Orbits of Matrix Tuples

Lieven Le BRUYN*

Abstract

In this paper we outline a procedure which can be seen as an approximation to the well known "hopeless" problem of classifying m-tuples ($m \geq 2$) of $n \times n$ matrices under simultaneous conjugation by GL_n. The method relies on joint work with C. Procesi, on the étale local structure of matrix-invariants and recent work [10], [11] on the nullcone of quiver-representations.

Résumé

Dans ce papier, nous présentons une procédure qui peut être considérée comme une approximation au problème bien connu et "sans espoir" de classification des m-uplets ($m \geq 2$) de matrices $n \times n$ sous l'action de conjugaison simultanée par GL_n. La méthode est basée sur un travail en commun avec C. Procesi, sur la structure étale locale des invariants de matrices et sur un travail récent de l'auteur sur le cône nilpotent des représentations de carquois.

Throughout, we fix an algebraically closed field of characteristic zero and call it $\mathbb{C}$. Let $X_{m,n} = M_n(\mathbb{C})^{\oplus m}$ be the affine space of m-tuples of $n \times n$ matrices with the action of GL_n given by simultaneous conjugation, that is

$$g.X = g.(x_1, \ldots, x_m) = (gx_1g^{-1}, \ldots, gx_mg^{-1})$$

for all $g \in GL_n$ and all $X \in X_{m,n}$. The first approximation to the orbit space of this action is the quotient variety $V_{m,n}$ which is determined by its coordinate ring which is the ring of invariant polynomial functions $\mathbb{C}[V_{m,n}] = \mathbb{C}[X_{m,n}]^{GL_n}$. The inclusion $\mathbb{C}[V_{m,n}] \subset \mathbb{C}[X_{m,n}]$ gives the quotient map

$$\pi : X_{m,n} \longrightarrow V_{m,n} = X_{m,n}//GL_n$$

Procesi [14] has shown that the coordinate ring $\mathbb{C}[V_{m,n}]$ is generated by traces in the generic matrices of degree at most n^2. From general invariant theory [13] we know that the points of $V_{m,n}$ classify the closed orbits in $X_{m,n}$. The correspondence being given by associating to a point $\zeta \in V_{m,n}$ the orbit of minimal dimension in the fiber $\pi^{-1}(\zeta)$.

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*Senior researcher at the NFWO — Departement Wiskunde UIA, B-2610 Wilrijk (Belgium)

A more algebraic interpretation is as follows. A point $X = (x_1, \dots, x_m) \in X_{m,n}$ determines an n-dimensional representation of $\mathbb{C} < X_1, \dots, X_m >$ by associating to X the algebra map

$$\varphi_X : \mathbb{C} < X_1, \dots, X_m > \mapsto M_n(\mathbb{C})$$

given by $X_i \mapsto x_i$. Two representations φ_X and φ_Y are isomorphic if and only if X and Y belong to the same orbit. By the Artin-Voigt theorem [6, II.2.7] the closed orbits correspond to the semi-simple n-dimensional representations. A general orbit is mapped under the quotient π to its semi-simplification, that is the direct sum of the Jordan-Hölder components.

Our first aim is to study the highly singular variety $V_{m,n}$ better. Assume that $\zeta \in V_{m,n}$ determines a semi-simple n-dimensional representation of the form

$$S_1^{\oplus e_1} \oplus \dots \oplus S_r^{\oplus e_r}$$

where the S_i are the distinct simple components of dimension k_i occuring with multiplicity e_i. We then say that ζ is of representation type

$$\tau = (e_1, k_1; \dots; e_r, k_r)$$

where this tuple is of course only determined upto permuting the indices. The algebraic notion of degeneration of representation types can be described combinatorially as follows. We say that $\tau' = (e'_1, k'_1; \dots; e'_{r'}, k'_{r'}) < \tau$ if there is a permutation σ on $\{1, \dots, r'\}$ such that there exist numbers

$$1 = j_0 < j_1 < j_2 < \dots < j_r = r'$$

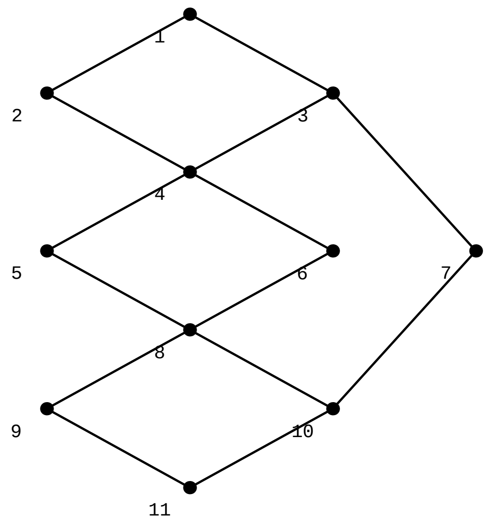
such that for every $1 \leq i \leq r$ we have

