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APPLICATIONS OF CURVED BERNSTEIN-GELFAND-GELFAND SEQUENCES

by

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Abstract. — I discuss applications of Bernstein-Gelfand-Gelfand sequences in conformal differential geometry.

***Résumé* (Applications des suites de Bernstein-Gelfand-Gelfand courbées).** — J'étudie des applications des suites de Bernstein-Gelfand-Gelfand en géométrie différentielle conforme.

One of the themes of mathematics in the twentieth century has been the growing realization that representation theory and geometry are closely related. There are at least two aspects to this. Firstly, there is the geometric study of representation theory that follows inevitably from the definition of a Lie group, where global methods of geometry and topology are applied to homogeneous spaces. Secondly, there is the increasing use of representation theory as a tool and language for the invariant analysis of geometric structure—that is to say, the local (pointwise) aspects of differential geometry. Although this second aspect also has a long history (the work of Cartan stretches back into the nineteenth century), I think it is fair to say that only in the last twenty years or so has representation theory really begun to gain ground as an alternative to the hands-on approach of local coordinate computations. One area which has motivated this shift is quaternionic geometry, the study of which only intensified relatively recently in the history of differential geometry, driven by many different forces, such as supersymmetry, the classification of metric holonomies, and the geometry of moduli spaces. Confronted by an unfamiliar geometry, geometers turned to the representation theory of $\mathbb{H}^* \cdot \mathrm{GL}(n, \mathbb{H})$ and its subgroups (such as $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)\,\mathrm{Sp}(n)$) as an efficient way to develop intuition.

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This paper concerns the role of representation theory in geometric structures, but, in keeping with the theme of the conference (and the author's interests), the emphasis will be on conformal and Riemannian geometry, rather than quaternionic geometry. The aim is demonstrate that even in these classical subjects, representation theoretic methods not only provide an efficient language when computations become unmanageable, but also that they lead to genuinely new insights and constructions. Nowhere has this been more true than in the study of invariant differential operators, and so I will focus on an area where much progress has been made in recent years: Bernstein-Gelfand-Gelfand sequences.

1. Parabolic geometries and the BGG sequences

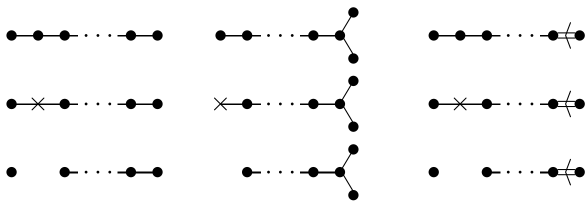
A parabolic geometry is a geometry modelled on a generalized flag variety G/P, where G is a semisimple Lie group and P is a parabolic subgroup. These compact homogeneous spaces arise naturally in representation theory as projectivized orbits of highest weight vectors in irreducible representations. A parabolic subgroup P has a decomposition $P = L \ltimes N$, where L is reductive and N is nilpotent—L is often called the Levi factor. It is convenient to describe parabolic subgroups by crossing nodes on the Dynkin diagram of G, so that if the crossed nodes (and adjoining lines) are deleted, the result is the Dynkin diagram of (the semisimple part of) L. Such diagrams will also denote the corresponding flag varieties. Strictly speaking, these diagrams describe complex geometries, whereas real flag varieties should be denoted by Satake diagrams with crosses. I shall ignore this distinction. Here are some examples, together with names for the corresponding parabolic geometries (or rather, for suitable real forms of these geometries).

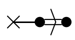
Quaternionic Even conformal Quaternionic CR

G semisimple Lie group

P parabolic subgroup

L Levi factor



The key example for this paper is $G = \mathrm{SO}(n+1,1)$, $P = \mathrm{CO}(n) \ltimes (\mathbb{R}^n)^*$, with $G/P \cong S^n$. This is the conformal sphere, identified as the "sky" in $(n+1,1)$-dimensional spacetime. The Dynkin diagram is shown above for $n \geqslant 6$ even. For $n = 1$–5, the diagrams are $\times$, $\times\ \times$, , , . The geometries for $n = 1, 2$ are projective and Möbius geometry respectively.

Geometries "modelled on" homogeneous spaces are most simply defined as Cartan geometries, i.e., one views the homogeneous geometry on G/P as a principal P-bundle

$G \to G/P$ equipped with the parallelism $TG \cong G \times \mathfrak{g}$; then a curved analogue of this is a principal P-bundle $\mathcal{G} \to M$ equipped with a Cartan connection.

