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ON THE FIRST VAFA-WITTEN BOUND FOR TWO-DIMENSIONAL TORI

by

Nicolae Anghel

Abstract. — In this paper we explicitly compute the first Vafa-Witten bound for a two-dimensional torus, namely the best uniform upper bound for the first eigenvalue of the family of twisted (by arbitrary vector potentials) Dirac operators on a flat two-torus. Starting with an arbitrary flat metric we give either an exact answer or a precise algorithm for producing an answer. As a by-product we develop a constructive way of implementing the projection map from the Poincaré upper half-plane onto the standard fundamental domain for its $\mathrm{SL}(2, \mathbf{Z})$-action.

Résumé **(Sur la première borne de Vafa-Witten pour les tores de dimension deux)**
Dans cet article nous calculons explicitement la première borne de Vafa-Witten pour un tore de dimension 2, c'est-à-dire la meilleure borne supérieure pour la première valeur propre de la famille d'opérateurs de Dirac couplés à des potentiels vectoriels arbitraires, définis sur un tore plat de dimension 2. Pour une métrique plate arbitraire nous donnons soit la solution exacte de ce problème soit un algorithme précis pour en produire une. Une conséquence de nos résultats est une réalisation constructive de la projection du demi-plan de Poincaré sur le domaine fondamental de l'action de $\mathrm{SL}(2, \mathbf{Z})$ sur celui-ci.

1. Introduction

Let M be a fixed compact Riemannian spin manifold with spinor bundle S and Dirac operator $\partial\!\!\!/$. For any Hermitian vector bundle E with metric connection A form the twisted Dirac operator $\partial\!\!\!/_A$ acting on $S \otimes E$. In a remarkable paper [**VW**], also [**A**], Vafa and Witten proved, among other things, that if the discrete eigenvalues of $\partial\!\!\!/_A$ are indexed by increasing absolute value,

$$|\lambda_1| \leq |\lambda_2| \leq \dots,$$

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then there is a bound C_1, which depends on M but not on the twisting data (E, A), such that

$$|\lambda_1| \leq C_1. \tag{1.1}$$

Subsequently, Moscovici [**M**] extended the inequality (1.1) to noncommutative geometric spaces, in the sense of Connes [**C**], which have finite topological type and satisfy rational Poincaré duality in K-theory.

Vafa and Witten [loc.cit.] also addressed the problem of finding the best bound C_1 in (1.1), if M is the d-dimensional torus $\mathbf{T}^d$ with angular variables $\phi^1, \phi^2, \ldots, \phi^d$, and flat metric $ds^2 = \sum_{i,j} g_{ij} d\phi^i d\phi^j$. They concluded that in this case the best C_1 is

$$\max_{\mathbf{a}\in\mathbf{R}^d} \min_{\mathbf{m}\in\mathbf{Z}^d} \sqrt{\sum_{i,j} g^{ij}(m_i - a_i)(m_j - a_j)}, \tag{1.2}$$

where $\left[g^{ij}\right]$ is the inverse of the constant positive definite matrix $[g_{ij}]$. For instance, if the metric tensor is diagonal with $g^{ij} = c_i \delta^{ij}$, then (1.2) equals $\sqrt{c_1 + c_2 + \cdots + c_d}/2$.

It is certainly desirable to have an explicit formula for (1.2), in terms of the matrix $\left[g^{ij}\right]$ or its invariants. This problem becomes geometrically intuitive if one views a d-dimensional flat torus as a quotient $\mathbf{R}^d/L$, where L is a lattice in $\mathbf{R}^d$ of maximal rank [**MH**]. If L has basis $\{v_1, v_2, \ldots, v_d\}$ then the metric is given by $g_{ij} = \langle v_i, v_j\rangle$, where $\langle , \rangle$ denotes the standard inner product in $\mathbf{R}^d$. It turns out that for some lattices the Vafa-Witten bound is easy to calculate while for others it is not.

To see just how this distinction arises we will look now at flat metrics on a torus from the viewpoint of homogeneous spaces. The space $\mathrm{Met}(\mathbf{T}^d)$ of flat metrics on $\mathbf{T}^d$ can be identified with the homogeneous space $\mathrm{GL}(d, \mathbf{R})/\,\mathrm{O}(d)$ [**B**] under the transformation

$$\mathrm{GL}(d, \mathbf{R})/\,\mathrm{O}(d) \ni \widehat{\Phi} \longmapsto [g_{ij}] \in \mathrm{Met}(\mathbf{T}^d), \tag{1.3}$$

where if $\Phi \in \mathrm{GL}(d, \mathbf{R})$ then $[g_{ij}]$ is given by

$$g_{ij} := \langle \Phi^{-1}\mathbf{e}_i, \Phi^{-1}\mathbf{e}_j\rangle,$$

$(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d)$ being the standard basis in the Euclidean space $\mathbf{R}^d$.

In other words, $[g_{ij}] = \left(\Phi^{-1}\right)^t \Phi^{-1}$, or equivalently $\left[g^{ij}\right] = \Phi\Phi^t$. It follows that under the identification (1.3) the first Vafa-Witten bound becomes

$$\max_{\mathbf{a}\in\mathbf{R}^d} \min_{\mathbf{m}\in\mathbf{Z}^d} \sqrt{\langle \Phi^t(\mathbf{m} - \mathbf{a}), \Phi^t(\mathbf{m} - \mathbf{a})\rangle}. \tag{1.4}$$

It is obvious (see also Proposition 2.2, c)) that a conformal change of the metric $[g_{ij}]$ by a factor r changes (1.2) by a factor of $1/\sqrt{r}$. As a result, it suffices to calculate (1.2) for metrics of fixed volume, or equivalently to replace $\mathrm{GL}(d, \mathbf{R})/\,\mathrm{O}(d)$ with $\mathrm{SL}(d, \mathbf{R})/\,\mathrm{SO}(d)$ in (1.4).

Notice now that (1.4) factors to the double coset space $\mathrm{SL}(d,\mathbf{Z})\backslash\mathrm{SL}(d,\mathbf{R})/\mathrm{SO}(d)$. Indeed, if $\Phi\in\mathrm{SL}(d,\mathbf{R})$ and $\Psi\in\mathrm{SL}(d,\mathbf{Z})$ then, for $\mathbf{a}\in\mathbf{R}^d$,

$$\min_{\mathbf{m}\in\mathbf{Z}^d}\sqrt{\langle(\Psi\Phi)^t(\mathbf{m}-\mathbf{a}),(\Psi\Phi)^t(\mathbf{m}-\mathbf{a})\rangle}=\min_{\mathbf{m}\in\mathbf{Z}^d}\sqrt{\langle\Phi^t\left(\mathbf{m}-\Psi^t\mathbf{a}\right),\Phi^t\left(\mathbf{m}-\Psi^t\mathbf{a}\right)\rangle}.$$

In conclusion, one might be satisfied with calculating (1.2) only for metrics corresponding to a fundamental domain representing the space $\mathrm{SL}(d,\mathbf{Z})\backslash\mathrm{SL}(d,\mathbf{R})/\mathrm{SO}(d)$, such as the Siegel domain [**R**].

