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DEPENDENCE OF THE DIRAC SPECTRUM ON THE SPIN STRUCTURE

by

Christian Bär

Abstract. — The theme is the influence of the spin structure on the Dirac spectrum of a spin manifold. We survey examples and results related to this question.

Résumé **(Dépendance du spectre de l'opérateur de Dirac par rapport à la structure spinorielle)**
Sur une variété spinorielle, nous étudions la dépendance du spectre de l'opérateur de Dirac par rapport à la structure spinorielle. Nous donnons ensuite un résumé des exemples et des résultats liés à cette question.

1. Introduction

The relation between the geometry of a Riemannian manifold and the spectrum of its Laplace operator acting on functions (or more generally, on differential forms), has attracted a lot of attention. This is the question how shape and sound of a space are related. A beautiful introduction into this topic can be found in [**12**]. When one passes from this "bosonic" theory to "fermions", i.e. when turning to spinors and the Dirac operator, a new object enters the stage, the *spin structure*. This is a global topological object needed to define spinors. The question arises how this piece of structure, in addition to the usual geometry of the manifold, influences the spectrum of the Dirac operator.

It has been known for a long time that even on the simplest examples such as the 1-sphere the Dirac spectrum does depend on the spin structure. We will discuss the 1-sphere, flat tori, 3-dimensional Bieberbach manifolds, and spherical space forms in some detail. For these manifolds the spectrum can be computed explicitly. For some of these examples an important invariant computed out of the spectrum, the η-invariant, also depends on the spin structure. On the other hand, under a certain assumption,

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the difference between the η-invariants for two spin structures on the same manifold must be an integer. Hence the two η-invariants are not totally unrelated.

We also look at circle bundles and the behavior of the Dirac spectrum under collapse. This means that one shrinks the fibers to points. The spin structure determines the qualitative spectral behavior. If the spin structure is projectable, then some eigenvalues tend to $\pm\infty$ while the others essentially converge to the eigenvalues of the basis manifold. If the spin structure if nonprojectable, then all eigenvalues diverge.

In most examples it is totally hopeless to try to explicitly compute the Dirac (or other) spectra. Still, eigenvalue estimates are very often possible. So far, these estimates have not taken into account the spin structure despite its influence on the spectrum. The reason for this lies in the essentially local methods such as the Bochner technique. In order to get better estimates taking the spin structure into account one first has to find new, truly spin geometric invariants. We discuss some of the first steps in this direction. Here the *spinning systole* is the relevant spin geometric input.

Finally we look at noncompact examples in order to check if the continuous spectrum is affected by a change of spin structure. It turns out that this is the case. There are hyperbolic manifolds having two spin structures such that for the first one the Dirac spectrum is discrete while it is all of $\mathbb{R}$ for the other one. The influence of the spin structure could hardly be any more dramatic.

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2. Generalities

Let us start by collecting some terminology and basic facts. A more thorough introduction to the concepts of spin geometry can e.g. be found in [**15, 9, 18**]. Let M denote an n-dimensional oriented Riemannian manifold with a spin structure P. This is a $\mathrm{Spin}(n)$-principal bundle which doubly covers the bundle of oriented tangent frames $P_{\mathrm{SO}}M$ of M such that the canonical diagram

$$\begin{tikzcd} P \times \mathrm{Spin}(n) \arrow[r] \arrow[d] & P \arrow[d] \arrow[dr] & \\ P_{\mathrm{SO}}M \times \mathrm{SO}(n) \arrow[r] & P_{\mathrm{SO}}M \arrow[r] & M \end{tikzcd}$$

commutes. Such a spin structure need not exist, e.g. complex projective plane $\mathbb{CP}^2$ has none. If M has a spin structure we call M a *spin manifold*. The spin structure of a spin manifold is in general not unique. More precisely, the cohomology $H^1(M;\mathbb{Z}_2)$ of a spin manifold acts simply transitively on the set of all spin structures.

Given a spin structure P one can use the spinor representation

$$\mathrm{Spin}(n) \to \mathrm{Aut}(\Sigma_n)$$

to construct the associated *spinor bundle* ΣM over M. Here Σ_n is a Hermitian vector space of dimension $2^{[n/2]}$ on which $\mathrm{Spin}(n)$ acts by unitary transformations. Hence ΣM is a Hermitian vector bundle of rank $2^{[n/2]}$. Sections in ΣM are called *spinor fields* or simply *spinors*. Note that unlike differential forms the definition of spinors requires the choice of a spin structure. The Levi-Civita connection on $P_{\mathrm{SO}}M$ can be lifted to P and therefore induces a covariant derivative ∇ on ΣM.

Algebraic properties of the spinor representation ensure existence of *Clifford multiplication*

$$T_pM \otimes \Sigma_pM \to \Sigma_pM, \quad X \otimes \psi \mapsto X \cdot \psi,$$

satisfying the relations

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi + 2\langle X, Y\rangle \psi = 0$$

for all $X, Y \in T_pM$, $\psi \in \Sigma_pM$, $p \in M$. Here $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric.

