From General Relativity to Group Representations

The Background to Weyl's Papers of 1925–26

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Abstract

Hermann Weyl's papers on the representation of semisimple Lie groups (1925-26) stand out as landmarks of twentieth century mathematics. The following essay focuses on how Weyl came to write these papers. It offers a reconstruction of his intellectual journey from intense involvement with the mathematics of general relativity to that of the representation of groups. In particular it calls attention to a 1924 paper by Weyl on tensor symmetries that played a pivotal role in redirecting his research interests. The picture that emerges illustrates how Weyl's broad philosophically inclined interests inspired and informed his creative work in pure mathematics.

Résumé

Les articles de Hermann Weyl sur la représentation des groupes de Lie semi-simples (1925-26) apparaissent comme des étapes majeures des mathématiques du vingtième siècle. En analysant ce qui a amené Weyl à écrire ces articles, cet essai présente une reconstruction de sa démarche intellectuelle, depuis les mathématiques de la relativité générale jusqu'à celles des représentations de groupes. Il attire notamment l'attention sur l'article de 1924 sur les symétries tensorielles, pivot de la réorientation de ses domaines de recherche. On voit aussi comment les larges intérêts et les motivations philosophiques de Weyl ont inspiré et enrichi sa créativité en mathématiques pures.

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Dieudonné once wrote that “progress in mathematics results, most of the time, through the imaginative fusion of two or more apparently different topics” [Dieudonné 1975, p. 537]. One of the most brilliant examples of progress by fusion is provided by Herman Weyl’s celebrated papers on the representation of semisimple Lie groups (1925-1926). For in them he fashioned a theory which embraced I. Schur’s recent work (1924) on the invariants and representations of the n-dimensional rotation group, which was conceived within the conceptual framework of Frobenius’ theory of group characters and representations, and E. Cartan’s earlier work (1894–1913) on semisimple Lie algebras, which was done within the framework of Lie’s theory of groups and had been unknown to Schur. Moreover, in fashioning his theory of semisimple groups, Weyl drew on a host of ideas from such historically disparate areas as Frobenius’ theory of finite group characters, Lie’s theory, tensor algebra, invariant theory, complex function theory (Riemann surfaces), topology and Hilbert’s theory of integral equations. Weyl’s papers were thus a paradigm of fusion, and they exerted a considerable influence on subsequent developments. They stand out as one of the landmarks of twentieth century mathematics.

It is not my purpose here to describe the rich contents of these remarkable papers nor to analyze their influence. This has been done by Chevalley and Weil [1957], by Dieudonné [1976], and, above all, by Borel [1986]. I wish to focus instead on how Weyl came to write these remarkable papers. In this connection Weyl wrote:

> “for myself I can say that the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups ...”[Weyl 1949, p. 400].

My goal is to attempt to explain what Weyl meant by this remark, that is, to reconstruct the historical picture of his intellectual journey from his involvement with the mathematics of general relativity to that of the representation of semisimple Lie groups. In particular, I want to call attention to a paper by Weyl [1924a], which in my opinion adds a fullness and clarity to the picture that would otherwise be lacking. The picture that emerges illustrates how Weyl’s broad philosophically inclined interests — in this instance in theoretical physics — inspired and informed his creative work in pure mathematics.[1]

[1] For another such instance, see [Scholz 1995] where Weyl’s interest in Fichte’s philosophy is related to his approach to the geometry of manifolds.

The Space Problem

Weyl's involvement with general relativity began in 1916, when, at age 31, he returned from military service to his position at the Eidgenössische Technische Hochschule (ETH) in Zürich. "My mathematical mind was as blank as any veteran's," he later recalled, "and I did not know what to do. I began to study algebraic surfaces; but before I had gotten far, Einstein's memoir came into my hand and set me afire."[2] By the summer of 1917 Weyl was lecturing on general relativity at the ETH. These lectures formed the starting point for his classic book, *Raum, Zeit, Materie*, which went through four editions during 1918–23,[3] and spawned many collateral publications by Weyl aimed at further developing the ideas and implications of his lectures. One of the outcomes of Weyl's reflections on general relativity was his introduction of what he called a "purely infinitesimal geometry."[4]

Weyl became convinced that Riemannian geometry, including the quasi Riemannian geometry of an indefinite metric $ds^2 = \sum_{ij} g_{ij} dx_i dx_j$, $g_{ij} = g_{ij}(x_1, \ldots, x_n)$, on which Einstein's theory was based, was not a consistently infinitesimal geometry. That is, in Riemannian geometry, a vector $v = (dx_1, \ldots, dx_n)$ in the tangent plane at point P of the manifold could only be compared with a vector $w = (dy_1, \ldots, dy_n)$ in the tangent plane at point Q in the relative sense of a path-dependent parallel transport from P to Q, but the lengths of v and w were absolutely comparable in the sense that

$$\frac{|v|}{|w|} = \sqrt{\frac{\sum_{i,j} g_{ij}(P) dx_i dx_j}{\sum_{i,j} g_{ij}(Q) dy_i dy_j}}.$$

These considerations led Weyl to a generalization of Riemannian geometries in which the lengths of v and w are not absolutely comparable. As in Riemannian geometry a nondegenerate quadratic differential form ds^2 of constant signature is postulated but metric relations are determined locally only up to a positive calibration (or gauge) factor λ and so are given by $ds^2 = \sum_{ij} \lambda g_{ij} dx_i dx_j$. Here λ varies from point to point in such a way that the comparison of the lengths of v at P and w at Q is also in general a path-dependent process.[5]

[2]Quoted by S. Sigurdsson [1991, p. 62] from Weyl's unpublished "Lecture at the Bicentennial Conference" (in Princeton).

[3]There were actually five editions, but the second (1919) was simply a reprint of the first [Scholz 1994, p. 205n].

[4]See Scholz [1994, 1995] for a detailed account of the historical context and evolution of Weyl's ideas on this theory during 1917–23 .

[5]For a complete definition of Weyl's geometry see [Scholz 1994, p. 213] and for a contemporary formulation see [Folland 1970]. Weyl's geometry represented the first of a succession of gauge theories that has continued into present-day physics [Vizgin 1989, p. 310].

Although Weyl's geometry was motivated by the above critique of Riemannian geometry, he discovered that he could use its framework to develop a unified field theory, that is, a theory embracing both the gravitational and the electromagnetic field. Hilbert had been the first to devise a unified theory within the framework of general relativity in 1915. Weyl's theory was presented in several papers during 1918–19 and in the third edition (1919) of *Raum, Zeit, Materie*. Einstein admired Weyl's theory for its mathematical brilliance, but he rejected it as physically impossible. Although Weyl respected Einstein's profound physical intuition and was accordingly disappointed by the negative reaction to his unified theory, Einstein's arguments did not convince him that his own approach was wrong. His belief in the correctness of his theory was bolstered by the outcome of his reconsideration, in publications during 1921–23, of the "space problem" first posed by Helmholtz in 1866. It was in connection with this problem that Weyl first began to appreciate the value of group theory for investigating questions of interest to him involving the mathematical foundations of physical theories.

In 1866 Helmholtz sought to deduce the geometrical properties of space from facts about the existence and motion of rigid bodies. He concluded that the distance between points (x, y, z) and $(x + dx, y + dy, z + dz)$ is $\sqrt{dx^2 + dy^2 + dz^2}$ and that space is indeed Euclidean. He returned to the matter in 1868, however, after learning from the work of Riemann and Beltrami about geometries of constant curvature. Using the properties of rigid bodies he had singled out earlier, he argued that Riemann's hypothesis that metric relations are given locally by a quadratic differential form is actually a mathematical consequence of these facts. Later, in 1887, Poincaré obtained Helmoltz's results for two-dimensional space by applying Lie's theory of groups and utilizing, in particular, the consideration of Lie algebras. Lie himself considered the problem in n dimensions by means of the consideration of Lie groups and algebras in 1892. The Lie-Helmholtz treatment of the space problem, however, was rendered obsolete by the advent of general relativity since, as Weyl put it:

> "Now we are ... dealing with a four-dimensional [continuum] with a metric based not on a positive definite quadratic form but rather one that is indefinite. What is more, we no longer believe in the metric homogeneity of this medium — the very foundation of the Helmholtzian metric — since the metric field is not something fixed but rather stands in causal dependency on matter" [Weyl 1921a, p. 263].