This is the first in a series of two papers addressing the problem of finding an explicit formula for the Vafa-Witten bound (1.2). In it we restrict ourselves to two-dimensional tori and work directly with a flat metric $[g_{ij}]$, whose inverse is $g^{11}=A$, $g^{12}=g^{21}=B$, $g^{22}=C$, where A,B,C are real numbers such that $A>0$, $C>0$, and $AC-B^2>0$. The computation of the Vafa-Witten bound in two dimensions is so classical in scope that it can be handled independently within several areas of mathematics: bilinear form theory, lattice theory, modular group theory. We choose to treat the problem using the framework of bilinear forms simply because this is how Vafa and Witten state their result. The lattice and modular group approaches to flat tori do appear, but only indirectly, either in some of the proofs or in the subsequent interpretations and comparisons. The second paper in the series, to appear elsewhere, will be dedicated to higher dimensional tori and will deal only with metrics corresponding to a Siegel domain.

We summarize now our main results, proven below in Theorem 2.5, Theorem 3.8, and Theorem 4.7.

a) If $\min\{A,C\}\geq 2|B|$, then the first Vafa-Witten bound equals

$$\frac{1}{2}\sqrt{\frac{AC(A+C-2|B|)}{AC-B^2}}$$

b) If $\min\{A,C\}<2|B|$, then the transformation (3.3) given in Section 3 below applied to the inverse of the metric tensor a certain number of times, number controlled by the size of $(AC-B^2)/(\min\{A,C\})^2$, reduces the problem to Case *a*).

c) Metrics corresponding to points in the standard fundamental domain F associated to the action of $\mathrm{SL}(2,\mathbf{Z})$ on the Poincaré upper half plane H do satisfy the inequality $\min\{A,C\}\geq 2|B|$, and so Case *a*) applies to them. Arbitrary metrics can then be investigated by noticing that the transformation (3.3) is the basic step of an algorithm that implements the quotient map

$$\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)\longrightarrow\mathrm{SL}(2,\mathbf{Z})\backslash\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2),$$

viewed as a map from H to F.

In addition, we show that the above results still hold if $\min\{A,C\}$ is compared to $|B|$ rather than $2|B|$ (Corollary 3.18).

2. The Particular Case $\min\{A, C\} \geq 2|B|$

Equip the two-dimensional torus $\mathbf{T}^2$ with a flat metric $[g_{ij}]$, whose inverse is $g^{11} = A$, $g^{12} = g^{21} = B$, $g^{22} = C$, where A, B, C are real numbers such that $A > 0$, $C > 0$, and $AC - B^2 > 0$. Then the first Vafa-Witten bound $\lambda_1 = \lambda_1(A, B, C)$ is given by

(2.1)
$$\lambda_1 = \max_{(a_1,a_2)\in\mathbf{R}^2} \min_{(m_1,m_2)\in\mathbf{Z}^2} \sqrt{A(m_1 - a_1)^2 + 2B(m_1 - a_1)(m_2 - a_2) + C(m_2 - a_2)^2}$$

In this section we will calculate λ_1 explicitly in the particular case $\min\{A, C\} \geq 2|B|$. We start with some obvious properties of $\lambda_1(A, B, C)$.

Proposition 2.2. — *If $\lambda_1(A, B, C)$ is defined by* (2.1) *then*

a) $\lambda_1(A, B, C)$ *is symmetric in A and C, i.e.,* $\lambda_1(A, B, C) = \lambda_1(C, B, A)$.

b) $\lambda_1(A, B, C) = \lambda_1(A, |B|, C)$

c) If $r > 0$, then $\lambda_1(rA, rB, rC) = \sqrt{r}\lambda_1(A, B, C)$

d) The set of pairs $(a_1, a_2) \in \mathbf{R}^2$ where $\lambda_1(A, B, C)$ occurs intersects $[0, 1]^2$ and is symmetric with respect to the point $(1/2, 1/2)$.

Proof. — Let $f_{A,B,C} : \mathbf{R}^2 \to [0, \infty)$, be given by

(2.3)
$$f_{A,B,C}(a_1, a_2) := \min_{(m_1,m_2)\in\mathbf{Z}^2} \left(A(m_1 - a_1)^2 + 2B(m_1 - a_1)(m_2 - a_2) + C(m_2 - a_2)^2\right)$$

Then the proposition follows from the following properties of $f_{A,B,C}$, respectively.

a) $f_{A,B,C}(a_1, a_2) = f_{C,B,A}(a_2, a_1)$

b) $f_{A,-B,C}(a_1, a_2) = f_{A,B,C}(a_1, -a_2)$

c) If $r > 0$, then $f_{rA,rB,rC} = rf_{A,B,C}$

d) $f_{A,B,C}(a_1 + 1, a_2 + 1) = f_{A,B,C}(a_1, a_2) = f_{A,B,C}(1 - a_1, 1 - a_2)$. □

Remark 2.4. — According to the above proposition in order to find $\lambda_1(A, B, C)$ it is enough to assume that $A \geq C$ and $B \geq 0$ (from *a*) and *b*)), to normalize the metric tensor such that $AC - B^2 = 1$ (from *c*)), and to look for $(a_1, a_2) \in [0, 1]^2$ maximizing $f_{A,B,C}$ only in a suitable "half" of $[0, 1]^2$, for instance $[0, 1] \times [0, 1/2]$ (from *d*).

Theorem 2.5. — *Assume that the torus $\mathbf{T}^2$ is equipped with a flat metric $[g_{ij}] \leftrightarrow (A, B, C)$ such that* $\min\{A, C\} \geq 2|B|$. *Then the first Vafa-Witten bound is given by the formula*

$$\lambda_1(A, B, C) = \frac{1}{2}\sqrt{\frac{AC(A + C - 2|B|)}{AC - B^2}} \tag{2.6}$$

Proof. — By Proposition 2.2 and Remark 2.4 it suffices to prove Formula 2.6 for $A \geq C \geq 2B \geq 0$ and $AC - B^2 = 1$. As a result, $B^2 \leq 1/3$. The theorem is then equivalent to showing that

$$\max_{(a_1,a_2)\in[0,1]\times[0,1/2]} f_{A,B,C}(a_1, a_2) = \frac{AC(A + C - 2B)}{4}, \tag{2.7}$$

where $f_{A,B,C}$ is the function given by Equation 2.3.