- $e_i k_i = \sum_{j=j_{i-1}+1}^{j_i} e'_{\sigma(j)} k'_{\sigma(j)}$
- $e_i \leq e_{\sigma(j)'}$ for all $j_{i-1} < j \leq j_i$

For example for $n = 3$ we have 5 representation types with a line degeneration pattern $(3, 1) < (2, 1; 1, 1) < (1, 1; 1, 1; 1, 1) < (1, 2; 1, 1) < (1, 3)$. However, things quickly become more complex. For $n = 4$ we have 11 representation types

type	τ
1	$(1,4)$
2	$(1,3;1,1)$
3	$(1,2;1,2)$
4	$(1,2;1,1;1,1)$
5	$(1,1;1,1;1,1;1,1)$
6	$(1,2;2,1)$
7	$(2,2)$
8	$(1,1;1,1;2,1)$
9	$(1,1;3,1)$
10	$(2,1;2,1)$
11	$(4,1)$

with corresponding Hasse diagram



With $V_{m,n}(\tau)$ we will denote the set of points ζ of $V_{m,n}$ of representation type τ. An application of the Luna slice theorem ([12] and [17]) gives the following result. The crucial observation in the proof is that the representation type determines the conjugacy class of the isotropy group of the corresponding closed orbit.

Proposition 1 ([8, II.1.1]) — *$\{V_{m,n}(\tau);\tau\}$ is a finite stratification of $V_{m,n}$ into locally closed irreducible smooth algebraic subvarieties.*

$V_{m,n}(\tau')$ lies in the Zariski closure of $V_{m,n}(\tau)$ if and only if $\tau' < \tau$.

Further, one can use the theory of trace identities [15] to describe the defining equations of these locally closed subvarieties $V_{m,n}(\tau)$. Thus, we may assume that we have a firm grip on these strata. Remains the difficulty of studying the orbit structure of the fiber $\pi^{-1}(\zeta)$ for ζ in a fixed stratum $V_{m,n}(\tau)$. That is, we want to describe the isomorphism classes of n-dimensional representations with a fixed Jordan-Hölder decomposition. Again, the first step is provided by the Luna slice machinary.

Let $X \in X_{m,n}$ be a point lying on the unique closed orbit in the fiber $\pi^{-1}(\zeta)$ for $\zeta \in V_{m,n}(\tau)$. Let G_X denote the isotropy group, then the tangent space at X, $T_X(X_{m,n}) \simeq X_{m,n}$ has a splitting as G_X-modules as

$$T_X(X_{m,n}) = T_X(GL_n.X) \oplus N_X$$

where $T_X(GL_n.X)$ is the tangent space to the orbit and N_X the corresponding normal space. By the Luna slice theorem we have in a neighborhood of $0 \in N_X$ the following commutative diagram

$$\begin{tikzcd} GL_n \times^{G_X} N_X \arrow[r, "\alpha"] \arrow[d] & X_{m,n} \arrow[d] \\ N_X//G_X \arrow[r, "\alpha'"] & V_{m,n} \end{tikzcd}$$

where α is determined by sending the class of (g,n) to $g.(X+n)$, where $GL_n \times^{G_X} N_X = (GL_n \times N_X)//G_X$ under the action $h.(g,n) = (gh^{-1}, h.n)$ and where both α and α' are étale maps.

It follows from this description (see for example [17, p.101]) that the fiber at ζ is isomorphic to

$$\pi^{-1}(\zeta) \simeq GL_n \times^{G_X} \mathrm{Null}(N_X, G_X)$$

as GL_n-varieties where we denote by $\mathrm{Null}(N_X, G_X)$ the nullcone of the G_X-action on the normalspace N_X, that is, if

$$N_X \xrightarrow{\pi'} N_X//G_X$$

the nullcone $\mathrm{Null}(N_X, G_X) = \pi'^{-1}(\pi'(0))$. In particular, we deduce that the orbit structure of the fibers $\pi^{-1}(\zeta)$ is the same along a stratum $V_{m,n}(\tau)$ and is fully understood provided we know the G_X-orbit structure in the nullcone $\mathrm{Null}(N_X, G_X)$.

