ for all $p > 0$. Moreover, $H^{0,0}(L) = W$ and $H^{0,1}(L) = (W \otimes V^*)/L$. Thus, the interesting groups are $H^{p,q}(L)$ where $q \geq 2$.

In the cases of present interest, $W = V$ and $L = \mathfrak{h} \subset V \otimes V^* = \mathfrak{gl}(V)$ is the Lie algebra of a connected Lie subgroup $H \subset \mathrm{GL}(V)$ which acts irreducibly on V. The lower corner of the bigraded complex takes the form

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & & & \\
\mathfrak{h}^{(2)} & & \mathfrak{h}^{(2)} \otimes V^* & & \cdots & & & & \\
& \searrow & & \searrow & & & & & \\
\mathfrak{h}^{(1)} & & \mathfrak{h}^{(1)} \otimes V^* & & \mathfrak{h}^{(1)} \otimes \Lambda^2(V^*) & & \cdots & & \\
& \searrow & & \searrow & & \searrow & & & \\
\mathfrak{h} & & \mathfrak{h} \otimes V^* & & \mathfrak{h} \otimes \Lambda^2(V^*) & & \mathfrak{h} \otimes \Lambda^3(V^*) & & \cdots \\
& \searrow & & \searrow & & \searrow & & \searrow & \\
V & & V \otimes V^* & & V \otimes \Lambda^2(V^*) & & V \otimes \Lambda^3(V^*) & & \cdots
\end{array}$$

where all of the slanted arrows are simply δ. It is worth remarking that all of these vector spaces are H-modules in an obvious way and that all of the maps δ are H-module maps. Thus, in particular, all of the Spencer cohomology groups are themselves H-modules.

A.2. The torsion-free condition. As explained in §1.2.2, $h^{0,2}(\mathfrak{h})$ is the number of independent first order equations on H-structures needed to express the condition of being torsion-free. Since H is assumed to act irreducibly on V, a result of [KoNa1] (based on previous work of Cartan and Weyl) asserts that if $\dim V \geq 3$, then $h^{0,2}(\mathfrak{h}) = 0$ if and only if H contains the identity component of an orthogonal group $O(V,Q)$ where Q is some non-degenerate quadratic form on V. Of course, the only connected groups satisfying this condition are the identity components of the groups $O(V,Q)$, $\mathrm{CO}(V,Q)$, $\mathrm{SL}(V)$, and $\mathrm{GL}(V)$.

However, it must be noted that when $\dim V = 4$ their classification admits an exception, namely $\mathrm{CSp}(V,\Omega)$. Presumably this is the only exception.

Also, their result assumes that $\dim V \geq 3$. When $\dim V = 2$, in addition to the subgroups which contain an orthogonal group, there is a one-parameter family of subgroups

$$H_\lambda = \left\{ e^{\lambda t}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} \simeq \mathbb{R}, \qquad \lambda > 0$$

which also have $h^{0,2}(\mathfrak{h}_\lambda) = 0$. All these exceptions turn up later in this report.

A.3. Non-uniqueness of torsion-free connections. Another classification result which will be important for this report is the determination of the subgroups H which act irreducibly on V and which have $\mathfrak{h}^{(1)} \neq 0$. In the case that the ground field is $\mathbb{C}$, this list can be found in Table A.

The derivation of this result has an interesting history. É. Cartan derived most of this list in his fundamental paper [Ca2], which classifies the primitive infinite transitive pseudogroups in the holomorphic category. Unfortunately, he missed the two “sporadic” cases at the end of Table A. This omission came to light in [KoNa2] when Kobayashi and Nagano classified the irreducible second-order transformation groups (see below). Interestingly, in 1893 Cartan knew of these two exceptions to his classification of 1909. In the closing paragraphs of [Ca1], he explicitly lists realizations as second-order transformation groups of the exceptional groups $E_6^{\mathbb{C}}$ and $E_7^{\mathbb{C}}$ on (complex) spaces of respective dimensions 16 and 27. Their respective isotropy representations are $\mathbb{C}^*\cdot\mathrm{Spin}(10,\mathbb{C})$ on $\mathbb{C}^{16}$ and $\mathbb{C}^*\cdot E_6^{\mathbb{C}}$ on $\mathbb{C}^{27}$, precisely the ones Cartan

TABLE A. Irreducible $\mathbb{C}$-groups $H \subset \mathrm{GL}(V)$ with $\mathfrak{h}^{(1)} \neq 0$. (Notation: $d = \dim W$ and the dimension restrictions prevent repetition and/or reducibility.)

H	V	$\mathfrak{h}^{(1)}$	Restrictions
$\mathrm{SL}(W)$	W	$\big(W \otimes S^2(W^*)\big)_0$	$d \geq 2$
$\mathrm{GL}(W)$	W	$W \otimes S^2(W^*)$	$d \geq 1$
$\mathrm{Sp}(W, \Omega)$	W	$S^3(W^*)$	$d \geq 4$
$\mathrm{CSp}(W, \Omega)$	W	$S^3(W^*)$	$d \geq 4$
$\mathrm{CO}(W, Q)$	W	W^*	$d \geq 3$
$\mathrm{GL}(W)$	$S^2(W)$	$S^2(W^*)$	$d \geq 3$
$\mathrm{GL}(W)$	$\Lambda^2(W)$	$\Lambda^2(W^*)$	$d \geq 5$
$\mathrm{GL}(W_1) \cdot \mathrm{GL}(W_2)$	$W_1 \otimes W_2$	$W_1^* \otimes W_2^*$	$d_1 \geq d_2 \geq 2$, $(d_1, d_2) \neq (2, 2)$
$\mathbb{C}^* \cdot \mathrm{Spin}(10, \mathbb{C})$	$\mathbb{S}_+ \simeq \mathbb{C}^{16}$	$\mathbb{S}_+^* \simeq \mathbb{C}^{16}$	
$\mathbb{C}^* \cdot E_6^{\mathbb{C}}$	$V \simeq \mathbb{C}^{27}$	$V^* \simeq \mathbb{C}^{27}$	

omitted in his 1909 classification.[3]

The following information will be needed in the section on non-metric holonomy:

For the first four entries of Table A, $\mathfrak{h}^{(k)} \neq 0$ for all $k \geq 0$ while, for all of the rest of the entries, $\mathfrak{h}^{(k)} = 0$ for all $k > 1$.

Most of the groups listed in Table A satisfy $h^{1,2}(\mathfrak{h}) = h^{2,2}(\mathfrak{h}) = 0$. The three exceptions are as follows:

(1) When $H = \mathrm{CSp}(W, \Omega)$ where Ω is a non-degenerate 2-form on a vector space W of dimension 4, then $h^{1,2}(\mathfrak{h}) = 5$. In fact, $H^{1,2}(\mathfrak{h})$ is isomorphic to the kernel of the map $\Lambda^2(W^*) \to \Lambda^4(W^*)$ defined by exterior multiplication by Ω.

[3] Given Cartan's abiding interest in the exceptional groups, this omission is particularly puzzling. It is perhaps of some significance that, although Cartan refers to the results on infinite groups from his 1909 paper frequently in his later works, he never (to my knowledge) mentions his classification of the irreducible second-order transformation groups again. This is in spite of his extensive later work on second-order geometries (such as projective and conformal geometries), real forms of the complex simple groups, Riemannian symmetric spaces, and bounded symmetric domains, each of which is a subject where second-order transformation groups could have quite naturally arisen.

TABLE B. Irreducible $\mathbb{R}$-groups $H \subset \mathrm{GL}(V)$ with $\mathfrak{h}^{(1)} \neq 0$. (Notation: $G_{\mathbb{F}}$ denotes any connected subgroup of $\mathbb{F}^*$ and the dimension restrictions prevent repetition and/or reducibility.)

H	V	Restrictions
$\mathbb{R}^*$	$\mathbb{R}$	
$\mathbb{C}^*$	$\mathbb{C}$	
$G_{\mathbb{R}}\cdot\mathrm{SL}(n,\mathbb{R})$	$\mathbb{R}^n$	$n \geq 2$
$G_{\mathbb{C}}\cdot\mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 2$
$G_{\mathbb{R}}\cdot\mathrm{Sp}(n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n \geq 2$
$G_{\mathbb{C}}\cdot\mathrm{Sp}(n,\mathbb{C})$	$\mathbb{C}^{2n}$	$n \geq 2$
$\mathrm{CO}(p,q)$	$\mathbb{R}^{p+q}$	$p+q \geq 3$
$\mathrm{CO}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 3$
$\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(q,\mathbb{R})$	$\mathbb{R}^{pq}$	$p \geq q \geq 2,\ (p,q) \neq (2,2)$
$\mathbb{C}^*\cdot\mathrm{SL}(p,\mathbb{C})\cdot\mathrm{SL}(q,\mathbb{C})$	$\mathbb{C}^{pq}$	$p \geq q \geq 2,\ (p,q) \neq (2,2)$
$\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{H})\cdot\mathrm{SL}(q,\mathbb{H})$	$\mathbb{R}^{4pq}$	$p \geq q \geq 1,\ (p,q) \neq (1,1)$
$\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{C})$	$\mathbb{R}^{p^2} \simeq H_p(\mathbb{C})$	$p \geq 3$
$\mathrm{GL}(p,\mathbb{R})$	$\mathbb{R}^{p(p+1)/2} \simeq S_p(\mathbb{R})$	$p \geq 3$
$\mathrm{GL}(p,\mathbb{C})$	$\mathbb{C}^{p(p+1)/2} \simeq S_p(\mathbb{C})$	$p \geq 3$
$\mathrm{GL}(p,\mathbb{H})$	$\mathbb{R}^{p(2p+1)} \simeq S_p(\mathbb{H})$	$p \geq 2$
$\mathrm{GL}(p,\mathbb{R})$	$\mathbb{R}^{p(p-1)/2} \simeq A_p(\mathbb{R})$	$p \geq 5$
$\mathrm{GL}(p,\mathbb{C})$	$\mathbb{C}^{p(p-1)/2} \simeq A_p(\mathbb{C})$	$p \geq 5$
$\mathrm{GL}(p,\mathbb{H})$	$\mathbb{R}^{p(2p-1)} \simeq A_p(\mathbb{H})$	$p \geq 3$
$\mathbb{R}^*\cdot\mathrm{Spin}(5,5)$	$\mathbb{R}^{16}$	
$\mathbb{R}^*\cdot\mathrm{Spin}(1,9)$	$\mathbb{R}^{16}$	
$\mathbb{C}^*\cdot\mathrm{Spin}(10,\mathbb{C})$	$\mathbb{C}^{16}$	
$\mathbb{R}^*\cdot E_6^1$	$\mathbb{R}^{27}$	
$\mathbb{R}^*\cdot E_6^4$	$\mathbb{R}^{27}$	
$\mathbb{C}^*\cdot E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	

(2) When $H = \mathrm{CO}(W,Q)$ for Q a non-degenerate quadratic form on a vector space W of dimension at least 4.

(3) When $H = \mathrm{GL}(W_1)\cdot\mathrm{GL}(W_2)$ where $\dim W_1 > \dim W_2 = 2$.

In [Ma], an attempt was made to extend Cartan's classification to the real field. However, this list is incomplete for two reasons. First, since he relied on Cartan's incomplete list, Matsushima's list does not contain any of the real forms of the two 'sporadic' cases. Second, for several of the entries on Cartan's complex list, Matsushima missed one or more of their real forms, particularly, the ones associated to the quaternions. This list was finally completed by Kobayashi and Nagano in [KoNa2] and is to be found in Table B.

Finally, because it will be of use in the last section of this report, I have included the list of irreducible second-order homogeneous spaces as complied by Kobayashi and Nagano [KoNo2]. I would also like to remind the reader of the following terminology. Let $G_0 \subset G$ be a closed subgroup of a Lie group G which contains no normal subgroup of G of positive dimension. Then G acts almost faithfully on the homogeneous space G/G_0 and G_0 is the stabilizer subgroup of $[e] \in G/G_0$. There is a natural isomorphism $T_{[e]}(G/G_0) \simeq \mathfrak{g}/\mathfrak{g}_0$ where $[e]$ is the identity coset and there is a natural representation $\rho : G_0 \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{g}_0)$ given by $\rho(g) = L'_g\big([e]\big)$ where $L_g : G/G_0 \to G/G_0$ is the obvious left action by $g \in G$.

Let $G_1 \subset G_0$ denote the kernel of ρ. If G_0 is discrete, the homogeneous space is said to be of *order zero*, while if G_0 has positive dimension, the order of G/G_0 is defined to be one more than the order of G/G_1. (It is easy to see that if $\dim G_1 = \dim G_0$ then the identity component of G_0 is a normal subgroup of G which lies in G_0, so this inductive definition actually works.)

Let $H = \rho(G_0) \subset \mathrm{Aut}(\mathfrak{g}/\mathfrak{g}_0)$ be the image subgroup. The homogeneous space G/G_0 is said to be *irreducible* if H acts irreducibly on $\mathfrak{g}/\mathfrak{g}_0$. In [Ca1] Cartan claimed that any irreducible homogeneous space (in the holomorphic category) is either of order one or two. Apparently, his proof is flawed, but the result (even in the real category) is correct anyway, as was verified by Kobayashi and Nagano.

The list of the irreducible first-order homogeneous spaces is long, including, in particular, all of the irreducible affine symmetric spaces [Be2]. However in contrast, the complete list over the reals of the irreducible second-order homogeneous spaces is rather short. It is due to Kobayashi and Nagano [KoNa2]. This list is reproduced here

TABLE C. The irreducible second-order homogeneous spaces

$H = G_0/G_1$	$\mathfrak{g}/\mathfrak{g}_0$	G	G/G_0
$\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(q,\mathbb{R})$ $\mathbb{C}^*\cdot\mathrm{SL}(p,\mathbb{C})\cdot\mathrm{SL}(q,\mathbb{C})$ $\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{H})\cdot\mathrm{SL}(q,\mathbb{H})$	$\mathbb{R}^{pq}$ $\mathbb{C}^{pq}$ $\mathbb{R}^{4pq}$	$\mathrm{SL}(p{+}q,\mathbb{R})$ $\mathrm{SL}(p{+}q,\mathbb{C})$ $\mathrm{SL}(p{+}q,\mathbb{H})$	$\mathrm{Gr}_p^{\mathbb{R}}(\mathbb{R}^{p+q})$ $\mathrm{Gr}_p^{\mathbb{C}}(\mathbb{C}^{p+q})$ $\mathrm{Gr}_p^{\mathbb{H}}(\mathbb{H}^{p+q})$
$\mathrm{CO}(p,q)$ $\mathrm{CO}(p,\mathbb{C})$	$\mathbb{R}^{p,q}$ $\mathbb{C}^{p}$	$\mathrm{SO}(p{+}1,q{+}1)$ $\mathrm{SO}(p{+}2,\mathbb{C})$	$\mathcal{N}_1(\mathbb{R}^{p+1,q+1})$ $\mathcal{N}_1(\mathbb{C}^{p+2})$
$\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{C})$	$H_p(\mathbb{C})$	$\mathrm{SU}(p,p)$	$\mathcal{H}_p(\mathbb{C}^{2p})$
$\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{R})$ $\mathbb{C}^*\cdot\mathrm{SL}(p,\mathbb{C})$ $\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{H})$	$A_p(\mathbb{R})$ $A_p(\mathbb{C})$ $A_p(\mathbb{H})$	$\mathrm{SO}(p,p)$ $\mathrm{SO}(2p,\mathbb{C})$ $\mathrm{SO}(2p,\mathbb{H})$	$\mathcal{N}_p(\mathbb{R}^{2p})$ $\mathcal{N}_p(\mathbb{C}^{2p})$ $\mathcal{N}_p(\mathbb{H}^{2p})$
$\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{R})$ $\mathbb{C}^*\cdot\mathrm{SL}(p,\mathbb{C})$ $\mathbb{R}^*\cdot\mathrm{SL}(p,\mathbb{H})$	$S_p(\mathbb{R})$ $S_p(\mathbb{C})$ $S_p(\mathbb{H})$	$\mathrm{Sp}(p,\mathbb{R})$ $\mathrm{Sp}(p,\mathbb{C})$ $\mathrm{Sp}(p,p)$	$\mathcal{L}_p(\mathbb{R}^{2p})$ $\mathcal{L}_p(\mathbb{C}^{2p})$ $\mathcal{L}_p(\mathbb{H}^{2p})$
$\mathbb{R}^*\cdot\mathrm{Spin}(5,5)$ $\mathbb{R}^*\cdot\mathrm{Spin}(1,9)$ $\mathbb{C}^*\cdot\mathrm{Spin}(10,\mathbb{C})$	$\mathbb{R}^{16}$ $\mathbb{R}^{16}$ $\mathbb{C}^{16}$	E_6^1 E_6^4 $E_6^{\mathbb{C}}$	$\mathbb{OP}^2$
$\mathbb{R}^*\cdot E_6^1$ $\mathbb{R}^*\cdot E_6^4$ $\mathbb{C}^*\cdot E_6^{\mathbb{C}}$	$\mathbb{R}^{27}$ $\mathbb{R}^{27}$ $\mathbb{C}^{27}$	E_7^1 E_7^3 $E_7^{\mathbb{C}}$	

as Table C.[4] As will be seen in later sections, this list is relevant to the determination of the non-metric holonomies. For my purposes, it is only necessary to observe that, for each of the irreducible second-order homogeneous spaces, one has the isomorphism of H-modules $\mathfrak{g}_1 \simeq (\mathfrak{g}/\mathfrak{g}_0)^*$ and that, except for the first two entries of Table C, one has $\mathfrak{h}^{(1)} \simeq (\mathfrak{g}/\mathfrak{g}_0)^*$.