1.1. Definition. — Let M be a manifold of the same dimension as G/P.

(i) A *Cartan geometry* of type $(\mathfrak{g}, P)$ on M is a principal P-bundle $\pi\colon \mathcal{G} \to M$, together with a P-equivariant $\mathfrak{g}$-valued 1-form $\eta\colon T\mathcal{G} \to \mathfrak{g}$ such that for each $u \in \mathcal{G}$, $\eta_u\colon T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism restricting to the canonical isomorphism between $T_u(\mathcal{G}_{\pi(u)})$ and $\mathfrak{p}$.

(ii) The *curvature* $K\colon \Lambda^2 T\mathcal{G} \to \mathfrak{g}$ of a Cartan geometry is defined by
$$K(U,V) = d\eta(U,V) + [\eta(U), \eta(V)].$$
It induces a curvature function $\kappa\colon \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ via
$$\kappa(u)(\xi,\chi) = K_u\big(\eta^{-1}(\xi), \eta^{-1}(\chi)\big) = [\xi,\chi] - \eta_u[\eta^{-1}(\xi), \eta^{-1}(\chi)],$$
where $u \in \mathcal{G}$ and the brackets are the Lie bracket in $\mathfrak{g}$ and the Lie bracket of vector fields on $\mathcal{G}$.

Cartan geometries usually arise from a more familiar geometric structure on M by a process of prolongation [**10, 26**]. In the case of conformal geometry this prolongation procedure is the famous construction of the normal Cartan connection.

Associated to a P-module $\mathbb{E}$ is a vector bundle $E = \mathcal{G} \times_P \mathbb{E}$. For example, the Cartan connection η identifies the tangent bundle of M with $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$. An important special case of this is a G-module $\mathbb{W}$, viewed as a P-module by restriction. In this case, $W = \mathcal{G} \times_P \mathbb{W} = \tilde{\mathcal{G}} \times_G \mathbb{W}$, where $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$. A Cartan connection on $\mathcal{G}$ induces a principal bundle connection of $\tilde{\mathcal{G}}$ and hence a covariant derivative on W. In various contexts, such linear representations of the Cartan connection have been called "tractor" or "local twistor" connections [**1, 2**].

When the Cartan connection is flat (e.g., on the homogeneous model), parallel sections of W can be identified, at least locally, with $\mathbb{W}$, so we have

$$0 \to \mathbb{W} \to \mathrm{C}^\infty(W) \to \mathrm{C}^\infty(T^*M \otimes W). \tag{1.1}$$

This is the beginning of a resolution, the (dualized, generalized) Bernstein-Gelfand-Gelfand resolution, of $\mathbb{W}$ (or, more accurately, of the sheaf of parallel sections of W).

A differential geometer or topologist, asked to extend (1.1), would immediately come up with a resolution by a complex of first order differential operators, the twisted de Rham complex:

$$\mathrm{C}^\infty(W) \to \mathrm{C}^\infty(T^*M \otimes W) \to \mathrm{C}^\infty(\Lambda^2 T^*M \otimes W) \to \mathrm{C}^\infty(\Lambda^3 T^*M \otimes W) \to \cdots$$

The problem with this resolution is that W is often quite a complicated bundle, and hence so are the bundles in this resolution. Bernstein, Gel'fand and Gel'fand [**5**] found a way to break up these bundles under the action of P using differential projections and hence obtain a resolution with much simpler bundles, but perhaps higher order differentials. (In fact, they only considered the Borel case, and their construction was generalized to arbitrary parabolics by Lepowsky [**22**].)

This paper is about the curved version of this construction, which began with the work of Baston [**3**] (see also [**7, 14, 16**]), who gave a double complex construction of the BGG resolution in the case that N is abelian, and argued that his construction could be generalized to curved geometries, with one crucial difference. Namely (as one might expect from the twisted de Rham sequence above) the curved BGG sequence is no longer a complex. Unfortunately, Baston's proofs were not entirely clear, and it was not until the work of Čap, Slovák, and Souček [**11**] that the curved BGG sequences were obtained for arbitrary parabolic geometries, and with full arguments.

My goal in this paper is to try to indicate why this is a significant achievement and why curved BGG sequences might be useful in practice. In other words, my focus will be on applications rather than the theory.