In order to achieve this goal we need to have a better representation theoretic description of the normal space N_X, of the isotropy group G_X and of its action on N_X. These facts can best be described in terms of quiver representations. Let us recall some definitions.

A **quiver** Q is a 4-tuple (Q_v, Q_a, t, h) where Q_v is a finite set $\{1, ..., k\}$ of vertices, Q_a a finite set of arrows φ between these vertices and $t, h : Q_a \to Q_v$ are two maps assigning to an arrow φ its tail t_φ and its head h_φ respectively. Note that we do not exclude loops or multiple arrows.

A **representation** V of a quiver Q consists of a family $\{V(i) : i \in Q_v\}$ of finite dimensional $\mathbb{C}$-vector spaces and a family $\{V(\varphi) : V(t_\varphi) \to V(h_\varphi); \varphi \in Q_a\}$ of linear maps between these vectorspaces, one for each arrow in the quiver. The dimension-vector $dim(V)$ of the representation V is the k-tuple of integers $(dim(V(i)))_i \in \mathbb{N}^k$. We have the natural notion of morphisms and isomorphisms between representations consisting of k-tuples of linear maps with obvious commutativity conditions.

For a fixed dimension-vector $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$ one defines the **representation space** $R(Q, \alpha)$ of the quiver Q to be the set of all representations V of Q with $V_i = \mathbb{C}^{\alpha_i}$ for all $i \in Q_v$. Because $V \in R(Q, \alpha)$ is completely determined by the linear maps $V(\varphi)$, we have a natural vector space structure

$$R(Q, \alpha) = \underset{\varphi \in Q_a}{\oplus} M_\varphi(\mathbb{C})$$

where $M_\varphi(\mathbb{C})$ is the vector space of all $\alpha_{h_\varphi} \times \alpha_{t_\varphi}$ matrices over $\mathbb{C}$.

We consider the vector space $R(Q, \alpha)$ as an affine variety with coordinate ring $\mathbb{C}[Q, \alpha]$ and function field $\mathbb{C}(Q, \alpha)$. There is a canonical action of the linear reductive group

$$GL(\alpha) = \prod_{i=1}^{k} GL_{\alpha_i}(\mathbb{C})$$

on the variety $R(Q, \alpha)$ by base change in the V_i. That is, if $V \in R(Q, \alpha)$ and $g = (g(1), ..., g(k)) \in GL(\alpha)$, then

$$(g.V)(\varphi) = g(h_\varphi) V(\varphi) g(t_\varphi)^{-1}$$

The $GL(\alpha)$-orbits in $R(Q, \alpha)$ are precisely the isomorphism classes of representations.

Let us return to our problem of describing the G_X-action on the nullcone $\mathrm{Null}(N_X, G_X)$. To a representation type $\tau = (e_1, k_1; \ldots; e_r, k_r)$ we associate a quiver Q_τ and a dimension vector α_τ in the following way.

- Q_τ is the quiver on r-vertices $\{v_1, \ldots, v_r\}$ with
  - $(m-1)k_i^2 + 1$ loops at vertex v_i
  - $(m-1)k_i k_j$ directed arrows from v_i to v_j
- $\alpha_\tau = (e_1, \ldots, e_r)$

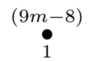
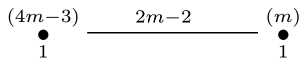
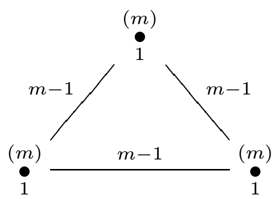
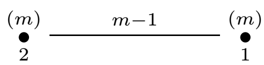
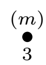
If $\zeta \in V_{m,n}(\tau)$ and X a point of the corresponding closed orbit it is easy to verify that the isotropy group $G_X \simeq GL(\alpha_\tau)$. Moreover, studying the isotypic decomposition of the normal space to the orbit as $GL(\alpha_\tau)$ spaces one can prove the following result, see [8] and [9]

Proposition 2 — *With notations as above we have that*

$$N_X = R(Q_\tau, \alpha_\tau)$$

as $G_X = GL(\alpha_\tau)$ vectorspaces.