To this end fix $(a_1,a_2)\in[0,1]\times[0,1/2]$. For $(m_1,m_2)\in\mathbf{Z}^2$,

$$\begin{aligned} A(m_1-a_1)^2 &+ 2B(m_1-a_1)(m_2-a_2)+C(m_2-a_2)^2 \\ &= C\left(\frac{AC-B^2}{C^2}(m_1-a_1)^2+\left(\frac{B}{C}(m_1-a_1)+(m_2-a_2)\right)^2\right) \\ &= \frac{1}{C}(m_1-a_1)^2+C\left(\frac{B}{C}(m_1-a_1)+(m_2-a_2)\right)^2 \\ &= \frac{1}{C}(m_1-b_1)^2+C\left(\frac{B}{C}m_1+m_2-b_2\right)^2, \end{aligned}$$

where

$$b_1=a_1 \quad\text{and}\quad b_2=\frac{B}{C}a_1+a_2.$$

Thus,

$$f_{A,B,C}(a_1,a_2)=\min_{(m_1,m_2)\in\mathbf{Z}^2}\left(\frac{1}{C}(m_1-b_1)^2+C\left(\frac{B}{C}m_1+m_2-b_2\right)^2\right). \tag{2.8}$$

By choosing an integer m_1 such that $|m_1-b_1|\le 1/2$, followed by an integer m_2 such that $|\frac{B}{C}m_1+m_2-b_2|\le 1/2$, one sees that

$$f_{A,B,C}(a_1,a_2)\le\frac{1}{4C}+\frac{C}{4}. \tag{2.9}$$

We claim now that $f_{A,B,C}(a_1,a_2)$ occurs for $(m_1,m_2)\in\{(0,0),(0,1),(1,0)\}$. Indeed, let (m_1^0,m_2^0) be an integer pair where $f_{A,B,C}(a_1,a_2)$ occurs. Then $|m_1^0-b_1|<1$, since otherwise (2.8) implies that

$$f_{A,B,C}(a_1,a_2)\ge\frac{1}{C},$$

which in conjunction with (2.9) gives $C^2\ge 3$. But then $1=AC-B^2\ge 3-1/3$, a contradiction. Since $b_1=a_1\in[0,1]$, it follows that $m_1^0\in\{0,1\}$.

If $m_1^0=0$, then

$$f_{A,B,C}(a_1,a_2)=\frac{b_1^2}{C}+\min_{m_2\in\mathbf{Z}}C(m_2-b_2)^2,$$

and so m_2^0 can be chosen from $\{0,1\}$, since $b_2=\frac{B}{C}a_1+a_2\in[0,1]$.

If $m_1^0=1$, then

$$f_{A,B,C}(a_1,a_2)=\frac{(1-b_1)^2}{C}+\min_{m_2\in\mathbf{Z}}C\left(m_2+\frac{B}{C}-b_2\right)^2,$$

and since $\frac{B}{C}-b_2=\frac{B}{C}(1-a_1)-a_2\in[-1/2,1/2]$, m_2^0 can be taken to be 0. The claim follows.

Maximizing $f_{A,B,C}$ on $[0,1]\times[0,1/2]$ becomes now a very geometric problem. Rewriting (2.8) as

$$f_{A,B,C}(a_1,a_2)=C\min_{(m_1,m_2)\in\mathbf{Z}^2}\left|m_1\left(\frac{1}{C},\frac{B}{C}\right)+m_2(0,1)-\left(\frac{1}{C}b_1,b_2\right)\right|^2,$$

we see that, up to a constant, $f_{A,B,C}(a_1,a_2)$ minimizes the square distance from $\left(\frac{1}{C}b_1,b_2\right)=\left(\frac{1}{C}a_1,\frac{B}{C}a_1+a_2\right)$ to the lattice spanned by the vectors $\left(\frac{1}{C},\frac{B}{C}\right)$ and $(0,1)$. The claim just proved amounts to the fact that this minimum can be attained only for three points on the lattice, $O(0,0)$, $U\left(\frac{1}{C},\frac{B}{C}\right)$, and $V(0,1)$ (see Fig.1), for all $(a_1,a_2)\in[0,1]\times[0,1/2]$.

Noticing further that under the transformation $(a_1,a_2)\to\left(\frac{1}{C}a_1,\frac{B}{C}a_1+a_2\right)$ the rectangular region $[0,1]\times[0,1/2]$ is mapped onto the parallelogram region spanned by the vectors $\left(\frac{1}{C},\frac{B}{C}\right)$ and $(0,1/2)$ (the shaded area in Fig.1), it becomes obvious that $f_{A,B,C}$ is maximized at the point in $[0,1]\times[0,1/2]$ corresponding to the point M in the parallelogram region equidistant from O, U, and V (see Fig.1). Thus M has coordinates $\left(\frac{A-B}{2},1/2\right)$, as the intersection point of the bisector lines of the sides OU and OV in the triangle OUV, with respective equations

$$\frac{1}{C}\left(x-\frac{1}{2C}\right)+\frac{B}{C}\left(y-\frac{B}{2C}\right)=0\quad\text{and}\quad y=\frac{1}{2}.$$

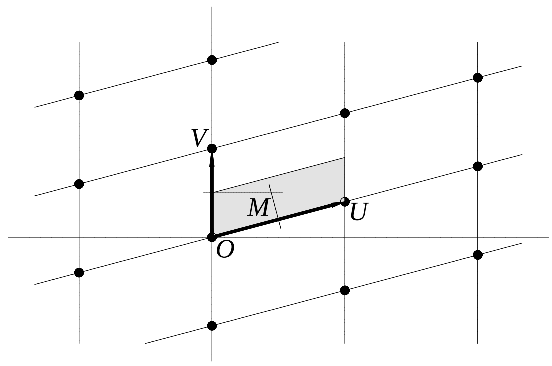

FIGURE 1

It is not hard to see that under the various hypotheses on A, B, and C, the point $M\left(\frac{A-B}{2},\frac{1}{2}\right)$ does belong to the shaded parallelogram region. In conclusion,

$$\max f_{A,B,C}=C\left(\left(\frac{A-B}{2}\right)^2+\left(\frac{1}{2}\right)^2\right).$$

The theorem follows. □

Remark 2.10. — It is clear why the method of proof employed in Theorem 2.5 does not extend to arbitrary metrics. In general, it is difficult to single out the lattice points necessary to calculate $f_{A,B,C}$ on $[0,1] \times \left[0, \frac{1}{2}\right]$. Instead, we will pursue an algorithmic way for computing the Vafa-Witten bound.