For applications, the main thing needed from the theory is an efficient algorithm for computing the bundles in the BGG sequence. These bundles are associated to Lie algebra homology groups, and most algorithms to compute them are based on Kostant's theorem [**20**], which states that this homology can be obtained by applying the affine action of Weyl group reflections to the highest weight vector of $\mathbb{W}$.

I want to explain how to do this, using some notation devised a few years ago for representations of parabolic subgroups.

Let G have rank m and let $\varepsilon_1, \dots \varepsilon_m$ be an orthonormal basis for the Cartan subalgebra $\mathfrak{h} \cong \mathfrak{h}^*$ so that the roots are given in the "standard" form that one finds in any book on Lie theory (e.g. [**19**]): for type A_m and G_2 it is more convenient to identify $\mathfrak{h}$ with $\mathbb{R}^{m+1}/\langle(1,1,\dots 1)\rangle$ or $\mathbb{R}^3/\langle(1,1,1)\rangle$. In terms of this basis, all but one or two of the simple roots of $\mathfrak{g}$ are of the form $\varepsilon_i - \varepsilon_{i+1}$.

The highest weights of irreducible representations may now be described by m-tuples $(\lambda_1, \dots \lambda_m)$—or $(m+1)$-tuples for A_m and G_2—where the entries are "integral" in the sense that the inner products with the coroots are integers (this usually means the entries are all integers, or all half-integers in some cases). In order to have a representation of G, this weight has to be to be G-*dominant*, i.e., the inner product of $\sum \lambda_i \varepsilon_i$ with the simple roots should be non-negative. In practice this means that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$ with some additional inequalities depending on the type of the Lie group. For representations of P, only P-dominance is needed, i.e., the inequality only needs to hold for roots of the Levi factor of P. In terms of the Dynkin diagram, whose nodes are in one-to-one correspondence with simple roots, this means the inequalities corresponding to crossed nodes need not hold.

If the node corresponding to $\varepsilon_i - \varepsilon_{i+1}$ is crossed, then the highest weight of a P-representation need not satisfy $\lambda_i \geqslant \lambda_{i+1}$. This can be indicated by writing the highest weights of P-representations as $(\lambda_1, \dots \lambda_i | \lambda_{i+1}, \dots \lambda_m)$. If this highest weight happens to satisfy $\lambda_i \geqslant \lambda_{i+1}$ anyway, then the corresponding P-representation is the irreducible P-subrepresentation generated by the highest weight vector of the G-representation associated to $(\lambda_1, \ldots \lambda_i, \lambda_{i+1}, \ldots \lambda_m)$.

This works well for the simple examples which arise in practice. If more nodes are crossed, then more bars are needed, although in general, one would have to dream up a different notation for the exceptional roots at the right hand end of the Dynkin diagram.

The Weyl group is generated by reflections in the hyperplanes orthogonal to the simple roots. The advantage of working in an orthonormal basis is that reflections are easy to handle. In particular, a (not necessarily simple) root $\varepsilon_i - \varepsilon_j$ acts by exchanging λ_i and λ_j. The affine action is obtained by conjugating this action with translation by the half sum of the positive roots. In practice this means that $(\varepsilon_i - \varepsilon_j).\lambda = \tilde{\lambda}$ where $\tilde{\lambda}_i = \lambda_j - (j-i)$ and $\tilde{\lambda}_j = \lambda_i + (j-i)$. Note that if $\lambda_i \geqslant \lambda_j$ (for $i < j$) then $\tilde{\lambda}_i < \tilde{\lambda}_j$.

A more systematic notation involves using the basis corresponding to the fundamental weights, perhaps indicating the coefficients by writing them above the nodes on the Dynkin diagram [**4**]. The problem with this notation is that non-simple root reflections are difficult to apply, which entails a two pass procedure to compute the BGG sequence (again, see [**4**]).

The zeroth bundle in the BGG sequence of a G-representation λ is associated to the irreducible P-subrepresentation with the same highest weight. The first bundle is associated to the direct sum of all irreducible P-representations whose highest weight can be obtained by applying a simple affine root reflection to λ. The second bundle is obtained by applying a second, not necessarily simple, affine root reflection to these weights and keeping the P-dominant weights which can be obtained from λ by a composite of two simple affine root reflections. One continues applying root reflections in this way, so that the length of the element of the Weyl group (in terms of simple roots) increases by one at each step. It is usually easy to see which reflections will give P-dominant weights: in simple examples there are often only one or two reflections which work, and the they will often automatically increase the length of the element of the Weyl group by one. Furthermore, the same sequence of reflections generates the BGG sequence of any generic G-representation. In conformal geometry, the sequence of root reflections begins $\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_4, \ldots$.