Let us give the quivers occuring in the case of m-tuples of 3×3 matrices. In the table below the upper vertex-indices give the number of loops, the under vertex-indices the components of the dimension-vector α_τ. The number l associated to an undirected edge between two vertices v and w indicates that there are l directed arrows from v to w and l arrows from w to v.

type	τ	(Q_τ, α_τ)
1	$(1,3)$	
2	$(1,2;1,1)$	
3	$(1,1;1,1;1,1)$	
4	$(2,1;1,1)$	
5	$(3,1)$	

In [9] the quotient varieties of quiver representations were studied. In particular it was shown that the rings of invariant polynomial functions are generated by the traces of oriented cycles in the quiver. Hence, a representation V will belong to $N(Q_\tau, \alpha_\tau)$, the nullcone of $R(Q_\tau, \alpha_\tau)$ if and only if the matrices obtained by multiplying along any cycle in the quiver are all nilpotent.

By the above results we have reduced the study of the orbit structure of $\pi^{-1}(\zeta)$ to that of the $GL(\alpha_\tau)$ orbits in $N(Q_\tau, \alpha_\tau)$. The theory of optimal one-parameter subgroups due to Kempf [4] and which is a refinement of the Hilbert-Mumford criterium to describe the nullcone can be used to obtain a remarkable stratification of the nullcones due to Hesselink [3]. For more details on the general theory we refer the reader to [18].

For arbitrary quiver-representations, the Hesselink stratification of the nullcone was studied in [10]. Again, we will describe the strata by associating to each potential stratum a new quiver situation. The question whether the stratum is non-empty is then rephrased into a representation theoretic problem for which an algorithm exists using the work of A. Schofield [16].

In general, the combinatorics underlying the strata is rather complex [10]. For the quivers Q_τ describing the étale local structure we can simplify things considerably primarely due to the fact that every relevant weight occurs in the weight space decomposition of $R(Q_\tau, \alpha_\tau)$ which in turn is a consequence of the fact that Q_τ is a symmetric quiver. In the terminology of [3] and [10] the main contrast with the general case considered in [10] is that every balanced coweight determines a saturated subset and hence a potential stratum. Here, we will not go into these definitions, but outline the required combinatorics from a practical point of view.

From now on, we fix a representation type $\tau = (e_1, k_1; \dots; e_r, k_r)$ with corresponding quiver Q_τ and dimension vector α_τ and we want to stratify the nullcone of the quiver-representations $N(Q_\tau, \alpha_\tau)$.

Denote $\sum e_i = z \leq n$ and $\mathscr{S}_z$ the set of all $s = (s_1, \dots, s_z) \in \mathbb{Q}^z$ which are disjoint union of strings of the form

$$\{p_i, p_i + 1, \dots, p_i + l_i\}$$

where $l_i \in \mathbb{N}$, all intermediate numbers $p_i + j$ with $j \leq l_i$ do occur as components in s with multiplicity $a_{ij} \geq 1$ and satisfy the condition

$$\sum_{0 \leq j \leq l_i} a_{ij}(p_i + j) = 0$$

for every string in s. For given z one can describe the set $\mathscr{S}_z$ easily. For fixed $s \in \mathscr{S}_z$ we distribute the s_i over the vertices (e_j of them to vertex v_j) in all possible ways modulo the action of the Weyl group $S_{e_1} \times \dots \times S_{e_r}$. That is, we can rearrange the

s_i's belonging to one vertex such that they are in decreasing order. This gives us a list $\mathcal{SH}_\tau$ which describes the potential strata. For example, if $\tau = (2,1;1,1)$ for m-tuples of 3×3-matrices with associated quiver Q_τ described before, then we have for $\mathcal{SH}_\tau$

s	s_1	s_2	s_3
a	1	0	-1
b	0	-1	1
c	1	-1	0
d	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$
e	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$
f	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
g	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
h	$\frac{1}{2}$	0	$-\frac{1}{2}$
i	0	$-\frac{1}{2}$	$\frac{1}{2}$
j	$\frac{1}{2}$	$-\frac{1}{2}$	0
k	0	0	0

Fix a maximal torus T_z in $GL(\alpha_\tau)$ and decompose the space $R(Q_\tau,\alpha_\tau)$ into weightspaces with respect to it

$$R(Q_\tau,\alpha_\tau) = \oplus_{\pi\in\mathbb{Z}^z} R(\tau)_\pi$$

For given $s \in \mathcal{SH}_\tau$ we can consider the subspaces

$$Y_s = \oplus_{\pi:(\pi,s)\geq 1} R(\tau)_\pi \text{ and } X_s = \oplus_{\pi:(\pi,s)=1} R(\tau)_\pi$$