[4] The reader may want to note that I have corrected the misprint $\mathrm{SU}^*(4n)$ in their fifth entry to $\mathrm{SO}^*(4n) = \mathrm{SO}(2n,\mathbb{H})$. Also, for a related result, see Ochiai [Oc].

2. THE METRIC CASES

The list of all of the subalgebras of $\mathfrak{gl}(V)$ which satisfy Berger's two criteria is not known. However, if one restricts attention to the irreducibly acting subalgebras of $\mathfrak{gl}(V)$, many techniques from representation theory become available. Using these techniques, Berger himself found a large list of irreducibly acting subalgebras of $\mathfrak{gl}(V)$ which satisfy his two criteria. This list falls naturally into two parts: The first part, the metric list, consists of the irreducibly acting subalgebras of $\mathfrak{gl}(V)$ which also fix some non-degenerate quadratic form Q on V. This part appears in Table 1. The second part, the non-metric list, will be treated in the next section.

Since $\mathfrak{so}(p,q)^{(1)} = 0$, one has $\mathfrak{h}^{(1)} = 0$ for any subgroup $H \subset \mathrm{SO}(p,q)$. Thus, each connection with holonomy $H \subset \mathrm{SO}(p,q)$ determines a N_H/H-parameter family of torsion-free H-structures and each torsion-free H-structure determines a *unique* torsion-free connection with holonomy a subgroup of H. In particular, the geometry of torsion-free H-structures in the metric case is essentially the same as the geometry of the torsion-free connections with holonomy conjugate to a subgroup of H.

I will now discuss what is known about the generality of the space of connections with these holonomies on a case-by-case basis.

2.1. $\mathrm{SO}(p,q)$. — This is the generic (pseudo-)Riemannian metric. Such a metric is locally determined by choosing the $n(n+1)/2$ components of the metric in a local coordinate system, subject only to the open condition that the quadratic form have the desired signature. It is easy to see that the Levi-Civita connection of a generic metric of signature (p,q) will have $\mathfrak{so}(p,q)$-full curvature and hence will have holonomy equal to $\mathrm{SO}(p,q)$. Now, the local diffeomorphisms depend on n functions of n variables, so it follows that the local 'moduli space' of connections with this holonomy is described by $n(n-1)/2$ (local) functions of n variables.

TABLE 1. Berger's metric list

n	H	Geometric Type
$p+q \geq 2$	$\mathrm{SO}(p,q)$	Generic Metric
$2p$	$\mathrm{SO}(p,\mathbb{C})$	Holomorphic Metrics
$2(p+q) \geq 4$	$\mathrm{U}(p,q)$	Kähler
$2(p+q) \geq 4$	$\mathrm{SU}(p,q)$	special Kähler
$4(p+q) \geq 8$	$\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$	Quaternionic Kähler
$4(p+q) \geq 8$	$\mathrm{Sp}(p,q)$	hyperKähler
$4p \geq 12$	$\mathrm{SO}(p,\mathbb{H})$	?
7	G_2	Associative
7	G_2'	split-Associative
14	$\mathrm{G}_2^{\mathbb{C}}$	Holomorphic Associative
8	$\mathrm{Spin}(7)$	Cayley
8	$\mathrm{Spin}(4,3)$	split-Cayley
16	$\mathrm{Spin}(7,\mathbb{C})$	Holomorphic Cayley
16	$\mathrm{Spin}(9)$	?
16	$\mathrm{Spin}(8,1)$	?
16	$\mathrm{Spin}(5,4)$	?

2.2. $\mathrm{SO}(p,\mathbb{C})$. — These structures are simply the holomorphic analogues of Riemannian metrics. The only essential difference is that there is now no signature to worry about. The Levi-Civita connection of the generic holomorphic metric on $\mathbb{C}^p \simeq \mathbb{R}^{2p}$ has holonomy $\mathrm{SO}(p,\mathbb{C})$ and, modulo local biholomorphism, these structures depend on $p(p-1)/2$ holomorphic functions of p complex variables.

2.3. $\mathrm{U}(p,q)$. — This is the generic Kähler (pseudo-)metric. Since $\mathrm{U}(p,q)$ is a subgroup of $\mathrm{GL}(p+q,\mathbb{C})$, a torsion-free $\mathrm{U}(p,q)$-structure has an underlying torsion-free almost complex structure. By the Newlander-Nirenberg theorem, torsion-free almost complex structures are locally flat, so one may assume that the underlying almost complex structure is locally the standard one on $\mathbb{C}^{p+q}$. Moreover, the metric tensor can be described on $U \subset \mathbb{C}^{p+q}$ in terms of a function f (known as the *Kähler* *potential*) by the formula

$$g = \frac{\partial^2 f}{\partial z^i\,\partial \bar z^j}\,dz^i\,d\bar z^j = H_{i\bar\jmath}(f)\,dz^i\,d\bar z^j\ ,$$

where f is only required to satisfy the open condition that its complex Hessian $H(f)$ have Hermitian type (p,q). Thus, the torsion-free $\mathrm{U}(p,q)$ structures depend locally on one function of $n = 2(p{+}q)$ variables. (The choice of local holomorphic coordinates z^i depends on $(p{+}q)$ holomorphic functions of $(p{+}q)$ complex variables. Since such coordinates are determined by their restriction to any totally real submanifold, it follows that this coordinate ambiguity depends on $2(p{+}q)$ real analytic functions of $(p{+}q)$ real variables. Thus, this coordinate ambiguity does not materially affect the generality count.)

For generic f, the Levi-Civita connection of the metric g will be $\mathfrak{u}(p,q)$-full, so that the holonomy of the metric will be all of $\mathrm{U}(p,q)$.

2.4. $\mathrm{SU}(p,q)$. — Since $\mathrm{SU}(p,q)$ is a subgroup of $\mathrm{SL}(p{+}q,\mathbb{C})$, such a structure has not only an underlying parallel complex structure, but a parallel holomorphic volume form as well. An easy consequence of the Newlander-Nirenberg theorem is that all torsion-free $\mathrm{SL}(p{+}q,\mathbb{C})$-structures are locally flat, so there exist local complex coordinates $z^1,\dots,z^{p+q}$ in which this volume form becomes $dz^1{\wedge}\cdots{\wedge}dz^{p+q}$. Specifying the metric tensor is then locally equivalent to choosing a function f on $U \subset \mathbb{C}^{p+q}$ and letting

$$g = \frac{\partial^2 f}{\partial z^i\,\partial \bar z^j}\,dz^i\,d\bar z^j\ ,$$

where the complex Hessian of f has Hermitian type (p,q) and moreover, satisfies the single second-order PDE

$$\det\left(\frac{\partial^2 f}{\partial z^i\,\partial \bar z^j}\right) = (-1)^q\ .$$

Locally, in the real analytic category, a solution f to this equation will be determined by the values of it and its normal derivative along a (non-null) hypersurface. Thus, the solutions of this equation depend locally on two functions of $(2p{+}2q{-}1)$ variables. As long as $p{+}q > 1$, the coordinate ambiguity is of lower generality, depending only on $2p{+}2q{-}2$ functions of $p{+}q$ real variables.

TABLE 2. Local generality of metric holonomies (modulo diffeomorphisms)

n	H	Local Generality
$p{+}q \geq 2$	$\mathrm{SO}(p,q)$	$\frac{1}{2}n(n{-}1)$ of n
$2p$	$\mathrm{SO}(p,\mathbb{C})$	$\frac{1}{2}p(p{-}1)^{\mathbb{C}}$ of $p^{\mathbb{C}}$
$2(p{+}q) \geq 4$	$\mathrm{U}(p,q)$	1 of n
$2(p{+}q) \geq 4$	$\mathrm{SU}(p,q)$	2 of $n{-}1$
$4(p{+}q) \geq 8$	$\mathrm{Sp}(p,q)$	$2(p{+}q)$ of $(2p{+}2q{+}1)$
$4(p{+}q) \geq 8$	$\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$	$2(p{+}q)$ of $(2p{+}2q{+}1)$
$4p \geq 8$	$\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$	$2p$ of $(2p{+}1)$
$8p \geq 16$	$\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$	$2p^{\mathbb{C}}$ of $(2p{+}1)^{\mathbb{C}}$
7	G_2	6 of 6
7	G_2'	6 of 6
14	$\mathrm{G}_2^{\mathbb{C}}$	$6^{\mathbb{C}}$ of $6^{\mathbb{C}}$
8	$\mathrm{Spin}(7)$	12 of 7
8	$\mathrm{Spin}(4,3)$	12 of 7
16	$\mathrm{Spin}(7,\mathbb{C})$	$12^{\mathbb{C}}$ of $7^{\mathbb{C}}$

For a generically chosen solution f to this equation, the Levi-Civita connection of g_f will have $\mathfrak{su}(p,q)$-full curvature, so that its holonomy will be the full group $\mathrm{SU}(p,q)$.

2.5. $\mathrm{Sp}(p,q)$. — This is the geometry of pseudo-Kähler metrics which also possess a parallel holomorphic symplectic form. According to the holomorphic version of Darboux' theorem, any holomorphic symplectic form has a local coordinate expression of the form

$$\Omega = dz^1 \wedge dz^{p+q+1} + \cdots + dz^{p+q} \wedge dz^{2p+2q} .$$

A reduction of this $\mathrm{Sp}(p{+}q,\mathbb{C})$-structure to a $\mathrm{Sp}(p,q)$-structure is specified by choosing another 2-form ω with appropriate algebraic properties. These properties are described as follows: The 2-form ω must be expressed in the form

$$\omega = \tfrac{i}{2}\, {}^t dz \wedge G \wedge d\overline{z}$$

where dz is the column of height $2(p+q)$ of the differentials of the coordinate functions and G is a Hermitian symmetric matrix of functions on some domain $U \subset \mathbb{C}^{2(p+q)}$ with the property that it has Hermitian type $(2p, 2q)$ at each point of U and, moreover satisfies the matrix equation $\overline{G}\, J_{p+q}\, G = J_{p+q}$ where

$$J_{p+q} = \begin{pmatrix} 0 & \mathrm{I}_{p+q} \\ -\mathrm{I}_{p+q} & 0 \end{pmatrix} .$$

In this case, the associated metric is given by $g = {}^t dz\, G\, d\overline{z}$.

It is not difficult to show that the resulting $\mathrm{Sp}(p, q)$-structure is torsion-free if and only if the 2-form ω is closed. Since the real 3-form $d\omega$ must be the real part of a 3-form of type $(2, 1)$, this constitutes at most $4(p+q)^2(2p+2q-1)$ independent equations for the coefficients of G. The algebraic conditions on G (analyzed more fully below) imply that such a matrix is determined locally by $(p + q)(2p+2q+1)$ unknowns, so the closure of ω is always an overdetermined system of first order PDE for these unknowns.

One might hope to avoid dealing with an overdetermined system by introducing the Kähler potential f, i.e, a function f so that $\omega = -i\, \partial\bar{\partial} f$, as was done in the study of torsion-free $\mathrm{SU}(p, q)$-structures. However, as an equation for f, this is expressed in terms of its complex Hessian $H(f)$ by the matrix equation $\overline{H(f)}\, J_{p+q}\, H(f) = J_{p+q}$ together with the open condition that $H(f)$ be of Hermitian type $(2p, 2q)$. This matrix equation expands to be $(p+q)(2p+2q-1)$ independent second order PDE for f and hence is still an overdetermined system as soon as $p+q > 1$.

In fact, it is not difficult to construct an exterior differential system which allows analysis of the equations directly without the introduction of a potential. Since this will serve as a model for other such calculations, I will consider this one in some detail.

First, recall that the groups in question are defined by

$$\mathrm{Sp}(p+q, \mathbb{C}) = \{\ A \in \mathrm{GL}(2p+2q, \mathbb{C}) \,\big|\, {}^t A\, J_{p+q}\, A = J_{p+q}\ \}$$

and

$$\mathrm{Sp}(p, q) = \{\ A \in \mathrm{Sp}(p+q, \mathbb{C}) \,\big|\, A\, H_{p,q}\, \overline{{}^t A} = H_{p,q}\ \}$$

where

$$H_{p,q} = \begin{pmatrix} \mathrm{I_p} & 0 & 0 & 0 \\ 0 & -\mathrm{I_q} & 0 & 0 \\ 0 & 0 & \mathrm{I_p} & 0 \\ 0 & 0 & 0 & -\mathrm{I_q} \end{pmatrix} .$$

By its very construction, there is an identification

$$\mathrm{Sp}(p{+}q,\mathbb{C})/\,\mathrm{Sp}(p,q) = \{\, A\,H_{p,q}\,{}^t\overline{A} \;\big|\; A \in \mathrm{Sp}(p{+}q,\mathbb{C})\,\}$$

and this latter subspace of $M_{2p+2q}(\mathbb{C})$ is exactly the set of Hermitian symmetric matrices G of Hermitian type $(2p, 2q)$ which satisfy $\overline{G}\,J_{p+q}\,G = J_{p+q}$, as is easy to verify.

Now set $X_{p,q} = \mathbb{C}^{2p+2q} \times \mathrm{Sp}(p{+}q,\mathbb{C})/\,\mathrm{Sp}(p,q)$ and note that there is a natural embedding of $X_{p,q}$ into the bundle of $(1,1)$-forms on $\mathbb{C}^{2p+2q}$ given by the assignment

$$\big(z, A\cdot \mathrm{Sp}(p,q)\big) \mapsto \Big(z, \tfrac{i}{2}\,{}^t dz \wedge A\,H_{p,q}\,{}^t\overline{A} \wedge \overline{dz}\Big) .$$

This identifies $X_{p,q}$ as the bundle whose sections correspond to the reductions of the standard $\mathrm{Sp}(p{+}q,\mathbb{C})$ structure on $\mathbb{C}^{2p+2q}$ to a $\mathrm{Sp}(p,q)$-structure. Such a section corresponds to a closed 2-form if and only if the 3-form $d\omega$ vanishes identically on it where ω is the 2-form

$$\omega = \tfrac{i}{2}\,{}^t dz \wedge A\,H_{p,q}\,{}^t\overline{A} \wedge \overline{dz} ,$$

which is clearly well-defined on $X_{p,q}$.

Thus, let $\mathcal{I}$ be the differential ideal on $X_{p,q}$ generated by the 3-form $d\omega$ and let the independence condition be given by the standard volume form on $\mathbb{C}^{2p+2q}$. Note that this ideal is homogeneous and that, to compute its characters, it suffices to work at a single point. Moreover, there exist integral elements at each point since the flat structures defined by constant sections are integral manifolds of $\mathcal{I}$.

Now, ω is clearly well-defined on $\mathbb{C}^{2p+2q} \times \mathrm{Sp}(p{+}q,\mathbb{C})$ as well, and, setting $\zeta = {}^tA\,dz$ and $\alpha = A^{-1}\,dA$, one finds that the formula for $d\omega$ takes the form

$$d\omega = -\tfrac{i}{2}\,{}^t\zeta \wedge \big(\alpha\,H_{p,q} + H_{p,q}\,{}^t\overline{\alpha}\big) \wedge \overline{\zeta} ,$$

showing that the system $\mathcal{I}$ is in linear form with constant coefficients (in this basis), making the calculation of the characters particularly easy. Since $\mathcal{I}$ is generated by a real 3-form, one has $s'_k \leq k-1$ for all k. However, the sum of the reduced characters clearly cannot be more than the dimension of the homogeneous space $\mathrm{Sp}(p+q,\mathbb{C})/\mathrm{Sp}(p,q)$, i.e., $(p+q)(2p+2q+1)$. Thus, if the lower characters are to have their maximal possible value, then one might expect

$$s'_i = \begin{cases} i-1, & 1 \leq i \leq 2p+2q+1 \\ 0, & 2p+2q+1 < i \leq 4(p+q) \end{cases} .$$

Not surprisingly, this actually turns out to be the case, as is easily calculated. Another calculation reveals that the space of integral elements at each point has dimension

$$2\binom{2p+2q+2}{3} = \sum_{i=1}^{2p+2q+1} i\, s'_i$$

so that Cartan's Test is satisfied, and the system is involutive. Since its last non-zero Cartan character is $s_{2p+2q+1} = 2(p+q)$, the Cartan-Kähler theorem implies that, in the analytic category, solutions depend on $2(p+q)$ functions of $2p+2q+1$ variables. In fact, a more precise statement can be made: Starting with a real analytic submanifold $N^{2p+2q+1} \subset \mathbb{C}^{2p+2q}$ which is in sufficiently general position with respect to the holomorphic symplectic structure and a real analytic 1-form α on N whose exterior derivative $d\alpha$ satisfies certain open conditions, there is an open neighborhood U of N on which there exists a unique, real analytic, closed section of $X_{p,q}$ which pulls back to N to become $d\alpha$.