For example, the BGG resolution of the trivial representation in conformal geometry starts as follows:

$$0 \to (0,0,0,\ldots 0) \to (0|0,0,\ldots 0) \to (-1|1,0,\ldots 0) \to (-2|1,1,0,\ldots 0) \to \cdots$$

Here, and throughout the paper, I adopt the usual convention of denoting a representation, an associated bundle and its (sheaf of) smooth sections by the corresponding highest weight. In conventional terms, the first entry of $(\lambda_1|\lambda_2, \ldots \lambda_m)$ is the conformal weight, while the remaining elements describe the representation of $\mathrm{SO}(n)$, the semisimple part of L. If $\sum_{i\geqslant 2} |\lambda_i| = k$ then the representation is a subspace of $\otimes^k \mathbb{R}^n$.

In particular, if $(\lambda_2, \dots \lambda_\ell) = (1, 1, \dots 1, 0, 0, \dots 0)$ with k ones, then the representation is just $\Lambda^k \mathbb{R}^n$. This notation is consistent with Fegan [**17**], except for the sign of the conformal weight.

The BGG sequence of the trivial representation given above is therefore simply the de Rham complex, *except* that in even dimensions, the middle-dimensional forms decompose into two components under the conformal group, namely, the selfdual and antiselfdual forms.

This is a complex even in the curved case.

2. The adjoint representation

The most "natural" representation of G is the adjoint representation $\mathfrak{g}$. The corresponding BGG sequence governs the deformation theory of the parabolic geometry. In conformal geometry, the adjoint representation is $(1, 1, 0, \dots 0)$ and the BGG sequence begins:

$$(1|1, 0, \dots 0) \to (0|2, 0, \dots 0) \to (-2|2, 2, 0, \dots 0) \to (-3|2, 2, 1, 0, \dots 0) \to \cdots$$

The first three sheaves arising here are: vector fields, symmetric traceless weightless bilinear forms (which are linearized conformal metrics) and Weyl tensors (which are linearized conformal curvatures). The operators are conformally invariant, so one can identify the first three as: the conformal Killing (or Ahlfors) operator, the linearized Weyl curvature operator (which is second order), and the conformal Bianchi operator. In the flat case, this is a complex, and its first three cohomology groups give the conformal vector fields, the formal tangent space to the moduli space of flat conformal structures, and the obstruction space for integrating deformations.

The above sequence is only completely correct in more than four dimensions. In three dimensions, the linearized Weyl curvature is replaced by the third order linearized Cotton-York operator, while in four dimensions, the bundle of Weyl tensors splits into selfdual and antiselfdual parts and the conformal Bianchi identity becomes second order. This decomposition is very important, because it means that on a selfdual conformal 4-manifold, part of the BGG sequence is a complex, namely the sequence

$$(1|1, 0) \to (0|2, 0) \to (-2|2, -2),$$

where $(-2|2, -2)$ denotes the sheaf of antiselfdual Weyl tensors. It is precisely this sequence which lies behind the deformation theory of selfdual conformal structures [**13, 21**].

Similar ideas should apply in other examples, such as quaternionic and quaternionic CR geometry.

3. Selfdual Einstein metrics

The remaining examples in the paper will mostly concern the $\mathrm{SO}(n+1,1)$-representation $(1,1,1,0,\ldots 0)$, with BGG sequence

$$(1|1,1,0,\ldots 0) \to (0|2,1,0,\ldots 0) \to (-1|2,2,0,\ldots 0) \to \cdots$$

or part of this sequence on a selfdual 4-manifold:

$$(1|1,-1) \to (0|2,-1) \to (-1|2,-2).$$

The sheaves appearing here are (antiselfdual) 2-forms of weight 1, weightless Cotton-York tensors and weight -1 Weyl tensors. In terms of the spinor bundles $\Sigma^{\pm} = (\frac{1}{2}|\frac{1}{2},\pm\frac{1}{2})$, the latter sequence is

$$S^2\Sigma^- \to S^3\Sigma^- \otimes \Sigma^+ \to S^4\Sigma^-$$

which is a complex on any selfdual 4-manifold. This is the elliptic complex appearing in Chapter 13 of **[6]**, where Arthur Lancelot Besse (with some help from his friends) gives a proof of the following theorem:

***3.1. Theorem* (Hitchin [18]).** — *Suppose M is a compact selfdual Einstein manifold with positive scalar curvature. Then M is isometric to S^4 with a round metric or $\mathbb{C}P^2$ with a Fubini-Study metric.*

Let me outline the proof. The operator from $S^4\Sigma^- \oplus S^2\Sigma^-$ to $S^3\Sigma^- \otimes \Sigma^+$ is a twisted Dirac operator and the Atiyah-Singer index formula computes its index to be $5\chi - 7\tau$ where χ is the Euler characteristic and τ is the signature of M. A Weitzenböck argument shows that the only contribution to this index is the dimension of the kernel of the operator $\overline{D}_2 \colon S^2\Sigma^- \to S^3\Sigma^- \otimes \Sigma^+$. Hence the dimension of $\ker \overline{D}_2$ is $5\chi - 7\tau = 10 - 2b_+$, where we use a vanishing theorem for half of the de Rham complex (i.e., part of another BGG sequence) to obtain $b_1 = b_- = 0$, so that $\chi = 2 + b_+$ and $\tau = b_+$.

A simple representation-theoretic argument shows that the map sending a 2-form α to the vector field dual to $*(d\alpha)$ maps $\alpha \in \ker \overline{D}_2$ to a Killing field. Since $b_1 = 0$, $d\alpha$ is nonzero for all nonzero α in $\ker \overline{D}_2$ and hence $\dim \mathrm{Isom}\, M \geqslant \dim \ker \overline{D}_2$. The Hitchin-Thorpe inequality shows that $10 - 2b_+ \geqslant 4$ and so our manifold has lots of isometries. Elementary arguments now show that it must be S^4 or $\mathbb{C}P^2$.

4. Applications in Einstein-Weyl geometry

One of my contributions to this subject has been some simple applications in Einstein-Weyl geometry. Weyl geometry is a generalization of Riemannian geometry, in the following sense: a Riemannian manifold is a conformal manifold together with a trivialization of the oriented real line bundle $L^1 = (1|0,\ldots 0)$, whereas a Weyl manifold is a conformal manifold together with a covariant derivative D on this line bundle. Such a covariant derivative induces a torsion-free conformal connection (a "Weyl connection") generalizing the Levi-Civita connection of a Riemannian metric. An *Einstein-Weyl manifold* is a Weyl manifold such that the symmetric tracefree part of the Ricci tensor of the Weyl connection vanishes.

Many concepts from Riemannian geometry generalize naturally to Weyl geometry, except that there is a new feature with no counterpart in the Riemannian case, namely the curvature of the covariant derivative on L^1, which is called the *Faraday curvature* F^D. (This term was coined by Tammo Diemer to reflect the origins of gauge theory and Weyl geometry in electromagnetism.)

The applications I will discuss are about controlling the Faraday curvature using the following trick: twist a complex of differential operators by D on some power of L^1 so that it is no longer a complex unless $F^D = 0$.

For example, dual to the de Rham complex of exterior derivatives is the complex of exterior divergences

$$\cdots \to (2-n|1,1,0,\dots 0) \to (1-n|1,0,\dots 0) \to (-n|0,\dots 0)$$

where n is the dimension of the manifold. On an oriented manifold, this can also be viewed as the end of the de Rham complex. Now twist this complex by D on L^{n-4} to give a sequence

$$(-2|1,1,0,\dots 0) \to (-3|1,0,\dots 0) \to (-4|0,\dots 0)$$

which will no longer be a complex unless $n = 4$ or $F^D = 0$. In particular F^D itself, being a 2-form, is a section of $(-2|1,1,0,\dots 0)$.

4.1. Proposition. — *Let δ^D denote the twisted exterior divergence on 2-forms and suppose $\delta^D F^D = 0$. Then $F^D = 0$ or $n = 4$.*

The proof is easy [**8**]: if $\delta^D F^D = 0$ then $0 = (\delta^D)^2 F^D = (n-4)\langle F^D, F^D\rangle$.

On an Einstein-Weyl manifold the (twice) contracted Bianchi identity shows that $\delta^D F^D = \frac{1}{n} D scal^D$, where $scal^D$ is the trace of the Ricci curvature (using the conformal structure), which is a section of L^{-2}. Hence if $scal^D$ is identically zero, then $F^D = 0$ or $n = 4$. The four dimensional case really does occur: hypercomplex 4-manifolds are Einstein-Weyl with $scal^D = 0$ (see [**23**]).