where $(\pi,s) = \sum \pi_i s_i$. Then, the projection map

$$\chi : Y_s \mapsto X_s$$

is a vectorbundle, the associated parabolic subgroup $P_s = \oplus_{(\pi,s)\geq 0} GL(\alpha_\tau)_\pi$ acts on Y_s and its Levi-subgroup $L_s = \oplus_{(\pi,s)=0} GL(\alpha_\tau)_\pi$ acts on X_s. There is a Zariski open (but possibly empty) subset V_s of X_s consisting of those points for which the one parameter subgroup corresponding to s is optimal (see [17] or [10] for details). The Hesselink stratification of the nullcone $N(Q_\tau,\alpha_\tau)$ is given by the locally closed smooth irreducible subvarieties

$$St(s) = GL(\alpha_\tau).U_s$$

where $U_s = \chi^{-1}(V_s)$ for those s such that $V_s \neq \emptyset$.

In [10] an algorithm is given to determine the sublist $\mathcal{SH}'_\tau$ of those s such that $U_s \neq \emptyset$. To do this we associate to each $s \in \mathcal{SH}$ a new quiver which we will call $Q(\tau)_s$. It is a finite subquiver of the infinite quiver Γ with vertices $\gamma_v = (Q_\tau)_v \times \mathbb{Z}$ and arrows : for each $\varphi : t_\varphi \longrightarrow h_\varphi$ in Q_τ there are $\mathbb{Z}$ arrows in Γ

$$\varphi_n : (t_\varphi, n) \longrightarrow (h_\varphi, n+1)$$

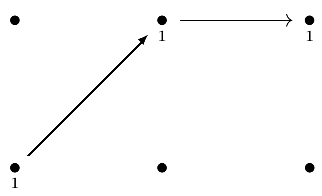
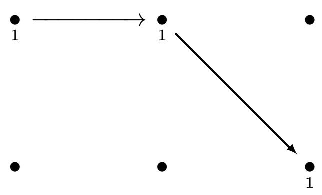
To find the dimension vector α_s for this subquiver $Q(\tau)_s$ decompose s in its disjoint strings

$$\{\underbrace{p_i, \ldots, p_i}_{a_0}, \underbrace{p_i+1, \ldots, p_i+1}_{a_1}, \ldots, \underbrace{p_i+k_i, \ldots, p_i+k_i}_{a_{k_i}}\}$$

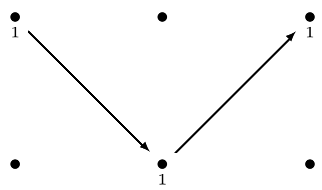
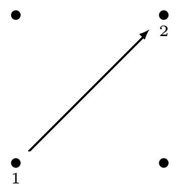
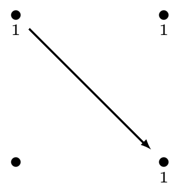
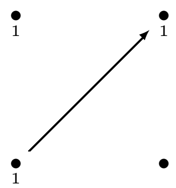
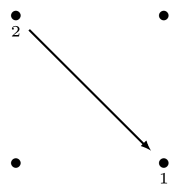
and for each segment i take a part of Γ consisting of k_i+1 columns say starting at integer t_i separated from the parts belonging to the other segments. The dimension vector for $(v, t_i + j)$ is the number of times $s_k = p_i + j$ belongs to vertex v.

We will illustrate this procedure in the case $\tau = (2,1;1,1)$. In the following table we give for every $s \in \mathcal{SH}_\tau$ the corresponding quiver $Q(\tau)_s$, the under-indices of the vertices give the components of the dimension vector α_s, a diagonal arrow stands for a collection of $m-1$ such arrows, a horizontal arrow for a collection of m such arrows, as is clear from the structure of the quiver Q_τ. In the final column we give the moduli-spaces which will be defined later.