Now, the ambiguity in the choice of holomorphic symplectic coordinates is given by one symplectic generating function and hence is one holomorphic function of $(2p+2q)$ complex variables. Note that this ambiguity is of strictly smaller degree than that of the integral manifolds of $\mathcal{I}$. Thus, modulo diffeomorphisms, the generic local torsion-free $\mathrm{Sp}(p,q)$-structure depends on $2(p+q)$ functions of $2p+2q+1$ variables.

Finally, using the fact that $\mathcal{I}$ is involutive, one can compute that the generic integral manifold of $\mathcal{I}$ yields a torsion-free $\mathrm{Sp}(p,q)$-structure whose Levi-Civita connection is $\mathfrak{sp}(p,q)$-full, so that the holonomy of such a structure is the full group $\mathrm{Sp}(p,q)$. The details of this calculation (which are very similar to the calculations done in [Br1] for the groups G_2 and $\mathrm{Spin}(7)$) are left to the reader.

Metrics with this holonomy are known as *hyperKähler* (at least in the case $H = \mathrm{Sp}(p)$). See [Bea] for a survey article on the global aspects of the subject.

Explicit examples of hyperKähler metrics can be constructed by a generalization of the classical symplectic reduction procedure in which reduction works on three symplectic forms simultaneously. For references on this procedure, see [Bes]. No examples of hyperKähler metrics on compact manifolds are known explicitly, though one can study them by twistor methods [HKLR].

2.6. $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$. — These structures turn out to be only slightly more general than the $\mathrm{Sp}(p,q)$ examples. In fact, they satisfy the same curvature identities except that there is one extra parameter, the scalar curvature, and it must be constant. These metrics are all Einstein metrics. They have the same degree of local generality as $\mathrm{Sp}(p,q)$-structures (as is not hard to prove).

However, globally, things are more restrictive. For example, it was shown in [PoSa] that there is no compact smooth example with holonomy $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$ with positive scalar curvature other than the locally symmetric examples. On the other hand, yet another generalization of the symplectic reduction procedure allows one to construct such structures on certain compact orbifolds in any dimension, [Bo].

Using twistor methods, [LeB] has shown how to construct families of complete metrics with this holonomy depending on functions of $p+q+2$ variables. Recently, LeBrun and Salamon [LeSa] have shown that in each dimension there are only a finite number of diffeomorphism types of compact manifolds which admit a metric with this holonomy.

2.7. $\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$. — This is one of the two metric possibilities omitted from Berger's original list. It is a subgroup of $\mathrm{SO}(2p,2p)\subset\mathrm{GL}(4p,\mathbb{R})$. Note that this case and the previous case are real forms of the same complex subgroup $\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$ of $\mathrm{SO}(4p,\mathbb{C})$. All the corresponding metrics are Einstein. Just as in the other cases, a Cartan-Kähler analysis shows that modulo diffeomorphism the (analytic) local torsion-free H-structures in this case depend on $2p$ functions of $2p+1$ variables and that the Levi-Civita connection of the generic such structure has $\mathfrak{sp}(p,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$-full curvature, so that its holonomy is the full group $\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$.

2.8. $\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$. — This is the other of the two metric possibilities omitted from Berger's original list. The groups $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ and $\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$ are real forms of this complex group. All the corresponding metrics are holomorphic Einstein metrics. A Cartan-Kähler analysis shows that modulo diffeomorphism the (analytic) local torsion-free H-structures in this case depend on $2p$ holomorphic functions of $2p{+}1$ holomorphic variables and that the Levi-Civita connection of the generic such structure has $\mathfrak{sp}(p,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$-full curvature, so that its holonomy is the full group $\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$.

2.8. $\mathrm{SO}(p,\mathbb{H})$. — As was recently pointed out by R. McLean [Mc], this group does not satisfy Berger's first criterion. In fact, $\mathcal{K}\big(\mathfrak{so}(p,\mathbb{H})\big)=0$ since the complexification of the inclusion $\mathfrak{so}(p,\mathbb{H})\subset\mathfrak{so}(2p,\mathbb{C})\subset\mathfrak{gl}(4p,\mathbb{R})$ is the diagonal inclusion $\mathfrak{so}(2p,\mathbb{C})\subset\mathfrak{so}(2p,\mathbb{C})\oplus\mathfrak{so}(2p,\mathbb{C})\subset\mathfrak{gl}(4p,\mathbb{C})$. Thus there are no torsion-free connections with this holonomy.

2.9. The Exceptional Cases. Of the remaining cases, often called the 'exceptional' holonomies, the three groups of type $\mathrm{Spin}(9{-}k,k)$ were eliminated independently by Alexeevski [Al] and Brown and Gray [BrGr], who showed that these groups did not actually satisfy Berger's second criterion.

In [Br1] it was shown that all of the other exceptional groups on Berger's list do, in fact, occur, and with the local generality stated in Table 2. Complete examples for the compact holonomies were constructed in [BrSa]. It is now known that a compact Riemannian 7-manifold with holonomy G_2 would necessarily have finite fundamental group, must have its first Pontrjagin class p_1 be non-zero, and cannot be a product. Quite recently, Joyce [Jo1] has constructed compact Riemannian 7-manifolds with holonomy G_2 and compact 8-manifolds with holonomy $\mathrm{Spin}(7)$.

2.10. Summary. — The results of this discussion are summarized in Table 2. This table gives the "generality" of the non-symmetric local connections with a given metric holonomy group once one reduces modulo the diffeomorphism group. The entry "m of q" means "m functions of q variables". A superscript $\mathbb{C}$ is used to denote the holomorphic category and the spurious entries from Berger's list have been removed.

3. THE NON-METRIC CASES

TABLE 3. Berger's non-metric list (modified) (Notation: $G_{\mathbb{F}}$ denotes any connected subgroup of $\mathbb{F}^*$ and the dimension restrictions prevent repetition and/or reducibility.)

H	V	Restrictions
$G_{\mathbb{R}}\cdot\mathrm{SL}(n,\mathbb{R})$	$\mathbb{R}^n$	$n \geq 2$
$G_{\mathbb{C}}\cdot\mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 1$
$G_{\mathbb{R}}\cdot\mathrm{SL}(n,\mathbb{H})$	$\mathbb{H}^n$	$n \geq 1$
$G_{\mathbb{R}}\cdot\mathrm{Sp}(n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n \geq 2$
$G_{\mathbb{C}}\cdot\mathrm{Sp}(n,\mathbb{C})$	$\mathbb{C}^{2n}$	$n \geq 2$
$\mathrm{CO}(p,q)$	$\mathbb{R}^{p+q}$	$p+q \geq 3$
$G_{\mathbb{C}}\cdot\mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 3$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(q,\mathbb{R})$	$\mathbb{R}^{pq}$	$p \geq q \geq 2,\ (p,q) \neq (2,2)$
$G_{\mathbb{C}}\cdot\mathrm{SL}(p,\mathbb{C})\cdot\mathrm{SL}(q,\mathbb{C})$	$\mathbb{C}^{pq}$	$p \geq q \geq 2,\ (p,q) \neq (2,2)$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{H})\cdot\mathrm{SL}(q,\mathbb{H})$	$\mathbb{R}^{4pq}$	$p \geq q \geq 1,\ (p,q) \neq (1,1)$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{C})$	$H_p(\mathbb{C}) \simeq \mathbb{R}^{p^2}$	$p \geq 3$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{R})$	$S_p(\mathbb{R}) \simeq \mathbb{R}^{p(p+1)/2}$	$p \geq 3$
$G_{\mathbb{C}}\cdot\mathrm{SL}(p,\mathbb{C})$	$S_p(\mathbb{C}) \simeq \mathbb{C}^{p(p+1)/2}$	$p \geq 3$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{H})$	$S_p(\mathbb{H}) \simeq \mathbb{R}^{p(2p+1)}$	$p \geq 2$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{R})$	$A_p(\mathbb{R}) \simeq \mathbb{R}^{p(p-1)/2}$	$p \geq 5$
$G_{\mathbb{C}}\cdot\mathrm{SL}(p,\mathbb{C})$	$A_p(\mathbb{C}) \simeq \mathbb{C}^{p(p-1)/2}$	$p \geq 5$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{H})$	$A_p(\mathbb{H}) \simeq \mathbb{R}^{p(2p-1)}$	$p \geq 3$

The second part of Berger's list consisted of a list of irreducibly acting groups which did *not* fix a quadratic form and which satisfied Berger's two criteria. Moreover, Berger stated a theorem to the effect that this list contained all but a finite number of such groups.

With some modifications, Berger's non-metric list is given in Table 3. The major modification is that I have collected all of the entries on Berger's original list which differ only by 'extension by scalars' into single entries in the table; the entries which contain a $G_{\mathbb{K}}$ actually represent several entries (see below). Also, some of these entries must be discarded for certain values of the (integer) parameters because the group in question either fixes a quadratic form, does not act irreducibly, or else is already accounted for somewhere else in the table.

Note that the list, as constituted, does not contain any examples (comparable to the 'exceptional' cases like $\mathrm{G}_2 \subset \mathrm{SO}(7)$ on the metric list) which occur in only one dimension. I will refer to any groups which act irreducibly, satisfy Berger's criteria, do not fix a non-degenerate quadratic form, and yet do not appear on this list as *exotic*. It is not obvious that exotic groups exist.

We will now examine what is known about each of the entries on the list. For convenience and because of the similarity of many of the arguments so grouped, I have collected the entries into 'families'.

3.1. The Affine Families. This is first group in the table and represents the possible 'affine' cases as one allows the 'ground field' to vary.

3.1.1. $G_{\mathbb{R}}\cdot \mathrm{SL}(n,\mathbb{R})$. — This contains two cases, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$ acting via their usual representations on $V = \mathbb{R}^n$. They will be referred to respectively as general and special $\mathbb{R}$-affine connections.

I will begin with the general affine case, where the corresponding H-structure is the whole of $\mathcal{F}$. Any coframing is a section of $\mathcal{F}$, so there is no loss of generality in choosing a closed coframing, i.e., so that $\eta = dx$ for some V-valued function x. Then a torsion-free connection is represented by a 1-form ϕ with values in n-by-n matrices which satisfies the condition $\phi{\wedge}\eta = d\eta = 0$. In other words,

$$\phi = \left(\phi^i_j\right) = \left(\Gamma^i_{jk}\, dx^k\right)$$

for arbitrary functions $\Gamma^i_{jk} = \Gamma^i_{kj}$. The generic choice of such Γ's will yield a connection with $\mathfrak{gl}(n,\mathbb{R})$-full curvature and hence with holonomy $\mathrm{GL}(n,\mathbb{R})$. The ambiguity in the choice of local coordinates x is given by n functions of n variables, so the generality of such local connections modulo the local diffeomorphisms is given by

$$N = n\big(\tbinom{n+1}{2} - 1\big) = \tfrac{1}{2}\, n(n+2)(n-1)$$

functions of n variables.

Now, I turn to the special affine case. Here, there is an underlying parallel volume form. Since volume forms have no local invariants, it follows that there are local coordinates $x^1, \ldots, x^n$ so that the parallel volume form is $dx^1{\wedge}\cdots{\wedge}\, dx^n$. Then, as before, a section of the $\mathrm{SL}(n,\mathbb{R})$-structure is simply $\eta = dx$. Now again, the 1-form corresponding to a torsion-free connection preserving this structure is given by a n-by-n matrix of 1-forms ϕ which satisfies the condition $\phi{\wedge}dx = 0$, but it must also satisfy $\operatorname{tr}\phi = 0$. This, of course is the extra condition on the Γ's that $\Gamma^i_{ij} = 0$ for all j. Moreover, the generic choice of Γ's satisfying these restrictions will yield a connection with $\mathfrak{sl}(n,\mathbb{R})$-full curvature and hence with holonomy $\mathrm{SL}(n,\mathbb{R})$. Thus, by the same analysis as above, it follows that, modulo diffeomorphisms, the local torsion-free connections with holonomy $\mathrm{SL}(n,\mathbb{R})$ depend on

$$N = n\big(\tbinom{n+1}{2} - 2\big) + 1 = \tfrac{1}{2}\,(n^2 + 2n - 2)(n-1)$$

functions of n variables.

3.1.2. $G_{\mathbb{C}}{\cdot}\,\mathrm{SL}(n,\mathbb{C})$. — This case divides into three subcases, depending on the dimension of the subgroup $G_{\mathbb{C}} \subset \mathbb{C}^*$. Set $V = \mathbb{C}^n$ and let $G_{\mathbb{C}}{\cdot}\,\mathrm{SL}(n,\mathbb{C})$ act in the usual way on V.

If $G_{\mathbb{C}} = \mathbb{C}^*$, then the fact that the underlying $\mathrm{GL}(n,\mathbb{C})$-structure is torsion-free implies, via the Newlander-Nirenberg Theorem, that it is integrable, and hence that a local section can be taken in the form $\eta = dz$. The associated connection 1-form of any compatible torsion-free connection is given in the form

$$\phi = \big(\phi^i_j\big) = \big(\Gamma^i_{jk}\, dz^k\big)$$

where $\Gamma^i_{jk} = \Gamma^i_{kj}$ are arbitrary functions (*not* necessarily holomorphic). Thus, the generality of such connections modulo (holomorphic) changes of coordinates is

$$N = 2n\tbinom{n+1}{2} = n^2(n+1)$$

(real) functions of $2n$ variables. The generic choice of such Γ's yields a connection with $\mathfrak{gl}(n,\mathbb{C})$-full curvature and hence with holonomy $\mathrm{GL}(n,\mathbb{C})$.

If $G_{\mathbb{C}} = \{1\}$, then the fact that the underlying $\mathrm{SL}(n,\mathbb{C})$-structure is torsion-free implies that it is integrable, and a local section can be taken in the form $\eta = dz$. The accompanying connection is given in the form

$$\phi = \big(\phi^i_j\big) = \big(\Gamma^i_{jk}\, dz^k\big)$$

where $\Gamma^i_{jk} = \Gamma^i_{kj}$ are arbitrary functions (again *not* necessarily holomorphic) subject to the extra condition that $\Gamma^i_{ij} = 0$ for all j. Thus, the generality of such connections modulo (holomorphic) changes of coordinates is

$$N = 2n\Big(\tbinom{n+1}{2} - 1\Big) = n(n+2)(n-1)$$

(real) functions of $2n$ variables. Note, that one must assume $n > 1$ in order to have an irreducible action in this case. Again the generic such connection will have holonomy $\mathrm{SL}(n,\mathbb{C})$.

Finally, if $G_{\mathbb{C}}$ is a 1-parameter subgroup of $\mathbb{C}^*$, then there is an angle θ in the interval $0 \le \theta < \pi$ so that $G_{\mathbb{C}}$ is the group of complex numbers of the form

$$e^{t(\cos\theta + i\sin\theta)}, \qquad t \in \mathbb{R}\ .$$

Using the integrability of the underlying almost complex structure, it is not difficult to show that one can always choose local holomorphic coordinates $z^1, \ldots, z^n$ so that there exists an $\mathbb{R}$-valued function f so that

$$\eta = e^{-(\sin\theta - i\cos\theta)f}\, dz\ .$$

Moreover, the connection matrix ϕ is of the form

$$\phi^i_j = \delta^i_j(\sin\theta - i\cos\theta)\, df + \Gamma^i_{jk}\, dz^k$$

where $\Gamma^i_{jk} = \Gamma^i_{kj}$. In order that ϕ take values in $\mathfrak{h}$, these functions must also satisfy the trace condition

$$\Gamma^i_{ik}\, dz^k = -2in\, e^{-i\theta}\, \partial f\ .$$

Moreover, it is easy to see that, for generic choices of f and Γ satisfying these relations, the connection ϕ will have $\mathfrak{h}$-full curvature and hence its holonomy will be $G_{\mathbb{C}}\cdot \mathrm{SL}(n,\mathbb{C})$. Thus, the space of such connections (modulo holomorphic diffeomorphisms) depends on

$$N = 2n\big(\binom{(n+1)}{2} - 1\big) + 1 = n(n+2)(n-1)+1$$

(real) functions of $2n$ variables. Note, by the way, that when $n = 1$ this family contains the subgroup $S^1\cdot \mathrm{SL}(1,\mathbb{C}) \subset G_{\mathbb{C}}\cdot \mathrm{SL}(1,\mathbb{C})$, which fixes a quadratic form, and hence has already been counted in the metric list.

3.1.3. $G_{\mathbb{R}}\cdot \mathrm{SL}(n,\mathbb{H})$. — This subfamily of affine structures contains two cases, namely $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{SL}(n,\mathbb{H})$. There is a twistor-theoretic approach to their analysis, due independently to Salamon and Bérard-Bergery. For more information on this, consult [Bes, §§14.66–76] and [Sa]. The term *hypercomplex* is often used to refer to $4n$-manifolds endowed with torsion-free $\mathrm{GL}(n,\mathbb{H})$-structures. For more information on the construction of examples, the reader can consult [Jo1] and [Bo].