I now discuss a more complicated example. Suppose $n \geqslant 4$ and M is conformally flat, or $n = 4$ and M is selfdual. Then the BGG sequence of $(1,1,1,0,\dots 0)$ or $(1,1,-1)$ is a complex. Twist this complex with D on L^{-3} to obtain:

$$(-2|1,1,0,\dots 0) \to (-3|2,1,0,\dots 0) \to (-4|2,2,0,\dots 0)$$

or

$$(-2|1,-1) \to (-3|2,-1) \to (-4|2,-2).$$

These sequences of first order differential operators will no longer form a complex unless $F^D = 0$ in the first case, or $F^D_- = 0$ (i.e., F^D is selfdual) in the second case. Now, exactly as in the previous proposition, apply the first operator $\mathcal{D}_0$ in these sequences to F^D or F^D_- to deduce that $\mathcal{D}_0 F^D = 0$ implies $F^D = 0$ in the first case, and that $\mathcal{D}_0 F^D_- = 0$ implies $F^D_- = 0$ in the second case.

On an Einstein-Weyl manifold the (once) contracted Bianchi identity shows that $\mathcal{D}_0 F^D = \delta^D W = 0$ in the first case (the Weyl curvature W vanishes) and that $\mathcal{D}_0 F^D_- = \delta^D W^- = 0$ in the second case (W is selfdual).

This yields a new proof of a result of Eastwood and Tod, and a new theorem about Einstein-Weyl 4-manifolds [**8**].

***4.2. Theorem* (Eastwood-Tod [15])**. — *Let M, D be a conformally flat Einstein-Weyl manifold in four or more dimensions. Then $F^D = 0$ so D is locally the Levi-Civita connection of an Einstein metric.*

4.3. Theorem. — *Let M, D be a conformally selfdual Einstein-Weyl manifold. Then F^D is also selfdual, so D is locally the Levi-Civita connection of an Einstein metric or the Obata connection of a hypercomplex structure.*

The second part of this theorem follows because if F^D is selfdual, $\delta F^D = 0$ and so $D\mathit{scal}^D = 0$. If $\mathit{scal}^D = 0$, then D is flat on $(0|1, -1)$ and M is locally hypercomplex—see Pedersen and Swann [**23**].

5. Recent developments

In the few months since the Luminy meeting, there have been some further developments, which I believe make BGG sequences more accessible to the conformal geometer, and, perhaps more significantly, substantially broaden the range of possible applications. These developments arise from the PhD thesis of Tammo Diemer [**12**], who has given a surprisingly simple construction of Bernstein-Gelfand-Gelfand resolutions which has the additional benefit of providing bilinear pairings generalizing the wedge product on the de Rham complex. Just as the BGG operators have higher order, in general, than the exterior derivative, so too the wedge product generalizes to bilinear differential pairings, no longer necessarily zero order, but still satisfying a Leibniz rule with respect to the BGG operators.

The search for such pairings was motivated by joint work to understand the relationship between different BGG sequences, called "helicity raising and lowering" in mathematical physics, or "the translation principle" in representation theory. The construction in [**12**] was given in terms of parabolic Verma modules (as in the original work of Bernstein, Gel'fand, Gel'fand and Lepowsky): we have presented the geometric (i.e., dualized) and curved version in [**9**]. In the curved case the Leibniz rule only holds up to curvature terms, and these led us to consider multilinear differential operators, which have a rather rich structure (they form an A_∞-algebra).

Here I would like to explain briefly some motivations and potential applications of bilinear differential pairings in conformal geometry.

Adjoints. — A differential operator $\mathcal{D}$ from $\mathrm{C}^\infty(E)$ to $\mathrm{C}^\infty(F)$ has a natural adjoint $\mathcal{D}^*$, from $\mathrm{C}^\infty(L^{-n}\otimes F^*)$ to $\mathrm{C}^\infty(L^{-n}\otimes E^*)$. On sections compactly supported in the interior of M we have $\int_M\langle\mathcal{D}e,\phi\rangle = \int_M\langle e,\mathcal{D}^*\phi\rangle$. This duality should manifest itself locally as the existence of a bilinear differential pairing $X(e,\phi)$, valued in vector densities, such that

$$div\, X(e,\phi) = \langle\mathcal{D}e,\phi\rangle - \langle e,\mathcal{D}^*\phi\rangle. \tag{5.1}$$