3. The General Case

In this section we are going to consider the case of an arbitrary flat metric $[g_{ij}] \leftrightarrow (A,B,C)$ on $\mathbf{T}^2$. It turns out that Theorem 2.5 still holds if $\min\{A,C\} \geq |B|$, while if $\min\{A,C\} < |B|$ the metric transformation (3.3) below will reduce the problem to one where Theorem 2.5 is applicable.

Now write $|B|/\min\{A,C\}$ uniquely as

$$\frac{|B|}{\min\{A,C\}} = \left[\frac{|B|}{\min\{A,C\}}\right] + \left\{\frac{|B|}{\min\{A,C\}}\right\}, \tag{3.1}$$

where $\left[\frac{|B|}{\min\{A,C\}}\right]$ is a non-negative integer and $\left\{\frac{|B|}{\min\{A,C\}}\right\}$ is a real number such that $-1/2 < \left\{\frac{|B|}{\min\{A,C\}}\right\} \leq 1/2$.

Definition 3.2. — With the above notations define the transformation $A \to \widetilde{A}$, $B \to \widetilde{B}$, $C \to \widetilde{C}$, by

$$\widetilde{A} = \min\{A,C\},\ \widetilde{B} = \left|\left\{\frac{|B|}{\min\{A,C\}}\right\}\right| \min\{A,C\},\ \widetilde{C} = \frac{AC - B^2 + \widetilde{B}^2}{\min\{A,C\}} \tag{3.3}$$

Remark 3.4. — The transformation given by (3.3) preserves the determinant quantity $AC - B^2$. This follows from the expression of $\widetilde{C}$. Also, $\widetilde{A} \geq 2\widetilde{B}$ and $\widetilde{B} = |B|$ if (and only if) $\min\{A,C\} \geq 2|B|$.

Theorem 3.5. — *For the torus $\mathbf{T}^2$ with an arbitrary flat metric $[g_{ij}] \leftrightarrow (A,B,C)$ the transformation $A \mapsto \widetilde{A}$, $B \mapsto \widetilde{B}$, $C \mapsto \widetilde{C}$, given by Definition 3.2, yields a new flat metric $[\widetilde{g}_{ij}] \leftrightarrow (\widetilde{A}, \widetilde{B}, \widetilde{C})$, and the two metrics have the same first Vafa-Witten bound, that is*

$$\lambda_1(\widetilde{A}, \widetilde{B}, \widetilde{C}) = \lambda_1(A,B,C) \tag{3.6}$$

Proof. — $[\widetilde{g}_{ij}]$ is a flat metric on $\mathbf{T}^2$ if $\widetilde{A} > 0$, and $\widetilde{A}\widetilde{C} - \widetilde{B}^2 > 0$. This is obvious, since from (3.3), $\widetilde{A} = \min\{A,C\}$ and $\widetilde{A}\widetilde{C} - \widetilde{B}^2 = AC - B^2$, cf. Remark 3.4.

Now, $\lambda_1(A,B,C) = \lambda_1(C,B,A) = \lambda_1(A,|B|,C)$, so there is no loss of generality in assuming that $B \geq 0$ and $A \leq C$, i.e., $\min\{A,C\} = A$. With this assumption we will prove (3.6) by showing that for any $(a_1,a_2) \in \mathbf{R}^2$,

$$f_{A,B,C}(a_1,a_2) = f_{\widetilde{A},\widetilde{B},\widetilde{C}}\left(a_1 + \left[\frac{B}{A}\right] a_2, \epsilon a_2\right), \tag{3.7}$$

where $\left[\frac{B}{A}\right]$, $\left\{\frac{B}{A}\right\}$, are given by (3.1) and $\epsilon = \begin{cases} 1 & , \text{ if } \left\{\frac{B}{A}\right\} \geq 0 \\ -1 & , \text{ if } \left\{\frac{B}{A}\right\} < 0. \end{cases}$ Indeed, since

$$A(m_1 - a_1)^2 + 2B(m_1 - a_1)(m_2 - a_2) + C(m_2 - a_2)^2$$
$$= A\left((m_1 - a_1) + \frac{B}{A}(m_2 - a_2)\right)^2 + \frac{AC - B^2}{A}(m_2 - a_2)^2$$
$$= A\left(\left(m_1 + \left[\frac{B}{A}\right] m_2 - a_1 - \left[\frac{B}{A}\right] a_2\right) + \left\{\frac{B}{A}\right\}(m_2 - a_2)\right)^2 + \frac{AC - B^2}{A}(m_2 - a_2)^2,$$

we have,

$$f_{A,B,C}(a_1, a_2) = \min_{(m_1,m_2)\in\mathbf{Z}^2} A(m_1 - a_1)^2 + 2B(m_1 - a_1)(m_2 - a_2) + C(m_2 - a_2)^2$$
$$= \min_{(m_1,m_2)\in\mathbf{Z}^2} \left(A\left(\left(m_1 + \left[\frac{B}{A}\right] m_2 - a_1 - \left[\frac{B}{A}\right] a_2\right) + \left\{\frac{B}{A}\right\}(m_2 - a_2)\right)^2 \right.$$
$$\left. + \frac{AC - B^2}{A}(m_2 - a_2)^2\right)$$
$$= \min_{(m_1,m_2)\in\mathbf{Z}^2} \left(A\left(\left(m_1 - a_1 - \left[\frac{B}{A}\right] a_2\right) + \epsilon\left|\left\{\frac{B}{A}\right\}\right|(m_2 - a_2)\right)^2 \right.$$
$$\left. + \frac{AC - B^2}{A}(m_2 - a_2)^2\right)$$
$$= \min_{(m_1,m_2)\in\mathbf{Z}^2} \left(\widetilde{A}\left(\left(m_1 - a_1 - \left[\frac{B}{A}\right] a_2\right) + \frac{\widetilde{B}}{\widetilde{A}}(\epsilon m_2 - \epsilon a_2)\right)^2 \right.$$
$$\left. + \frac{\widetilde{A}\widetilde{C} - \widetilde{B}^2}{\widetilde{A}}(\epsilon m_2 - \epsilon a_2)^2\right)$$
$$= \min_{(m_1,m_2)\in\mathbf{Z}^2} \left(\widetilde{A}\left(\left(m_1 - a_1 - \left[\frac{B}{A}\right] a_2\right) + \frac{\widetilde{B}}{\widetilde{A}}(m_2 - \epsilon a_2)\right)^2 \right.$$
$$\left. + \frac{\widetilde{A}\widetilde{C} - \widetilde{B}^2}{\widetilde{A}}(m_2 - \epsilon a_2)^2\right)$$
$$= \min_{(m_1,m_2)\in\mathbf{Z}^2} \left(\widetilde{A}\left(m_1 - a_1 - \left[\frac{B}{A}\right] a_2\right)^2 + 2\widetilde{B}\left(m_1 - a_1 - \left[\frac{B}{A}\right] a_2\right)(m_2 - \epsilon a_2) \right.$$
$$\left. + \widetilde{C}(m_2 - \epsilon a_2)^2\right) = f_{\widetilde{A},\widetilde{B},\widetilde{C}}\left(a_1 + \left[\frac{B}{A}\right] a_2, \epsilon a_2\right). \qquad \square$$