When $n = 1$, the subgroup $\mathrm{SL}(1,\mathbb{H}) \simeq \mathrm{SU}(2)$ is compact and so fixes a quadratic form; hence this one case has already been treated and so will be set aside in this analysis.

First, I need to fix some notation. Let $V = \mathbb{H}^n$ and regard V as the space of column vectors of height n with quaternion entries. The representation of $\mathrm{GL}(n,\mathbb{H})$ on V is then defined by matrix multiplication on the left while scalar multiplication takes place on the right. It will be useful to have an explicit identification of $\mathbb{H}^n$ in $\mathbb{H}^n$, corresponds to the usual scalar multiplication by i in $\mathbb{C}^{2n}$. For various reasons, I have chosen to make the identification

$$v_0 - v_1\, j = \begin{pmatrix} v_0 \\ \overline{v_1} \end{pmatrix}$$

for all $v_0, v_1 \in \mathbb{C}^n$. By this identification, $\mathrm{GL}(n,\mathbb{H})$ is embedded as a subgroup of $\mathrm{GL}(2n,\mathbb{C})$ so that

$$A - B\, j = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}$$

where A and B are n-by-n complex matrices. Moreover, under this identification, R_j becomes the linear transformation $w \mapsto J_n\,\overline{w}$ where

$$J_n = \begin{pmatrix} 0_n & \mathrm{I_n} \\ -\mathrm{I_n} & 0_n \end{pmatrix} ,$$

so that

$$\mathrm{GL}(n,\mathbb{H}) = \left\{ \ A \in \mathrm{GL}(2n,\mathbb{C}) \,\middle|\, \overline{A}\,J_n = J_n\,A \ \right\} \ .$$

A $\mathrm{GL}(n,\mathbb{H})$-structure $\mathcal{B}$ on a $4n$-manifold M defines two skew-commuting almost complex structures R_i, $R_j : TM \to TM$, i.e., linear bundle maps which satisfy $(R_i)^2 = (R_j)^2 = -1$ and $R_iR_j = -R_jR_i$. (I am using R_i and R_j to remind the reader that quaternion vector spaces are *right* vector spaces.) In fact, any local section of $\mathcal{B}$ with domain $U \subset M$ is, by definition an $\mathbb{H}^n$-valued 1-form $\eta : TU \to \mathbb{H}^n$ which is an isomorphism restricted to each fiber T_xU and the maps R_i and R_j are then uniquely defined (independent of the choice of section η) by the equations $\eta(R_iv) = \eta(v)\,i$ and $\eta(R_jv) = \eta(v)\,j$. Conversely, given two skew-commuting almost complex structures R_i, $R_j : TM \to TM$, the local $\mathbb{H}^n$-valued coframings $\eta : TU \to \mathbb{H}^n$ which satisfy $\eta(R_iv) = \eta(v)\,i$ and $\eta(R_jv) = \eta(v)\,j$ are the sections of a unique $\mathrm{GL}(n,\mathbb{H})$-structure on M.

An important difference between this family and the first two affine families is that $\mathfrak{sl}(n,\mathbb{H})^{(1)} = \mathfrak{gl}(n,\mathbb{H})^{(1)} = 0$. This follows since, when one complexifies the inclusions $\mathfrak{gl}(n,\mathbb{H}) \subset \mathfrak{gl}(2n,\mathbb{C}) \subset \mathfrak{gl}(4n,\mathbb{R})$, the resulting inclusion of complex Lie algebras is simply the diagonal inclusion[5]

$$\mathfrak{gl}(2n,\mathbb{C}) \subset \mathfrak{gl}(2n,\mathbb{C}) \oplus \mathfrak{gl}(2n,\mathbb{C}) \subset \mathfrak{gl}(4n,\mathbb{C}) \ .$$

Thus, a torsion-free $\mathrm{SL}(n,\mathbb{H})$-structure or $\mathrm{GL}(n,\mathbb{H})$-structure on a $4n$-manifold has only one compatible torsion-free connection, an observation originally due to Obata, for whom this connection is named.

Another important feature of this case is the characterization of the torsion of such structures. Letting $\mathfrak{h}_i$ (respectively, $\mathfrak{h}_j$) denote the subalgebra of $\mathfrak{gl}(4n,\mathbb{R})$ which

[5] It is a general fact that, given two faithful representations $\rho_i : \mathfrak{h} \to \mathfrak{gl}(V_i)$, their sum in $\mathfrak{gl}(V_1{\oplus}V_2)$ satisfies $\big((\rho_1{\oplus}\rho_2)(\mathfrak{h})\big)^{(1)} = 0$.

consists of the $\mathbb{R}$-linear endomorphisms of $\mathbb{H}^n$ commuting with right multiplication by i (respectively, j), then one has $\mathfrak{gl}(n,\mathbb{H}) = \mathfrak{h}_i \cap \mathfrak{h}_j$ and hence a canonical mapping

$$H^{0,2}\big(\mathfrak{gl}(n,\mathbb{H})\big) \longrightarrow H^{0,2}\big(\mathfrak{h}_i\big) \oplus H^{0,2}\big(\mathfrak{h}_j\big)$$

which, after some calculation (see below), is seen to be an isomorphism.[6] Consequently, a $\mathrm{GL}(n,\mathbb{H})$-structure is torsion-free if and only if the two associated skew-commuting almost complex structures R_i and R_j are themselves torsion-free, i.e., integrable.

Now, let $\mathcal{B} \to M^{4n}$ be a torsion-free $\mathrm{GL}(n,\mathbb{H})$-structure. Let η be a local section of $\mathcal{B}$ with associated connection 1-form θ. Then the first structure equation $d\eta = -\theta \wedge \eta$ holds. Write $\eta = \eta_0 - \eta_1\, j$ and $\theta = \theta_0 - \theta_1\, j$ where η_0 and η_1 are 1-forms with values in $\mathbb{C}^n$ while θ_0 and θ_1 are 1-forms with values in $\mathfrak{gl}(n,\mathbb{C})$. Then the first structure equation expands to

$$d\begin{pmatrix} \eta_0 \\ \overline{\eta_1} \end{pmatrix} = -\begin{pmatrix} \theta_0 & -\overline{\theta_1} \\ \theta_1 & \overline{\theta_0} \end{pmatrix} \wedge \begin{pmatrix} \eta_0 \\ \overline{\eta_1} \end{pmatrix} .$$

It follows that the $\mathbb{C}^{2n}$-valued coframing $\eta_0 \oplus \overline{\eta_1}$ (which is, of course, R_i-linear) is a section of an integrable $\mathrm{GL}(2n,\mathbb{C})$-structure on M. By the Newlander-Nirenberg theorem, the domain of η can be covered by open sets U on which there exists a coordinate chart $z : U(\subset M) \to \mathbb{C}^{2n}$ so that

$$\begin{pmatrix} \eta_0 \\ \overline{\eta_1} \end{pmatrix} = G^{-1}\, dz$$

for some function $G : U \to \mathrm{GL}(2n,\mathbb{C})$. Since $\eta(R_j v) = \eta(v)\, j$ for all $v \in TU$, it follows that

$$\begin{pmatrix} \eta_0(R_j v) \\ \overline{\eta_1(R_j v)} \end{pmatrix} = \begin{pmatrix} \overline{\eta_1(v)} \\ -\overline{\eta_0(v)} \end{pmatrix} = J_n \overline{\begin{pmatrix} \eta_0(v) \\ \overline{\eta_1(v)} \end{pmatrix}} ,$$

from which it follows that

$$dz(R_j v) = \overline{\overline{G}\, J_n\, G^{-1}\, dz(v)} .$$

[6] It is *not* true for arbitrary subalgebras $\mathfrak{g}$ and $\mathfrak{h}$ of $\mathfrak{gl}(V)$ that the canonical map $H^{0,2}\big(\mathfrak{g}\cap\mathfrak{h}\big) \longrightarrow H^{0,2}\big(\mathfrak{g}\big) \oplus H^{0,2}\big(\mathfrak{h}\big)$ is an isomorphism. Both injectivity and surjectivity fail in general.

Define a $\mathrm{GL}(2n,\mathbb{C})$-valued function J on $z(U)$ by the rule $J\circ z = \overline{G}\,J_n\,G^{-1}$. Then $dz\circ R_j = \overline{J\circ z\,dz}$. Note that J takes values in the smooth submanifold $\mathcal{J}_n \subset \mathrm{GL}(2n,\mathbb{C})$ of real dimension $4n^2$ defined by

$$\begin{aligned}\mathcal{J}_n &= \left\{\, \overline{G}\,J_n\,G^{-1} \,\middle|\, G \in \mathrm{GL}(2n,\mathbb{C}) \,\right\}\\ &= \left\{\, A \in \mathrm{GL}(2n,\mathbb{C}) \,\middle|\, A\,\overline{A} = -I_{2n} \,\right\}\\ &\simeq \mathrm{GL}(2n,\mathbb{C})/\,\mathrm{GL}(n,\mathbb{H})\ .\end{aligned}$$

Conversely, any map $J : z(U) \to \mathcal{J}_n$ determines a $\mathrm{GL}(n,\mathbb{H})$-structure on U for which R_i and R_j are defined by the equations $dz\circ R_i = i\,dz$ and $dz\circ R_j = \overline{J\circ z\,dz}$. The condition that this $\mathrm{GL}(n,\mathbb{H})$-structure be torsion-free is then expressible in terms of some system of partial differential equations on the map J, to be determined presently, and which are equivalent to the condition that the almost complex structure R_j so defined should be integrable.

First, though, I want to examine the effect of the choice of the local coordinate z on the resulting function J. If one were to choose a different coordinate chart $w : U \to \mathbb{C}^{2n}$ satisfying $dw(R_i v) = i\,dw(v)$, then there would exist a biholomorphism $\varphi : z(U) \to w(U)$ so that $w = \varphi\circ z$. By the Chain Rule, if $dw = \varphi'\,dz$ then

$$dw(R_j v) = \overline{\overline{\varphi'}\,J\circ z\,(\varphi')^{-1}\,dw(v)}\ .$$

It follows that classifying the torsion-free $\mathrm{GL}(n,\mathbb{H})$-structures up to local diffeomorphism on smooth $4n$-manifolds is equivalent to classifying locally defined maps $J : U(\subset \mathbb{C}^{2n}) \to \mathcal{J}_n$ (which satisfy the system of partial differential equations to be defined below) up to the equivalence relation $\big(U,J\big) \simeq \big(\varphi(U),J_\varphi\big)$ where $\varphi : U \to \varphi(U) \subset \mathbb{C}^{2n}$ is any biholomorphism and $J_\varphi : \varphi(U) \to \mathcal{J}_n$ is defined by

$$J_\varphi\big(\varphi(z)\big) = \overline{\varphi'(z)}\,J(z)\big(\varphi'(z)\big)^{-1}\ .$$

Note that this formulation reduces the coordinate 'ambiguity' in the problem from the full diffeomorphism group of $\mathbb{R}^{4n}$ (which depends on $4n$ smooth functions of $4n$ variables) to the biholomorphism group of $\mathbb{C}^{2n}$ (which only depends on $4n$ real-analytic functions of $2n$ variables).

For the purposes of studying the local torsion-free $\mathrm{GL}(n,\mathbb{H})$-structures, it clearly suffices to study the case in which U is already an open subset of $\mathbb{C}^{2n}$ and $z : U \to \mathbb{C}^{2n}$ is just the identity map, so I will assume this from now on.

I claim that the condition that J define a torsion-free $\mathrm{GL}(n,\mathbb{H})$-structure on U is simply the condition

$$\partial\bigl(J\,dz\bigr) = 0\ .$$

(Note that this condition is invariant under the biholomorphism pseudo-group defined above.) This is most easily seen as follows: Locally on U one can write $J = \overline{G}\,J_n\,G^{-1}$ for some function $G : U \to \mathrm{GL}(2n,\mathbb{C})$. Then the $\mathbb{C}^{2n}$-valued 1-form

$$\begin{pmatrix} \eta_0 \\ \hline \eta_1 \end{pmatrix} = G^{-1}\,dz$$

defines, via $\eta = \eta_0 - \eta_1\,j$, an $\mathbb{H}$-linear 1-form on U which may be regarded as a section of a unique $\mathrm{GL}(n,\mathbb{H})$-structure on U. Differentiating both sides of this equation yields

$$d\begin{pmatrix} \eta_0 \\ \hline \eta_1 \end{pmatrix} = -G^{-1}\,dG \wedge \begin{pmatrix} \eta_0 \\ \hline \eta_1 \end{pmatrix}\ .$$

If one assumes that the associated $\mathrm{GL}(n,\mathbb{H})$-structure be torsion-free, then there must exist 1-forms θ_0 and θ_1 with values in $\mathfrak{gl}(n,\mathbb{C})$ satisfying

$$d\begin{pmatrix} \eta_0 \\ \hline \eta_1 \end{pmatrix} = -\begin{pmatrix} \theta_0 & -\overline{\theta_1} \\ \hline \theta_1 & \overline{\theta_0} \end{pmatrix} \wedge \begin{pmatrix} \eta_0 \\ \hline \eta_1 \end{pmatrix}\ .$$

Comparing terms in the last two equations, the only possibility for the connection θ is seen to be

$$\begin{pmatrix} \theta_0 & -\overline{\theta_1} \\ \hline \theta_1 & \overline{\theta_0} \end{pmatrix} = G^{-1}\overline{\partial}G + J_n^{-1}\,\overline{G^{-1}}\,\partial\overline{G}\,J_n\ .$$

Manipulating the equality

$$G^{-1}\,dG \wedge \begin{pmatrix} \eta_0 \\ \hline \eta_1 \end{pmatrix} = \Bigl(G^{-1}\overline{\partial}G + J_n^{-1}\,\overline{G^{-1}}\,\partial\overline{G}\,J_n\Bigr) \wedge \begin{pmatrix} \eta_0 \\ \hline \eta_1 \end{pmatrix}$$

by expanding dG into $\partial G + \overline{\partial}G$, canceling equal terms, and then premultiplying by $\overline{G}\,J_n$, this equation simplifies to $\partial(J\,dz) = 0$. These steps are reversible also, so this latter condition is the necessary and sufficient condition for the connection θ defined above to be torsion-free, which is what I wanted to show.

Since $\mathcal{J}_n$ is a smooth manifold of dimension $4n^2$, locally the choice of a J depends on $4n^2$ functions of $4n$ variables. However, the equations $\partial(J\,dz) = 0$, being equivalent (as mentioned previously) to the integrability of the almost complex structure R_j, constitute $4n^2(2n{-}1)$ independent first-order partial differential equations for the map J. Thus, as soon as $n > 1$, this is an overdetermined system of PDE for J.

It is not difficult to see that this system of equations is elliptic for each value of n, and an immediate consequence of this is that any C^1 solution $J : U \to \mathcal{J}_n$ of the system $\partial(J\,dz) = 0$ is actually real-analytic. This justifies further analysis of these equations via Cartan-Kähler methods.

One attempt to analyze these equations is to 'introduce a potential'. By the local exactness of the ∂-complex associated to a complex manifold, it follows that, at least locally, for every solution J of the equations $\partial(J\,dz) = 0$, there exists a $\mathbb{C}^{2n}$-valued function $F : U \to \mathbb{C}^{2n}$ satisfying $\partial F = J\,dz$, i.e., $F_z = J$. Thus, the original quasi-linear first order equations on J become the non-linear first order equations

$$F_z\,\overline{F_z} = -\mathrm{I}_{2n}\ .$$

Of course, this constitutes $4n^2$ first order equations for the $4n$ unknown components of the potential function F. However, a short calculation shows that this first order system is not involutive as soon as $n > 1$.

However, I claim that the original system $\partial(J\,dz) = 0$ is involutive, with Cartan characters $s_i = 4n^2$ for $1 \le i \le 2n{+}1$ and $s_i = 0$ for $2n{+}2 \le i \le 4n$. It will follow from this claim that the general solution depends on $4n^2$ functions of $2n{+}1$ variables. Indeed, the Cartan-Kähler analysis will show that along any real-analytic submanifold $S^{2n+1} \subset \mathbb{C}^{2n}$ with the property that each tangent space T_sS is minimally complex (i.e., $T_sS \cap i(T_sS)$ is a complex line for all $s \in S$), the analytic function $J : S \to \mathcal{J}_n$ can be specified arbitrarily subject to satisfying a certain open condition on the pairs $\big(J(s), T_sS\big)$ (asserting that the subspace T_sS be in 'general position' with respect to the associated $R_j(s) : T_s\mathbb{C}^{2n} \to T_s\mathbb{C}^{2n}$).

Because it illustrates several features of the use of exterior differential systems, I want to sketch the details of this case.