Following the Helmholtz-Lie tradition, Weyl conceived of space (including therewith the possibility of space-time) as an n-dimensional differentiable manifold $\mathcal{M}$ with metric relations determined by the properties of congruences which are conceived in terms of groups. Thus at each point $P \in \mathcal{M}$ the rotations at P are assumed to form a continuous group of linear transformations $\mathfrak{G}_P$, and since the volume of parallelepipeds is assumed to be preserved by rotations, the $\mathfrak{G}_P$ are taken as subgroups of $\mathbf{SL}\ (T_P(\mathcal{M}))$. Metrical relations in a neighborhood $\mathcal{U}$ of P are then based on the assumption that all rotations at $P' \in \mathcal{U}$ can be obtained from a single linear congruence transformation A taking P to P' by composition with the rotations at P; that is, every $T' \in \mathfrak{G}_{P'}$ is of the form $T' = ATA^{-1}$ so that $\mathfrak{G}_{P'} = A\mathfrak{G}_P A^{-1}$. By "passing continuously" from P to any point Q of the manifold $\mathcal{M}$, Weyl was led to the assumption that all the groups $\mathfrak{G}_P$ are congruent to a group $\mathfrak{G} \subset \mathbf{SL}(n)$ with Lie algebra $\mathfrak{g} \subset \mathbf{sl}(n)$. Thus, whereas in the Lie-Helmholtz treatment of the space problem, the homogeneity of space entails the identity of the rotation groups at diverse points, in Weyl's formulation the rotation groups have differing "orientations," although they share the same abstract Lie algebra.

Within this mathematical context Weyl stipulated two postulates: (1) the nature of space imposes no restriction on the metrical relationship; (2) the affine connection is uniquely determined by the metrical relationship. His interesting mathematical interpretation of these two postulates led to the conclusion that the Lie algebra $\mathfrak{g}$ must possess the following properties:

a) For all $X \in \mathfrak{g}$, tr $X = 0$ (*i.e.*, $\mathfrak{g} \subset \mathbf{sl}(n, \mathbb{R})$);

b) dim $\mathfrak{g} = \frac{1}{2}n(n-1)$;

c) For any $X_1, \ldots, X_n \in \mathfrak{g}$ with matrix form $X_k = (a_{ij}^{(k)})$ with regard to some basis, if Col i of X_j = Col j of X_i for all $i, j = 1, ..., n$, then $X_i = 0$ for all $i = 1, ..., n$.

In the fourth edition of *Raum, Zeit, Materie*, where Weyl first presented his analysis of the space problem [Weyl 1921a, §18], he pointed out that the Lie algebras $\mathfrak{g}_Q$ of all orthogonal groups with respect to a nonsingular quadratic form Q satisfy (a)–(c) and he conjectured as "highly probable" the following theorem which he had confirmed for $n = 2, 3$:

Theorem 1. — *The only Lie algebras satisfying (a)–(c) are the orthogonal Lie algebras $\mathfrak{g}_Q$ corresponding to a nondegenerate quadratic form Q.*

Weyl's conjectured theorem thus implied the locally Pythagorean nature of space. Weyl pointed out that when $\mathfrak{g}$ does correspond to an orthogonal Lie algebra, the quadratic form Q is only determined up to a constant of proportionality [Weyl 1921a, p. 146]. Although he did not say it explicitly at this point, the truth of Theorem 1 would thus imply that his generalization of quasi-Riemannian geometry, his purely infinitesimal geometry, was also compatible with the conclusions of his analysis of the space problem.

Within a few months of completing the fourth edition of *Raum, Zeit, Materie*, Weyl had obtained a proof of Theorem 1, which he submitted for publication in April 1921 [Weyl 1921b] and announced in a general talk in September 1921 [Weyl 1922]. With the proof of Theorem 1 his analysis of the space problem was complete. Weyl saw it as confirmation of the legitimacy of his geometrical approach to relativity theory — his purely infinitesimal geometry with its concomitant unified field theory. He was also mindful of the fact that it had been achieved by utilizing the theory of groups: "The establishment by group theory is hence a new support for my conviction that this geometry, as geometry of the world, is the basis for the interpretation of physical field phenomena, rather than, as with Einstein, the more restrictive Riemannian [geometry]" [Weyl 1922, p. 344]. Indeed, Weyl was so taken up with Theorem 1 that he even likened the "confirmation by logic" of the correctness of his approach to the space problem afforded by Theorem 1 to the factual confirmation of the correctness of Einstein's relativistic approach to gravitation afforded by the observed advance of the perihelion of mercury [Weyl 1921b, p. 269].

During the spring of 1922 Weyl lectured on the space problem in Spain, and a version of his lectures was then published as a monograph [1923a], which he regarded as a supplement to *Raum, Zeit, Materie* since "the deeper conception of the space problem using group theory" was only sketched there. In the eighth lecture, which sketches a proof of Theorem 1, Weyl wrote:

> "While almost all deeper mathematical theories — such as, *e.g.*, the wonderful theory of algebraic number fields — have little to signify within the great philosophical continuum of knowledge and while, on the other hand, what mathematics can contribute to enlighten the general problem of knowledge mostly stems from the surface of mathematics, here we have the rare case that a problem which is fundamental to all knowledge of reality, as is the space problem, gives rise to deeply penetrating mathematical questions."[Weyl 1923a, p. 61]

Within the context of the space problem Weyl had discovered group theory as a powerful tool for dealing with fundamental questions inspired by general relativity and leading to "deeply penetrating mathematical questions." Although he described it as a rare occurrence, as we shall see, this was not the last time that his involvement with the fundamentals of general relativity led to important mathematical questions of a group-theoretic nature and, ultimately, to his papers on the representation of Lie groups.

Before proceeding to consider these further occurrences, however, there is one additional, historically important, consequence of Weyl's involvement with the space problem that needs to be mentioned. In 1922 the fourth edition of *Raum, Zeit, Materie* was translated into French and read by Elie Cartan, who, since 1921, had become interested in Einstein's theory. Unaware that Weyl had already proved the conjectured Theorem 1, Cartan provided a proof of his own [Cartan 1922]. Strictly speaking, Cartan did not prove Theorem 1. Instead, he reformulated Weyl's somewhat vaguely articulated version of the space problem in terms of his own approach to geometry based on moving frames and differential forms. Cartan's approach evolved into the modern theory of G-structures.[6] Within that framework, however, Cartan's formulation of the space problem ultimately reduced to the problem of determining all linear Lie algebras $\mathfrak{g}$ satisfying Weyl's conditions (a) and (b) and, in lieu of the rather mysterious condition (c), the condition that $\mathfrak{g}$ leaves no proper subspaces invariant.[7] By a theorem Cartan had proved in [Cartan 1909, p. 912] it followed that $\mathfrak{g}$ must be semisimple. Since Cartan had already determined all such linear Lie algebras which leave no vector spaces invariant [Cartan 1913, 1914], it was, as he noted, just a matter of checking which of these Lie algebras satisfy the dimension condition (b), to arrive at Weyl's conclusion that $\mathfrak{g}$ must be an orthogonal Lie algebra.[8]

Expressed in modern terms, what Cartan had done in [Cartan 1913] was to determine all irreducible representations of a complex semisimple Lie algebra, and in [Cartan 1914] he did the same for real Lie algebras. However Cartan did not conceive of his work within the conceptual framework of group representations. He conceived of his work as solving the problem of determining all groups of projective transformations which "leave nothing planar" invariant — a problem of importance from the Kleinian view of geometry as the study

[6]See in this connection [Scheibe 1988, p. 66] and [Scholz 1994, p. 225]. Scheibe argues that if what Weyl had in mind is made more precise in accord with what his writings seem to suggest, then it is not equivalent to Cartan's formulation, but the theorem Cartan proved implies the theorem Scheibe reconstructs from Weyl's vague statements [Scheibe 1988, pp. 68–69].

[7]This property of the $\mathfrak{g}$ satisfying (a)–(c) of Weyl's Theorem 1 actually follows readily from propositions Weyl deduced from (a)–(c) [Weyl 1923a,c], although he did not expressly take note of this fact.

[8]Both Cartan and Weyl realized that it suffices to consider the problem for complex Lie algebras. In his announcement Cartan indicated that a detailed solution of the problem in a generalized form was contained in [Cartan 1923].

and classification of groups acting on manifolds. Historically the conceptual framework of group representations and characters came from Frobenius' theory as developed for finite groups during 1896–1903, and it was Weyl who first brought Cartan's work within that framework in his papers of 1925–26.