The *Dirac operator* acting on spinors is defined as the composition of ∇ with Clifford multiplication. Equivalently, if $e_1, \dots, e_n$ is an orthonormal basis of T_pM, then

$$(D\psi)(p) = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi.$$

The Dirac operator is a formally self-adjoint elliptic differential operator of first order. If the underlying Riemannian manifold M is complete, then D, defined on compactly supported smooth spinors, is essentially self-adjoint in the Hilbert space of square-integrable spinors. General elliptic theory ensures that the spectrum of D is discrete if M is compact and satisfies Weyl's asymptotic law

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^n} = \frac{2^{[n/2]} \cdot \mathrm{vol}(M)}{(4\pi)^{\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2}+1\right)},$$

where $N(\lambda)$ is the number of eigenvalues whose modulus is $\leq \lambda$. This implies that the series

$$\eta(s) = \sum_{\lambda\neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s}$$

converges for $s \in \mathbb{C}$ if the real part of s is sufficiently large. Here summation is taken over all nonzero eigenvalues λ of D, each eigenvalue being repeated according to its multiplicity. It can be shown that the function $\eta(s)$ extends to a meromorphic function on the whole complex plane and has no pole at $s = 0$. Evaluation of this meromorphic extension at $s = 0$ gives the η-invariant,

$$\eta := \eta(0).$$

If M is complete but noncompact, then D may also have eigenvalues of infinite multiplicity, cumulation points of eigenvalues, and continuous spectrum.

3. The baby example

In order to demonstrate the dependence of the Dirac spectrum on the choice of spin structure the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ can serve as a simple but nonetheless illustrative example. Since the frame bundle $P_{\mathrm{SO}}S^1$ is trivial we can immediately write down the trivial spin structure $P = S^1 \times \mathrm{Spin}(1)$. Note that $\mathrm{Spin}(1) = \mathbb{Z}_2$ and $\Sigma_1 = \mathbb{C}$. The associated spinor bundle is then also trivial and 1-dimensional. Hence spinors are simply $\mathbb{C}$-valued functions on S^1. The Dirac operator is nothing but

$$D = i\frac{d}{dt}.$$

Elementary Fourier analysis shows that the spectrum consists of the eigenvalues

$$\lambda_k = k$$

with corresponding eigenfunctions $t \mapsto e^{-ikt}$, $k \in \mathbb{Z}$. Since the spectrum is symmetric about zero, the η-series, and in particular, the η-invariant vanishes,

$$\eta = 0.$$

From $H^1(S^1;\mathbb{Z}_2) = \mathbb{Z}_2$ we see that S^1 has a second spin structure. It can be described as $\tilde{P} = ([0,2\pi] \times \mathrm{Spin}(1))/\sim$ where $\sim$ identifies 0 with 2π while it interchanges the two elements of Spin(1). Let us call this spin structure the *nontrivial spin structure* of S^1. Spinors with respect to this spin structure no longer correspond to functions on S^1, i.e. to 2π-periodic functions on $\mathbb{R}$, but rather to 2π-anti-periodic complex-valued functions on $\mathbb{R}$,

$$\psi(t + 2\pi) = -\psi(t).$$

This time the eigenvalues are

$$\lambda_k = k + \frac{1}{2},$$

$k \in \mathbb{Z}$, with eigenfunctions $t \mapsto e^{-i(k+\frac{1}{2})t}$. Again, the spectrum is symmetric about 0, hence $\eta = 0$. Vanishing of the η-invariant is in fact not surprising. One can show that always $\eta = 0$ for an n-dimensional manifold unless $n \equiv 3 \bmod 4$.

The example S^1 has shown that the eigenvalues of the Dirac operator definitely do depend on the choice of spin structure. Even the dimension of the kernel of the Dirac operator is affected by a change of spin structure. For the trivial spin structure of S^1 it is 1 while it is zero for the nontrivial spin structure.

We conclude this section with a remark on extendability of spin structures because this sometimes causes confusion. If M is a Riemannian spin manifold with boundary ∂M, then a spin structure on M induces one on ∂M. To see this consider the frame bundle $P_{\mathrm{SO}}\partial M$ of the boundary as a subbundle of $P_{\mathrm{SO}}M$ restricted to the boundary by completing a frame for ∂M with the exterior unit normal vector to a frame for M. Now the inverse image of $P_{\mathrm{SO}}\partial M$ under the covering map $P \to P_{\mathrm{SO}}M$ defines a spin structure on ∂M.

Look at the case that M is the disc with S^1 as its boundary. Since the disk is simply connected it can have only one spin structure. Hence only one of the two spin structures of S^1 extends to the disc. The tangent vector to the boundary S^1 together with the unit normal vector forms a frame for the disk which makes one full rotation when going around the boundary one time. It is therefore a loop in the frame bundle of the disk whose lift to the spin structure does not close up. Thus the induced spin structure on the boundary is the nontrivial spin structure of S^1 while the trivial spin structure does not bound. Hence from a cobordism theoretical point of view the trivial spin structure is nontrivial and vice versa.

