Schematic Algebras and the Auslander-Gorenstein Property

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Abstract

Noncommutative algebraic geometry studies a certain quotient category R-qgr of the category of graded R-modules which for commutative R is equivalent to the category of quasi-coherent sheaves by a famous theorem of Serre. For a large class of graded algebras, the so-called schematic algebras, we are able to construct a kind of scheme such that the coherent sheaves on it are equivalent to R-qgr. We give a brief survey on the results so-far on schematic algebras and include some new results on cohomological properties of Auslander-Gorenstein algebras which might be useful in determining the strength of the schematic property.

Résumé

En géométrie algébrique noncommutative on étudie un certain quotient R-qgr de la catégorie des R-modules gradués, qui pour R commutatif, est équivalente à la catégorie des faisceaux quasi-cohérents par un théorème bien connu de Serre. Pour une grande classe d'algèbres, les algèbres schématiques, nous pouvons construire une sorte de schéma sur lequel les faisceaux cohérents forment une catégorie équivalente à R-qgr. Nous rappelons les résultats connus sur les algèbres schématiques et donnons quelques résultats nouveaux sur les propriétés cohomologiques des algèbres de Auslander-Gorenstein.

1 Preliminaries and introduction

Let k be an algebraically closed field of characteristic zero and consider a k-algebra R of the following kind : R is connected (i.e. R is positively graded and $R_0 = k$), Noetherian and generated in degree 1. The category of graded R-modules will be denoted by R-gr and the two-sided ideal $\oplus_{i\geq 1} R_i$ by R_+. A graded R-module M is said to be torsion if each of its elements is annihilated by some power of R_+ : $\forall m \in M\ \exists n \in \mathbb{N}\ :\ (R_+)^n m = 0$. The set $\{(R_+)^n\ :\ n \in \mathbb{N}\}$ is an idempotent filter (cf. [11]). Hence we have a localization functor Q_{κ_+} available. Moreover, the

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corresponding kernel functor κ_+ (the one which sends a graded R-module to its torsion submodule : $\kappa_+(M) = \{m \in M \mid \exists n \in \mathbb{N} \ : \ (R_+)^n m = 0\}$) is stable, i.e. the injective envelope of a torsion module is again torsion. The quotient category (R, κ_+)-gr consists of all κ_+-closed graded R-modules, i.e. those graded R-modules which satisfy $M \cong \mathrm{Hom}_R((R_+)^n, M)$ for all n. More details about the quotient category may be found in [11] or in [6], but we will provide an alternative description in the next section.

If R is commutative, then algebraic geometry is a very powerful tool for studying R. Whence the question arises whether one can do something similar for a non-commutative R. The answer is positive if R is a PI-algebra, since the method of the prime spectrum generalizes well ([13]). The recent discovery of interesting algebras not possessing enough prime ideals (like the so-called Sklyanin algebras ([8, 9, 5, 4])) motivated us to start from a rather unusual description of the projective scheme of a commutative algebra, not stressing the prime ideals but the complementary multiplicatively closed sets.

The projective scheme associated to a *commutative* algebra R is a pair $(X, \mathcal{O}_X)$, X being a topological space and $\mathcal{O}_X$ a sheaf of *graded* rings on X. Each homogeneous element f of R defines an open set $X(f)$. Open sets of this kind form a basis and a finite number of them suffices to cover X. There is a functor F from the category of graded R-modules R-gr to the category of quasi-coherent sheaves on X such that $F(R) = \mathcal{O}_X$ and the sections of $F(M)$ (M any graded R-module) on the open set $X(f)$ is just the localization of M at the multiplicatively closed set $\{f^n \ : \ n \in \mathbb{N}\}$. The global sections-functor G maps a quasi-coherent sheaf $\mathcal{F}$ to its sections on the total space X : $G(\mathcal{F}) = \Gamma(X, \mathcal{F})$. The composition $F \circ G$ is the identity, but the functor $\Gamma \overset{\mathrm{def}}{=} G \circ F$ is merely left exact. An important theorem of Serre ([7]) states that these two functors induce an equivalence between the category of quasi-coherent sheaves on X and the quotient category R-qgr of R-gr in which graded R-modules are being identified if there is a map between them whose kernel and cokernel are torsion.