Weyl learned of Cartan's work when the latter sent him his announcement [Cartan 1922] of a solution to the space problem. In Weyl's reply, dated October 5, 1922, after pointing out that he had already given a proof of Theorem 1, he wrote:

> "Untraveled on the paved roads of the general theory of continuous groups, which have been laid out and constructed thanks to your masterly skill, I have on my own beat a steep inconvenient footpath through much underbrush to my goal. I have no doubt that your method corresponds better to the nature of the matter; still, I see that you also cannot arrive at the goal without distinguishing many cases."[9]

The general consensus seems to be that Weyl, impressed by Cartan's papers on Lie algebras, studied them carefully and that this study, combined with an interest in the theory of invariants piqued by some critical remarks by the mathematician E. Study (discussed below) led, through the inspiration provided by a paper on invariants by I. Schur [1924] (also discussed below), to his celebrated papers of 1925–26 on the representation of Lie groups. While there is much truth in such a portrayal of events, it does overlook Weyl's deep seated, philosophically inclined interest in the mathematical foundations of theoretical physics; in particular, it fails to fully account for Weyl's own statement that "the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups. ..." Weyl's involvement with the space problem was certainly an instance of his interest in the mathematical substance underlying relativity theory, and it led him to E. Cartan's work. But the space problem was not the only focal point of this interest. In what follows, I will attempt to give a clearer notion of how other manifestations of his interest in finding the proper mathematical basis for the mathematical apparatus of general relativity increased his involvement with the theory of groups and, in particular, with the theory of their representations and how this in turn made Cartan's work all the more relevant.

[9] I am grateful to B.L. van der Waerden, who called these letters to my attention many years ago and sent me copies after obtaining consent of the holders — H. Cartan in the case of Weyl's letters and the ETH in the case of E. Cartan's letters.

In this connection, it should be kept in mind that Weyl's above-quoted reply to Cartan was written when he had only the original proof of his theorem, which he disliked because it was complicated and lacked an overall unifying idea [Weyl 1922, p. 344]. He compared it deprecatingly to tightrope dancing [Weyl 1921b, p. 269], and in his popular lectures on the space problem, including those in Spain in the spring of 1922, he declined for this reason to sketch the proof. By the spring of 1923, however, when he published a German version of the lectures in Spain, he included a proof (as Appendix 12) because he had obtained what he felt were "far reaching simplifications" to his original proof [Weyl 1923a, p. v] so that, although still complicated in detail due to the need to distinguish many cases, it was now guided by a single idea, which in fact Weyl pushed further in [Weyl 1923c], where he wrote in conclusion:

> "Our game on the chessboard of matrix schemes has been played to its end. As complicated in details as it may be, it — including the first part, which was already laid out in ...[Weyl 1923a] ... Append. 12 — rests ... upon a single constructional idea which determined each step and was tenaciously carried out to the end."

It is interesting to observe that in Weyl's presentation of his new proof, he used another "roadway" analogy in comparing his and Cartan's proofs, but now with a different slant: "By contrast with Cartan's proof mine does not take the detour of the investigation of abstract groups. It is based on the classical theory of the individual linear mapping going back to Weierstrass" [Weyl 1923a, p. 88]. So now Cartan's solution involves a "detour" because it draws upon the theory of semisimple Lie algebras, whereas Weyl's approach is more direct and elementary, depending only on "the classical theory" of the Jordan canonical form of a matrix implicit in Weierstrass' theory of elementary divisors.

That is not to say that Weyl did not appreciate by this time — early 1923 — the impressive results and deep theory developed by Killing and Cartan. Indeed, immediately prior to the above quotation, Weyl characterized Cartan's solution to the space problem by writing:

> "An entirely different proof has been given by Cartan ... based on [his] ... earlier comprehensive and deep investigations ... on the theory of continuous groups, in which he achieved a far reaching solution to the problem of determining all abstract groups and their realization through linear operations .... Now he only needed to seek out among the groups determined by him those which satisfy my stipulations."

These words indicate that Weyl certainly understood the gist of what Cartan had done in his papers and appreciated the profundity of the mathematics. But as far as the space problem was concerned, the extensive detour required by Cartan's approach was not deemed appropriate by Weyl, who was still fascinated by his own approach. It is not clear he had found reason enough to take on the nontrivial task of mastering the details of Cartan's papers so as to put them to his own use. Eventually he did — and he was perhaps the first mathematician to do so — but the motivation to do so seems to have come not from the space problem but from tensor algebra.

Tensor Algebra and Symmetries

The formal apparatus of relativity theory consisted in large part of the calculus of tensors. This apparatus had evolved out of the work of mathematicians, notably Christoffel, Lipschitz and Ricci, interested in developing the theory of the transformation of quadratic differential forms suggested by Riemann's speculations on the foundations of geometry.[10] The principal source of the resulting theory upon which Einstein and Grossman drew in developing the mathematical side of general relativity starting in 1913 was the monographic paper by Ricci and Levi-Civita, "Méthodes de calcul différentiel absolu et leurs applications" [1900], which more or less summed up what had been achieved during 1868–1900. To this they added the term "tensor," the notion of mixed tensors and (in 1916) Einstein's now-familiar summation convention, but essentially they took over the apparatus of the absolute differential calculus of Ricci and Levi-Civita.

In *Raum, Zeit, Materie*, Weyl also credited the Ricci-Levi-Civita paper [1900] for the systematic development of tensor calculus,[11] but it was he, who, drawing upon his Göttingen background, recast tensor calculus in its essentially modern form. For one thing, Weyl treated tensor algebra — tensor analysis in a fixed tangent plane — independently as a preliminary to tensor analysis, and in developing tensor algebra he did so within the geometrically flavored context of vector spaces, which had grown out of Hilbert's work on integral equations as developed by Erhard Schmidt. It is within the context of tensor algebra as developed by Weyl in the pages of *Raum, Zeit, Materie* that the formal apparatus of relativity theory gave rise to fundamental mathematical questions. As I will attempt to show, Weyl's concern with these questions was a major factor in the considerations that ultimately led to his papers of

[10]The history of the tensor calculus from its origins in up to its application to general relativity is traced in [Reich 1994].

[11]See note 4 to p. 53 of the fourth edition [Weyl 1921a].

1925–26. To make these questions intelligible I will first sketch out the basics of Weyl's approach to tensors.

Let $\mathcal{V}$ denote an n-dimensional vector space over the real or complex field equipped with a nondegenerate quadratic form $Q(v, w)$, $v, w \in \mathcal{V}$ defining a scalar product.[12] Then if $e_1, \ldots, e_n$ is a basis for $\mathcal{V}$ we may express $v \in \mathcal{V}$ in the form

$$v = \sum_{i=1}^{n} x^i e_i. \tag{1}$$

The x^i are called *contravariant* coordinates of v since if $\bar{e}_1, \ldots, \bar{e}_n$ is another basis related to the first by

$$\bar{e}_i = \sum_{k=1}^{n} m_i^k e_k, \tag{2}$$

then if M denotes the matrix with (i, k) entry m_i^k, we have $v = \sum_{i=1}^{n} \bar{x}^i \bar{e}_i$ where $x^i = \sum_{k=1}^{n} m_k^i \bar{x}^k$ so that, expressed in matrix form (which Weyl did not use)

$$\bar{x} = \left(M^T\right)^{-1} x. \tag{3}$$

The vector v is also uniquely determined by the n values $y_i = Q(v, e_i)$, which are called *covariant* coordinates of v with respect to the basis $e_1, \ldots, e_n$ since they transform according to

$$\bar{y} = My, \tag{4}$$

and thus with the same coefficient matrix as in the basis change (2). Nowadays the y_i would be regarded as coordinates of the element v^* in the dual space $\mathcal{V}^*$ defined by $v^*(w) = Q(v, w)$. That is, the y_i are the coordinates of v^* with respect to the basis $e^1, \ldots, e^n$ of $\mathcal{V}^*$ dual to $e_1, \ldots, e_n$.

For Weyl tensors are uniquely determined by multilinear forms. For example, the mixed tensor of rank 3 denoted by T_{ij}^k by Einstein and covariant in the indices i, j and contravariant in the index k is conceived by Weyl as determined by a multilinear form $f = f(u, v, w)$, where if x^i and y^j are the contravariant coordinates of u and v respectively and z_k the covariant coordinates of w, then

$$f = \sum_{i,j,k} T_{ij}^k x^i y^j z_k. \tag{5}$$

[12]Weyl does not speak of $\mathcal{V}$ as a vector space but rather as an n-dimensional affine space. Also of course Q is not necessarily positive definite.