4. Flat tori and Bieberbach manifolds

The case of higher-dimensional flat tori is very similar to the 1-dimensional case. There are 2^n different spin structures on $T^n = \mathbb{R}^n/\Gamma$ where Γ is a lattice in $\mathbb{R}^n$. Let $b_1, \ldots, b_n$ be a basis of Γ, let $b_1^*, \ldots, b_n^*$ be the dual basis for the dual lattice Γ^*. Spin structures can then be classified by n-tuples $(\delta_1, \ldots, \delta_n)$ where each $\delta_j \in \{0, 1\}$ indicates whether or not the spin structure is twisted in direction b_j. The spectrum of the Dirac operator can then be computed:

Theorem 4.1 **(Friedrich [14])**. — *The eigenvalues of the Dirac operator on $T^n = \mathbb{R}^n/\Gamma$ with spin structure corresponding to $(\delta_1, \ldots, \delta_n)$ are given by*

$$\pm 2\pi \left| b^* + \frac{1}{2} \sum_{j=1}^{n} \delta_j b_j^* \right|$$

where b^* runs through Γ^* and each b^* contributes multiplicity $2^{[n/2]-1}$.

Again the spectrum depends on the choice of spin structure. In particular, eigenvalue 0 occurs only for the trivial spin structure given by $(\delta_1, \ldots, \delta_n) = (0, \ldots, 0)$. Since again the spectrum is symmetric about zero, the η-invariant vanishes, $\eta = 0$, for all spin structures.

This changes if one passes from tori to more general compact connected flat manifolds, also called *Bieberbach manifolds*. They can always be written as a quotient $M = G \backslash T^n$ of a torus by a finite group G. In three dimensions, $n = 3$, there are 5 classes of compact oriented Bieberbach manifolds besides the torus. Their Dirac spectra have been calculated by Pfäffle **[20]** for all flat metrics. This time one finds examples with asymmetric spectrum and the η-invariant depends on the choice of spin structure.

Theorem 4.2 **(Pfäffle [20])**. — *The η-invariant of the 3-dimensional compact oriented Bieberbach manifolds besides the torus are given by the following table:*

G	total # spin structures	η-invariant for # spin structures		
$\mathbb{Z}_2$	8	$\eta = 0$ for 6	$\eta = 1$ for 1	$\eta = -1$ for 1
$\mathbb{Z}_3$	2		$\eta = 4/3$ for 1	$\eta = -2/3$ for 1
$\mathbb{Z}_4$	4	$\eta = 0$ for 2	$\eta = 3/2$ for 1	$\eta = -1/2$ for 1
$\mathbb{Z}_6$	2		$\eta = 5/3$ for 1	$\eta = -1/3$ for 1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	$\eta = 0$ for 4		

TABLE 1

Note that the η-invariant does not depend on the choice of flat metric even though the spectrum does. Depending on G there is a 2-, 3- or 4-parameter family of such metrics on M.

5. Spherical space forms

The Dirac spectrum on the sphere S^n with constant curvature has been computed by different methods in [**6, 21, 22**]. The eigenvalues are

$$\pm\left(\frac{n}{2}+k\right), \tag{1}$$

$k \in \mathbb{N}_0$, with multiplicity $2^{[n/2]} \cdot \begin{pmatrix} k+n-1 \\ k \end{pmatrix}$. For $n \geq 2$ the sphere is simply connected, hence has only one spin structure. Therefore let us look at spherical space forms $M = \Gamma \backslash S^n$ where Γ is a finite fixed point free subgroup of $\mathrm{SO}(n+1)$. Spin structures correspond to homomorphisms $\varepsilon : \Gamma \to \mathrm{Spin}(n+1)$ such that

$$\begin{tikzcd} & \mathrm{Spin}(n+1) \arrow[d] \\ \Gamma \arrow[ur, "\varepsilon"] \arrow[r] & \mathrm{SO}(n+1) \end{tikzcd}$$

commutes. Since any eigenspinor on M can be lifted to S^n all eigenvalues of M are also eigenvalues of S^n, hence of the form (1). To know the spectrum of M one must compute the multiplicities μ_k of $\frac{n}{2}+k$ and μ_{-k} of $-(\frac{n}{2}+k)$. They can be most easily expressed by encoding them into two power series, so-called *Poincaré series*

$$F_+(z) = \sum_{k=0}^{\infty} \mu_k z^k,$$
$$F_-(z) = \sum_{k=0}^{\infty} \mu_{-k} z^k.$$

To formulate the result recall that in even dimension $2m$ the spinor representation is reducible and can be decomposed into two half spinor representations

$$\mathrm{Spin}(n) \to \mathrm{Aut}(\Sigma_{2m}^{\pm}),$$

$\Sigma_{2m} = \Sigma_{2m}^{+} \oplus \Sigma_{2m}^{-}$. Denote their characters by $\chi^{\pm} : \mathrm{Spin}(2m) \to \mathbb{C}$.

***Theorem 5.1* (Bär [6]).** — *Let $M = \Gamma\backslash S^n$, $n = 2m-1$, be a spherical space form with spin structure given by $\varepsilon : \Gamma \to \mathrm{Spin}(2m)$. Then the eigenvalues of the Dirac operator are $\pm(\frac{n}{2}+k), k \geq 0$, with multiplicities determined by*

$$\begin{aligned} F_+(z) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi^-(\varepsilon(\gamma)) - z\cdot\chi^+(\varepsilon(\gamma))}{\det(1_{2m} - z\cdot\gamma)}, \\ F_-(z) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi^+(\varepsilon(\gamma)) - z\cdot\chi^-(\varepsilon(\gamma))}{\det(1_{2m} - z\cdot\gamma)}. \end{aligned}$$

Note that only odd-dimensional spherical space forms are of interest because in even dimensions real projective space is the only quotient and in this case it is not even orientable.