If M is graded R-module, then $F(M)$ being a sheaf implies that $\Gamma(M)$ may be described as the inverse limit of the sections of $F(M)$ on a cover of X. In particular, if $f_1, \ldots, f_n$ are homogeneous elements of R such that $U_i X(f_i) = X$, then $\Gamma(M)$ is isomorphic to

$$\left\{ \left(\frac{m_i}{f_i^{n_i}} \right)_i \in \oplus_{i=1}^n M_{f_i} \ : \ \frac{f_j^{n_i} m_i}{(f_j f_i)^{n_i}} = \frac{f_i^{n_j} m_j}{(f_i f_j)^{n_j}} \text{ in } M_{f_i f_j} = \Gamma(X(f_i) \cap X(f_j), F(M)) \right\}$$

The functor $\Gamma = G \circ F$ has an entirely module-theoretical description coming from torsion theory ([11]) :

$$\Gamma(M) = \varinjlim_n \mathrm{Hom}_R((R_+)^n, M)$$

Moreover, the above mentioned quotient category also makes sense if R is not commutative and Serre's Theorem justifies the idea that R-qgr is the fundamental object to study in noncommutative geometry ([1, 3]).

Suppose now that R is *non-commutative*. We want to describe the objects of R-qgr by means of localizations of ordinary graded R-modules in the same way as above. However, in a non-commutative algebra, one needs Ore-sets for localizing and consequently, their existence is not guaranteed. Thus we have to confine to algebras possessing "enough" Ore-sets; these are the schematic algebras :

Definition — *R is schematic if there exists a finite number of two-sided homogeneous Ore-sets of R, say $S_1, \dots, S_n$, such that*

$$\forall (s_i)_{i=1,\dots,n} \in \prod_{i=1}^{n} S_i,\ \exists\, m \in \mathbb{N}\ :\ (R_+)^m \subseteq \sum_{i=1}^{n} Rs_i$$

The origin of this definition lies in the commutative case : we have that the "Ore-sets" generated by homogeneous elements f_i of R satisfy the above rule if and only if $\cup_i X(f_i) = X$.

Besides the commutative algebras, many interesting graded algebras are schematic ([14]) :

- algebras which are finite modules over their center.
- homogenizations of enveloping algebras and Weyl-algebras.
- 3-dimensional Sklyanin-algebras.
- several algebras of quantum-type.

Finding counterexamples is easy after noting that for a schematic algebra R all $Ext^n_R(k_R, R_R)$ are torsion (cf. [15]). For instance, the subalgebra S of $k < x, y > /(yx - xy - x^2)$ generated by y and xy is not schematic since $Ext^1_S(k_S, S_S)$ is not torsion (cf. [10]).

If we suppose that R is schematic, then our aim seems to be close at hand : since the multiplicatively closed set $S \vee T$ generated by two Ore-sets S and T is again an Ore-set, one might think that

$$\Gamma(M) \cong \left\{ \left(\frac{m_i}{s_i}\right)_i \in \oplus_i S_i^{-1} M\ :\ \frac{m_i}{s_i} = \frac{m_j}{s_j} \text{ in } (S_i \vee S_j)^{-1} M \right\}$$