In view of the remark following (4), f may be regarded as a multilinear form on $\mathcal{V}\times\mathcal{V}\times\mathcal{V}^*$ with $w^*=\sum_k y_k e^k$ from which (5) follows with $T_{ij}^k = f(e_i, e_j, e^k)$. Thus Weyl in effect identified the above type tensors with the vector space $\mathcal{L}$ of all such multilinear forms f, which agrees with the present-day formulation according to which

$$\mathcal{L} \cong (\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}^*)^* \cong \mathcal{V}^*\otimes\mathcal{V}^*\otimes\mathcal{V}. \tag{6}$$

In view of (6) the reader may wish to identify the tensor defined by (5) with the element

$$\sum_{i,j,k} T_{ij}^k\, e^i\otimes e^j\otimes e_k \in \mathcal{V}^*\otimes\mathcal{V}^*\otimes\mathcal{V}. \tag{7}$$

The representation of the tensor determined by f with respect to any basis is then known by the rules of linear algebra. That is, suppose a basis change is given by the matrix M defined by (2). Then the representation of f in the barred variables is

$$f = \sum_{i,j,k} \bar{T}_{ij}^k \bar{x}^i \bar{y}^j \bar{z}_k, \tag{8}$$

where the $\bar{T}_{ij}^k$ are obtained from (3)–(4) by substituting $x = M^T\bar{x}$, $y = M^T\bar{y}$ and $z = M^{-1}\bar{z}$ in (5). The result is:

$$\bar{T}_{\alpha\beta}^{\gamma} = \sum_{i,j,k} T_{ij}^k m_\alpha^i m_\beta^j n_k^\gamma, \tag{9}$$

where n_k^γ denotes the (k,γ) entry of M^{-1}. Before Weyl such a rank three tensor would have been defined as the "totality" of a system of functions $T_{ij}^k = T_{ij}^k(P)$, P a point in the underlying manifold, which transform by the rule laid down in (9), where $M{=}M(P)$ is the Jacobian matrix of a variable change in the underlying manifold.[13]

The above presentation of the algebra of tensors was novel on Weyl's part but was a reworking of earlier notions. However, Weyl also introduced a new notion — that of a *tensor density* — in his paper [Weyl 1918] and in *Raum, Zeit, Materie*.[14] He was motivated by the consideration of an invariant integral $I = \int W(x)dx$ where $x = (x_1, \ldots, x_n)$. Given a variable change

[13]See, *e.g.*, [Einstein and Grossmann 1913]. The same approach is found in [Ricci and Levi-Civita 1900, §2], although not applied to mixed tensors which were first introduced by Einstein and Grossmann [Reich 1994, p. 194].

[14]In the fourth edition [Weyl 1921a, see §13]. Pauli [1921, p. 32, n.16] credits Weyl with this notion and cites Weyl's paper [1918] — see §5 — and the third edition of *Raum, Zeit, Materie*; I am grateful to John Stachel for calling this to my attention.

$x = \varphi(y)$, a scalar function $W = W(x)$ transforms from W to $\bar{W}$ where $\bar{W}(y) = W(\varphi(x))$. In the integral I, however, where $W(x)$ can be regarded as giving the density of the manifold at x so that I represents its mass, we have $I = \int W(\varphi(y))|\partial x/\partial y| dy$, where $\partial x/\partial y$ denotes the Jacobian determinant of $x = \varphi(y)$. Hence the function W, as a scalar density function, transforms by the rule $W \to \bar{W}$ where $\bar{W}(y) = W(\varphi(x))|\partial x/\partial y|$. For tensors Weyl introduced the analogous notion of a tensor density. Expressed in the tensor algebra notation presented above, tensor densities are also identified with multilinear forms, such as the form f given in (5), but the rules of transformation are different. To obtain the representation (9) of the tensor density defined by f in the new coordinate system, instead of using (3) and (4), one uses

$$\bar{x} = |\det(M^T)^{-1}|(M^T)^{-1}x \tag{10}$$

on the contravariant variables and

$$\bar{y} = |\det M|My \tag{11}$$

on the covariant variables. "By contrasting tensors and tensor-densities," Weyl wrote in *Raum, Zeit, Materie*, "it seems to me that we have rigorously grasped the difference between *quantity* and *intensity*, so far as the difference has a physical meaning ..." [Weyl 1921a, p. 109]. Weyl's notion of tensor densities is still a part of general relativity today.[15]

The introduction of the concept of a tensor density seems to have prompted the following mathematical question. Although it is very "Weylian" in nature, it was first posed by Weyl's student at the ETH, Alexander Weinstein.[16] Weinstein, who had proof-read the third edition (1919) of *Raum, Zeit, Materie*, observed that all of the transformations (3)-(4) and (10)-(11) underlying Weyl's version of tensor algebra involve a matrix M' which is a function of the matrix M of the basis change (2), namely, if we assume without any real loss of generality that $\det M > 0$, $M' = (M^T)^{-1}$ in (3), $M' = M$ in (4), $M' = \det((M^T)^{-1})(M^T)^{-1}$ in (10), and $M' = (\det M)M$ in (11). In all of these cases, he observed, the law of matrix composition is preserved, *i.e.*,

$$(M_1M_2)' = M_1'M_2'. \tag{12}$$

[15] See, *e.g.*, [Misner et al. 1973, p. 501], [Møller 1972, p. 310].

[16] I owe my awareness of Weinstein's paper to A. Borel [1986, p. 54]. Weinstein was one of Weyl's few students, and one he regarded highly. He went on to distinguish himself as an analyst. See in this connection the biographical sketch by Diaz in *Weinstein Selecta*, see [Weinstein 1923].

Just as Weyl asked: what is the mathematical basis for the locally pythagorean nature of space in relativity theory, so now Weinstein asked: what is the mathematical basis for the transformation rules (3)-(4), (10)-(11) of tensor algebra? That is: "Are there other rules $M \to M'$ satisfying (12) and hence other sorts of tensors?" Weinstein proved the answer is "no" in the sense that (3)-(4), (10)-(11) are the only "elementary rules"; all others are composed out of these. Hence there are no essentially new types of tensors to consider. He called his result the "fundamental theorem of tensor calculus."

As with the space problem, so here too Weinstein's question involved a group, this time the group of all matrices of positive determinant. At the advice of Weyl, Weinstein proved his result by working on the Lie algebra level. By virtue of (12), of course, today we would say that Weinstein was studying degree n representations of this group, but Weinstein made no reference to such a theory. That is not surprising. Frobenius had developed a representation theory for finite groups in 1896–1904, but nothing comparable in scope had been done for continuous groups. Some things had been done which, in retrospect, can be seen as contributions to such a theory, although it is quite conceivable that neither Weyl nor Weinstein were aware of this fact at the time Weinstein worked on his dissertation, which was submitted for publication on February 22, 1922.[17] In addition to the above-mentioned work of E. Cartan, which, as we have seen, Weyl seems to have first learned about in October 1922, there was the doctoral dissertation of Frobenius' student Issai Schur [1901] devoted to the study of the type of representation of $\mathbf{GL}(n,\mathbb{C})$ that occurs in the theory of invariants. Schur's dissertation will be discussed further on. It was probably not known to Weyl until 1924. In any case, Weyl discovered a completely different, conceptually simpler way to connect representations of $\mathbf{GL}(n,\mathbb{C})$ with those of the symmetric group than that developed by Schur. As we shall see, this discovery was a by-product of his own interest in the mathematical underpinnings of tensor algebra and ultimately led him to his own "fundamental theorem" about tensors and to the involvement with the representation of continuous groups that culminated in his papers of 1925–26.

The aspect of tensor algebra that proved significant in this connection had to do with the symmetry properties of tensors. In relativistic physics and in geometry the tensors that arose were not totally general; they came with specific symmetry properties. Thus in the pages of *Raum, Zeit, Materie* [Weyl 1921a], the stress tensor S_{ik} is seen to be a symmetric tensor of rank 2 (§8), and the four-dimensional relativistic electromagnetic intensity vector F_{ik} of

[17] The paper was published in *Mathematische Zeitschrift* [Weinstein 1923] and also separately as Weinstein's doctoral dissertation at the ETH.

§20 is a skew-symmetrical tensor of rank 2. The symmetry properties of the Riemann curvature tensor R_{ijkl} (§17) are more complex, being given by

$$(i)\quad R_{jikl} = R_{ijlk} = -R_{ijkl};\quad (ii)\quad R_{klij} = R_{ijkl};$$

$$(iii)\quad R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

With such examples in mind, Weyl wrote emphatically at the beginning of §7 on "Symmetrical Properties of Tensors," that: "the character of a quantity is not in general described fully, if it is stated to be a tensor of such and such an order [*i.e.*, rank], but *symmetrical characteristics* have to be added" [Weyl 1921a, p. 54].

Weyl realized that permutations could be used to characterize symmetry properties in general. Consider, for example, a covariant tensor of rank 3, T_{ijk}, which following Weyl, we regard as a multilinear function

$$f = f(x, y, z) = \sum_{i,j,k} T_{ijk} x^i y^j z^k. \tag{13}$$

If S is some permutation of the x, y and z variables, let f_S denote the form that arises from f by permuting the variable series according to S. Then f is symmetric if $f_S = f$ for all S and skew-symmetric if $f_S = (\mathrm{sgn}\ S)f$, where as usual the sign of S is ± 1 according to whether S is an even or odd permutation. Weyl concluded his discussion of tensor symmetry by observing that the most general form of a symmetry condition is expressible by one or more equations of the form

$$\sum_S e_S f_S = 0, \tag{14}$$

where the e_S are numbers and S runs over all possible permutations of the variables.