Two things may happen when transforming (A, B, C) into $(\widetilde{A}, \widetilde{B}, \widetilde{C})$:

a) either, $\min\{\widetilde{A}, \widetilde{C}\} \geq 2\widetilde{B}$, in which case Theorem 2.5 and Theorem 3.5 combine to give $\lambda_1(A, B, C)$, or *b*) $\min\{\widetilde{A},\widetilde{C}\} < 2\widetilde{B}$, in which case one can apply the metric transformation again to $(\widetilde{A},\widetilde{B},\widetilde{C})$ and hope to land in Case *a*). Notice that in Case *b*) $\min\{\widetilde{A},\widetilde{C}\} = \widetilde{C}$, since from (3.3), $\widetilde{A} \geq 2\widetilde{B}$.

The nice thing is that by applying the metric transformation (3.3) over and over sufficiently many times one reaches a metric for which Theorem 2.5 holds. The bad thing is that the required number of tries varies with the expression $(AC - B^2)/(\min\{A,C\})^2$, and so an exact formula for λ_1 in terms of A, B, C is unavailable. The rest of this section will be devoted to substantiating these claims.

Theorem 3.8. — *Starting with an arbitrary flat metric* $[g_{ij}] \leftrightarrow (A,B,C)$ *on* $\mathbf{T}^2$ *define the sequence of flat metrics* $\left[{}^k g_{ij}\right] \leftrightarrow (A_k, B_k, C_k)$, $k \geq 0$, *inductively by*

$$(A_0, B_0, C_0) = (A,B,C) \text{ and } (A_{k+1}, B_{k+1}, C_{k+1}) = (\widetilde{A_k}, \widetilde{B_k}, \widetilde{C_k}),\ \ k \geq 0.$$

Assume that $\min\{A,C\} < 2|B|$, *and let* n *be the least non-negative integer such that*

$$\frac{AC - B^2}{(\min\{A,C\})^2} \geq \frac{3}{4}\frac{1}{9^n}. \tag{3.9}$$

Then $\min\{A_{n+1}, C_{n+1}\} \geq 2B_{n+1}$, *and therefore*

$$\lambda_1(A,B,C) = \frac{1}{2}\sqrt{\frac{A_{n+1}C_{n+1}(A_{n+1}+C_{n+1}-2B_{n+1})}{AC - B^2}}.$$

Proof. — Again, without loss of generality we may assume that $A \geq C$ and $B \geq 0$. Notice that if $\min\{A_k, C_k\} \geq 2B_k$ for some k, then (3.3) implies that

$$A_{k+1} = \min\{A_k, C_k\},\ \ B_{k+1} = B_k,\ \text{ and } C_{k+1} = \max\{A_k, C_k\},$$

and so $\min\{A_{k+1}, C_{k+1}\} \geq 2B_{k+1}$.

Assume now, by contradiction, that $\min\{A_{n+1}, C_{n+1}\} < 2B_{n+1}$. From the hypothesis, the above observation, and Remark 3.4, it follows that

$$C_k = \min\{A_k, C_k\} < 2B_k, \quad \text{for all} \quad 0 \leq k \leq n+1. \tag{3.10}$$

Since $C_{k+1} = \frac{(AC-B^2)+B_{k+1}^2}{C_k}$, Equation 3.10 implies that

$$AC - B^2 < 2B_{k+1}C_k - B_{k+1}^2, \quad \text{for all} \quad 0 \leq k \leq n. \tag{3.11}$$

However,

$$B_{k+1} = \left|\left\{\frac{B_k}{C_k}\right\}\right| C_k \leq \frac{1}{2}C_k, \tag{3.12}$$

and so $2B_{k+1}C_k - B_{k+1}^2 \leq \frac{3}{4}C_k^2$. This, combined with (3.11) gives

$$AC - B^2 < \frac{3}{4}C_k^2, \quad \text{for all} \quad 0 \leq k \leq n. \tag{3.13}$$

We claim that in fact (3.13) implies that $AC - B^2 < \frac{3}{4}\frac{1}{9^n}C^2$, which contradicts the hypothesis (3.9). We will prove this claim by means of the following lemma:

Lemma 3.14. — *The hypothesis being the same as in Theorem 3.8, if for some k, $1 \le k \le n$ and for some α, $0 < \alpha \le \frac{3}{4}$, $AC - B^2 < \alpha C_k^2$, then $AC - B^2 < \frac{\alpha}{9} C_{k-1}^2$.*

Proof of Lemma 3.14. — The formula $C_k = \frac{(AC-B^2)+B_k^2}{C_{k-1}}$, in conjunction with $B_k \le \frac{1}{2} C_{k-1}$, which follows from (3.12), yields the inequality

$$C_k \le \frac{(AC - B^2) + \frac{1}{4} C_{k-1}^2}{C_{k-1}}. \tag{3.15}$$

Since by hypothesis, $\frac{\sqrt{AC-B^2}}{\sqrt{\alpha}} < C_k$, (3.15) implies that

$$(AC - B^2) - \frac{C_{k-1}}{\sqrt{\alpha}} \sqrt{AC - B^2} + \frac{1}{4} C_{k-1}^2 > 0. \tag{3.16}$$