Let us use Theorem 5.1 to compute the η-invariant of spherical space forms. We get immediately for the *θ-functions*

$$\begin{aligned} \theta_\pm(t) &:= e^{-(n/2)t}\cdot F_\pm(e^{-t}) \\ &= \frac{e^{-(m+1/2)t}}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi^{\mp}(\varepsilon(\gamma)) - e^{-t}\cdot\chi^{\pm}(\varepsilon(\gamma))}{\det(1_{2m} - e^{-t}\cdot\gamma)}. \end{aligned}$$

The coefficient of t^0 in the Laurent expansion at $t=0$ is given by

$$\begin{aligned} LR_0(\theta_+) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma-\{1_{2m}\}}\frac{\chi^-(\varepsilon(\gamma)) - \chi^+(\varepsilon(\gamma))}{\det(1_{2m} - \gamma)} \\ &\quad + LR_0\left(\frac{e^{-(m+1/2)t}}{|\Gamma|}\cdot\frac{2^{m-1} - e^{-t}\cdot 2^{m-1}}{\det(1_{2m} - e^{-t}\cdot 1_{2m})}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} LR_0(\theta_-) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma-\{1_{2m}\}}\frac{\chi^+(\varepsilon(\gamma)) - \chi^-(\varepsilon(\gamma))}{\det(1_{2m} - \gamma)} \\ &\quad + LR_0\left(\frac{e^{-(m+1/2)t}}{|\Gamma|}\cdot\frac{2^{m-1} - e^{-t}\cdot 2^{m-1}}{\det(1_{2m} - e^{-t}\cdot 1_{2m})}\right). \end{aligned}$$

Hence we obtain for $\theta := \theta_+ - \theta_-$

$$\begin{aligned} LR_0(\theta) &= LR_0(\theta_+) - LR_0(\theta_-) \\ &= \frac{2}{|\Gamma|}\sum_{\gamma\in\Gamma-\{1_{2m}\}}\frac{\chi^-(\varepsilon(\gamma)) - \chi^+(\varepsilon(\gamma))}{\det(1_{2m} - \gamma)}. \end{aligned}$$

The same argument shows that the poles of θ_+ and θ_- cancel, hence θ is holomorphic at $t = 0$ with

$$\theta(0) = \frac{2}{|\Gamma|} \sum_{\gamma \in \Gamma - \{1_{2m}\}} \frac{(\chi^- - \chi^+)(\varepsilon(\gamma))}{\det(1_{2m} - \gamma)}.$$

Now we observe that

$$\theta_+(t) = \sum_{k=0}^{\infty} \mu_k e^{-(n/2+k)t} = \sum_{\lambda > 0} e^{-\lambda t},$$

and similarly for θ_-. Application of the *Mellin transformation* yields

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \theta(t) t^{s-1} dt.$$

Therefore

$$\eta = \lim_{s \to 0} \frac{1}{\Gamma(s)} \int_0^{\infty} \theta(t) t^{s-1} dt = \mathrm{Res}_{s=0} \left(\int_0^{\infty} \theta(t) t^{s-1} dt \right).$$

Since θ decays exponentially fast for $t \to \infty$ the function $s \mapsto \int_1^{\infty} \theta(t) t^{s-1} dt$ is holomorphic at $s = 0$. Thus

$$\eta = \mathrm{Res}_{s=0} \left(\int_0^{1} \theta(t) t^{s-1} dt \right) = \theta(0).$$

We have proved

Theorem 5.2. — *Let $M = \Gamma \backslash S^{2m-1}$ be a spherical space form with spin structure given by $\varepsilon : \Gamma \to \mathrm{Spin}(2m)$. Then the η-invariant of M is given by*

$$\eta = \frac{2}{|\Gamma|} \sum_{\gamma \in \Gamma - \{1_{2m}\}} \frac{(\chi^- - \chi^+)(\varepsilon(\gamma))}{\det(1_{2m} - \gamma)}.$$

Example 5.3. — We take a look at real projective space $\mathbb{RP}^{2m-1}$, i.e. $\Gamma = \{1_{2m}, -1_{2m}\}$. If we view $\mathrm{Spin}(2m)$ as sitting in the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{2m})$, compare [**18**], then we can define the "volume element"

$$\omega := e_1 \cdot e_2 \cdot \cdots \cdot e_{2m} \in \mathrm{Spin}(2m) \subset \mathrm{Cl}(\mathbb{R}^{2m})$$

where $e_1, \ldots, e_{2m}$ denotes the standard basis of $\mathbb{R}^{2m}$. It is not hard to see that under the map $\mathrm{Spin}(2m) \to \mathrm{SO}(2m)$ the volume element ω is mapped to -1_{2m}. Hence the two preimages of -1_{2m} in $\mathrm{Spin}(2m)$ are $\pm\omega$. To specify a spin structure we may define

$$\varepsilon(-1_{2m}) := \omega$$

or

$$\varepsilon(-1_{2m}) := -\omega.$$

One checks

$$1 = \varepsilon(1_{2m}) = \varepsilon((-1_{2m})^2) = \varepsilon(-1_{2m})^2 = (\pm\omega)^2 = (-1)^m.$$

Hence $\mathbb{RP}^{2m-1}$ is not spin if m is odd ($m \geq 3$) whereas it has two spin structures if m is even.