First of all, the introduction of the holomorphic coordinate system z, has allowed a reduction from considering the sections of the bundle $\mathcal{F}/\operatorname{GL}(n,\mathbb{H})$ over $\mathbb{C}^{2n}$ (which correspond to arbitrary $\operatorname{GL}(n,\mathbb{H})$-structures on the $4n$-manifold $\mathbb{C}^{2n}$ to considering the sections for which the associated almost complex structure R_i is the standard complex structure on $\mathbb{C}^{2n}$. In other words, the bundle $X = \mathbb{C}^{2n} \times \mathcal{J}_n$ over $\mathbb{C}^{2n}$ is canonically embedded into the structure bundle $\mathcal{F}/\operatorname{GL}(n,\mathbb{H})$ over $\mathbb{C}^{2n}$. Let $J^1(\mathbb{C}^{2n}, X)$ denote the space of 1-jets of sections of X over $\mathbb{C}^{2n}$. It is a manifold of dimension $4n{+}4n^2{+}16n^3$. Let $X^{(1)} \subset J^1(\mathbb{C}^{2n}, X)$ denote the submanifold consisting of the 1-jets of sections of X which satisfy the equation $\partial(J\,dz) = 0$. The codimension of this submanifold is $4n^2(2n{-}1)$ since this is the number of independent equations in $\partial(J\,dz) = 0$.

As I remarked previously, the system of equations $\partial(J\,dz) = 0$ is invariant under the pseudo-group $\mathbf{B}_{2n}$ of local biholomorphisms of $\mathbb{C}^{2n}$. The description of the equivalence relation $\big(U, J\big) \simeq \big(\varphi(U), J\big)$ above induced by $\mathbf{B}_{2n}$ shows how to lift the action of $\mathbf{B}_{2n}$ to an action on X which commutes with its projection to $\mathbb{C}^{2n}$. It follows that $\mathbf{B}_{2n}$ also acts on the space of k-jets of sections of X in a natural way and this action clearly preserves $X^{(1)}$. A straightforward calculation in local coordinates (essentially a dimension count) shows that the sub-pseudo-group $\mathbf{B}^2_{2n} \subset \mathbf{B}_{2n}$, consisting of those biholomorphisms which fix the origin in $\mathbb{C}^{2n}$ to first order, acts transitively on the fiber of $X(1) \to X$ which lies over the 0-jet at the origin of the constant section $J \equiv J_0$. In particular, since X is clearly homogeneous under the action of $\mathbf{B}_{2n}$, it follows that $X^{(1)}$ is as well.

Now let I denote pullback to $X^{(1)}$ of the contact system on $J^1(\mathbb{C}^{2n}, X)$. The local sections of the bundle $X^{(1)} \to \mathbb{C}^{2n}$ which are integral manifolds of I are, by construction, the 1-jet graphs of local solutions to the equation $\partial(J\,dz) = 0$.

Now, I claim that the differential ideal $\mathcal{I}$ generated by I together with the independence condition Ω got by pulling a volume form on $\mathbb{C}^{2n}$ up to $X^{(1)}$ is involutive. To see this, first note that the ideal $\mathcal{I}$ is a Pfaffian system in good form and then note that, because there are admissible integral manifolds (given, for example, by the constant sections), the torsion of the system $\mathcal{I}$ must vanish somewhere, but then, because cI is invariant under the transitive action of $\mathbf{B}_{2n}$ on $X^{(1)}$, it follows that the torsion of the system must vanish everywhere. Since $\mathcal{I}$ is generated by a Pfaffian system I of rank $4n^2$, it follows that $s_i' \leq \min\{4n^2, s_{i-1}'\}$ for all $1 \leq i \leq 4n$. Since the sum $s_1' + \cdots + s_{4n}'$ must be $4n^2(2n{+}1)$ (the dimension of the fibers of $X^{(1)} \to X$), it follows that the minimal configuration of the s_i' would be to have $s_i' = 4n^2$ for $1 \leq i \leq 2n{+}1$ and $s_i' = 0$ for $2n{+}2 \leq i \leq 4n$. Now, by homogeneity, it suffices to compute the reduced characters of $\mathcal{I}$ for the integral element tangent to the constant section $J \equiv J_0$ at the origin and these computations show that this minimal configuration does, in fact, obtain. A final, somewhat tedious, calculation shows that the space of admissible integral elements of $\mathcal{I}$ at this point has dimension

$$4n^2(n+1)(2n+1) = \sum_{i=1}^{2n+1} i\, s_i' \ ,$$

so that Cartan's Test is verified and the system is in involution, as I claimed. Moreover, along a $(2n{+}1)$-submanifold $S \subset \mathbb{C}^{2n}$ which is in general position with respect to the complex structure on $\mathbb{C}^{2n}$, any (real analytic) choice of a section J of X which satisfies certain zeroth order open conditions will be the restriction to S of a unique solution of $\partial(J\, dz) = 0$ on an open neighborhood of S.

Since the ambiguity in the choice of the local holomorphic coordinate z depends only on functions of $2n$ variables, it follows that, modulo local diffeomorphisms, the space of local torsion-free $\mathrm{GL}(n,\mathbb{H})$-structures depends on $4n^2$ functions of $2n{+}1$ variables. Since such structures have unique compatible connections, it follows that the space of local torsion-free connections on $4n$-manifolds with holonomy conjugate to some subgroup of $\mathrm{GL}(n,\mathbb{H})$ depends on $4n^2$ functions of $2n{+}1$ variables.

There remains the problem of showing that the holonomy of the canonical connection of the generic torsion-free $\mathrm{GL}(n,\mathbb{H})$-structure actually has holonomy equal to $\mathrm{GL}(n,\mathbb{H})$. This can be done as follows: The first Bianchi identity in the above notation takes the form

$$\begin{pmatrix} \Theta_0 & -\Theta_1 \\ \overline{\Theta_1} & \overline{\Theta_0} \end{pmatrix} \wedge \begin{pmatrix} \eta_0 \\ \overline{\eta_1} \end{pmatrix} = 0$$

where I have set $\Theta = d\theta + \theta{\wedge}\theta = \Theta_0 - \Theta_1\, j$ in conformity with the above notation. This can be solved in component form as follows (the index range is $1 \leq i,j,k,\ell \leq n$):

$$(\Theta_0)^i_{\bar{j}} = A^i_{\bar{j}\bar{k}\bar{\ell}}\, \eta^k_0 \wedge \eta^\ell_1 + B^i_{\bar{j}k\ell}\, \overline{\eta^k_0} \wedge \overline{\eta^\ell_1} + C^i_{\ell k \bar{j}} \left(\eta^k_0 \wedge \overline{\eta^\ell_0} + \eta^k_1 \wedge \overline{\eta^\ell_1} \right)$$

$$(\Theta_1)^i_j = D^i_{jk\ell}\, \overline{\eta^k_0} \wedge \overline{\eta^\ell_1} + C^i_{j\bar{k}\bar{\ell}}\, \eta^k_0 \wedge \eta^\ell_1 + B^i_{\bar{k}\ell j} \left(\eta^k_0 \wedge \overline{\eta^\ell_0} + \eta^k_1 \wedge \overline{\eta^\ell_1} \right)$$

where each of $A^i_{jk\ell}$ and $D^i_{jk\ell}$ is a collection of complex functions on the domain of the coframing η which is symmetric in the three lower indices while $B^i_{\bar{j}k\ell}$ and $C^i_{j\bar{k}\bar{\ell}}$ have the symmetries $B^i_{\bar{j}k\ell} = B^i_{\bar{j}\ell k}$ and $C^i_{j\bar{k}\bar{\ell}} = C^i_{j\bar{\ell}\bar{k}}$. This corresponds to the fact that the space $\mathcal{K}\big(\mathfrak{gl}(n,\mathbb{H})\big)$ is a vector space of real dimension $2n \cdot \binom{2n+2}{3}$.

Now, the involutivity of the system $\partial(J\,dz) = 0$ has, as one of its consequences, that any element of $\mathcal{K}\big(\mathfrak{gl}(n,\mathbb{H})\big)$ can occur as the curvature of the canonical connection associated to some torsion-free $\mathrm{GL}(n,\mathbb{H})$-structure. Since choosing sufficiently generic values for the components of A, B, C, and D at a point will clearly yield a curvature tensor which does not lie in any proper subspace $\mathfrak{p} \otimes \Lambda^2(V^*)$ of $\mathfrak{gl}(n,\mathbb{H}) \otimes \Lambda^2(V^*)$, it follows that the holonomy of the generic torsion-free $\mathrm{GL}(n,\mathbb{H})$-structure is equal to $\mathrm{GL}(n,\mathbb{H})$, as desired.

The other holonomy group to be understood in this case is $\mathrm{SL}(n,\mathbb{H})$. Since this case is very similar to that of $\mathrm{GL}(n,\mathbb{H})$, I will only sketch the argument. One starts with a local section η of the bundle $\mathcal{B}$ and uses the Newlander-Nirenberg theorem together with the fact that all integrable $\mathrm{SL}(2n,\mathbb{C})$-structures are locally flat to show that one can find local coordinate charts $z : U \to \mathbb{C}^{2n}$ so that

$$\begin{pmatrix} \eta_0 \\ \overline{\eta_1} \end{pmatrix} = G^{-1}\,dz$$

where G now takes values in $\mathrm{SL}(2n,\mathbb{C})$. The corresponding function J then takes values in the codimension 1 submanifold $\mathcal{J}^0_n \subset \mathcal{J}_n$ defined as

$$\mathcal{J}^0_n = \left\{\, A \in \mathrm{SL}(2n,\mathbb{C}) \;\middle|\; A\overline{A} = -I_{2n} \,\right\} \ .$$

For any open subset $U \subset \mathbb{C}^{2n}$, the condition that a map $J : U \to \mathcal{J}^0_n$ determine a torsion-free $\mathrm{SL}(n,\mathbb{C})$-structure on U is again just that $\partial(J\,dz) = 0$. Cartan-Kähler analysis shows that this system of partial differential equations for maps $J : U \to \mathcal{J}^0_n$ is involutive with Cartan characters

$$s_i = \begin{cases} 0 & \text{for } i < 2n, \\ 2n-1 & \text{for } i = 2n, \\ 4n^2-2n & \text{for } i = 2n+1, \\ 0 & \text{for } i > 2n+1. \end{cases}$$

It follows that, up to diffeomorphism, the local torsion-free $\mathrm{SL}(n,\mathbb{H})$-structures on $\mathbb{R}^{4n}$ depend on $4n^2-2n$ functions of $2n+1$ variables.

The space $\mathcal{K}\big(\mathfrak{sl}(n,\mathbb{H})\big)$ is calculated as above except with the extra condition that $\operatorname{tr}\Theta = 0$, which is equivalent to the $n(2n{+}1)$ conditions

$$A^i_{\bar{\imath}\bar{\jmath}\bar{k}} + \overline{B^i_{\bar{\imath}\bar{\jmath}k}} = C^i_{\bar{\jmath}\bar{k}\bar{\imath}} - \overline{C^i_{k\bar{\jmath}\bar{\imath}}} = 0 \ .$$

Again, it can be shown that $\mathcal{K}^\bullet\big(\mathfrak{sl}(n,\mathbb{H})\big)$ is dense in $\mathcal{K}\big(\mathfrak{sl}(n,\mathbb{H})\big)$. Moreover, because the ideal $\mathcal{I}$ is involutive, the generic element of $\mathcal{K}\big(\mathfrak{sl}(n,\mathbb{H})\big)$ can occur as the curvature of a local torsion-free $\mathrm{SL}(n,\mathbb{H})$-structure. Thus, the generic torsion-free $\mathrm{SL}(n,\mathbb{H})$-structure has holonomy equal to $\mathrm{SL}(n,\mathbb{H})$.

Finally, since the normalizer of both $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{SL}(n,\mathbb{H})$ in $\mathrm{GL}(4n,\mathbb{R})$ is the group $\mathrm{GL}(n,\mathbb{H})\cdot\mathrm{SL}(1,\mathbb{H})$ (where the second factor acts on $\mathbb{H}^n$ by right multiplication by scalars), it follows that a torsion-free connection on M^{4n} whose holonomy is $\mathrm{GL}(n,\mathbb{H})$ (respectively, $\mathrm{SL}(n,\mathbb{H})$) is actually compatible with a 3-parameter (respectively 4-parameter) family of distinct $\mathrm{GL}(n,\mathbb{H})$-structures (respectively, $\mathrm{SL}(n,\mathbb{H})$-structures) on M.

3.2. The Conformal Families. The second group of entries in Table 3 are the ones corresponding to groups which preserve a quadratic form up to a factor.

3.2.1. $\mathrm{CO}(p,q)$. — These are the conformal groups of various signatures and can clearly occur as holonomy. The underlying $\mathrm{CO}(p,q)$-structures are (of course) torsion-free and depend locally on $\binom{n+1}{2}{-}1$ functions of $n = p{+}q$ variables. Once a conformal structure is chosen, the space of compatible torsion-free connections is an affine space modeled on the space of sections of a bundle of rank n over M. The generic torsion-free connection for any conformal structure is easily seen to have holonomy $\mathrm{CO}(p,q)$. It follows that the space of such connections modulo the action of the diffeomorphism group is determined locally by the choice of $\binom{n+1}{2}{-}1$ functions of n variables, as expected.

3.2.2. $G_{\mathbb{C}}\cdot\mathrm{SO}(p,\mathbb{C})$. — Here, in order to not preserve a quadratic form, the scalar group $G_{\mathbb{C}}$ must not be trivial. When $G_{\mathbb{C}} = \mathbb{C}^*$, the underlying complex conformal structure must be torsion-free and hence holomorphic. Thus, it depends on $\binom{p+1}{2} - 1$ holomorphic functions of p complex variables. However, the compatible torsion-free connections are an affine space modeled on the space of (not necessarily holomorphic) sections of a vector bundle of complex rank p over M. Thus the space of such connections depends on $n = 2p$ functions of n variables. It is easy to see that the generic torsion-free connection compatible with a given holomorphic conformal structure has $\mathfrak{h}$-full curvature and hence its holonomy is equal to the full group $G_{\mathbb{C}}\cdot \mathrm{SO}(p,\mathbb{C})$. Thus, the space of connections with this holonomy modulo the diffeomorphism group depends on n functions of n variables.

The more interesting and complicated case is when $H = \mathrm{T}_\theta\cdot \mathrm{SO}(p,\mathbb{C})$ where T_θ is the 1-dimensional group of complex numbers of the form

$$e^{t(\cos\theta + i\sin\theta)}, \qquad t\in\mathbb{R}$$

for some real number θ in the range $0 \le \theta < \pi$. In this case $\mathfrak{h}^{(1)} = 0$, so a torsion-free H-structure has a unique compatible connection. Moreover, a torsion-free $\mathrm{T}_\theta\cdot \mathrm{SO}(p,\mathbb{C})$-structure $\mathcal{B}$ has an underlying torsion-free $\mathbb{C}^*\cdot \mathrm{SO}(p,\mathbb{C})$-structure which is therefore a holomorphic conformal structure. It follows that locally one can find a section η of $\mathcal{B}$ of the form

$$\eta = e^{-i(\cos\theta + i\sin\theta)f}\,\omega$$

where f is some smooth real-valued function on the domain of the coframing and ω is a holomorphic section of the underlying $\mathbb{C}^*\cdot \mathrm{SO}(p,\mathbb{C})$-structure. Now, since ω is holomorphic, there exists a unique holomorphic 1-form ψ with values in $\mathfrak{so}(p,\mathbb{C})$ so that $d\omega = -\psi{\wedge}\omega$. Of course, this implies that

$$d\eta = -(ie^{i\theta}df\, I_p + \psi)\wedge\eta\ .$$

Setting $\partial f = a\omega$ where a is a row of functions yields $d\eta = -\phi{\wedge}\eta$ where

$$\phi = \psi + e^{i\theta}\, i(\bar\partial f - \partial f) - 2ie^{i\theta}(\omega\, a - {}^t a\, {}^t\omega)\ .$$

Since ϕ has values in $\mathfrak{h}$, it represents the unique torsion-free connection compatible with $\mathcal{B}$.

For any underlying holomorphic conformal structure, choosing the real normalizing factor f sufficiently generically (for example, requiring $i\partial\overline{\partial} f > 0$) yields a connection whose curvature is $\mathfrak{h}$-full, so that its holonomy is equal to H. It follows that, modulo diffeomorphisms, the space of torsion-free connections with holonomy H is locally dependent on the choice of one arbitrary function of $n = 2p$ variables.