We can look at Equation 3.16 as a quadratic polynomial

$$P(t) := t^2 - \frac{C_{k-1}}{\sqrt{\alpha}} t + \frac{1}{4} C_{k-1}^2$$

which for $t = \sqrt{AC - B^2}$ takes a positive value. The roots of this quadratic polynomial are

$$t_{1,2} = \frac{C_{k-1}}{2\sqrt{\alpha}} (1 \pm \sqrt{1 - \alpha}).$$

As a result, either $\sqrt{AC - B^2} < \frac{C_{k-1}}{2\sqrt{\alpha}}(1 - \sqrt{1-\alpha})$ or $\sqrt{AC - B^2} > \frac{C_{k-1}}{2\sqrt{\alpha}}(1 + \sqrt{1-\alpha})$. We will show that $\sqrt{AC - B^2} > \frac{C_{k-1}}{2\sqrt{\alpha}}(1 + \sqrt{1-\alpha})$ cannot happen. Indeed, if this happened then a use of (3.13) would give $\frac{\sqrt{3}}{2} > \frac{1}{2\sqrt{\alpha}}(1 + \sqrt{1-\alpha})$. However, it is easy to see that if $0 < \alpha \le \frac{3}{4}$ then the opposite inequality holds: $\frac{\sqrt{3}}{2} \le \frac{1}{2\sqrt{\alpha}}(1 + \sqrt{1-\alpha})$. Thus,

$$\sqrt{AC - B^2} < \frac{C_{k-1}}{2\sqrt{\alpha}} (1 - \sqrt{1 - \alpha})$$

which for $0 < \alpha \le \frac{3}{4}$ implies $\sqrt{AC - B^2} < \frac{\sqrt{\alpha}}{3} C_{k-1}$, or equivalently $AC - B^2 < \frac{\alpha}{9} C_{k-1}^2$. ☐

Going back to the proof of Theorem 3.8, since $AC - B^2 < \frac{3}{4} C_n^2$, a repeated use of Lemma 3.14 gives $AC - B^2 < \frac{3}{4} \frac{1}{9^n} C^2$, a violation of (3.9). ☐

Remark 3.17. — Theorem 3.8 shows that for arbitrary metrics, $\lambda_1(A, B, C)$ can be calculated in at most $n + 1$ steps, where n is given by (3.9). In practice, fewer steps are required, and in fact we will show in the following corollary that λ_1 can be calculated in p steps, if p is the least integer such that $\min\{A_p, C_p\} \ge |B_p|$.

Corollary 3.18. — *Let $[g_{ij}] \leftrightarrow (A, B, C)$ be an arbitrary flat metric on the torus $\mathbf{T}^2$.*

a) If $\min\{A, C\} \ge |B|$, then the first Vafa-Witten constant is given by

$$\lambda_1(A, B, C) = \frac{1}{2} \sqrt{\frac{AC(A + C - 2|B|)}{AC - B^2}}.$$

b) If $\min\{A,C\} < |B|$, *define the sequence of flat metrics* $\left[{}^k g_{ij}\right] \leftrightarrow (A_k, B_k, C_k)$, $k \geq 0$, *on* $\mathbf{T}^2$ *by*

$$(A_0, B_0, C_0) = (A, B, C), \quad (A_{k+1}, B_{k+1}, C_{k+1}) = (\widetilde{A_k}, \widetilde{B_k}, \widetilde{C_k}), \ k \geq 0.$$

There is a (least) integer p *such that* $\min\{A_p, C_p\} \geq B_p$, *and then*

$$\lambda_1(A,B,C) = \frac{1}{2}\sqrt{\frac{A_pC_p(A_p + C_p - 2B_p)}{AC - B^2}}$$

Moreover, $p \leq n+1$, *where* n *is the least integer such that*

$$\frac{AC - B^2}{(\min\{A,C\})^2} \geq \frac{1}{2}\frac{1}{11^n}. \tag{3.19}$$

Proof. — The point of *a*) is that one can extend Theorem 2.5 at no cost from the case $\min\{A,C\} \geq 2|B|$ to the broader case $\min\{A,C\} \geq |B|$. To this end, assume that $2|B| > \min\{A,C\} \geq |B|$, which becomes $2B > C \geq B$ if we require, as we may, $B \geq 0, A \geq C$. Define now the sequence $(A_k, B_k, C_k)_{k=0}^{\infty}$ as in Theorem 3.8. Then $2B > C \geq B$ implies that $A_1 = C$, $B_1 = C - B$, and $C_1 = A + C - 2B$. Notice that

$$\min\{A_1, C_1\} \geq B_1 \quad \text{and} \quad A_1C_1(A_1 + C_1 - 2B_1) = AC(A + C - 2B).$$

By repeating this argument we conclude that

$$\min\{A_k, C_k\} \geq B_k \quad \text{and} \quad A_kC_k(A_k + C_k - 2B_k) = AC(A + C - 2B), \ k \geq 0.$$

By Theorem 3.8, for $k = n+1$, with n given by (3.9), we have $\min\{A_{n+1}, C_{n+1}\} \geq 2B_{n+1}$, and then

$$\lambda_1(A,B,C) = \frac{1}{2}\sqrt{\frac{A_{n+1}C_{n+1}(A_{n+1} + C_{n+1} - 2B_{n+1})}{AC - B^2}} = \frac{1}{2}\sqrt{\frac{AC(A + C - 2B)}{AC - B^2}}$$

b) For the proof of *b*) we can use theorem 3.8, since $\min\{A,C\} < |B|$ is merely a subcase of $\min\{A,C\} < 2|B|$. Being mindful of *a*) we can adjust the proof of Theorem 3.8 so that we stop the iterations after reaching an index k satisfying the weaker inequality $\min\{A_k, C_k\} \geq B_k$. The net gain is a slightly better a priori stopping condition than (3.9), namely (3.19). □

It is natural to ask whether or not the stopping index p of Corollary 3.18, *b*) is independent of the metric. The answer is no, as demonstrated by the following example.

Example 3.20. — The stopping index p of Corollary 3.18,b) can be arbitrarily large. For any non-negative integer m the assignment (a_m, b_m, c_m) given by

$$\begin{aligned} a_m =& \frac{2+\sqrt{2}}{4}(3+2\sqrt{2})^m + \frac{2-\sqrt{2}}{4}(3-2\sqrt{2})^m \\ b_m =& \frac{\sqrt{2}}{4}(3+2\sqrt{2})^m - \frac{\sqrt{2}}{4}(3-2\sqrt{2})^m \\ c_m =& \frac{2-\sqrt{2}}{4}(3+2\sqrt{2})^m + \frac{2+\sqrt{2}}{4}(3-2\sqrt{2})^m \end{aligned} \tag{3.21}$$

defines a metric for which the stopping index is exactly m. Moreover, $\lambda_1(a_m, b_m, c_m) = \sqrt{2}/2$.