The volume element ω acts on the two half spinor spaces via multiplication by ± 1. Hence

$$\begin{aligned} \eta &= \pm \frac{2}{2} \cdot \frac{2 \cdot 2^{m-1}}{2^{2m}} \\ &= \pm 2^{-m} \end{aligned}$$

We summarize

Corollary 5.4. — *For $n \geq 2$ real projective space $\mathbb{RP}^n$ is spin if and only if $n \equiv 3$ mod 4, in which case it has exactly two spin structures. The η-invariant for the Dirac operator is given by*

$$\eta = \pm 2^{-m}, \quad n = 2m - 1,$$

where the sign depends on the spin structure chosen.

See also [**15, 16**] where the η-invariant of all twisted signature operators on spherical space forms is determined and used to compute their K-theory.

6. Eigenvalue estimates

Up to very recently all known lower eigenvalue estimates for the Dirac operator did not take into account the spin structure despite its influence on the spectrum that we have encountered in the examples. This is due to the fact that they are all based on the Bochner technique, hence on a *local* computation. To find estimates which can see the spin structure one needs to define new, truly spin geometric invariants. Such invariants have been proposed by Ammann [**2, 3**] in the case of a 2-torus. Recall the definition of the *systole* of a Riemannian manifold (M, g)

$$\mathrm{sys}_1(M, g) = \inf\{\mathrm{length}(\gamma) \mid \gamma \text{ is a noncontractible loop}\}.$$

In case M is a torus there is a canonical spin structure, the trivial spin structure P_0. Hence the set of spin structures can be identified with $H^1(M; \mathbb{Z}_2)$ by identifying P_0 with 0. It then makes sense to evaluate a spin structure P on the homology class of a loop γ yielding an element in $\mathbb{Z}_2 = \{1, -1\}$. This value $P([\gamma])$ specifies whether or not the spin structure P is twisted along γ. Ammann defines the *spinning systole*

$$\text{spin-sys}_1(M, g, P) = \inf\{\mathrm{length}(\gamma) \mid \gamma \text{ is a loop with } P([\gamma]) = -1\}.$$

Hence the infimum is taken only over those loops along which the spin structure twists. In case the spin structure is trivial, $P = P_0$, the spinning systole is infinite.

***Theorem 6.1* (Ammann [3])**. — *Let g be a Riemannian metric on the 2-torus whose Gauss curvature K satisfies $\|K\|_{L^1(T^2,g)} < 4\pi$. Let P be a spin structure on T^2.*

Then for all eigenvalues λ of the Dirac operator the estimate

$$\lambda^2 \geq \frac{C(\|K\|_{L^1(T^2,g)}, \|K\|_{L^2(T^2,g)}, \mathrm{area}(T^2,g), \mathrm{sys}_1(T^2,g))}{\text{spin-sys}_1(T^2,g,P)^2}$$

holds where $C(\|K\|_{L^1(T^2,g)}, \|K\|_{L^2(T^2,g)}, \mathrm{area}(T^2,g), \mathrm{sys}_1(T^2,g)) > 0$ *is an explicitly given expression.*

The estimate is sharp in the sense that for some flat metrics equality is attained. In a similar way Ammann defines the *nonspinning systole* and proves an analogous estimate for which equality is attained for all flat metrics. The proofs are based on a comparison of the Dirac spectra for the metric g with the one for the conformally equivalent flat metric g_0. Remember that by Theorem 4.1 the spectrum for g_0 is explicitly known. Most of the work is then done to control the oscillation of the function which relates the two conformally equivalent metrics g and g_0 in terms of the geometric data occurring in $C(\|K\|_{L^1(T^2,g)}, \|K\|_{L^2(T^2,g)}, \mathrm{area}(T^2,g), \mathrm{sys}_1(T^2,g))$. This way it is also possible to derive upper eigenvalue estimates, see **[2, 3]** for details.

In the same paper **[3]** Ammann also studies the question how far Dirac spectra for different metrics on a compact manifold can be away from each other. If P_1 and P_2 are two spin structures on a Riemannian manifold (M,g), then there is a unique $\chi \in H^1(M;\mathbb{Z}_2)$ taking P_1 to P_2. On $H^1(M;\mathbb{Z}_2)$ there is a canonical norm, the *stable norm* (or L^∞-norm). Ammann shows that if the Dirac eigenvalues λ_j of (M,g,P_1) and λ'_j of (M,g,P_2) are numbered correctly, then