The adjointness of $\mathcal{D}$ and $\mathcal{D}^*$ would then follow from the divergence theorem. If $\mathcal{D}$ and $\mathcal{D}^*$ have order k, then $X(\cdot,\cdot)$ has order $k-1$. In particular, for first order differential operators, the pairing is algebraic. In general one might not expect $X(\cdot,\cdot)$ to be well defined, since adding the exterior divergence of a bivector density does not alter the product rule (5.1). It is a rather pleasant surprise then, that there often is a natural choice. For example, for the conformal Laplacian $\Delta = tr\, D^2 - \frac{n-2}{4(n-1)}scal^D id$, acting on sections of $L^{(2-n)/2}$, we have

$$div\big((D\phi)\psi - \phi(D\psi)\big) = (\Delta\phi)\psi - \phi(\Delta\psi)$$

and the pairing inside the divergence is conformally invariant. The conformal Laplacian appears in a singular BGG sequence, which is not covered by the general results above, but a similar observation holds for the conformal Hessian, $\mathcal{H}_0 = sym_0(D^2+r^D)$, where r^D is the (normalized) Ricci tensor; this is the first operator in the BGG sequence:

$$(1|0,0,\dots0)\to(-1|2,0,\dots0)\to(-2|2,1,\dots0)\to\cdots.$$

Helicity raising and lowering. — It has been known for some time that solutions of twistor equations (i.e., elements in the kernel of the first operator in a BGG sequence) may be paired with solutions of conformally invariant field equations to give solutions of other such equations. Many examples have been computed in which this pairing is zero or first order, especially using Penrose twistors in four dimensions [**25**].

If one wants to study fields with sources, then it is no longer enough to be able to manipulate *solutions* of (source-free) field equations; one must compute the extra source terms that arise in these pairings. It is also natural to ask what happens if the twistor field is no longer required to solve the twistor equation. Clearly, one would like some sort of product rule:

$$\mathcal{D}_k(\phi\cdot F) = (\mathcal{D}_0\phi)\cdot F + \phi\cdot(\mathcal{D}_kF)$$

where the $\mathcal{D}_k$'s denote the kth operators in some BGG sequences and the dots denote some pairings. Again examples have been known for some time where the pairings are all zero order. For instance, $F\in(3-n|2,2,0,\dots0)$ and $\omega\in(1|1,1,0,\dots0)$ may be contracted to give a bivector density, and there is a simple Leibniz rule for the exterior divergence:

$$\delta\langle F,\omega\rangle = \langle\delta F,\omega\rangle + \langle F,\mathcal{D}_0\omega\rangle, \tag{5.2}$$

where the second δ denotes the conformally invariant divergence of Weyl tensors of weight $3-n$. This has been used by Penrose to discuss his quasi-local mass construction [**24**]: bivector densities can be integrated over cooriented codimension two surfaces and (5.2) provides a conservation law in source-free regions if $\mathcal{D}_0\omega = 0$.

Several years ago, Tammo Diemer and I began to wonder whether there was a systematic theory behind this. We noticed that in general we would need higher order pairings, and found some examples involving first order pairings, such as the following:

$$\begin{aligned}\delta\big(F(D\phi,\cdot,\cdot) - \frac{1}{2}(\delta^D F)(\cdot,\cdot)\phi\big) &= \phi\,\mathrm{div}^D(S\mathrm{div}\,F) - (S\mathrm{div}\,F)(D\phi,\cdot)\\ &\quad + \langle\mathcal{H}_0\phi, F\rangle + \frac{1}{2}\langle W, F\rangle\phi\end{aligned}$$

where $F \in (2-n|2,1,0,\ldots 0)$, $\phi \in (1|0,0,\ldots 0)$, $S\mathrm{div}\,F$ is the symmetric divergence in $(1-n|2,0,\ldots 0)$, $\delta^D F$ is the skew divergence in $(1-n|1,1,0,\ldots 0)$, and W is the Weyl curvature.