Unfortunately, this statement is *not* true, mainly because two subsequent Ore-localizations do not commute : $S_i^{-1}R \otimes_R S_j^{-1}R$ is not necessarily isomorphic to $S_j^{-1}R \otimes_R S_i^{-1}R$. The solution to this problem is a refinement of the inverse system : indeed we do have that $\Gamma(M)$ is isomorphic to the set of those tuples $(\frac{m_i}{s_i})_i$ in $\oplus_i S_i^{-1}M$ such that

$$1 \otimes \frac{m_i}{s_i} = \frac{1}{s_j} \otimes \frac{m_j}{1} \text{ in } S_j^{-1}(S_i^{-1}M) \ \ \textit{and}\ \ \frac{1}{s_i} \otimes \frac{m_i}{1} = 1 \otimes \frac{m_j}{s_j} \text{ in } S_i^{-1}(S_j^{-1}M)$$

If one wants to state a precise analogue of Serre's Theorem, then either one has to work with a strange Grothendieck topology (the "intersection" of two open sets depending on the ordering) or either the sections of $F(M)$ get more complicated. In both cases, one can define sheaves, specify quasi-coherent sheaves and obtain the desired equivalence between the category of quasi-coherent sheaves and R-qgr ([16]). We prefer the former approach since it is useful in cohomology.

2 Cohomology in the quotient category

Let R be any Noetherian connected k-algebra. Define a category $\mathscr{C}$ with the same objects as R-gr, the category of graded R-modules. We will write $\pi(M)$ when considering the graded R-module M as an object of $\mathscr{C}$. Morphisms in $\mathscr{C}$ are defined as follows :

$$Hom_{\mathscr{C}}(\pi(M), \pi(N)) = \varinjlim_{M'} Hom_{R\text{-gr}}(M', N/\kappa_+(N))$$

where M' runs over the category of submodules of M such that M/M' is torsion. Consequently, π is an exact functor from R-gr to $\mathscr{C}$. Moreover, π has a right adjoint $\omega : \mathscr{C} \to R$-gr, in the sense that for all $\mathscr{N} \in \mathscr{C}$

$$Hom_{R\text{-gr}}(M, \omega(\mathscr{N})) \cong Hom_{\mathscr{C}}(\pi(M), \mathscr{N})$$

These two functors establish an equivalence between $\mathscr{C}$ and the quotient category (R, κ_+)-gr. Since $\mathscr{C}$ has enough injectives, we may define H^i, the i-th right derived functor of $\mathrm{Hom}_{\mathscr{C}}(\pi(R), -)$. In order to calculate $H^i(\pi(M))$, we should start with an injective resolution of $\pi(M)$ in $\mathscr{C}$, apply the functor $\mathrm{Hom}_{\mathscr{C}}(\pi(R), -)$ and take homology on the i-th place. We get an injective resolution of $\pi(M)$ in $\mathscr{C}$ if we apply the functor π to an injective resolution $E^\bullet$ of M in R-gr. Moreover, since

$$\mathrm{Hom}_{\mathscr{C}}(\pi(R), \pi(E^i)) \cong \mathrm{Hom}_{R\text{-gr}}(R, \omega\pi(E^i)) \cong (\omega\pi(E^i))_0$$

we get that $H^i(\pi(M)) \cong h^i(\omega\pi(E^\bullet)_0)$ $\forall i \in \mathbb{N}$. If one defines the shifted module $M[n]$ as the module M with gradation $(M[n])_p = M_{n+p}$, one obtains the graded cohomology-groups by :

$$\underline{H}^i(\pi(M)) \overset{\text{def}}{=} \oplus_{n\in\mathbb{Z}} H^i(\pi(M[n]))$$

In particular, $\underline{H}^0(\pi(M)) \cong \omega\pi(M) \cong Q_{\kappa_+}(M)$. These graded cohomology groups are again graded R-modules and from the reasoning above we obtain that

$$\underline{H}^i(\pi(M)) \cong h^i(\omega\pi(E^\bullet))$$

The complex $\omega\pi(E^\bullet)$, the homology of which we want to calculate, may be described in an easier way, using the fact that κ_+ is stable. Indeed, stability implies that each graded injective R-module E may be written as a direct sum $I \oplus Q$ where I is graded torsion and Q is graded torsionfree. Moreover, both I and Q are graded injective and $Q_{\kappa_+}(E) \cong Q$. We may then rewrite the injective resolution $E^\bullet$ of M as :

$$0 \longrightarrow M \longrightarrow I^0 \oplus Q^0 \xrightarrow{f_0} I^1 \oplus Q^1 \xrightarrow{f_1} I^2 \oplus Q^2 \xrightarrow{f_2} \cdots$$