Proof. — Although it may not look so, the assignment (3.21) is the simplest example with the property that

$$(\widetilde{a_m}, \widetilde{b_m}, \widetilde{c_m}) = (a_{m-1}, b_{m-1}, c_{m-1}) \quad \text{and} \quad \min\{a_m, c_m\} < b_m, \quad \text{for } m \geq 1. \tag{3.22}$$

Indeed, according to the transformation (3.3), $\widetilde{a_m} = \min\{a_m, c_m\}$, which for convenience can be taken to be c_m, for all m. Thus, (3.22) gives $c_m = a_{m-1}$. Also, $\widetilde{b_m} = \left|\left\{\frac{b_m}{c_m}\right\}\right| c_m$, since b_m must be positive. Thus,

$$\left|\left\{\frac{b_m}{a_{m-1}}\right\}\right| a_{m-1} = b_{m-1}, \tag{3.23}$$

and (3.23) will certainly hold if

$$\frac{b_m}{a_{m-1}} = 2 + \frac{b_{m-1}}{a_{m-1}}, \quad \text{or} \quad b_m = 2a_{m-1} + c_{m-1}. \tag{3.24}$$

(The simpler choice of integer, 1 instead of 2, in (3.24) will not work, since (3.22) requires $b_m \leq 2a_m$). Finally, the invariance of the quantity $a_m c_m - b_m^2$ under the transformation (3.3) suggests that one might want to set $a_m c_m - b_m^2 = 1$, which gives $a_m = 4a_{m-1} + 4b_{m-1} + c_{m-1}$. Therefore, we obtain the linear recurrent system, for $m \geq 1$,

$$\begin{aligned} a_m &= 4a_{m-1} + 4b_{m-1} + c_{m-1} \\ b_m &= 2a_{m-1} + b_{m-1} \\ c_m &= a_{m-1} \end{aligned} \tag{3.25}$$

We want to subject the above system to a simple initial condition (a_0, b_0, c_0) for which $\min\{a_0, c_0\} \geq b_0$, for instance $(a_0, b_0, c_0) = (1, 0, 1)$. Then the solution of system (3.25) with this initial condition is precisely (3.21).

Indeed, the matrix of this system,

$$M = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

having eigenvalues -1, $3 \pm 2\sqrt{2}$, with eigenvectors $(-1,1,1)$, $(1 \pm \sqrt{2}, 1, -1 \pm \sqrt{2})$, is diagonalizable and $M = P\Delta P^{-1}$, where

$$P = \begin{bmatrix} -1 & 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 & 1 \\ 1 & -1+\sqrt{2} & -1-\sqrt{2} \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3+2\sqrt{2} & 0 \\ 0 & 0 & 3-2\sqrt{2} \end{bmatrix}.$$

Therefore, the solution of the system (3.25) is

$$\begin{bmatrix} a_m \\ b_m \\ c_m \end{bmatrix} = P\Delta^m P^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which amounts exactly to (3.21). The first five triples (a_m, b_m, c_m) are $(1,0,1)$, $(5,2,1)$, $(29,12,5)$, $(169,70,29)$, and $(985,408,169)$. □

4. The Homogeneous Space Viewpoint

In this section we interpret our previous results by looking at flat metrics on $\mathbf{T}^2$ the homogeneous way, as objects in $\mathrm{GL}(2,\mathbf{R})/\,\mathrm{O}(2)$. As indicated in the Introduction it suffices to analyze metrics of determinant 1, i.e., elements of the space $\mathrm{SL}(2,\mathbf{R})/\,\mathrm{SO}(2)$.

Recall first some classical results about $\mathrm{SL}(2,\mathbf{R})$ [**L, T**]. $\mathrm{SL}(2,\mathbf{R})/\,\mathrm{SO}(2)$ can be identified canonically with the Poincaré upper half plane $H := \{z \in \mathbf{C} \mid \Im(z) > 0\}$, via the transformation

$$\mathrm{SL}(2,\mathbf{R})/\,\mathrm{SO}(2) \ni \widehat{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \longmapsto \frac{ai+b}{ci+d} \in H. \tag{4.1}$$

Iwasawa decomposition in $\mathrm{SL}(2,\mathbf{R})$ [**L**] shows that the inverse of (4.1) is

$$H \ni z = x + iy \longmapsto \widehat{\begin{bmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{bmatrix}} \in \mathrm{SL}(2,\mathbf{R})/\,\mathrm{SO}(2). \tag{4.2}$$

Under these identifications the natural left action of $\mathrm{SL}(2,\mathbf{Z})$ on $\mathrm{SL}(2,\mathbf{R})/\,\mathrm{SO}(2)$ translates to the following action of $\mathrm{SL}(2,\mathbf{Z})$ on H,

$$\mathrm{SL}(2,\mathbf{Z}) \times H \ni \left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, z \right) \longmapsto \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot z := \frac{\alpha z + \beta}{\gamma z + \delta} \in H. \tag{4.3}$$

Thus $\mathrm{SL}(2,\mathbf{Z})\backslash\,\mathrm{SL}(2,\mathbf{R})/\,\mathrm{SO}(2)$ identifies with $\mathrm{SL}(2,\mathbf{Z})\backslash H$. Recall now that the standard fundamental domain for the action of $\mathrm{SL}(2,\mathbf{Z})$ on H is (see Fig.2)

$$F := \{z \in H \mid -1/2 < \Re(z) \leq 1/2,\ |z| \geq 1,\ \text{and if } |z| = 1,\ \text{then } \Re(z) \geq 0\}.$$

From (4.2) and the discussion preceding (1.4) we see now that for a 'metric' $z = x + iy \in H$ the inverse of the metric tensor is given, with the notations from Section 2, by the quantities

$$A = y + x^2 y^{-1},\ B = xy^{-1},\ C = y^{-1}. \tag{4.4}$$

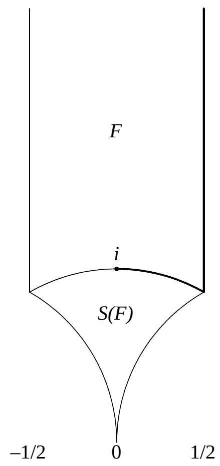

FIGURE 2

Equivalently,

$$x = \frac{B}{C},\ y = \frac{1}{C}. \tag{4.5}$$

Therefore, the first Vafa-Witten bound defines an automorphic form λ_1 on H, given by

$$\lambda_1(z) = \max_{\mathbf{a}\in\mathbf{R}^2} \min_{\mathbf{m}\in\mathbf{Z}^2} \sqrt{(y+x^2y^{-1})(m_1-a_1)^2 + 2xy^{-1}(m_1-a_1)(m_2-a_2) + y^{-1}(m_2-a_2)^2}, \quad z = x+iy \in H \tag{4.6}$$