With these examples, and others, in mind, it was natural to look for a result along the following lines:

5.1. Theorem. — *Let (M,η) be a normal parabolic geometry modelled on G/P and let $\mathbb{W}_1$, $\mathbb{W}_2$ and $\mathbb{W}_3$ be finite dimensional G-modules with a nontrivial G-equivariant linear map $\mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{W}_3$. Then there are nontrivial bilinear differential pairings*

$$\begin{array}{ccc} \mathrm{C}^\infty(H_k(W_1)) \times \mathrm{C}^\infty(H_\ell(W_2)) & \longrightarrow & \mathrm{C}^\infty(H_{k+\ell}(W_3)) \\ (\alpha,\beta) & \longmapsto & \alpha \sqcup \beta \end{array}$$

which satisfy a Leibniz rule with curvature terms:

$$\mathcal{D}_{k+\ell}(\alpha\sqcup\beta) = (\mathcal{D}_k\alpha)\sqcup\beta + (-1)^k\alpha\sqcup(\mathcal{D}_\ell\beta) - \langle\mathcal{K},\alpha,\beta\rangle + \langle\alpha,\mathcal{K},\beta\rangle - \langle\alpha,\beta,\mathcal{K}\rangle,$$

where H_* denote the Lie algebra homology bundles appearing in the BGG sequences, $\mathcal{K} \in \mathrm{C}^\infty(H_2(M))$ is the Lie algebra homology class of the curvature of the Cartan connection, and $\langle\cdot,\cdot,\cdot\rangle$ are some trilinear differential operators. Furthermore, $\mathcal{D}_2\mathcal{K} = 0$, and the composite of two operators in a BGG sequence is given by $\mathcal{D}_{k+1}\circ\mathcal{D}_k\phi = \mathcal{K}\sqcup\phi$.

This theorem was proven in the flat case (using Verma modules) in [**12**], and, in general, in [**9**]. The cup product $\sqcup$ and trilinear operators are given quite explicitly and generalize to the multilinear differential operators mentioned earlier. Potential applications include quadratic and higher degree obstructions in the deformation theory of parabolic geometries discussed in section 2.

I want to end by suggesting another answer to a question which several people asked me at Luminy: what is the point of studying complexes which are no longer complexes? In other words, the objection is that since $\mathcal{D}^2$ is not zero, there is no cohomology theory any more. I have attempted to indicate some answers in the above applications, but the work in [**9**] has also suggested the following construction.

Let $\mathcal{P} \to M$ be a principal G-bundle with a principal connection and let ∇ be the induced covariant derivative on the adjoint bundle $\mathfrak{g}_M = \mathcal{P} \times_G \mathfrak{g}$. Then there is a twisted de Rham sequence

$$\mathrm{C}^\infty(\mathfrak{g}_M) \to \mathrm{C}^\infty(T^*M \otimes \mathfrak{g}_M) \to \mathrm{C}^\infty(\Lambda^2 T^*M \otimes \mathfrak{g}_M) \to \cdots$$

which is not a complex in general: the composite of two twisted exterior derivatives $(d^\nabla)^2$ is given by the wedge product with the curvature $R^\nabla \in \mathrm{C}^\infty(\Lambda^2 T^*M \otimes \mathfrak{g}_M)$, where the wedge product is contracted by the Lie bracket.

Denote the sheaf $\mathrm{C}^\infty(\Lambda T^*M \otimes \mathfrak{g}_M)$ by A and consider the supersymmetric coalgebra of sA, i.e., the subspace of symmetric elements of $\bigoplus_{k\geqslant 0} \otimes^k A$ in the graded sense, where the grading on A is shifted by one, and the coproduct is inherited from the obvious coproduct on $\bigoplus_{k\geqslant 0} \otimes^k A$. Then $(R^\nabla, d^\nabla, \wedge)$ define linear maps $\otimes^k A \to A$ for $k = 0, 1, 2$. The sum of these may be extended to a coderivation ∂ and one readily finds that $\partial^2 = 0$, since $d^\nabla R^\nabla = 0$, $(d^\nabla)^2 = R^\nabla \wedge (\cdot)$, the wedge product and d^∇ satisfy a Leibniz rule, and the wedge product satisfies the graded Jacobi identity. In other words, even though $(d^\nabla)^2$ is not zero, the triple $(R^\nabla, d^\nabla, \wedge)$ has square zero in a generalized sense. If E is an associated bundle, then the same trick applies to $E \rtimes \mathfrak{g}_M$, with trivial Lie bracket on E.

A similar construction works for the BGG sequence, except that the contracted cup product does not satisfy the graded Jacobi identity: instead one needs to extend $(\mathcal{K}, \mathcal{D}, \sqcup)$ by multilinear differential operators. This is an example of an L_∞-algebra.

In all these cases a natural question arises: what is the homology of the differential coderivation ∂?

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