Weyl's emphasis on the symmetry properties of tensors and the manner in which he conceived of them, *i.e.*, in terms of permutations and (14) naturally suggest questions about the mathematical basis of tensor symmetry. Here are some questions suggested by the above presentation, and eventually posed by Weyl. Suppose $\mathcal{C}$ is a symmetry class of tensors determined by one or more symmetry relations of the form (14). What are the possibilities for $\mathcal{C}$? That is, what is the mathematical basis for understanding the possibilities for $\mathcal{C}$? Also, is there an analog for $\mathcal{C}$ of the following properties P', P'' which hold, respectively, for symmetric and skew symmetric tensors:

Property P′. *If f^* is an arbitrary covariant tensor of rank ν, then the tensor $f = (\frac{1}{\nu!})\sum_S f^*_S$ is symmetric. Furthermore, all symmetric tensors of rank ν are so expressible since if f is such a tensor then $f = (\frac{1}{\nu!})\sum_S f_S$.*

Property P$''$. *If f^* is an arbitrary covariant tensor of rank ν, then the tensor $f = (\frac{1}{\nu!})\sum_S(\text{sgn } S)f_S^*$ is skew symmetric. Furthermore all skew symmetric tensors of rank ν are so expressible since if f is such a tensor, then $f = (\frac{1}{\nu!})\sum_S(\text{sgn } S)f_S$.*

Although Weyl did not explicitly mention properties P', P'' in *Raum, Zeit, Materie*, it is doubtful they escaped his notice. Indeed, he used the fact that any skew symmetric tensor f is expressible as $f = (\frac{1}{\nu!})\sum_S(\text{sgn } S)f_S$ to show that (for $\nu = 3$) every such f is expressible as a linear combination of the special skew symmetric "volume tensors" (defined by Weyl using determinants) which have become the standard basis for the subspace of skew symmetric tensors [Weyl 1921a, p. 55].

As we shall see, Weyl posed and answered the above questions in a paper submitted in January 1924 [Weyl 1924a]. I suspect he may have had them in mind much earlier, but his resolution of them — or at least his publication of these results — may have been prompted by an episode involving the mathematician Eduard Study (1862-1930) which occurred in 1923.

Response to Study

Study was an idiosyncratic, somewhat cantankerous mathematician whose primary mathematical research interest was in the theory of invariants and its geometrical applications. For a while in the late 1880's and early 1890's, he became a part of Lie's school, being charged by Lie with the task of relating his theory of transformation groups to the theory of invariants. During this period his work on ternary invariants even led him to conjecture, in a letter to Lie, what amounts to the complete reducibility theorem for semisimple Lie groups — the theorem Weyl first succeeded in proving in his papers of 1925-26. But Study finally abandoned his efforts to deal with groups on the "abstract" level of Lie's theory and concentrated instead on more concrete problems, including the study of the invariants of groups other than the general linear group. In particular he studied the invariants of the orthogonal group in [Study 1897].

At the beginning of 1923 Study published a book on the theory of invariants [Study 1923], and in the lengthy introduction he chastised those working on relativity theory for their neglect of the tools of the theory of invariants in favor of tensor calculus. He pointed out that for over fifty years a highly developed theory of invariants with respect to the general linear group had been in existence and, citing his own work on orthogonal invariants, he noted that an invariant theory of other groups had also been indicated. But "with the majority of authors there is nothing to indicate that they live in an age in which the theory of groups is in full bloom" (p. 3). "In short," Study continued (p.4), "they are behind the times, and not just a little. Even with an otherwise knowledgeable writer one can read for example the following: 'Many will be appalled at the deluge of formulas and indices with which the leading ideas are inundated. It is certainly regrettable that we have to enter into the purely formal aspect in such detail and to give it so much space but, nevertheless, it cannot be avoided'." That quotation, although not identified as such, was drawn from Weyl's book, *Raum, Zeit, Materie*.[18]

Study went on to criticize Weyl for accepting the formalism of the tensor calculus as an unavoidable, necessary evil. That is not to say that Study was against formalism. Quite the contrary! He believed the formal aspects of mathematics were important, but the formalism must be of the right kind: "Mathematics is neither the art of calculation nor the art of avoiding calculations. To mathematics, however, belongs the art of avoiding superfluous calculations and carrying out the necessary ones adroitly. In this regard, one can learn from the older authors" (p. 4). What Study had principally in mind was the symbolical method of the theory of invariants which went back to Aronhold and Clebsch. This method reduced the problem of determining invariants to the far easier problem of determining symbolical or vector invariants. In sum (to use Study's own analogy): mathematicians had thought that in the tensor calculus they were borrowing from the garden of their neighbor the physicist the seeds of the golden apples of the Hesperides but were contenting themselves with a harvest of potatoes! The neglected theory of invariants and in particular the symbolical method, Study implied, would prove to be far more valuable.

It will be helpful for what is to follow to briefly indicate the nature of the theory of invariants in Study's time and the gist of the symbolical method. Let $\mathfrak{G}$ denote a group of nonsingular linear transformations of variables $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n), \ldots$ In the classical theory $\mathfrak{G}$ was $\mathbf{GL}(n, \mathbb{C})$, but by Study's time other groups such as the orthogonal group $\mathbf{O}(n, \mathbb{C})$ were also considered, thanks largely to Study's efforts. Invariants are defined with respect to one or more base forms (*Grundformen*), which are homogeneous polynomials of specific type in one or more variables series $x, y, \ldots$ with unspecified coefficients. Consider, for example, as base form the bilinear form $f(a; x, y) = \sum_{i,j=1}^{n} a_{ij} x_i y_j$. Then each $T \in \mathfrak{G}$ induces a linear

[18] I am grateful to Erhard Scholz for informing me that Study was using the first edition of 1918 or its 1919 reprint as second edition. The quotation is from p. 111. In subsequent editions published before 1923 the passage was changed and is not as vulnerable to Study's criticism. See p. 123 of the third edition and p. 137 of the fourth edition [Weyl 1921a].

transformation $M(T)$ of $a = (a_{11}, ..., a_{nn})$ as follows. The variable change $x = Tx'$, $y = Ty'$, transforms $f(a; x, y)$ into $f'(a', x', y') = f(a, Tx', Ty')$ and the relation between the coefficients a_{ij} and a'_{ij} is given by a nonsingular linear transformation: $a' = M(T)a$. An invariant of the form $f(a; x, y)$ is any homogeneous polynomial $I(a) = I(a_{11}, \ldots, a_{nn})$ for which $I(a') = (\det T)^{\mu} I(a)$ for all $a' = M(T)a$, *i.e.*, for all $T \in \mathfrak{G}$. Here, in the traditional presentation $T \to M(T)$ is not quite a representation of $\mathfrak{G}$ since $M(T_1T_2) = M(T_2)M(T_1)$, but this can be remedied by considering $T \to M(T^{-1})$. In effect this is the representation determined by the action of $\mathfrak{G}$ on the vector space of all bilinear forms. The symbolical method uses the polarization process of Aronhold to transform each invariant $I(a)$ into a symbolical or vector invariant $i(\alpha, \beta, \ldots)$, *i.e.*, a homogeneous polynomial in vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$,..., such that $i(T\alpha, T\beta, \ldots) = (\det T)^{\mu} i(\alpha, \beta, \ldots)$.[19] Since the original invariant can be recaptured from the vector invariant, the problem of determining the invariants of $\mathfrak{G}$ with respect to the form $f(a; x, y)$ reduces to the simpler problem of finding vector invariants. In [1897] Study determined all vector invariants of the orthogonal group, thereby in principle solving the problem of determining all the invariants of the orthogonal group with respect to a set of base forms.

Weyl replied to Study's criticism in two papers. The first reply was explicit and was contained in a paper submitted at the end of October 1923 [Weyl 1923c]. This paper was intended as the first of a series of papers in which Weyl proposed to deal with mathematical topics of interest to all mathematicians and to emphasize clarifying known results rather than presenting new ones. One such topic Weyl dealt with was that of determining the invariants, in the sense of the symbolical method of determining vector invariants, for the "classical groups," the symplectic group being treated here for the first time. Thus he tacitly accepted invariant theory and the symbolical method as a part of basic mathematics, but in a footnote referring to Study's criticism, he rejected Study's suggestion that invariant theory and, in particular, the symbolical method belonged in a treatise on relativity theory. Even if he possessed Study's great command of the theory of invariants, Weyl declared: "I would not apply the symbolical method in my book 'Space, Time, Matter' and not a single word would have been said about the completeness theorems of invariant theory. Everything in its proper place!"