3.3. The Symplectic Families. The third group on the list consists of various modifications of the symplectic group.

3.3.1. $\mathrm{Sp}(p,\mathbb{R})$. — In this case, $n = 2p > 2$ and the representation of $\mathrm{Sp}(p,\mathbb{R})$ on $V = \mathbb{R}^{2p}$ is the standard one. As usual, let

$$J_p = \begin{pmatrix} 0 & \mathrm{I}_p \\ -\mathrm{I}_p & 0 \end{pmatrix} ,$$

and recall that $\mathrm{Sp}(p,\mathbb{R})$ is the subgroup of $\mathrm{GL}(2p,\mathbb{R})$ consisting of those matrices A which satisfy ${}^tA\,J\,A = J$. The Lie algebra of $\mathrm{Sp}(p,\mathbb{R})$ is the vector space $\mathfrak{sp}(p,\mathbb{R})$ of $2p$-by-$2p$ matrices a with the property that Ja is symmetric. In fact, the map $a \mapsto Ja$ induces an isomorphism $\mathfrak{sp}(p,\mathbb{R}) \xrightarrow{\sim} S^2(V^*)$. Under this isomorphism, the map $\mathfrak{h}\otimes V^* \to V\otimes\Lambda^2(V^*)$ simply becomes the natural map $S^2(V^*)\otimes V^* \to V\otimes\Lambda^2(V^*)$. Its kernel $\mathfrak{sp}(p,\mathbb{R})^{(1)}$ is isomorphic to $S^3(V^*)$ while its cokernel $H^{0,2}\big(\mathfrak{sp}(p,\mathbb{R})\big)$ is isomorphic to $\Lambda^3(V^*)$. This latter isomorphism corresponds to the fact that an $\mathrm{Sp}(p,\mathbb{R})$-structure $\mathcal{B}$ on a manifold M^{2p} is torsion-free if and only if the 3-form $d\Omega$ vanishes, where Ω, the canonical 2-form associated to the $\mathrm{Sp}(p,\mathbb{R})$-structure $\mathcal{B}$, is of the form $\Omega = \frac{1}{2}{}^t\eta J\eta$ in the domain of any local section η of $\mathcal{B}$.

A torsion-free $\mathrm{Sp}(p,\mathbb{R})$-structure on a manifold M^{2p} is simply a symplectic structure. By the Darboux Theorem, all such structures are locally isomorphic to the standard one. Thus, locally one can always choose a section of $\mathcal{B}$ of the form $\eta = dx$, where $x : U \to V$ is a local coordinate chart. Consequently, $\Omega = \frac{1}{2}\,{}^t dx\,J\,dx$. An associated torsion-free connection matrix must have the form $\phi = J\psi$, where ψ is a 1-form with values in symmetric matrices and its components can be written in the form $\psi_{ij} = P_{ijk}\,dx^k$ where P is symmetric in its lower indices. The generic choice of such a P will yield a connection with $\mathfrak{sp}(p,\mathbb{R})$-full curvature and hence with holonomy $\mathrm{Sp}(p,\mathbb{R})$. Since the local symplectomorphisms (i.e., canonical transformations) are generated by a single function of $n = 2p$ variables, it follows that, modulo the diffeomorphism group, the space of local connections with holonomy $\mathrm{Sp}(p,\mathbb{R})$ depends on $\binom{n+2}{3}-1$ functions of n variables.

3.3.2. $\mathrm{CSp}(p,\mathbb{R})$. — In Berger's original list, along with the symplectic case, there was included what might be called 'conformally symplectic' connections, i.e., connections whose holonomy was $\mathrm{CSp}(p,\mathbb{R}) = \mathbb{R}^+\cdot \mathrm{Sp}(p,\mathbb{R})$.

However, it turns out that, for $p \geq 3$, the group $\mathrm{CSp}(p,\mathbb{R})$ cannot occur as holonomy of a torsion-free connection, for it does not satisfy Berger's first criterion. In fact $\mathcal{K}\bigl(\mathfrak{csp}(p,\mathbb{R})\bigr) = \mathcal{K}\bigl(\mathfrak{sp}(p,\mathbb{R})\bigr)$ when $p \geq 3$. Here is how one can see this.

Suppose that ∇ were a torsion-free connection on M^{2p} with holonomy $\mathrm{CSp}(p,\mathbb{R})$. Any section η of the corresponding torsion-free $\mathrm{CSp}(p,\mathbb{R})$-structure would then have an associated connection matrix of the form $\phi = \rho\, I_{2p} + \psi$ where ρ is a single 1-form and ψ has values in the Lie algebra of $\mathfrak{sp}(p,\mathbb{R})$. The identity

$$d\eta = -\bigl(\rho\, I_{2p} + \psi\bigr)\wedge\eta$$

would then imply the identity $d\Omega = -2\rho{\wedge}\Omega$ where $\Omega = \frac{1}{2}\,{}^t\eta\, J\,\eta$. Computing the exterior derivative of this relation yields $0 = d\rho{\wedge}\Omega$. When $p \geq 3$, this implies that $d\rho = 0$, since Ω is a 2-form of half-rank p. However, this in turn implies that the curvature form of the connection form ϕ is the same as the curvature form of ψ, i.e., it takes values in the subalgebra $\mathfrak{sp}(p,\mathbb{R})$. Thus, the holonomy of the connection ∇ lies in $\mathrm{Sp}(p,\mathbb{R})$.

The situation when $p = 2$ is quite different, in two ways. First, as the reader can easily check, $H^{0,2}\bigl(\mathfrak{csp}(2,\mathbb{R})\bigr) = 0$, so that *every* $\mathrm{CSp}(2,\mathbb{R})$-structure on a 4-manifold is torsion-free. Second, $\mathrm{CSp}(2,\mathbb{R})$ *does* satisfy Berger's first criterion (as well as the second criterion).

A choice of a $\mathrm{CSp}(2,\mathbb{R})$-structure on a 4-manifold is equivalent to the choice of a non-degenerate 2-form well-defined up to non-zero scalar multiples. Now, if Ω is a generic non-degenerate 2-form on M^4, then there is a unique 1-form ρ_Ω which satisfies $d\Omega = -2\rho_\Omega{\wedge}\Omega$. Moreover, if $\tilde\Omega = \lambda\,\Omega$ for some function $\lambda \neq 0$, then $\rho_{\tilde\Omega} = \rho_\Omega - \frac{1}{2}d\lambda/\lambda$, so that $d\rho_{\tilde\Omega} = d\rho_\Omega$. In particular, the 2-form $d\rho_\Omega$ depends only on the underlying $\mathrm{CSp}(2,\mathbb{R})$-structure to which Ω belongs. This 2-form vanishes if and only if Ω has a (local) non-zero multiple which is closed. The generic compatible torsion-free connection for a $\mathrm{CSp}(2,\mathbb{R})$-structure whose invariant 2-form ρ is non-zero has $\mathfrak{csp}(2,\mathbb{R})$-full curvature and hence has its holonomy equal to $\mathrm{CSp}(2,\mathbb{R})$.

Moreover, the generic $\mathrm{CSp}(2,\mathbb{R})$-structure has no local symmetries, so a simple count shows that, modulo local diffeomorphism, the local $\mathrm{CSp}(2,\mathbb{R})$-structures depend on one function of four variables. Since $\mathfrak{csp}(2,\mathbb{R})^{(1)} = \mathfrak{sp}(2,\mathbb{R})^{(1)} \simeq \mathbb{R}^{20}$, it follows that, modulo local diffeomorphisms, the connections on 4-manifolds with holonomy $\mathrm{CSp}(2,\mathbb{R})$ depend on 21 functions of four variables.

3.3.3. $\mathrm{Sp}(p,\mathbb{C})$. — In this case, the situation is much the same as in the real case. A torsion-free $\mathrm{Sp}(p,\mathbb{C})$-structure on a manifold M^{4p} is a holomorphic symplectic structure, and by the holomorphic version of the Darboux Theorem, these are all flat. The main difference is that the ambiguity in the connection is not required to be holomorphic, while the arbitrary function which parametrizes the holomorphic symplectomorphism is required to be holomorphic. Moreover, since $\mathcal{K}^\bullet(\mathfrak{h})$ is dense in $\mathcal{K}(\mathfrak{h}) = \mathcal{B}^{1,2}(\mathfrak{h})$, it follows that the generic compatible connection has holonomy equal to all of $\mathrm{Sp}(p,\mathbb{C})$. Thus, modulo the diffeomorphism group, the space of local connections with holonomy $\mathrm{Sp}(p,\mathbb{C})$ depends on $2\binom{2p+2}{3}$ functions of $n = 4p$ variables.

3.3.4. $\mathrm{CSp}(p,\mathbb{C})$. — An argument similar to that in the real case applies in the complex case to show that, when $p > 2$, any connection whose holonomy lies in $\mathrm{CSp}(p,\mathbb{C})$ must actually have its holonomy lie in $\mathrm{Sp}(p,\mathbb{C})$. In the case $p = 2$, however, the full group $\mathrm{CSp}(2,\mathbb{C})$ is possible. The complex case is slightly different of course, because the group $H^{0,2}\big(\mathfrak{csp}(2,\mathbb{C})\big)$ is non-zero. However, because of the connection ambiguity, it is not difficult to see that, modulo diffeomorphisms, the local connections on $\mathbb{C}^4 = \mathbb{R}^8$ with holonomy $\mathrm{CSp}(2,\mathbb{C})$ depend on 40 functions of 8 variables.

3.4. The Segre Families. The fourth family of groups in Table 3 consists of the ones which are representable as tensor product representations. There is no uniform terminology for these groups, some authors call these 'paraconformal' others call these 'almost Grassmannian'. I first heard them called 'Segre' structures, and so have adopted this name for them.

3.4.1. $G_{\mathbb{R}} \cdot \mathrm{SL}(p, \mathbb{R}) \cdot \mathrm{SL}(q, \mathbb{R})$. — Let V be the space of p-by-q matrices with real entries and define an action of $\mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R})$ on $V \simeq \mathbb{R}^{pq}$ by the the rule

$$(A, B) \cdot v = A\, v\, B^{-1} \qquad \text{for all } (A, B) \in \mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R}),\ v \in V\ .$$

This action is not effective; its ineffective subgroup is the subgroup of matrices of the form (rI_p, rI_q). Most of this ineffective subgroup can be removed by restricting attention to the subgroup $G \subset \mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R})$ consisting of those pairs (A, B) satisfying $\det(A) \det(B) = 1$. (Since this discussion is local, I will ignore any problems caused by the remaining finite ineffective subgroup.) I will denote the image subgroup in $\mathrm{GL}(V) \simeq \mathrm{GL}(pq, \mathbb{R})$ by $\mathbb{R}^+ \cdot \mathrm{SL}(p, \mathbb{R}) \cdot \mathrm{SL}(q, \mathbb{R})$, and let $\mathrm{SL}(p, \mathbb{R}) \cdot \mathrm{SL}(q, \mathbb{R})$ denote the obvious codimension 1 subgroup.

I will assume that $p \geq q \geq 2$ and that $(p, q) \neq (2, 2)$ since the cases excluded by these inequalities have already been discussed. The discussion of the remaining cases divides naturally into two types: The ones where $q = 2$ and the ones where $q > 2$. The reason for this is that, when $q > 2$, any torsion-free $\mathbb{R}^+ \cdot \mathrm{SL}(p, \mathbb{R}) \cdot \mathrm{SL}(q, \mathbb{R})$-structure is necessarily flat, while this is not true when $q = 2$.

On general principles, this follows a computation showing that, first of all $\mathfrak{h}^k = 0$ for $k > 1$ and that, when $q > 2$, the groups $H^{k,2}(\mathfrak{h})$ vanish for $k = 1, 2$. It then follows that there are no formal obstructions to flatness for a torsion-free H-structure. Since these H-structures are of finite type, it then follows that there are no local obstructions at all.

Here is how a direct proof of the flatness of Segre structures in the case $q > 2$ can be constructed. Suppose that $\mathcal{B} \to M^{pq}$ is a torsion-free $\mathbb{R}^+ \cdot \mathrm{SL}(p, \mathbb{R}) \cdot \mathrm{SL}(q, \mathbb{R})$-structure. Choose a local section η of $\mathcal{B}$. Then the assumption that $\mathcal{B}$ be torsion-free implies that there exists a 1-form α with values in $\mathfrak{gl}(p, \mathbb{R})$ and a 1-form β with values in $\mathfrak{gl}(q, \mathbb{R})$ so that the pair (α, β) satisfies $\operatorname{tr}\alpha + \operatorname{tr}\beta = 0$ and so that the following structure equation holds:

$$d\eta = -\alpha \wedge \eta - \eta \wedge \beta\ .$$

(The trace condition ensures that the 1-form (α, β) actually takes values in the Lie algebra of $\mathbb{R}^+ \cdot \mathrm{SL}(p, \mathbb{R}) \cdot \mathrm{SL}(q, \mathbb{R})$.) Differentiating this equation and setting

$$A = d\alpha + \alpha \wedge \alpha$$

$$B = d\beta + \beta \wedge \beta$$

yields the relation $A{\scriptstyle\wedge}\eta = \eta{\scriptstyle\wedge}B$. Moreover, the trace condition on the connection implies that $\operatorname{tr} A + \operatorname{tr} B = 0$.

Keeping in mind the relation $\operatorname{tr} A + \operatorname{tr} B = 0$ and applying a little linear algebra, these relations can be shown to imply that there exists a unique triple (ψ, A_0, B_0) where ψ is a 1-form with values in V^* ($= q$-by-p matrices), A_0 is a 2-form with values in $\mathfrak{sl}(p,\mathbb{R})$, and B_0 is a 2-form with values in $\mathfrak{sl}(q,\mathbb{R})$ so that

$$\begin{aligned} A &= A_0 - \eta\wedge\psi \\ B &= B_0 - \psi\wedge\eta \\ 0 &= A_0\wedge\eta = \eta\wedge B_0 \ . \end{aligned}$$

The assumptions $p > q \geq 2$ imply $p > 2$ and, in this situation, the equation $\eta{\scriptstyle\wedge}B_0 = 0$ implies that $B_0 = 0$. Moreover, if $q > 2$, then the equation $A_0{\scriptstyle\wedge}\eta = 0$ implies that $A_0 = 0$.

Assume that $q > 2$. Differentiating the relations $d\alpha + \alpha{\scriptstyle\wedge}\alpha = -\eta{\scriptstyle\wedge}\psi$ and $d\beta + \beta{\scriptstyle\wedge}\beta = -\psi{\scriptstyle\wedge}\eta$ (and again using the assumption that $p \geq q > 2$) yields

$$d\psi = -\psi\wedge\alpha - \beta\wedge\psi \ .$$

Of course, these relations altogether imply that the $\mathfrak{sl}(p{+}q,\mathbb{R})$-valued matrix

$$\omega = \begin{pmatrix} \beta & \psi \\ \eta & \alpha \end{pmatrix}$$

satisfies $d\omega = -\omega{\scriptstyle\wedge}\omega$. Thus, the given $\mathbb{R}^+\cdot \mathrm{SL}(p,\mathbb{R})\cdot \mathrm{SL}(q,\mathbb{R})$-structure is locally equivalent to the flat one induced on the Grassmannian manifold $\mathrm{Gr}_p\big(\mathbb{R}^{p+q}\big)$, as was claimed. In fact, if $\pi_1(M) \simeq 1$, there exists a smooth map $g : \mathcal{B} \to \mathrm{SL}(p{+}q)$ satisfying $\omega = g^{-1}\,dg$ which covers a smooth local diffeomorphism $f : M \to \mathrm{Gr}_p\big(\mathbb{R}^{p+q}\big)$ inducing a local equivalence of flat $\mathbb{R}^+\cdot \mathrm{SL}(p,\mathbb{R})\cdot \mathrm{SL}(q,\mathbb{R})$-structures.

Using this local flatness result, the study of the local connections with holonomy $\mathbb{R}^+\cdot \mathrm{SL}(p,\mathbb{R})\cdot \mathrm{SL}(q,\mathbb{R})$ or $\mathrm{SL}(p,\mathbb{R})\cdot \mathrm{SL}(q,\mathbb{R})$ becomes quite simple.

First, consider the case of a connection ∇ on a manifold M whose holonomy is $H = \mathbb{R}^+\cdot \mathrm{SL}(p,\mathbb{R})\cdot \mathrm{SL}(q,\mathbb{R})$. Since the underlying H-structure $\mathcal{B}$ is flat, it follows that it is possible to choose a local section η of $\mathcal{B}$ so that $\eta = dx$ where $x : U \to V$

is a local coordinate chart. The $\mathfrak{h}$-valued 1-form corresponding to this section is a pair of 1-forms (α,β) where α takes values in $\mathfrak{gl}(p,\mathbb{R})$, β takes values in $\mathfrak{gl}(q,\mathbb{R})$, $\operatorname{tr}\alpha+\operatorname{tr}\beta=0$, and

$$0 = d\eta = -\alpha\wedge\eta-\eta\wedge\beta\ .$$

By straightforward linear algebra, it follows that there exists a unique function s on U with values in V^* so that $\alpha = dx\,s$ and $\beta=-s\,dx$.

Conversely, any choice of a V^*-valued function s on U yields a torsion-free connection $(\alpha,\beta)=(\,dx\,s,\,-s\,dx\,)$ on U which is compatible with the flat H-structure. It is not difficult to show that, by choosing s to be sufficiently generic, one can arrange that the curvature of the resulting connection be $\mathfrak{h}$-full, so that the holonomy of such a connection will be equal to the full group H. Thus, modulo diffeomorphisms, the space of local connections on $\mathbb{R}^{pq}$ with holonomy $\mathbb{R}^+\cdot\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(q,\mathbb{R})$ depends on pq arbitrary functions of pq variables.