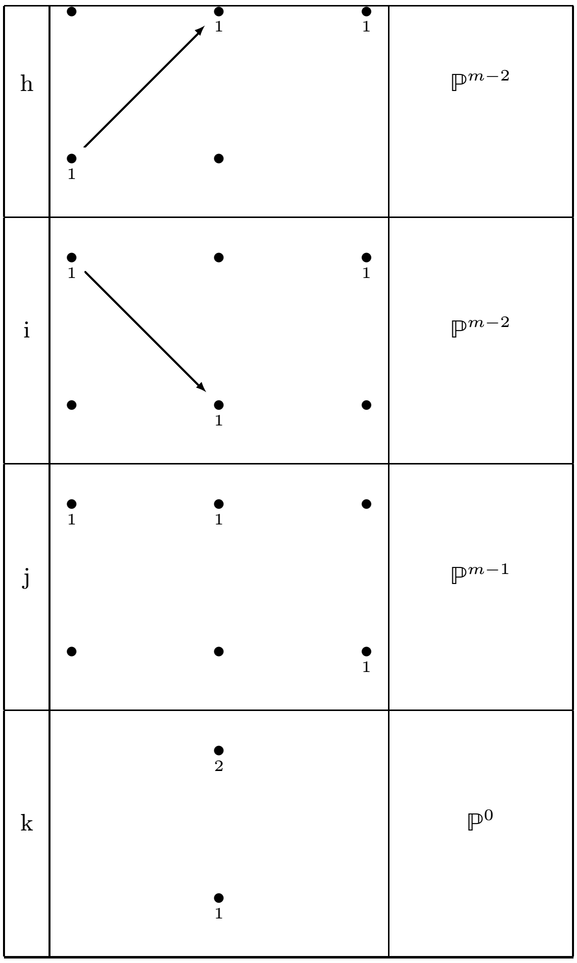
strata-quivers for $\tau = (2,1;1,1)$

s	$(Q(\tau)_s, \alpha_s)$	moduli
a		$\mathbb{P}^{m-2} \times \mathbb{P}^{m-1}$
b		$\mathbb{P}^{m-2} \times \mathbb{P}^{m-1}$

strata-quivers for $\tau = (2,1;1,1)$

c		$\mathbb{P}^{m-2} \times \mathbb{P}^{m-2}$
d		
e		$\mathbb{P}^{m-2} \times \mathbb{P}^{m-1}$
f		$\mathbb{P}^{m-2} \times \mathbb{P}^{m-1}$
g		

strata-quivers for $\tau = (2,1;1,1)$



In addition to assigning the quiver-situation $(Q(\tau)_s, \alpha_s)$ to a potential stratum s we will associate to it a character χ_s which is determined by associating to the vertex $(v, t_i + j)$ the number $n_{ij} = d.(p_i + j)$ where d is the least common multiple of the numerators of the p_k's determining the strings of s.

Above we have seen that the Hesselink stratum corresponding to s is nonempty if and only if $V_s \neq \emptyset$ and this is the open subset of the level-quiver representations $R(Q(\tau)_s, \alpha_s)$ for which a semi-invariant corresponding to the character χ_s does not vanish.

One of the advantages of reducing to this quiver situation is that we can view points of $R(Q(\tau)_s, \alpha_s)$ as objects in the Abelian category of all representations of

$Q(\tau)_s$, that is, the category of modules over the path algebra $\mathbb{C}Q(\tau)_s$. Therefore, we can associate to the character χ_s (which is determined by the integers (n_{ij}) defined above) an additive function on the Grothendieck group of the path algebra

$$\theta_s : K_0(\mathbb{C}Q(\tau)_s) \longrightarrow \mathbb{Z}$$

which is determined by sending the class of a representation of dimension-vector $\beta = (b_{ij})$ to $\sum n_{ij} b_{ij}$.

Using the analogy with vector bundles on projective varieties, A. King [5] defines a representation V of $Q(\tau)_s$ to be θ-semistable (for any additive function θ on the Grothendieck group) if $\theta(V) = 0$ and every sub-representation $V' \subset V$ satisfies $\theta(V') \geq 0$. Similarly, a representation V is called θ-stable if the only subrepresentations V' with $\theta(V') = 0$ are 0 and V. Using [5, Prop.3.1] we then have

Proposition 3 — *V_s is the open subset of $R(Q(\tau)_s, \alpha_s)$ which are θ_s-semistable.*

Hence, in order to verify whether $x \in R(Q(\tau)_s, \alpha_s)$ lies in V_s it suffices to know the dimension vectors of all subrepresentations of x and verify that their values under θ_s are ≥ 0. If $V_s \neq \emptyset$ it is open in $R(Q(\tau)_s, \alpha_s)$ and it suffices to know the dimension vectors of subrepresentations of a general representation.

Precisely this problem had to be addressed by A. Schofield [16] in his solution of some conjectures of V. Kač on the generic decomposition. Recall that V. Kač showed [2] that the dimension vectors of indecomposable quiver-representations form an infinite root system with associated generalized Cartan matrix the symmetrization of the Ringel form or the Euler inner product. This form encodes a lot of information on representations. If V resp. W are representations of dimension-vector α resp. β then

$$\epsilon(\alpha, \beta) = \dim \operatorname{Hom}(V, W) - \dim \operatorname{Ext}(V, W)$$

For fixed dimension vector β and any quiver Q, there is an open subset of representations V in $R(Q, \beta)$ such that the dimension vectors of its indecomposable components are constant, say β_i. Then,

$$\beta = \beta_1 + \dots + \beta_l$$

is called the canonical decomposition of β into Schur roots β_i (Schur roots are roots γ such that there is an open set of indecomposable representations in $R(Q, \gamma)$).