Weyl's paper [1923c] is sometimes seen as revealing an awakening interest in the theory of invariants, which in turn encouraged his work on the representation of Lie groups. However, this does not quite agree with Weyl's

[19] In his book [Weyl 1946, pp. 5–6, 243–245] Weyl gives a clear exposition of the polarization process and its role in the symbolical method.

own words quoted at the beginning that his study of *both* the representation and invariants of groups was motivated by his interest in the mathematical substance behind the formal apparatus of relativity theory. In my opinion, in order to understand Weyl's move towards an interest in group representation theory, it is more enlightening to consider what I regard as his second reply to Study's criticism.

Weyl's second reply was implicit — Study is nowhere mentioned by name — and came about six weeks later in a paper submitted in January 1924 "On the symmetry of tensors and the scope of the symbolical method in the theory of invariants" [Weyl 1924a].[20] The paper has two parts. In part one, on tensor symmetries, Weyl answered the questions on tensor symmetries formulated above. In part two, he applied these results to a question in the theory of invariants that may well have been prompted by his encounter with Study. Let me explain.

In part two Weyl considered the kind of invariant theoretic question that would be of interest to a relativist. As we have seen, a typical problem considered in the theory of invariants would be that of determining the invariants of the general linear group with the base form being the general covariant tensor f of rank $\nu = 3$ given in (13). In modern terms, this is the study of the polynomial invariants of the representation of the general linear group induced by its action on the 3-fold tensor product $\mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}^*$. Formulated as such, this would be a standard invariant-theoretic problem. But, as was pointed out when discussing Weyl's treatment of tensor symmetries in *Raum, Zeit, Materie*, he stressed the fact that in physics and geometry tensors come endowed with specific symmetry properties. Echoing this sentiment, Weyl wrote in the first part of [Weyl 1924a, p. 472]: "For every tensor which arises, a category characterized by symmetry relations must be specified *a priori*, inside of which the tensor is to be thought of as freely variable."

So suppose that we consider instead of the general tensor of rank 3, the tensors of that rank with prescribed symmetry properties as given by equations of the form (14). Then such tensors transform among themselves by variable changes of, say, elements in the general linear group. As in the standard situations of invariant theory, the transformation of the coefficients of these tensors is linear and we may consider the invariants with respect to these linear transformations. In other words, if $\mathcal{W} \subset \mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}^*$ consists of the tensors satisfying some symmetry relations of the form (14), then $\mathcal{W}$ is a a representation module in its own right, and we may consider the invariant

[20] In Weyl's *Gesammelte Abhandlungen II*, this paper is misleadingly placed (with the date of submission omitted) after Weyl's two notes of November 1924 [Weyl 1924b,c] announcing his principal results on the representation of semisimple Lie groups.

polynomials of the associated representation. This is what Weyl proposed to consider, albeit expressed in the older terminology. These are the type of invariants that might arise within relativistic physics, where physical quantities are given by tensors with specific symmetry properties.

Now it turns out, as Weyl observed, that the symbolical method breaks down in this case. That is, because f is not the "general" rank 3 covariant tensor but is restricted by symmetry conditions (14), the link between its invariants and the symbolical ones that is fundamental to the symbolical method is severed. As a simple illustration of this fact, consider the general skew symmetric rank two tensor $f = a_{12}x^1y^2 - a_{12}x^2y^1$, which has the linear invariant $I(a) = a_{12}$. The symbolical method associates with I the expression $i(\alpha, \beta) = \alpha_1\beta_2$, where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$. But i is not a vector invariant. Thus an ordinary invariant need not give rise in the usual manner to a symbolical one, and so direct application of the method fails.

Weyl, however, perceived a way to salvage the symbolical method. Suppose, for example, that f belongs to the class of skew symmetric tensors. Then by virtue of property P'', f is obtained from a completely general tensor f^* of the same rank, and by virtue of this fact, Weyl could see how to push through the symbolical method by utilizing f^*. To illustrate this point we consider again the above skew symmetric tensor f of rank 2. By property P'', f can be obtained from the completely general rank two tensor $f^* = a_{11}x^1y^1 + a_{12}x^1y^2 + a_{21}x^2y^1 + a_{22}x^2y^2$:

$$f = \frac{1}{2}f^* - \frac{1}{2}f^*_{(12)} = \frac{1}{2}(a_{12} - a_{21})x^1y^2 - \frac{1}{2}(a_{12} - a_{21})x^2y^1 \equiv \bar{a}_{12}x^1y^2 - \bar{a}_{12}x^2y^1.$$

Thus f is expressible in terms of the coefficients of the completely general f^*. As a consequence the linear invariant $I = \bar{a}_{12} = \frac{1}{2}(a_{12} - a_{21})$ is expressible in terms of the coefficients of f^* and is an invariant with respect to f^* as base form. Thus the symbolical method, which requires that the coefficients of the base form be completely unconstrained, may now be applied to $I = \frac{1}{2}(a_{12} - a_{21})$ to obtain the (skew symmetric) vector invariant

$$i(\alpha, \beta) = \frac{1}{2}\alpha_1\beta_2 - \alpha_2\beta_1 = \frac{1}{2}\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}.$$

Weyl could see how to do the same sort of thing for any symmetry class of tensors, because he could generalize properties P', P'' to any such class. This constitutes the first part of his paper. There he proved the following result:

Theorem 2. — *Let $f(x, y, z, \cdots) = \sum_{ijk\cdots} T_{ijk\cdots}x^iy^jz^k \cdots$ denote a tensor of rank ν. Then:*

a) If $\mathcal{W}$ is the class of such tensors f which satisfy several symmetry relations of the form (14), there is a single such relation which characterizes $\mathcal{W}$.
b) Given any symmetry class $\mathcal{W}$, there exist constants c_S, $S \in \mathfrak{S}_\nu$, such that if f^* is arbitrary, then $f = \sum_{S\in\mathfrak{S}_\nu} c_S f^*_S \in \mathcal{W}$. Moreover every $f \in \mathcal{W}$ is obtained in this manner since if $f \in \mathcal{W}$, then $f = \sum_{S\in\mathfrak{S}_\nu} c_S f_S \in \mathcal{W}$.

How did Weyl obtain these results? As he tells us: "By applying the representation theory of Frobenius to the group of all permutations one easily obtains a complete insight into the possible symmetry characteristics of tensors" [Weyl 1924a, pp. 468–9]. Exactly how Weyl hit upon Frobenius' theory is not known, but the left hand side of the general symmetry relation (14) of *Raum, Zeit, Materie*, $\sum_{S\in\mathfrak{S}_\nu} e_S f_S$, certainly suggests looking at the group algebra of the symmetric group $\mathfrak{S}_\nu$, and the structure of this algebra was known to be related to the representations of $\mathfrak{S}_\nu$. In [1924a] Weyl wrote (14) in the equivalent form

$$\sum_{S\in\mathfrak{S}_\nu} k_S f_{S^{-1}} = 0 \tag{15}$$

and observed that one can associate to the form f an element $\mathbf{f}$ in the group algebra $\mathcal{H}$ of the symmetric group $\mathfrak{S}_\nu$, namely

$$\mathbf{f} = \sum_{S\in\mathfrak{S}_\nu} f_S S. \tag{16}$$

Since the $\mathbf{f}$ given in (16) depends upon the values of the variables $x, y, z, \cdots$ defining f, (16) actually defines a family of elements in $\mathcal{H}$. Weyl glossed over this point, but his results go through nonetheless.[21] Direct calculation then shows that (15) is equivalent to $\mathbf{kf} = \mathbf{0}$, where $\mathbf{k} = \sum_{S\in\mathfrak{S}_\nu} k_S S$, and, for example, part (b) of Theorem 2 can be deduced from the following result about the group algebra $\mathcal{H}$:

Theorem 3. — *Given $\mathbf{k} = \sum_{S\in\mathbf{S}_\nu} k_S S \in \mathcal{H}$, there is an element $\mathbf{c} = \sum_{S\in\mathbf{S}_\nu} c_S S \in \mathcal{H}$ such that $\mathbf{kf} = \mathbf{0}$ if and only if $\mathbf{f} = \mathbf{cf}^*$ for some $\mathbf{f}^*$ in $\mathcal{H}$. Moreover, $\mathbf{cf} = \mathbf{f}$ for all $\mathbf{f}$ satisfying $\mathbf{kf} = \mathbf{0}$.*

Theorem 3 was proved using the fact that the group algebra of the symmetric group decomposes into a sum of complete matrix algebras[22] — the group algebra version of Frobenius' complete reducibility theorem for finite

[21] Weyl later touched on this point in his exposition of tensor symmetries and the group algebra $\mathcal{H}$ in his book on group theory and quantum mechanics [Weyl 1931, p. 283].
[22] That is, the linear associative algebra of all $m \times m$ matrices for some $m \in \mathbb{Z}^+$.

groups. By virtue of this decomposition into complete matrix algebras the proof of Theorem 3 is reduced to basic linear algebra, as Weyl showed. The same is true of part (a) of Theorem 2.