Next, consider the case of a connection ∇ on a manifold M whose holonomy is $H=\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(q,\mathbb{R})$. Since any torsion-free $\mathbb{R}^+\cdot H$-structure is locally flat, it follows from the above analysis that a torsion-free H-structure always has a local section of the form

$$\eta = e^{(p+q)f}\,dx$$

for some function f on the domain U of the section. Moreover, it is easy to see that, associated to the section η, there is a unique torsion-free connection 1-form with values in the Lie algebra of H, namely, (α,β), where

$$\alpha = dx\,F - q\,df\,I_p\,,\qquad \beta=-F\,dx+p\,df\,I_q\ ,$$

where F is the unique q-by-p matrix of functions on U which satisfies

$$pq\,df = \operatorname{tr}\bigl(\,F\,dx\,\bigr)\ .$$

Thus, the H-structure determined by an arbitrary choice of the function f is always torsion-free. Since the space of closed sections of the underlying $\mathbb{R}^+\cdot H$-structure depends only on constants, it follows that the arbitrary function f is determined up to a finite dimensional ambiguity. Thus, modulo diffeomorphism, the space of torsion-free H-structures depends on one arbitrary function of pq variables.

Finally, for any sufficiently generic choice of the function f, the formula given above defines a connection with $\mathfrak{h}$-full curvature, so that it has holonomy H. It follows that, modulo diffeomorphisms, connections with holonomy $\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(q,\mathbb{R})$ depend on one function of $n = pq$ variables.

Now, in the case where $q = 2$, things are somewhat different since it is no longer true that the torsion-free H-structures are locally flat or locally conformally flat. However, by an analysis that is not difficult but too long to include here, it can be shown that modulo diffeomorphism these local torsion-free structures on $2p$-manifolds depend on functions of $p+1$ variables modulo diffeomorphism. However, in the case $H = \mathbb{R}^+\cdot\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$, once one fixes a torsion-free H-structure, the ambiguity in the choice of compatible torsion-free connection is still $2p$ functions of $2p$ variables and the generic such choice will have $\mathfrak{h}$-full curvature, so its holonomy will be all of H. Thus, the general connection with holonomy H depends on $2p$ functions of $2p$ variables modulo diffeomorphism. In the case $H = \mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$, all these torsion-free structures arise by choosing a volume form on a manifold endowed with a torsion-free $\mathbb{R}^+\cdot\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$-structure and then reducing the structure group to match the volume form. Any choice of volume form will yield a torsion-free H-structure and for a given $\mathbb{R}^+\cdot\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$-structure, the generic choice of volume form will yield an H-structure whose canonical torsion-free connection has $\mathfrak{h}$-full curvature, so its holonomy will be all of H. Thus, the general connection with holonomy H depends on one function of $2p$ variables modulo diffeomorphism.

3.4.2. $G_{\mathbb{C}}\cdot\mathrm{SL}(p,\mathbb{C})\cdot\mathrm{SL}(q,\mathbb{C})$. — This case is very much like the previous case except that it takes place in the holomorphic category rather than the smooth category. Rather than go through all of the details, I will just describe the results.

Using the inclusion $G_{\mathbb{C}}\cdot\mathrm{SL}(p,\mathbb{C})\cdot\mathrm{SL}(q,\mathbb{C}) \subset \mathrm{GL}(pq,\mathbb{C})$, one sees that, for each H in this family, every torsion-free H-structure on a manifold M^{2pq} has a canonical underlying integrable almost complex structure and so one can regard M as a complex manifold of complex dimension pq. Again, when $p \geq q > 2$, all torsion-free

$\mathbb{C}^*\cdot \mathrm{SL}(p,\mathbb{C})\cdot \mathrm{SL}(q,\mathbb{C})$-structures are locally flat with the standard model being the complex Grassmannian $\mathrm{Gr}_p^{\mathbb{C}}(\mathbb{C}^{p+q})$. When $q=2$, this local flatness fails, but the general torsion-free $\mathbb{C}^*\cdot \mathrm{SL}(p,\mathbb{C})\cdot \mathrm{SL}(2,\mathbb{C})$-structure depends on holomorphic functions of $p{+}1$ complex variables modulo biholomorphism.

In the case where $H=\mathbb{C}^*\cdot \mathrm{SL}(p,\mathbb{C})\cdot \mathrm{SL}(q,\mathbb{C})$, once a torsion-free H-structure is fixed, there is still the ambiguity of the choice of connection which lies in the smooth sections of a bundle isomorphic to the holomorphic cotangent bundle of M. The general choice of such a connection will have $\mathfrak{h}$-full curvature and will thus have holonomy H. Thus, modulo diffeomorphism, the torsion-free connections with this holonomy depend on $2pq$ (arbitrary) functions of $2pq$ (real) variables. Note that this count holds whether $q>2$ or not.

At the other extreme, where $H=\mathrm{SL}(p,\mathbb{C})\cdot \mathrm{SL}(q,\mathbb{C})\subset \mathrm{SL}(pq,\mathbb{C})$, one has $\mathfrak{h}^{(1)}=0$. It follows that there is a unique torsion-free connection compatible with a torsion-free H-structure and it supports a parallel holomorphic volume form. Conversely, starting with a torsion-free $\mathbb{C}^*\cdot \mathrm{SL}(p,\mathbb{C})\cdot \mathrm{SL}(q,\mathbb{C})$-structure on a complex manifold denoted M^{pq}, any choice of holomorphic volume form will yield a torsion-free H-structure and the generic choice will yield one whose canonical connection has $\mathfrak{h}$-full curvature, so that its holonomy will be all of H. Thus, the torsion-free connections with holonomy H depend on one holomorphic function of pq complex variables modulo diffeomorphism.

In the middle ground are the groups of the form $H=\mathrm{T}\cdot \mathrm{SL}(p,\mathbb{C})\cdot \mathrm{SL}(q,\mathbb{C})$ where T is any one-parameter subgroup of $\mathbb{C}^*$. Since $\mathfrak{h}^{(1)}=0$ for these groups, each such structure comes equipped with a unique compatible torsion-free connection. These structures can be constructed by starting with a torsion-free $\mathbb{C}^*\cdot \mathrm{SL}(p,\mathbb{C})\cdot \mathrm{SL}(q,\mathbb{C})$-structure $\mathcal{B}\to M$ and then choosing a smooth section of a 'reduced' determinant bundle (of real fiber rank 1) over M. Any such choice works and the generic choice yields a connection with $\mathfrak{h}$-full curvature. Thus, the general torsion-free connection with this holonomy depends on one arbitrary function of $2pq$ variables, modulo diffeomorphism.

3.4.3. $G_{\mathbb{R}}\cdot \mathrm{SL}(p,\mathbb{H})\cdot \mathrm{SL}(q,\mathbb{H})$. — Again, this case is very much like the two previous cases. Rather than go through all of the details, I will just describe the results. The main guiding principle is that the subgroup $\mathbb{R}^*\cdot \mathrm{SL}(p,\mathbb{H})\cdot \mathrm{SL}(q,\mathbb{H}) \subset \mathrm{GL}(4pq,\mathbb{R})$ is the real form of the subgroup $\mathbb{C}^*\cdot \mathrm{SL}(2p,\mathbb{C})\cdot \mathrm{SL}(2q,\mathbb{C}) \subset \mathrm{GL}(4pq,\mathbb{C})$.

When $p \geq q > 1$, any torsion-free $\mathbb{R}^*\cdot \mathrm{SL}(p,\mathbb{H})\cdot \mathrm{SL}(q,\mathbb{H})$-structure $\mathcal{B}$ on a manifold M^{4pq} is locally flat with the standard model being the quaternionic Grassmannian $\mathrm{Gr}^{\mathbb{H}}_p(\mathbb{H}^{p+q})$. When $q = 1$, this local flatness fails, but the general torsion-free $\mathbb{R}^*\cdot \mathrm{SL}(p,\mathbb{H})\cdot \mathrm{SL}(2,\mathbb{H})$-structure depends on functions of $2p{+}1$ variables modulo diffeomorphism. (For more information about this case, including the construction of specific examples using Lie group and/or twistor methods, see [Jo1] and [Bo].)

In the case where $H = \mathbb{R}^*\cdot \mathrm{SL}(p,\mathbb{H})\cdot \mathrm{SL}(q,\mathbb{H})$, once a torsion-free H-structure is fixed, there is still the ambiguity of the choice of connection which lies in the smooth sections of a bundle isomorphic to the cotangent bundle of M. The general choice of such a connection will have $\mathfrak{h}$-full curvature and will thus have holonomy H. Thus, modulo diffeomorphism, the torsion-free connections with this holonomy depend on $4pq$ (arbitrary) functions of $4pq$ variables. Note that this count holds whether $q > 1$ or not.

The other possibility in this case is $H = \mathrm{SL}(p,\mathbb{H})\cdot \mathrm{SL}(q,\mathbb{H}) \subset \mathrm{SL}(4pq,\mathbb{R})$, where one has $\mathfrak{h}^{(1)} = 0$, so there is a unique torsion-free connection compatible with a torsion-free H-structure and it supports a parallel holomorphic volume form. Conversely, starting with a torsion-free $\mathbb{R}^*\cdot \mathrm{SL}(p,\mathbb{H})\cdot \mathrm{SL}(q,\mathbb{H})$-structure on a manifold M^{4pq}, any choice of volume form will yield a torsion-free H-structure and the generic choice will yield one whose canonical connection has $\mathfrak{h}$-full curvature, so that its holonomy will be all of H. Thus, the torsion-free connections with holonomy H depend on one arbitrary function of $4pq$ variables modulo diffeomorphism.

3.5. The Quadratic Representation Families. This family constitutes all of the remaining groups on Berger's non-metric list. Before going into details, I will describe the basic results:

Each of the entries in the remainder of Table 3 which are of the form $G_{\mathbb{R}}\cdot \mathrm{SL}(p,\mathbb{F})$ represent one of two possible groups, either $\mathbb{R}^+\cdot \mathrm{SL}(p,\mathbb{F})$ or $\mathrm{SL}(p,\mathbb{F})$, acting on either $V = S_p(\mathbb{F})$, $V = A_p(\mathbb{F})$, or, in the case $\mathbb{F} = \mathbb{C}$, the space $V = H_p(\mathbb{C})$.

First, consider the entries of the form $H = \mathbb{R}^+\cdot \mathrm{SL}(p,\mathbb{F})$. Assuming the restrictions on p listed in Table 3, any torsion-free H-structure $\mathcal{B} \to M$ turns out to be locally flat, i.e., locally isomorphic to the corresponding second-order homogeneous space listed in Table C. Moreover, the space of compatible connections on $\mathcal{B}$ is an affine space modeled on the vector space $\mathcal{A}^1(M)$ of 1-forms on M. (This happens because, in each case, $\mathfrak{h}^{(1)}$ is isomorphic to V^* as an H-module.) The generic such connection has holonomy equal to H. Thus, modulo diffeomorphism, the space of local connections with holonomy $\mathbb{R}^+\cdot \mathrm{SL}(p,\mathbb{F})$ depends on $n = \dim V$ functions of n variables.

Second, consider the cases $H = \mathrm{SL}(p,\mathbb{F})$. Again, assuming the restrictions on p listed in the table, any torsion-free $\mathrm{SL}(p,\mathbb{F})$-structure turns out to be locally conformally flat, i.e., up to a conformal factor, locally isomorphic to the flat $\mathrm{SL}(p,\mathbb{F})$-structure on V itself. This conformal factor can be arbitrarily chosen and, for each such choice, there is a unique compatible connection. For the generic choice of conformal factor, the holonomy of the corresponding connection is equal to $\mathrm{SL}(p,\mathbb{F})$. Thus, modulo diffeomorphism, the space of connections with holonomy $\mathrm{SL}(p,\mathbb{F})$ depends on 1 function of n variables.

Each of the two remaining entries in Table 3 is of the form $G_{\mathbb{C}}\cdot \mathrm{SL}(p,\mathbb{C})$ and represents three possibilities, either $\mathbb{C}^*\cdot \mathrm{SL}(p,\mathbb{C})$, $\mathrm{SL}(p,\mathbb{C})$, or $\mathrm{T}\cdot \mathrm{SL}(p,\mathbb{C})$ (where $\mathrm{T} \subset \mathbb{C}^*$ is any 1-parameter subgroup) acting on either $V = S_p(\mathbb{C})$ (if $p \geq 3$) or $V = A_p(\mathbb{C})$ (if $p \geq 5$).

When $H = \mathbb{C}^*\cdot \mathrm{SL}(p,\mathbb{C})$, every torsion-free H-structure $\mathcal{B} \to M$ is locally flat and has an integrable underlying complex structure. Moreover, the space of (smooth) compatible connections on $\mathcal{B}$ is an affine space modeled on the vector space $\mathcal{A}^{1,0}(M)$ of (smooth) 1-forms of type $(1,0)$ on M. (This happens because, in each case, $\mathfrak{h}^{(1)}$ is isomorphic to V^* as an H-module.) The generic such connection has holonomy equal to H. Thus, modulo diffeomorphism, the space of local connections with holonomy $\mathbb{C}^*\cdot \mathrm{SL}(p,\mathbb{C})$ depends on $n = \dim_{\mathbb{R}} V$ arbitrary functions of n variables.

Next consider $H = \mathrm{SL}(p,\mathbb{C})$ and a torsion-free H-structure $\mathcal{B} \to M$. By the previous paragraph, the underlying $\mathbb{C}^*\cdot H$-structure $\mathcal{B}\cdot\mathbb{C}^*$ is flat and hence $\mathcal{B}$ is conformal to the flat H-structure. By the torsion-free assumption, the conformal factor is holomorphic with respect to the underlying integrable almost complex structure. Conversely, starting with the flat H-structure and choosing an arbitrary holomorphic conformal factor, the resulting H-structure is torsion-free, possessing a unique compatible connection. For a generically chosen holomorphic conformal factor, the resulting connection has holonomy equal to H. Thus, modulo diffeomorphism, the space of local connections with holonomy $\mathrm{SL}(p,\mathbb{C})$ depends on one holomorphic function of $n = \dim_{\mathbb{C}} V$ complex variables.

Finally, consider the case where $H = \mathrm{T}\cdot\mathrm{SL}(p,\mathbb{C})$ where T is any 1-parameter subgroup of $\mathbb{C}^*$ and one is given a torsion-free H-structure $\mathcal{B} \to M$. Since the underlying $\mathbb{C}^*\cdot H$-structure is flat, it follows that $\mathcal{B}$ is conformal to a flat structure by a conformal factor which is "real" in the appropriate sense (this sense depends on T and will be made explicit below). Conversely, starting with the flat H-structure and choosing an arbitrary conformal factor satisfying this reality condition, the resulting H-structure is torsion-free, and has a unique compatible connection. For a generically chosen conformal factor satisfying the appropriate reality condition, the resulting connection has holonomy equal to H. Thus, modulo diffeomorphism, the space of local connections with holonomy $\mathrm{T}\cdot\mathrm{SL}(p,\mathbb{C})$ depends on one arbitrary function of $n = \dim_{\mathbb{R}} V$ variables.

The analysis in each of these cases is essentially the same. The important algebraic fact is that, in each of the seven cases, the group $H \subset \mathrm{GL}(V)$ which arises when one sets $\mathrm{G}_{\mathbb{F}} = \mathbb{F}^*$ satisfies $\mathfrak{h}^{(1)} = V^*$ and

$$H^{1,2}(\mathfrak{h}) = H^{2,2}(\mathfrak{h}) = \mathfrak{h}^{(2)} = 0\ .$$

This is sufficient to prove that, for such H, any torsion-free H-structure is locally flat. Moreover, in each of these cases, there is a unique second-order homogeneous space (see Table C) carrying an invariant torsion-free H-structure.

To save space, I am only going to treat one of these subfamilies in detail, namely, the first one. The diligent reader can repeat this analysis in each of the six remaining subfamilies if necessary.

3.5.1. $G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{C}) \subset \mathrm{GL}\big(H_p(\mathbb{C})\big)$. — I will consider this case in some detail since it will be used as a model for the other cases, whose analysis is very similar and will only be sketched. This case consists of two groups, $\mathbb{R}^+\cdot\mathrm{SL}(p,\mathbb{C})$ and $\mathrm{SL}(p,\mathbb{C})$, acting on the p^2-dimensional real vector space $V = H_p(\mathbb{C})$ of p-by-p Hermitian symmetric matrices. The action of these groups on V is best described by letting $A \in \mathrm{GL}(p, \mathbb{C})$ act on the Hermitian symmetric matrix h by the rule $A{\cdot}h = A\,h\,{}^t\bar{A}$. This action is, of course, irreducible, but is not effective, the ineffective subgroup being the set of matrices of the form $e^{i\theta}\,I_p$. For this reason, I restrict to the subgroup $\mathbb{R}^+{\cdot}\,\mathrm{SL}(p, \mathbb{C})$, which acts almost effectively.

For simplicity in computations, it is worth remarking that the dual $G_{\mathbb{R}}{\cdot}\,\mathrm{SL}(p, \mathbb{R})$-module V^* can also be identified with $H_p(\mathbb{C})$ as a vector space, but with the action $A{\cdot}s = ({}^t\bar{A})^{-1}\,s\,A^{-1}$. Note that, in this form, the canonical pairing $V \times V^* \to \mathbb{R}$ can be written in the form $(h, s) \mapsto \mathrm{tr}(hs)$.