Kač asked for a combinatorial description of the set of Schur roots and of the canonical decomposition in terms of the Ringel form. Solutions to these problems were presented by A. Schofield [16] and depend heavily on being able to describe the dimension vectors of sub-representations of a general representation. Denote with

$$\beta \hookrightarrow \alpha$$

that a general representation of dimension-vector α has a sub-representation of dimension-vector β. Schofield gave an inductive way to find the dimension-vectors of these generic sub-representations using the Ringel form

$$\beta \hookrightarrow \alpha \quad \text{iff} \quad \max_{\beta' \hookrightarrow \beta} - \epsilon(\beta', \alpha - \beta) = 0$$

For example, the description of the Schur roots [16, Th.6.1] is then : α is Schur iff for all $\beta \hookrightarrow \alpha$ we have $\epsilon(\beta, \alpha) - \epsilon(\alpha, \beta) > 0$. A combinatorial description of the canonical decomposition was also given in [16].

These facts enable us to give the promised algorithmic description of the actually occuring strata in the Hesselink stratification of $N(Q_\tau, \alpha_\tau)$. For example, we can use this algorithm to show that in our example of $\tau = (2,1;1,1)$ all strata do occur when $m \geq 3$ and the only types which give empty strata for $m = 2$ are types d and g. A similar phenomen also happens in the general case, for m sufficiently large all potential strata will indeed occur.

Having determined which strata make up the nullcone

$$N(Q_\tau, \alpha_\tau) = \cup_{s \in \mathcal{S}\mathcal{H}'_\tau} St(s)$$

we still have to determine the $GL(\alpha_\tau)$-orbitstructure of one such stratum $St(s)$. From [3, Th.4.7] we deduce the existence of a natural morphism

$$GL(\alpha_\tau) \times^{P_s} V_s \longrightarrow St(s)$$

which is an isomorphism of $GL(\alpha_\tau)$-varieties. Hence, the stratum $St(s)$ is an open subvariety of a vectorbundle over the flag variety $GL(\alpha_\tau)/P_s$. Further, there is a natural one-to-one correspondence between the $GL(\alpha_\tau)$-orbits in $St(s)$ and the P_s-orbits in U_s. Moreover, under the natural projection map

$$U_s \xrightarrow{\chi} V_s \hookrightarrow R(Q(\tau)_s, \alpha_s)$$

points lying in the same P_s-orbit in U_s are mapped to points lying in the same $GL(\alpha_s)$ orbit in V_s. Therefore, we have an induced projection map

$$\mathrm{Orb}(P_s, U_s) \xrightarrow{\chi} M(Q(\tau)_s, \alpha_s; \theta_s)$$

from the orbit-space of U_s under P_s to the 'moduli' space of θ_s-semi stable representations of Q_s of dimension-vector α_s, see [5] for some results on these moduli spaces. We will mean in this section by $M(Q(\tau)_s, \alpha_s; \theta_s)$ the orbit-space of V_s under action of $GL(\alpha_s)$. Some easy examples of moduli spaces were given in the table above. The moduli spaces for the types d and g in that table are more complex, they are $\mathbb{P}^0$ when $m = 3$ and moduli spaces of certain Grassman varieties under action of GL_2 for higher m.

Recapitulating our discussion, we have the following procedure to determine the orbits of m-tuples of $n \times n$ matrices under simultaneous conjugation :

- Stratify the quotient variety $V_{m,n} = X_{m,n}//GL_n$ according to the different representation types τ into locally closed smooth irreducible subvarieties $V_{m,n}(\tau)$ and describe these by trace functions.
- For any point $\zeta \in V_{m,n}(\tau)$ the GL_n-orbit structure of the fiber $\pi^{-1}(\zeta)$ is in natural one-to-one correspondence with the $GL(\alpha_\tau)$-orbit structure of the nullcone $N(Q_\tau, \alpha_\tau)$ of the quiver situation describing the étale local structure of $V_{m,n}$ near ζ.
- The nullcone has a Hesselink stratification in locally closed irreducible smooth subvarieties $St(s)$ where s belongs to a finite list $\mathcal{SH}'_\tau$ which can be obtained from studying the semi-stable representations for a specific character θ_s and quiver situation $(Q(\tau)_s, \alpha_s)$ associated to a potential strata s.
- The $GL(\alpha_\tau)$-orbits in such a stratum $St(s)$ are in natural one-to- one correspondence with the orbits of the associated parabolic subgroup P_s in V_s and we have a projection morphism $\mathrm{Orb}(P_s, V_s) \mapsto M(Q(\tau)_s, \alpha_s; \theta_s)$ to the moduli space of the associated quiver situation.
- Study the structure of these moduli spaces using representation theory of quivers and finally describe the fibers of the projection map. This last problem is open and probably very hard except for small values of n.