So Weyl had discovered the value of Frobenius' theory of group characters and representations for answering questions from tensor algebra as well as questions about the scope of the symbolical method. If this paper is viewed as a response to Study, then the message would seem to be: I have discovered in the representation theory of groups something of far greater importance than the essentially formal symbolical method. Not only does it enable me to gain insight into questions about the foundations of tensor algebra that interest me, it also provides insight into the scope of the symbolical method itself.

Once again, Weyl had discovered in the theory of groups the means to answer questions motivated by relativity theory. In the case of the space problem, however, he had utilized only the basic elements of Lie's theory — primarily the fact that one could deal with certain group theoretic questions more easily by working with the infinitesimal group or Lie algebra. The same can be said for Weinstein's work on his "Fundamental Theorem of Tensor Calculus." Now for the first time, Weyl had gone beyond the elements of group theory to achieve his goal. He had discovered the power of the theory of representations for answering the questions that intrigued him.

The Group–Theoretic Foundation of Tensor Calculus

The paper by Frobenius that was especially relevant here was his paper on the primitive idempotents of the group algebra of the symmetric group [Frobenius 1903]. It was written after he learned of the work published by Alfred Young in 1901 and 1902. Young's work dealt with the theory of invariants, but Frobenius could see that it related to the group algebra of the symmetric group and that Young had in effect obtained a formula for the primitive idempotents. Since the primitive idempotents determine the irreducible representations and their characters, Frobenius deemed it worthwhile to rederive Young's formulae and to relate them to his theory of group characters. I should mention that Frobenius himself preferred to present his work on group representations without the explicit use of the "hypercomplex numbers" of the group algebra and this was true of the above mentioned paper as well. As he explained there: "It is less significant that I abstain from the use of hypercomplex numbers, since, as convenient as they occasionally are, they do not always serve to make the presentation more transparent" [Frobenius 1903, p. 266]. Weyl, however, could read between the lines and interpret Frobenius' theory in terms of the group algebra, which was precisely the point of view that was relevant to his study of tensor symmetries.

Weyl realized that the element $\mathbf{c}$ of the group algebra in Theorem 3 is an idempotent. He also realized that if $\mathbf{c}$ is an idempotent, then the totality $\mathcal{W}$ of all tensors obtained from the operation $f^* \to f = \sum_{S\in\mathfrak{S}_\nu} c_S f_S^*$ of part (b) of Theorem 2 is a symmetry class — with symmetry equation (15) given by $\mathbf{k} = \mathbf{1} - \mathbf{c}$, $\mathbf{1}$ being the identity element of the symmetric group. Since the primitive idempotents determine the irreducible representations of the symmetric group $\mathfrak{S}_\nu$, it would be natural to wonder whether if $\mathbf{c}$ is a primitive idempotent, and hence given by Young's formula, the representation of the special linear group which it generates by means of the representation module $\mathcal{W}$ would be one of the irreducible representations classified by Cartan according to highest dominant weights in [Cartan 1913]. Indeed, although Cartan did not couch his results in terms of tensors, to anyone with a background in tensor algebra, it would have been clear that the representation modules he constructed could be regarded as consisting of tensors.

Weyl eventually discovered how to associate with a given dominant weight a Young tableau such that the primitive idempotent $\mathbf{c}$ it defines generates, in the above sense, the symmetry class $\mathcal{W}$ of tensors which is an irreducible representation module for that weight. Consider, for example, $\mathfrak{G} = \mathbf{SL}(4, \mathbb{C})$ and the irreducible representation module of highest weight $\pi = \sum_{i=1}^{3} p_i\pi_i$, where the π_i are the fundamental dominant weights and $p_i \geq 0$ in accordance with Cartan's theory. If we set $m_1 = p_1 + p_2 + p_3$, $m_2 = p_2 + p_3$, $m_3 = p_3$, then $m_1 \geq m_2 \geq m_3$, and the m_i define the shape of a Young tableau, with first row of length m_1, second row of length m_2 and third of length m_3, which corresponds to the symmetric group $\mathfrak{S}_\nu$ with $\nu = \sum_{i=1}^{3} m_i$. For example, if $\pi = 2\pi_1 + \pi_2 + 3\pi_3$, so $m_1 = 6, m_2 = 4, m_3 = 3$ and $\nu = 13$, then

$$\mathcal{T} \quad = \quad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & & \\ 11 & 12 & 13 & & & \end{array}$$

is a Young tableau of the given shape. If $\mathcal{R}$ denotes the subgroup of elements of $\mathfrak{S}_{13}$ which permute the numbers in the rows of $\mathcal{T}$ among themselves and if $\mathcal{C}$ is defined analogously with respect to the columns of $\mathcal{T}$, then the corresponding Young-Frobenius primitive idempotent $\mathbf{c}$ turns out to be a constant multiple of $\mathbf{e} = \sum_{P\in\mathfrak{S}_\nu} \zeta(P)P$, where $\zeta(P) = \operatorname{sgn} C$ if $P = RC$ with $R \in \mathcal{R}$ and $C \in \mathcal{C}$ and $\zeta(P) = 0$ otherwise.[23] Since a constant multiple does not

[23]The constant multiple is $\frac{f}{\nu!}$, where f is the degree of the irreducible representation of

change the symmetry class defined by part (b) of Theorem 2, an irreducible module of weight $\pi = 2\pi_1 + \pi_2 + 3\pi_3$ consists of all the tensors obtained from the operation $f^* \to f = \sum_{S\in\mathfrak{S}_{13}} \zeta(S) f^*_S$ applied to the general tensor f^*. In other words, the tensors characterized by the symmetry relation $\sum_S k_S f_{S^{-1}} = 0$, where $\mathbf{k} = \mathbf{1} - \mathbf{c}$, form an irreducible module of hightest weight $2\pi_1 + \pi_2 + 3\pi_3$. Noting that Cartan had not indicated this connection between symmetry classes of tensors and irreducible representations, Weyl expressed the conviction that it was through this connection "that the entire matter is first placed in the right light."[24]

The fact that tensors with specific symmetry characteristics are the basis of all the irreducible representations of $\mathbf{SL}(n,\mathbb{C})$ had an intriguing implication for Weyl: If the complete reducibility theorem of Frobenius' theory were true for $\mathbf{SL}(n,\mathbb{C})$, it would mean that tensors — with specific symmetry properties — are the building blocks for all representations of $\mathbf{SL}(n,\mathbb{C})$. Weyl perhaps realized that Lie, prompted by Study's above-mentioned conjectures, had conjectured the truth of what amounts to the complete reducibility theorem for $\mathbf{SL}(n,\mathbb{C})$ in the third and final volume of his *Theory of Transformation Groups* [Lie 1893, pp. 785–6]; but, in any case, it was a paper by Frobenius' student, Issai Schur, that put Weyl in a position to prove it. On January 10, 1924, and hence at about the same time as Weyl submitted his paper [1924a] on tensor symmetries, Schur presented a paper [1924] to the Berlin Academy in which he pointed out how Frobenius' theory of group representations, including the complete reducibility theorem, could be extended to the rotation group of n-dimensional space, $\mathbf{SO}(n,\mathbb{R})$. He also referred to his dissertation [Schur 1901] in which he had studied representations $A \to R(A)$ of $\mathbf{GL}(n,\mathbb{C})$ for which the entries of the matrix $R(A)$ are polynomials in the entries of the matrix A, and this may have been how Weyl learned of it.[25] These are the sort of representations that come up implicitly in the classical theory of invariants and that is why Schur was interested in them. Making critical use of the polynomial nature of the entries of $R(A)$, he showed how to associate with each irreducible representation $R(A)$ an irreducible representation of a symmetric group. As we have seen, such a connection can be given by

$\mathfrak{S}_{13}$ associated to $\mathbf{c}$. Frobenius [1903, p. 265] and Weyl [1925, p. 573] gave different ways of defining f directly in terms of $\mathfrak{S}_\nu$.

[24] Weyl made this comment in his first communication on these matters [Weyl 1924b, p. 462]; they are discussed in somewhat more detail in [Weyl 1925, pp. 558–9, 571–3]. Complete details, in the context of $\mathbf{GL}(n,\mathbb{C})$, are given in his lectures [Weyl 1934, pp. 21ff.].