The first non-trivial case would be $p = 2$. However, in this case, these representations of $\mathrm{SL}(2, \mathbb{C})$ and $\mathbb{R}^+{\cdot}\,\mathrm{SL}(2, \mathbb{C})$ on $V \simeq \mathbb{R}^4$ have already been treated as $\mathrm{SO}(3, 1)$ and $\mathbb{R}^+{\cdot}\,\mathrm{SO}(3, 1) = \mathrm{CO}(3, 1)$, so I will assume $p \geq 3$ from now on. Under this assumption, the $G_{\mathbb{R}}{\cdot}\,\mathrm{SL}(p, \mathbb{R})$-modules V and V^* are not isomorphic.

First suppose that $H = \mathbb{R}^+{\cdot}\,\mathrm{SL}(p, \mathbb{C})$. I am now going to show that any torsion-free H-structure is flat. This should be expected for the following reason. The complexification $V^{\mathbb{C}}$ of V can be identified with the space of all p-by-p complex matrices in such a way that the complexification of the subgroup $\mathbb{R}^+{\cdot}\,\mathrm{SL}(p, \mathbb{C}) \subset \mathrm{GL}(V)$ is the subgroup $\mathbb{C}^*{\cdot}\,\mathrm{SL}(p, \mathbb{C}){\cdot}\,\mathrm{SL}(p, \mathbb{C}) \subset \mathrm{GL}(V^{\mathbb{C}})$ discussed above in the study of $\mathbb{C}$-Segre structures and it has already been shown that a torsion-free $\mathbb{C}$-Segre structure is flat when $p = q \geq 3$. This indicates that the same should be true of a torsion-free $\mathbb{R}^+{\cdot}\,\mathrm{SL}(p, \mathbb{C})$-structure $\mathcal{B}$ on M^{p^2} when $p \geq 3$ and, indeed, this is exactly what happens.

Briefly, if the V-valued 1-form η is a local section of a torsion-free H-structure $\mathcal{B}$, then the torsion-free assumption implies that there exists a 1-form α with values in the Lie algebra of H (i.e., α takes values in the space of p-by-p complex matrices with real trace) satisfying the equation

$$d\eta = -\alpha \wedge \eta + \eta \wedge {}^t\overline{\alpha}\ .$$

Setting $A = d\alpha + \alpha{\wedge}\alpha$ as usual, the first Bianchi identity says that $A{\wedge}\eta + \eta{\wedge}{}^t\overline{A} = 0$. This, coupled with the relation $\mathrm{tr}(A) + \mathrm{tr}(\overline{A}) = 0$ allows one to prove that there exists a 1-form ψ with values in V^* so that $A = -\eta{\wedge}\psi$. Now differentiating the equation

$$d\alpha = -\alpha \wedge \alpha - \eta \wedge \psi$$

yields the relation $\eta{\scriptstyle\wedge}\big(d\psi - {}^t\overline{\alpha}{\scriptstyle\wedge}\psi + \psi{\scriptstyle\wedge}\alpha\big) = 0$. This relation, together with the relation $\psi = {}^t\overline{\psi}$, implies

$$d\psi = {}^t\overline{\alpha}\wedge\psi - \psi\wedge\alpha\ .$$

These three equations combine into the single equation $d\omega = -\omega{\scriptstyle\wedge}\omega$ where

$$\omega = \begin{pmatrix} -{}^t\overline{\alpha} & -i\,\psi \\ i\,\eta & \alpha \end{pmatrix}\ .$$

Of course, the 1-form ω takes values in a Lie algebra isomorphic to $\mathfrak{su}(p,p)$. Since ω satisfies the Maurer-Cartan structure equation $d\omega = -\omega{\scriptstyle\wedge}\omega$, it follows that the structure $\mathcal{B}$ is locally isomorphic to that induced on the second-order homogeneous space $\mathcal{H}_p\big(\mathbb{C}^p\big) = \mathrm{SU}(p,p)/P$, where P is the appropriate parabolic subgroup. This proves the flatness of $\mathcal{B}$.

In particular, if ∇ is a connection on M^{p^2} with holonomy $\mathbb{R}^+\!\cdot\mathrm{SL}(p,\mathbb{C})$ and associated torsion-free $\mathbb{R}^+\!\cdot\mathrm{SL}(p,\mathbb{C})$-structure $\mathcal{B}$, then every point of M lies in an open set on which there exists a closed section η of $\mathcal{B}$. The corresponding connection 1-form α satisfies $\alpha{\scriptstyle\wedge}\eta - \eta{\scriptstyle\wedge}{}^t\overline{\alpha} = 0$. A little algebra now shows that there exists a function s with values in V^* so that $\alpha = \eta\,s$.

Conversely, if $x : U \to V$ is a local coordinate system on U and and one uses $\eta = dx$ to define an H-structure $\mathcal{B}$ on U, then for any V^*-valued function s on U, the $\mathcal{B}$-compatible torsion-free connection on U defined by the 1-form $\alpha = dx\,s$ will have holonomy in $\mathbb{R}^+\!\cdot\mathrm{SL}(p,\mathbb{C})$. Moreover, for sufficiently generic s, the holonomy of the resulting connection will be all of $\mathbb{R}^+\!\cdot\mathrm{SL}(p,\mathbb{C})$ since its curvature will be full. Thus it follows that, modulo the local diffeomorphisms, the torsion-free connections with holonomy $\mathbb{R}^+\!\cdot\mathrm{SL}(p,\mathbb{C})$ depend on p^2 functions of p^2 variables.

Now consider the subgroup $H = \mathrm{SL}(p,\mathbb{C}) \subset \mathbb{R}^+\!\cdot\mathrm{SL}(p,\mathbb{C})$. Suppose that $\mathcal{B} \to M$ is a torsion-free $\mathrm{SL}(p,\mathbb{C})$-structure. Since the underlying $\mathbb{R}^+\!\cdot\mathrm{SL}(p,\mathbb{C})$-structure defined by $\mathcal{B}' = \mathcal{B}\cdot\mathbb{R}^+$ is locally flat, the above analysis shows that every point of M lies in an open neighborhood U on which there is a section of the corresponding H-structure of the form

$$\eta = e^{2f}\,dx$$

where $x : U \to V$ is a local coordinate system, and f is a (real valued) function on U. It is now a simple matter to compute that the unique connection 1-form α with values in $\mathfrak{sl}(p, \mathbb{C})$ which satisfies the structure equation $d\eta = -\alpha{\wedge}\eta + \eta{\wedge}{}^t\overline{\alpha}$ is given by

$$\alpha = -df\, I_p + \eta\, F$$

where F is the unique V^*-valued function which satisfies $\mathrm{tr}(\eta\, F) = p\, df$.

Conversely, for a sufficiently generic function f, the resulting connection α has its holonomy equal to $\mathrm{SL}(p, \mathbb{C})$. In fact, its curvature at a generic point has all $p^2{-}1$ component 2-forms linearly independent. Thus, modulo local diffeomorphisms, the local connections on $\mathbb{R}^{p^2}$ with holonomy $\mathrm{SL}(p, \mathbb{C})$ depend on one function of p^2 variables.

4. SOME EXOTIC CASES

TABLE 4. Some exotic irreducible holonomies (Notation: $G_{\mathbb{F}}$ denotes any connected subgroup of $\mathbb{F}^*$.)

H	V
$H_\lambda \cdot \mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^4$
$\mathrm{CO}(2) \cdot \mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^4$
$H_\lambda \cdot \mathrm{SU}(2)$	$\mathbb{R}^4$
$\mathbb{C}^* \cdot \mathrm{SU}(2)$	$\mathbb{R}^4$
$G_{\mathbb{R}} \cdot \rho_3\big(\mathrm{SL}(2,\mathbb{R})\big)$	$\mathbb{R}^4$
$G_{\mathbb{C}} \cdot \rho_3\big(\mathrm{SL}(2,\mathbb{C})\big)$	$\mathbb{C}^4$
$G_{\mathbb{R}} \cdot \mathrm{Spin}(5,5)$	$\mathbb{R}^{16}$
$G_{\mathbb{R}} \cdot \mathrm{Spin}(1,9)$	$\mathbb{R}^{16}$
$G_{\mathbb{C}} \cdot \mathrm{Spin}(10,\mathbb{C})$	$\mathbb{C}^{16}$
$G_{\mathbb{R}} \cdot E_6^1$	$\mathbb{R}^{27}$
$G_{\mathbb{R}} \cdot E_6^4$	$\mathbb{R}^{27}$
$G_{\mathbb{C}} \cdot E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$

In this final section, I will list the known exotic irreducible holonomies, i.e., the irreducibly acting groups which do not appear on Berger's general list (Table 3) but which can occur as holonomy. As of this writing, it is not known that this list is complete.

4.1. Product groups in dimension 4.

4.1.1. $H_\lambda \cdot \mathrm{SU}(2)$. — Whether this group for $\lambda > 0$ can occur as the holonomy of a torsion-free connection on a 4-manifold remains open as of this writing. (In the case $\lambda = 0$, the group H is simply $\mathrm{U}(2)$, so this corresponds to Kähler geometry and this certainly does occur.) These groups all satisfy Berger's criteria, but the exterior differential systems analysis has not yet been completed and no examples appear to be known.

4.1.2. $\mathbb{C}^* \cdot \mathrm{SU}(2)$. — This case can be thought of as the 'conformal Hermitian' case. This means that a torsion-free H-structure in this case is just a choice, on a complex 2-manifold M, of a positive $(1,1)$-form Ω defined up to a (real) multiple. Note that there is no assumption that Ω be closed, as this is not necessary (or even well-defined) in this case.

It follows that these structures depend on three arbitrary functions of four variables modulo diffeomorphism. Moreover, an elementary calculation shows that the generic such choice yields an H-structure whose connection has $\mathfrak{h}$-full curvature, so that its holonomy is all of H.

4.1.3. $H_\lambda \cdot \mathrm{SL}(2,\mathbb{R})$. — Whether this group for $\lambda > 0$ can occur as the holonomy of a torsion-free connection on a 4-manifold remains open as of this writing. This case should be thought of as a different real form of the group $H_\lambda \cdot \mathrm{SU}(2)$ and so the two problems might be related. These groups all satisfy Berger's criteria, but the exterior differential systems analysis has not yet been completed and no examples appear to be known.

4.1.4. $\mathrm{CO}(2) \cdot \mathrm{SL}(2,\mathbb{R})$. — This is a different real form of the 'conformal Hermitian' case treated above. An analysis analogous to the one done there shows that these structures depend on three arbitrary functions of four variables modulo diffeomorphism. Moreover, an elementary calculation shows that the generic such choice yields an H-structure whose connection has $\mathfrak{h}$-full curvature, so that its holonomy is all of H.

4.2. The cubic representation family.

4.2.1. $\mathrm{G}_\mathbb{R} \cdot \rho_3\big(\mathrm{SL}(2,\mathbb{R})\big)$. — This pair of groups was first studied in [Br2]. Since there is a thorough analysis there, I will not reproduce it here. I will simply report that for the case $H = \rho_3\big(\mathrm{SL}(2,\mathbb{R})\big) \subset \mathrm{SL}(4,\mathbb{R})$, the torsion-free connections with this holonomy essentially depend on one constant modulo diffeomorphism while for the case $H = \mathbb{R}^+\cdot\rho_3\bigl(\mathrm{SL}(2,\mathbb{R})\bigr) \subset \mathrm{GL}(4,\mathbb{R})$, the generic torsion-free connections with this holonomy depend on 4 arbitrary functions of 3 variables modulo diffeomorphism.

Further work on the geometry of these connections has been done in [Sc], to which I refer the reader for more information, particularly about completeness aspects, etc.

4.2.2. $\mathrm{G}_{\mathbb{C}}\cdot\rho_3\bigl(\mathrm{SL}(2,\mathbb{C})\bigr)$. — These groups are the holomorphic analogs of the previous case. There are two possible groups here and the results are in every way analogous to the previous case. The one subtlety is that the only possible 'scalar' groups are $G_{\mathbb{C}} = 1$ or $G_{\mathbb{C}} = \mathbb{C}^*$, the case where $G_{\mathbb{C}}$ is a 1-parameter subgroup of $\mathbb{C}^*$ turns out not to be possible. Again, I refer the reader to the above-mentioned references for further details.

4.3. The conformal Spin(10)**-family.** This family consists of several groups, with the analysis being essentially the same in all cases. I will not present any of the details, since this would require developing enough algebra of the spin representations to explain the calculations which lead to the results, itself a space- and time-consuming task which is out of proportion to the interest in the examples.

There are six groups in this family plus a one-parameter family of groups of the form $\mathrm{T}\cdot\mathrm{Spin}(10,\mathbb{C})$ where T is any 1-parameter subgroup of $\mathbb{C}^*$.

The three maximal groups, i.e., the ones H for which $\mathrm{G}_{\mathbb{F}} = \mathbb{F}^*$, all have the property that $\mathfrak{h}^{(1)} \simeq V^* \simeq \mathbb{F}^{16}$ and satisfy the conditions

$$H^{1,2}(\mathfrak{h}) = H^{2,2}(\mathfrak{h}) = 0\ .$$

Any torsion-free $\mathbb{R}^+\cdot\mathrm{Spin}(5,5)$-structure (respectively, $\mathbb{R}^+\cdot\mathrm{Spin}(1,9)$- or $\mathbb{C}^*\cdot\mathrm{Spin}(10,\mathbb{C})$-structure on a manifold of dimension 16 (respectively, 16 or 32) is therefore locally flat and has a second-order homogeneous model of the form E_6^1/P (respectively, E_6^4/P or $E_6^{\mathbb{C}}/P$) where P is a maximal parabolic subgroup. The ambiguity in a choice of compatible torsion-free connection is n functions of n variables where n is, respectively, 16, 16, or 32. The generic choice of connection will have $\mathfrak{h}$-full curvature and so will have holonomy H.

The three minimal groups, i.e., the ones H for which $\mathrm{G}_{\mathbb{F}} = 1$, all have the property that $\mathfrak{h}^{(1)} \simeq 0$ and so any torsion-free H-structure $\mathcal{B}$ has a unique compatible torsion-free connection. Moreover, by the above discussion, $\mathcal{B}$ is 'conformally flat' and can be constructed by starting with the flat $\mathbb{F}^* \cdot H$-structure and choosing a volume form (holomorphic in the case $\mathbb{F} = \mathbb{C}$). The generic choice of volume form (holomorphic in the case $\mathbb{F} = \mathbb{C}$) will yield an H-structure $\mathcal{B}$ whose canonical connection has $\mathfrak{h}$-full curvature and so will have holonomy H. Thus, in the first two cases, the general torsion-free connection with holonomy H depends on one arbitrary function of 16 real variables while in the third case it depends on one holomorphic function of 16 complex variables.

Finally, consider the family of groups of the form $H = \mathrm{T} \cdot \mathrm{Spin}(10, \mathbb{C})$ where T is any 1-parameter subgroup of $\mathbb{C}^*$. These all have the property that $\mathfrak{h}^{(1)} \simeq 0$, so a torsion-free H-structure $\mathcal{B}$ on a manifold M^{32} has a unique compatible torsion-free connection. Moreover, these structures are constructed by taking an arbitrary smooth reduction from a locally flat $\mathbb{C}^* \cdot \mathrm{Spin}(10, \mathbb{C})$-structure on M. Since H has codimension 1 in this group, it follows that this depends on a choice of one arbitrary function of 32 variables. It can be checked that the generic such reduction yields an H-structure whose canonical connection has $\mathfrak{h}$-full curvature, so that it has holonomy H.

4.4. The E_6-family. This case is treated in a manner in every way analogous to the Spin(10)-family just treated, so I will just summarize the results.

For the three maximal groups, H is either $\mathbb{R}^+ \cdot E_6^1$, $\mathbb{R}^+ \cdot E_6^4$, or $\mathbb{C}^* \cdot E_6^C$, the torsion-free H-structures are all locally flat, being modeled by second order homogeneous spaces which are quotients of the appropriate form of E_7 and the corresponding local torsion-free connections depend on 27 arbitrary functions of 27 variables in the first two cases and 27 holomorphic functions of 27 complex variables in the third case.

For the three minimal groups, H is either E_6^1, E_6^4, or E_6^C, the corresponding torsion-free H-structures are all locally conformally flat, being constructed from the flat structures by choosing an arbitrary volume form (holomorphic in the third case). Each such structure has a unique compatible torsion-free connection, with the generic choice of volume form yielding a connection with $\mathfrak{h}$-full curvature. Thus, the corresponding local torsion-free connections depend on one arbitrary function of 27 variables in the first two cases and one holomorphic function of 27 complex variables in the third case.

Finally, for the groups of the form $H = \mathrm{T} \cdot E_6^{\mathbb{C}}$, the torsion-free H-structures are all got locally by taking an arbitrary reduction from the flat $\mathbb{C}^* \cdot E_6^{\mathbb{C}}$-structure. Thus, the general connection with this holonomy depends on one arbitrary function of 54 real variables.

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