$$|\lambda_j - \lambda'_j| \leq 2\pi\|\chi\|_{L^\infty}.$$

7. Collapse of circle bundles

Another instance where the choice of spin structure has strong influence on the spectral behavior occurs when one looks at circle bundles and their collapse to the basis. To this extent let (M, g_M) be a compact Riemannian spin manifold with an isometric and free circle action. For simplicity we suppose that the fibers have constant lengths. We give the quotient $N := S^1\backslash M$ the unique Riemannian metric g_N for which the projection $M \to N$ is a Riemannian submersion. By rescaling the metric g_M along the fibers while keeping it unchanged on the orthogonal complement to the fibers we obtain a 1-parameter family of Riemannian metrics g_ℓ on M with respect to which $(M, g_\ell) \to (N, g_N)$ is a Riemannian submersion and the fibers are of length $2\pi\ell$. *Collapse* of this circle bundle now means that we let $\ell \to 0$, i.e. we shrink the fibers to a point. Then (M, g_ℓ) tends to (N, g_N) in the Gromov-Hausdorff topology. In the physics literature this is also referred to as *adiabatic limit.* The question now is how the spectrum behaves. In particular, do eigenvalues of (M, g_ℓ) tend to those of (N, g_N)?

For the answer we have to study the spin structure P on M. The isometric circle action on M induces a circle action on the frame bundle $P_{\mathrm{SO}}M$. This S^1-action may or may not lift to P. In case it lifts we call P *projectable*, otherwise we call it *nonprojectable*. If P is projectable, then it induces a spin structure on N.

Theorem 7.1 **(Ammann-Bär [4])**. — *Let P be projectable and let N carry the induced spin structure. Denote the Dirac eigenvalues of (N, g_N) by μ_j. Then the Dirac eigenvalues $\lambda_{j,k}(\ell)$, $j, k \in \mathbb{Z}$, of (M, g_ℓ), if numbered correctly, depend continuously on ℓ and for $\ell \to 0$ the following holds:*

– *For all j and k we have*

$$\ell \cdot \lambda_{j,k}(\ell) \to k.$$

In particular, $\lambda_{j,k}(\ell) \to \pm\infty$ for $k \neq 0$.

– *If $\dim(N)$ is even, then*

$$\lambda_{j,0}(\ell) \to \mu_j.$$

– *If $\dim(N)$ is odd, then*

$$\lambda_{2j-1,0}(\ell) \to \mu_j, \lambda_{2j,0}(\ell) \to -\mu_j.$$

Roughly, some eigenvalues tend to $\pm\infty$ while the others converge to the eigenvalues of the bases (and their negatives for odd-dimensional basis). This can be applied to the Hopf fibration $S^{2m+1} \to \mathbb{CP}^m$. If m is odd, then the unique spin structure on S^{2m+1} is projectable and one can use Theorem 7.1 to compute the spectrum of complex projective space. If m is even, then the spin structure on S^{2m+1} is not projectable. Indeed $\mathbb{CP}^m$ is not spin in this case. The behavior of the spectrum is in this case described by the following

Theorem 7.2 **(Ammann-Bär [4])**. — *Let P be nonprojectable. Then the Dirac eigenvalues $\lambda_{j,k}(\ell)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z} + (1/2)$, of (M, g_ℓ), if numbered correctly, depend continuously on ℓ and for $\ell \to 0$ the following holds: For all j and k we have*

$$\ell \cdot \lambda_{j,k}(\ell) \to k.$$

In particular, $\lambda_{j,k}(\ell) \to \pm\infty$ for all k and j.

Both cases occur e.g. for Heisenberg manifolds. They are circle bundles over flat tori. The proofs are based on a Fourier decomposition along the fibers. For the case varying fiber length see [**1**], for a very recent paper containing a quite general treatment of collapse see [**19**].

8. η-invariant

We have already seen in examples that the η-invariant does depend on the spin structure. However it turns out that the η-invariants for different spin structures on the same Riemannian manifold M are not totally unrelated. Recall that for two spin structures P_1 and P_2 there is a unique $\chi \in H^1(M;\mathbb{Z}_2)$ mapping P_1 to P_2. We call χ *realizable as a differential form* if there exists a 1-form ω such that

$$\exp\left(2\pi i \int_\gamma \omega\right) = \chi([\gamma])$$

for all loops γ. This is equivalent to the vanishing of χ on the mod-2-reduction of all torsion elements in $H_1(M;\mathbb{Z})$. See [**2**] for this and other characterizations.

***Theorem 8.1* (Dahl [13]).** — *Let P_1 and P_2 be two spin structures on the compact Riemannian manifold M. Suppose the element $\chi \in H^1(M;\mathbb{Z}_2)$ mapping P_1 to P_2 is realizable as a differential form. Then*

$$\eta_{M,P_1} - \eta_{M,P_2} \in \mathbb{Z}.$$

Be careful that some conventions and in particular the definition of the η-invariant in [**13**] differ from ours. One can check that in the case of 3-dimensional Bieberbach manifolds the assumption on $\chi \in H^1(G\backslash T^3;\mathbb{Z}_2)$ is always fulfilled for $G = \mathbb{Z}_3$ and for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. It is fulfilled for some but not all $\chi \in H^1(G\backslash T^3;\mathbb{Z}_2)$ in case $G = \mathbb{Z}_2$ and $G = \mathbb{Z}_4$.

From $H_1(\mathbb{RP}^n;\mathbb{Z}) = \mathbb{Z}_2$ one sees that the nontrivial element of $H^1(\mathbb{RP}^n;\mathbb{Z}_2)$ is not realizable as a differential form. In fact, otherwise Theorem 8.1 would contradict Corollary 5.4. This example shows that this assumption on χ cannot be dispensed with.