Note that $f_n(I^n) \subseteq I^{n+1}$, since the image of a torsion element under a graded R-module homomorphism is again torsion. Applying $\omega \circ \pi$ yields a complex

$$0 \longrightarrow Q_{\kappa_+}(M) \longrightarrow Q^0 \xrightarrow{g_0} Q^1 \xrightarrow{g_1} Q^2 \xrightarrow{g_2} \cdots$$

where $g_i = Q_{\kappa_+}(f_i)$ is the composition of the maps $Q^i \hookrightarrow E^i \xrightarrow{f_i} E^{i+1} \to Q^{i+1}$. Thus $\underline{H}^j(\pi(M))$ is the j^{th} homology-group of the complex (Q^i, g_i).

In algebraic geometry, it is shown that these H^i coincide with the derived functors of the global sections functor on the category of sheaves, and the latter coincide with the more amenable Čech cohomology groups. If R is a schematic algebra, we can define (generalized) Čech cohomology groups as the homology of the complex

$$0 \longrightarrow \oplus_i S_i^{-1}M \longrightarrow \oplus_{(i,j)} S_i^{-1}R \otimes_R S_j^{-1}M \longrightarrow \cdots$$

We have shown in [15] (without intermediate step) that these Čech cohomology groups coincide with the functors H^i. The point is that Čech cohomology vanishes on graded injective modules. Besides providing a more computable way to the H^i (see the example in [15]), this equality has some interesting consequences :

1. the cohomology of R as a left R-module is the same as that of R as a right R-module.
2. if R' is a quotient of R, then it makes no difference whether we calculate the cohomology of R' as an R-module or as an R'-module.
3. if R is a finite module over its center $Z(R)$, then for each graded R-module we have $\underline{H}^i_R(M) \cong \underline{H}^i_{Z(R)}(M)$ as $Z(R)$-modules since R may be covered with Ore-sets contained in $Z(R)$ and hence the Čech complex we use coincides with the one in [12].

Moreover, if the schematic algebra R has finite global dimension, then the cohomology groups of any finitely generated graded R-module are finite dimensional ([2, 15]). In particular, applying the functor $\omega \circ \pi$ to an exact sequence of finitely generated graded R-modules yields a left exact sequence of graded R-modules whose parts of degree n are exact if n is large enough.

In the next lemma we collect some useful results from [2] :

Lemma — *Let $E^\bullet(M)$ be a graded injective resolution of the finitely generated graded R-module M. Then*

1. *for all $i \geq 1$: $\underline{H}^i(\pi(M)) \cong h^{i+1}(I^\bullet(M))$, the $i+1^{st}$ homology of the subcomplex $I^\bullet(M)$ of $E^\bullet(M)$.*
2. *if the resolution is moreover a minimal one, then $\underline{\mathrm{Hom}}(k, f_{j-1}(E^{j-1})) = \underline{\mathrm{Hom}}(k, E^j)$. Consequently, $\underline{\mathrm{Hom}}(k, E^j) \subseteq \mathrm{Ker} f_j$ and $\underline{\mathrm{Ext}}^j_R(k, M) \cong \underline{\mathrm{Hom}}_R(k, I^j)$*

Suppose now that R has finite global dimension d and satisfies the **Gorenstein** condition. This means that $\underline{\mathrm{Ext}}^i(k,R) = 0$ if $i \neq d$ and $\underline{\mathrm{Ext}}^d(k,R) \cong k[l]$ for some l in $\mathbb{Z}$. We know from [2] that the homology groups of such an algebra R are completely similar to those of projective $d-1$ space:

- $\underline{H}^0(\pi(R)) \cong R$
- $\underline{H}^j(\pi(R)) = 0$ for all $j \notin \{0, d-1\}$
- $\underline{H}^{d-1}(\pi(R)) \cong R^*[l]$ where $R^* = \oplus_n \mathrm{Hom}_k(R_{-n}, k)$ is the graded dual of R.