[25] Although Schur's dissertation was a brilliant piece of work, it was only published as a separate pamphlet (as was required of dissertations). Many dissertations (*e.g.*, Weinstein's) were also published in journals.

the above primitive idempotent **c** of Young and Frobenius, but Schur, whose work predates that of Young and Frobenius, made the connection in a different, more complicated way.[26] However he showed how it could be used to obtain many beautiful theorems, including a complete reducibility theorem for the polynomial representations $R(A)$.

In neither his dissertation nor his paper [1924] was Schur's primary goal an extension of Frobenius' theory to continuous groups. In [Schur 1924] he was concerned with a counting problem in the theory of invariants that had been solved in the classical case of the in- and covariants of binary forms by Cayley in 1856 and for the invariants of finite groups by T. Molien in 1897.[27] Molien had used the representation theory of finite groups to solve his problem, and Schur realized he could do the same for the invariants of the rotation group by extending Frobenius' theory to this group. His method of extension was based upon a technique introduced by Adolph Hurwitz and involved replacing summation over a finite group with invariant integration over a compact Lie group. Hurwitz had used the technique to extend Hilbert's basis theorem to orthogonal invariants — a new result — but he also used it to give another proof for invariants with respect to $\mathbf{SL}(n,\mathbb{C})$. The application of the technique to $\mathbf{SL}(n,\mathbb{C})$, which is not compact, involved an idea which Weyl later dubbed the "unitarian trick." Weyl saw how to use the same sort of trick to establish the complete reducibility theorem for $\mathbf{SL}(n,\mathbb{C})$, thereby showing that tensors (with prescribed symmetry conditions) are the building blocks for all possible representations.

The paper Weyl presented to the Göttingen Academy of Sciences in November 1924 [Weyl 1924b] announcing this discovery (as well as others), was entitled "Das gruppentheoretische Fundament der Tensorrechnung," and in it he opined that "the true group theoretic foundation of the tensor calculus" was to be found in the above-mentioned consequence of the complete reducibility theorem for $\mathbf{SL}(n,\mathbb{C})$. In the first part of his famous series of papers on the representations of semisimple Lie groups [Weyl 1925, pp. 545–6], which bore the same title as the Göttingen paper, Weyl put the matter as follows. Tensors, he explained, are examples of what he called "linear quantities." Consider for example the mixed tensors of rank three T_{ij}^k discussed earlier at (5). Each such T_{ij}^k may be regarded as an N-tuple (with $N = n^3$). The change of basis (2) corresponding to a matrix $M \in \mathbf{SL}(n,\mathbb{C})$ brings with it the variable changes $x^i \to \bar{x}^i$, $y^j \to \bar{y}^j$, $z_k \to \bar{z}_k$ which leads to an expression (8) for f

[26] After Schur learned of Weyl's approach, he returned to the subject of his dissertation and developed another way to make the connection that was simpler than his original approach [Schur 1927, pp. 70, 72ff.].

[27] The history of this counting problem is treated in my paper [Hawkins 1986].

in terms of the barred variables, and implicitly defines the linear transformation $R(M) : T^k_{ij} \to \bar{T}^k_{ij}$. It is easily seen that $R(M_2M_1) = R(M_2)R(M_1)$. For this reason the T^k_{ij} constitute a linear quantity. In general, according to Weyl, a linear quantity is an N-tuple $(a_1, \ldots, a_N)$ which transforms by a linear transformation $R(M) : (a_1, \ldots, a_N) \to (\bar{a}_1, \ldots, \bar{a}_N)$ such that $R(M_2M_1) = R(M_2)R(M_1)$. Of course linear quantities are just representations of $\mathbf{SL}(n, \mathbb{C})$, but Weyl recast the notion here in a form that was more congenial to the mathematical context from which he was coming — the mathematics of relativity theory. Thus the complete reducibility theorem for $\mathbf{SL}(n, \mathbb{C})$ became in this language the theorem that the only linear quantities are the tensors. It was in this form that it was particularly meaningful for him.

Weyl regarded this theorem as "the proper group theoretic justification of the tensor calculus" [Weyl 1925, p. 546]. In other words, he had obtained through the theory of groups, and in particular through the theory of group representations — as augmented by his own contributions — what he felt was a proper mathematical understanding of tensors, tensor symmetries, and the reason they represent the source of all linear quantities that might arise in mathematics or physics. Once again, he had come to appreciate the importance of the theory of groups — and now especially the theory of group representations — for gaining insight into mathematical questions suggested by relativity theory. Unlike his work on the space problem or Weinstein's work on the fundamental theorem of the tensor calculus, however, Weyl now found himself drawing upon far more than the rudiments of group theory. His study of tensor symmetries had drawn upon Frobenius' theory of group representations and his own "fundamental theorem" for tensors had involved him with the continuous analog of Frobenius' theory. And of course Cartan had showed that the space problem could also be resolved with the aid of results about representations. In short, the representation theory of groups had proved itself to be a powerful tool for answering the sort of mathematical questions that grew out of Weyl's involvement with relativity theory.

Frobenius had more or less developed all the essentials of the theory of representations for finite groups, but that was not at all the case for continuous groups, notwithstanding the important contributions contained in the work of Cartan and Schur. Their work certainly suggested to Weyl the potential richness of a continuous analog of Frobenius' theory, but it did not constitute a coherent theory. Schur, who was unaware of Cartan's work, had concentrated on two specific groups and had emphasized the role of group characters, whereas Cartan dealt with all semisimple groups but on the infinitesimal level using his theory of weights and without any complete reducibility theorem. Having become convinced of the importance of group representation theory, Weyl went on to extend his results about $\mathbf{SL}(n, \mathbb{C})$ so as to create in his brilliant papers of 1925–26 a coherent theory of the representation of all semisimple groups within which the approaches of Schur and Cartan were linked together for the first time.

Epilogue

Two additional points need to be made.

1. Weyl was not the only mathematician interested in tensor algebra who saw a connection with the representation theory of the symmetric group. As early as 1919 J. A. Schouten (1883–1971) studied the problem of expressing a tensor as a sum of "irreducible" tensors with symmetry properties. To this end he utilized the group algebra of the symmetric group and Frobenius' theory of group characters (in a formula for the principal idempotents). He was, however, unaware of Frobenius' paper [1903] or Young's work and independently developed notions akin to that involving Young tableaux. Schouten's work was expounded (with complete references to his earlier publications) in his book [Schouten 1924, VII], which Weyl cites in [Weyl 1924b, p. 462, n.2]. There is no evidence that Weyl knew of Schouten's work earlier and was influenced by it. Schouten actually submitted a note [1923] illustrating his method (on an example suggested by Study!) to the *Rendiconti del Circolo Matematico di Palermo* a year before Weyl's paper [1924a] on tensor symmetries was submitted to the same journal. Had Weyl known of this note he most certainly would have cited it in his own.

2. I have suggested that Weyl wrote his papers of 1925–26 on representation theory with the conviction that the theory was a powerful instrument for answering questions suggested by theoretical physics. Weyl acted on that conviction shortly after he completed the above papers. This time however group representations were utilized to deal with questions arising from the new quantum mechanics initiated by the work of Heisenberg and Schrödinger in 1924–25. By 1927 their work had led to further developments by theoretical physicists such as Born, Pauli, Jordan and Dirac and, from the mathematician's side, by von Neumann. Weyl seems to have assimilated and mastered these developments as rapidly as he had mastered relativity theory a decade earlier. Thus in a paper, "Quantenmechanik und Gruppentheorie", we find him posing the question: "How do I arrive at the matrix, the Hermitean form, which represents a given quantity in a physical system of known constitution?"[Weyl 1927, p. 90] To explain, precisely, what Weyl meant by this

would take us too far afield. But his further words should have a familiar ring to them by now: "Here with the help of group theory I believe I have succeeded in arriving at a deeper insight into the true nature of things" (p. 91). By "group theory" Weyl meant representation theory. This time it was to the study of unitary projective (or ray) representations of the abelian Lie group $\mathbb{R}^{2f}$ that he turned for the deeper insight. Once again the mathematics generated by the question went on to have a fruitful life of its own.[28]

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[28] See Mackey [1988, pp. 140ff.] for an interesting account of Weyl's question and its historical background as well as a discussion of Weyl's answer in terms of group representations and the subsequent mathematics it generated.

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[29] The date of receipt of this paper is given as 17 February 1922, but it would seem to be a typographical error. For example, throughout the paper Weyl cites (in text as well as in footnotes) the German version [1923a] of his lectures in Spain (originally presented in French and Castilian). The lectures themselves did not occur until March 1922, and the preface to [1923a] is dated April, 1923. I therefore assume the correct date of receipt was 17 February 1923.

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