The proof of Dahl's theorem is based on a suitable application of the Atiyah-Patodi-Singer index theorem [**5**] to the cylinder over M. The main idea is to write the difference of η-invariants as a linear combination of indices, hence of integers. This index theorem was the reason to introduce the η-invariant in the first place.

9. Noncompact hyperbolic manifolds

In contrast to spaces of constant sectional curvature ≥ 0 there is no hope to be able to explicitly compute the Dirac spectrum on a space of constant negative curvature. In [**8, 10, 11, 17**] the dimension of the kernel of the Dirac operator on hyperbolic Riemann surfaces is considered. For hyperelliptic metrics it can be computed for all spin structures and it varies with the spin structure.

So far we only have considered *compact* manifolds whose Dirac spectrum is always discrete. Let us now discuss noncompact hyperbolic manifolds with an eye to the question whether or not the continuous spectrum also depends on the choice of spin structure.

A *hyperbolic manifold* is a complete connected Riemannian manifold of constant sectional curvature -1. Every hyperbolic manifold M of finite volume can be decomposed disjointly into a relatively compact M_0 and finitely many cusps $\mathcal{E}_j$,

$$M = M_0 \sqcup \bigsqcup_{j=1}^{k} \mathcal{E}_j$$

where each $\mathcal{E}_j$ is of the form $\mathcal{E}_j = N_j \times [0, \infty)$. Here N_j denotes a connected compact

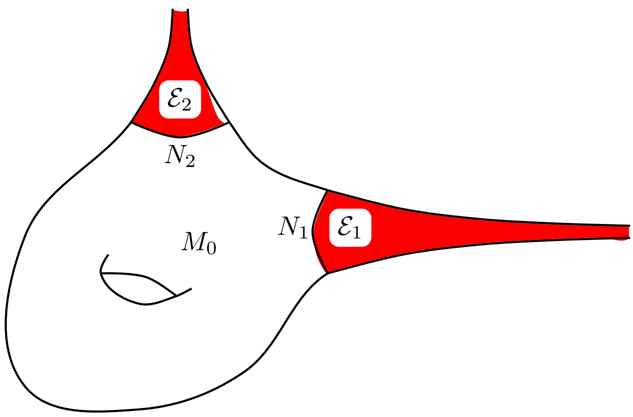

FIGURE 1

manifold with a flat metric g_{N_j}, a Bieberbach manifold, and $\mathcal{E}_j$ carries the warped product metric $g_{\mathcal{E}_j} = e^{-2t} \cdot g_{N_j} + dt^2$. If M is 2- or 3-dimensional and oriented, then N_j is a circle S^1 or a 2-torus T^2 respectively. We call a spin structure on M *trivial along the cusp* $\mathcal{E}_j$ if its restriction to N_j yields the trivial spin structure on N_j. Otherwise we call it *nontrivial along* $\mathcal{E}_j$.

Now it turns out that only two extremal cases occur for the spectrum of the Dirac operator, it is either discrete as in the compact case or it is the whole real line. And it is the spin structure which is responsible for the choice between the two cases.

***Theorem 9.1* (Bär [7]).** — *Let M be a hyperbolic 2- or 3-manifold of finite volume equipped with a spin structure.*

If the spin structure is trivial along at least one cusp, then the Dirac spectrum is the whole real line

$$\mathrm{spec}(D) = \mathbb{R}.$$

If the spin structure is nontrivial along all cusps, then the spectrum is discrete.

In fact, this theorem also holds in higher dimensions. The proof is based on the fact that the essential spectrum of the Dirac operator is unaffected by changes in compact regions. Hence one only needs to look at the cusps and they are given in a very explicit form. A separation of variables along the cusps yields the result. Of course, Theorem 9.1 does not say anything about existence of spin structures on M being trivial or nontrivial along the various cusps. This can be examined by topological methods and the answer for hyperbolic surfaces is given in the table

Hyperbolic surface of finite area		
# of cusps	existence of spin structure with discrete spectrum	existence of spin structure with $\mathrm{spec}(D) = \mathbb{R}$
0	YES	NO
1	YES	NO
≥ 2	YES	YES

TABLE 2

while the 3-dimensional case is given by

Hyperbolic 3-manifold of finite volume		
# of cusps	existence of spin structure with discrete spectrum	existence of spin structure with $\mathrm{spec}(D) = \mathbb{R}$
0	YES	NO
1	YES	NO
≥ 2	YES	depends on M

TABLE 3

The tables show that hyperbolic 2- or 3-manifolds of finite volume with one end behave like compact ones, the Dirac spectrum is always discrete. A surface with two or more ends always admits both types of spin structures. This is not true for 3-manifolds. Discrete spectrum is always possible but the case $\mathrm{spec}(D) = \mathbb{R}$ only sometimes. If the hyperbolic 3-manifold is topologically given as the complement of a link in S^3 (and this construction is one of the main sources for hyperbolic 3-manifolds of finite volume), then this question can be decided.