We want to prove the converse : that algebras whose cohomology groups have this shape are Gorenstein. This result is useful for schematic algebras, because for them we have a down-to-earth description of the cohomology groups.

Theorem 1 — *Let R be a Noetherian connected k-algebra with finite global dimension. Suppose there exists a natural number d and an integer l such that $\underline{H}^0(\pi(R)) \cong R$, $\underline{H}^j(\pi(R)) = 0$ for all $j \notin \{0, d-1\}$ and $\underline{H}^{d-1}(\pi(R)) \cong R^*[l]$. Then $d = \mathrm{gl.dim}(R)$ and R is Gorenstein.*

Proof. Consider a minimal injective resolution $E^\bullet$ of R and let $E^j = Q^j \oplus I^j$ as before. Let $\tau(I^j)$ be the socle $\underline{\mathrm{Hom}}_R(k, I^j)$. The lemma entails that $\underline{\mathrm{Ext}}^j(k,R) \cong \tau(I^j)$ and that $\tau(I^j) \subseteq \mathrm{ker} f_j$ for all $j \geq 1$. It is well-known that $\underline{H}^0(R) \cong R$ entails that $\underline{\mathrm{Ext}}^j(k,R) = 0$ for $j \in \{0,1\}$ and that $I^0 = I^1 = 0$. Consider the exact sequence

$$0 \longrightarrow I^2 \xrightarrow{f_2} I^3 \xrightarrow{f_3} \ldots \longrightarrow I^{d-1} \xrightarrow{f_{d-1}} I^d$$

and its subcomplex of socles. Now f_2 is injective and $\tau(I^2) \subseteq \mathrm{ker} f_2$, hence $\tau(I^2) = 0$ and also I^2 is zero since I^2 is the injective hull of its socle. Therefore, f_3 is injective and we may repeat this process, yielding that $I^2 = I^3 = \ldots = I^{d-1} = 0$. We conclude that $\underline{\mathrm{Ext}}^j(k,R) = 0$ for all $j \leq d-1$. The following sequence is now exact :

$$0 \longrightarrow R^*[l] \longrightarrow I^d \xrightarrow{f_d} I^{d+1} \longrightarrow \ldots$$

Again $\tau(I^d) \subseteq \mathrm{Ker} f_d$, whence $\tau(I^d) = \tau(R^*[l])$ and consequently $I^d = R^*[l]$. In the same way, we get $\tau(I^{d+1}) = 0$, hence $I^{d+1} = 0$ and it is clear that we may repeat this argument. We conclude that $\underline{\mathrm{Ext}}^d(k,R) = R^*[l]$ and $\underline{\mathrm{Ext}}^j(k,R) = 0$ for all $j > d$. Finally, $\mathrm{gl.dim}\, R = \mathrm{p.dim}_R\, k = sup\{i \mid \underline{\mathrm{Ext}}^i(k,R) \neq 0\} = d$. ☐

At this moment, we do not know the strongness of the hypothesis that a certain algebra is schematic. For instance, is a schematic algebra with finite global dimension automatically Gorenstein? We hope that the above procedure will lead us to an answer. It would also be interesting to know that the cohomological dimension of a schematic algebra (i.e. the least integer n such that $H^i(\pi(M)) = 0$ for all $M \in R$-gr and all $i > 0$) is still bounded by the number of open sets in a cover.