Theorem 4.7. — *a) When restricted to the fundamental domain F the automorphic form λ_1 given by Equation 4.6 admits the explicit expression*

$$\lambda_1(z) = \frac{1}{2y}\sqrt{\frac{(x^2+y^2)\left((|x|-1)^2+y^2\right)}{y}}, \quad z = x+iy \in F.$$

b) If $\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \in \mathrm{SL}(2,\mathbf{Z})$ and $z = x+iy \in F$, then

$$\lambda_1\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = \frac{1}{2y}\sqrt{\frac{(x^2+y^2)\left((|x|-1)^2+y^2\right)}{y}}.$$

Proof. — *a*) follows immediately from Theorem 2.5 and Equation 4.4, if we show that for $z = x+iy \in F$, $\min\{y+x^2y^{-1}, y^{-1}\} \geq 2|x|y^{-1}$, or equivalently $\min\{x^2+y^2, 1\} \geq 2|x|$. But the latter inequality is obvious, since on F, $x^2+y^2 \geq 1$ and $|x| \leq \frac{1}{2}$.

$b)$ is a simple consequence of $a)$, (4.3), and the fact, noted in the Introduction, that the first Vafa-Witten bound is invariant under the action of $\mathrm{SL}(2,\mathbf{Z})$ on $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)$. □

Remark 4.8. — The automorphic form λ_1 is also invariant under the transformation $z \to -\overline{z}$ of H, which clearly does not come from the $\mathrm{SL}(2,\mathbf{Z})$ action on H.

We conclude the paper by explaining how the transformation (3.2), $(A,B,C) \mapsto (\widetilde{A},\widetilde{B},\widetilde{C})$, implements the map $H \to F$, induced by the projection $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2) \to \mathrm{SL}(2,\mathbf{Z})\backslash\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)$. Strictly speaking it does not, since in our desire to work with a nonnegative $\widetilde{B}$ we incorporated in (3.2) an operation foreign to $\mathrm{SL}(2,\mathbf{Z})$, namely the one mentioned in Remark 4.8. But one can easily redefine (3.2) to stay inside $\mathrm{SL}(2,\mathbf{Z})$.

Definition 4.9. — If (A,B,C) is as in Section 2, redefine the transformation $A \mapsto \widetilde{A}$, $B \mapsto \widetilde{B}$, $C \mapsto \widetilde{C}$ of (3.3) by

$$\widetilde{A} = \min\{A,C\},\ \widetilde{B} = -\left\{\frac{B}{\min\{A,C\}}\right\}\min\{A,C\},\ \widetilde{C} = \frac{AC - B^2 + \widetilde{B}^2}{\min\{A,C\}}. \tag{4.10}$$

Clearly, in (3.3) and (4.10) $\widetilde{A}$ and $\widetilde{C}$ remain the same, while the $\widetilde{B}$'s may differ by at most a sign. Therefore, all the results in Section 3 remain valid if one replaces (3.3) with (4.9).

For the purpose of stating the next result let us introduce two transformations on H induced by elements of $\mathrm{SL}(2,\mathbf{Z})$:

$$S(z) = -\frac{1}{z} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot z \quad \text{and} \quad T(z) = z+1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot z.$$

Algorithm 4.11. — *The map $\phi : H \to F$, given by $\phi(z) = w$ iff $z \in H$, $w \in F$, and there is $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2,\mathbf{Z})$ such that $\dfrac{\alpha z+\beta}{\gamma z+\delta} = w$, can be constructed according to the following algorithm:*

Step 1. If $z = x+iy$ satisfies $\min\{x^2+y^2,1\} \geq 2|x|$, then exactly one of z, $S(z)$, $T(z)$, or $TS(z)$ belongs to F. Call it w.

Step 2. If $\min\{x^2+y^2,1\} < 2|x|$, make sure that $x^2+y^2 \geq 1$, eventually by replacing z with $S(z)$ to achieve that. Then, replace the new z with $\dfrac{-(x-n)+iy}{(x-n)^2+y^2} = ST^{-n}(z)$, where n is the unique integer such that $x = n+\epsilon$, for some $-1/2 < \epsilon \leq 1/2$.

Step 3. Repeat Step 2 for the new z until one gets a $z = x+iy$ such that $\min\{x^2+y^2,1\} \geq 2|x|$. This can be achieved in at most $p+1$ steps of type 2, where p is the least integer such that for the original z from Step 1,

$$\frac{y}{\min\{x^2+y^2,1\}} \geq \frac{\sqrt{3}}{2}\frac{1}{3^p}.$$

Then, apply Step 1 to this last z.

Proof. — To justify Step 1, notice first that $\min\{x^2+y^2,1\} \geq 2|x|$ is equivalent with $|x| \leq 1/2$, $(x-1)^2+y^2 \geq 1$, $(x+1)^2+y^2 \geq 1$ (Fig.2). Thus, $z \in \overline{F} \cup S(\overline{F})$. The conclusion then follows by looking at what S, T do to $\overline{F} \setminus F$, and the fact that $S^2 = I$.

Step 2 is precisely an implementation of the transformation (4.10) at the level of points in H, via (4.4) and (4.5).

Finally, Step 3 is the translation of Theorem 3.8 to points of H, again based on the dictionary provided by Equations 4.4 and 4.5. □

Remark 4.12. — In the literature, the map $\phi : H \to F$ is proven to exist, in connection with showing that F is a fundamental domain. We are not aware of any place which gives a constructive definition of it.

Remark 4.13. — Studying the expression of the automorphic form λ_1 given in Theorem 4.7, a) one concludes that the first Vafa-Witten bounds associated to variable metrics of determinant 1 admit an absolute minimum of $\sqrt{2}/\sqrt[4]{27}$, corresponding to $x = \pm 1/2$ and $y = \sqrt{2}/2$, or $A = C = 2\sqrt{3}/3$, $B = \pm\sqrt{3}/3$. The lattice spanned by the vectors $\left(\frac{1}{C}, \frac{B}{C}\right)$ and $(0,1)$ is in this case the hexagonal lattice, which provides the thinnest lattice covering of the plane [**CS**].

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N. Anghel, Department of Mathematics, University of North Texas, Denton, TX 76203
E-mail : `anghel@unt.edu`