Theorem 9.2 **(Bär [7])**. — *Let $K \subset S^3$ be a link, let $M = S^3 - K$ carry a hyperbolic metric of finite volume.*

If the linking number of all pairs of components (K_i, K_j) of K is even,

$$\mathrm{Lk}(K_i, K_j) \equiv 0 \ mod\ 2,$$

$i \neq j$, then the spectrum of the Dirac operator on M is discrete for all spin structures.

If there exist two components K_i and K_j of K, $i \neq j$, with odd linking number, then M has a spin structure such that the spectrum of the Dirac operator satisfies

$$\mathrm{spec}(D) = \mathbb{R}.$$

The condition on the linking numbers is very easy to verify in given examples. Since we compute modulo 2 orientations of link components are irrelevant. If the link is given by a planar projection, then modulo 2, $\mathrm{Lk}(K_i, K_j)$ is the same as the number of over-crossings of K_i over K_j.

Example 9.3. — The complements of the following links possess a hyperbolic structure of finite volume. All linking numbers are even. Hence the Dirac spectrum on those hyperbolic manifolds is discrete for all spin structures.

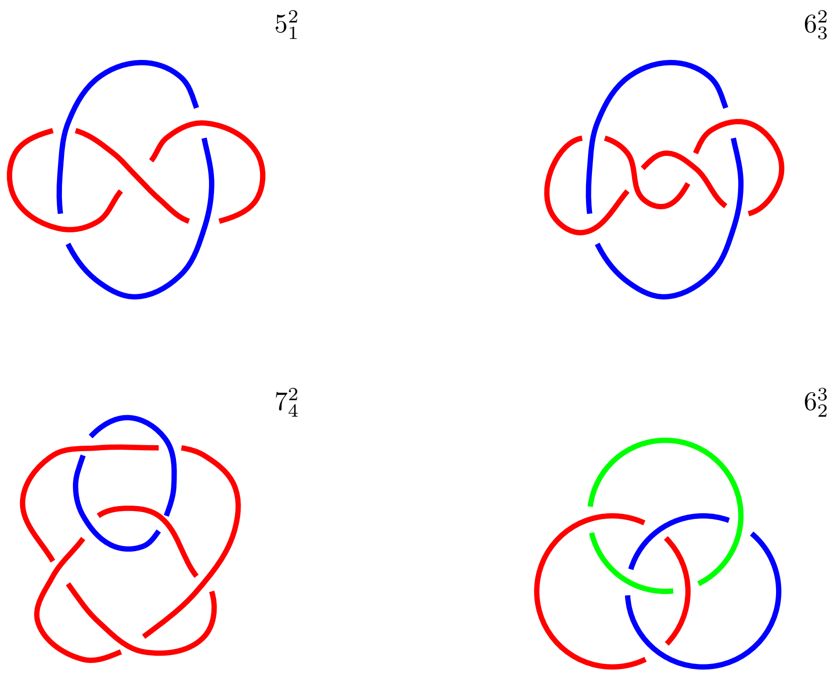

FIGURE 2

This example includes the Whitehead link (5^2_1) and the Borromeo rings (6^3_2).

Example 9.4. — The complements of the following links possess a hyperbolic structure of finite volume. There are odd linking numbers. Hence those hyperbolic manifolds have a spin structure for which the Dirac spectrum is the whole real line.

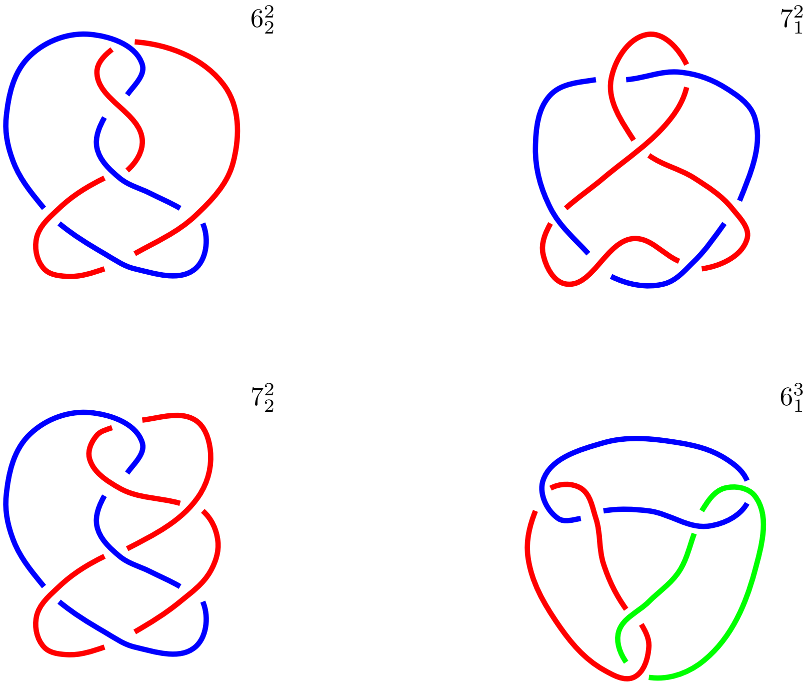

FIGURE 3

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C. Bär, Universität Hamburg, FB Mathematik, Bundesstr. 55, D-20146 Hamburg
E-mail : baer@math.uni-hamburg.de • *Url :* http://www.math.uni-hamburg.de/home/baer/