We conclude with a result which has been found independently by A. Yekutieli and J. Zhang (cf. [17]) : if the connected Noetherian k-algebra R is Gorenstein, then Serre-duality holds for R. The proof is just a restatement of the classical proof (for sheaves) in terms of the quotient category. We switch to the more suggestive notation $\mathcal{M} = \pi(M)$ and $\mathcal{O} = \pi(R)$.

Theorem 2 — *Let R be a Gorenstein-algebra of finite global dimension d (with $\underline{\mathrm{Ext}}^d(k, R) = k[l]$). Then :*

1. *The natural pairing* $\mathrm{Hom}(\mathcal{M}, \mathcal{O}[-l]) \times H^{d-1}(\mathcal{M}) \to H^{d-1}(\mathcal{O}[-l]) \cong k$ *is a perfect pairing of finite-dimensional vectorspaces for any finitely generated graded R-module M.*
2. $\forall i \geq 0$ *: $Ext^i(\mathcal{M}, \mathcal{O}[-l]) \cong (H^{d-1-i}(\mathcal{M}))'$, where $'$ denotes the dual vectorspace.*

Proof. Since R is Gorenstein we have $\omega\pi(R) = R$ and consequently $\mathrm{Hom}_{\mathscr{C}}(\mathcal{M}, \mathcal{O}[-l]) \cong \mathrm{Hom}_{R\text{-gr}}(M, R[-l]) \cong (\underline{\mathrm{Hom}}_R(M, R))_{-l}$. A morphism $f \in (\underline{\mathrm{Hom}}_R(M, R))_{-l}$ yields a map $H^{d-1}(f) : H^{d-1}(\mathcal{M}) \to \underline{H}^{d-1}(\mathcal{O})_{-l} = H^{d-1}(\mathcal{O}[-l])$ which induces the natural pairing. This pairing is perfect if M is a direct sum of shifts of R because $H^{d-1}(\mathcal{O}[q]) = \mathrm{Hom}_k(R_{-l-q}, k)$ and $\mathrm{Hom}(\mathcal{O}[q], \mathcal{O}[-l]) = R_{-l-q}$. If M is arbitrary, we consider a projective resolution $F_2 \to F_1 \to M \to 0$. On one hand, applying the left exact contravariant functor $\mathrm{Hom}_{\mathscr{C}}(-, \mathcal{O}[-l]) \circ \pi$ yields an exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{O}[-l]) \longrightarrow \underline{\mathrm{Hom}}(\pi(F_1), \mathcal{O}[-l]) \longrightarrow \underline{\mathrm{Hom}}(\pi(F_2), \mathcal{O}[-l])$$

On the other hand, if we apply the functor $H^{d-1}(-)'$, we get another exact sequence

$$0 \longrightarrow H^{d-1}(\mathcal{M})' \longrightarrow H^{d-1}(\pi(F))' \longrightarrow H^{d-1}(\pi(F_2))'$$

The 5-lemma entails that the natural map $\mathrm{Hom}(\mathcal{M}, \mathcal{O}[-l]) \to H^{d-1}(\mathcal{M})'$ is indeed an isomorphism. For the second statement, we note that both $Ext^i(-, \mathcal{O}[-l])$ and $(H^{d-1-i}(-))'$ are contravariant δ-functors $\mathscr{C} \to k - mod$ which are isomorphic for $i = 0$. If we show that they are both coeffaceable, then they are isomorphic by general machinery. Thus fix a homomorphism $\oplus_{j=1}^n R[-q] \to M$ (n and q large enough) such that M is κ_+-torsion over the image. Then $Ext^i(\oplus_{j=1}^n \mathcal{O}[-q], \mathcal{O}[-l]) = \oplus_{j=1}^n H^i(\mathcal{O}[-l+q])$ and $H^{d-1-i}(\oplus_{j=1}^n(\mathcal{O}[-q]))$ are both zero for $i > 0$. $\square$

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