As an easy application of the above methods let us study the orbits of m-tuples of 2×2 matrices. To the best of my knowledge only the case of couples of 2×2 matrices has been studied in the literature [1] and [7].

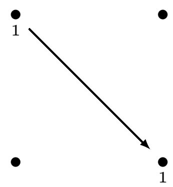
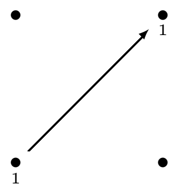
There are three representation types with the following local quiver situations

type	τ	(Q_τ, α_τ)
a	$(1,2)$	$\overset{(4m-3)}{\underset{1}{\bullet}}$
b	$(1,1;1,1)$	$\overset{(m)}{\underset{1}{\bullet}} \overset{m-1}{\text{———}} \overset{(m)}{\underset{1}{\bullet}}$
c	$(2,1)$	$\overset{(m)}{\underset{2}{\bullet}}$

Type a has only one potential stratum corresponding to $s = (0)$ and with associated quiver situation

which obviously has just $\mathbb{P}^0$ as moduli space. This corresponds to the fact that these points of $V_{m,2}$ have a unique closed orbit as their fiber.

For type b the following list gives us the potential strata and associated quiver-situations together with their moduli spaces. An arrow denotes $m-1$ directed arrows between the indicated vertices.

s_1	s_2	$(Q(\tau)_s, \alpha_s)$	M
$\frac{1}{2}$	$-\frac{1}{2}$		$\mathbb{P}^{m-2}$
$-\frac{1}{2}$	$\frac{1}{2}$		$\mathbb{P}^{m-2}$
0	0		$\mathbb{P}^0$

In this case $\mathscr{SH}_\tau = \mathscr{SH}'_\tau$ for all $m \geq 2$, that is all potential strata do indeed occur. Moreover, in these cases $U_s = V_s$ and so the required orbits $\mathrm{Orb}(P_s, U_s) = M(Q(\tau)_s, \alpha_s, \theta_s)$ which easily can be seen to be the indicated projective spaces. Hence, for $\zeta \in V_{m,2}(1,1;1,1)$ the fiber $\pi^{-1}(\zeta)$ consists of the unique closed orbit (corresponding to the $\mathbb{P}^0$) and two families $\mathbb{P}^{m-2}$ of non-closed orbits. In the $m = 2$ case studied by Kraft and Friedland there are two non-closed orbits in the fiber.

Finally, for $\tau = (2,1)$, the fiber is isomorphic to the nullcone of m-tuples of 2×2 matrices. We have the following strata-information (the arrow denotes m directed arrows)

s_1	s_2	$(Q(\tau)_s, \alpha_s)$	M
$\frac{1}{2}$	$-\frac{1}{2}$	$\underset{1}{\bullet} \longrightarrow \underset{1}{\bullet}$	$\mathbb{P}^{m-1}$
0	0	$\underset{2}{\bullet}$	$\mathbb{P}^0$

Hence, the fiber $\pi^{-1}(\zeta)$ consists of the closed orbit together with a $\mathbb{P}^{m-1}$-family of non-closed orbits. Again, we recover the $\mathbb{P}^1$ family of non-closed orbits in the $m = 2$ case found in [7] and [1].

Also the case of m tuples of 3×3 matrices can be fully worked out. We leave the details to the interested reader and mention here only the results

- For type 1 points the fiber consists of one orbit.
- For type 2 points the fiber consists of the closed orbit together with two $\mathbb{P}^{2m-3}$ families of non-closed orbits.
- For type 3 points the fiber consists of the closed orbit together with twelve $\mathbb{P}^{m-2} \times \mathbb{P}^{m-2}$ families and one $\mathbb{P}^{m-2}$ family of non-closed orbits.
- For type 4 points we have described the relevant data before.
- For type 5 points we have to study the nullcone of m-tuples of 3×3 matrices for which we refer to [11].

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