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CURVATURE AND SMOOTH TOPOLOGY IN DIMENSION FOUR

by

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Abstract. — Seiberg-Witten theory leads to a delicate interplay between Riemannian geometry and smooth topology in dimension four. In particular, the scalar curvature of any metric must satisfy certain non-trivial estimates if the manifold in question has a non-trivial Seiberg-Witten invariant. However, it has recently been discovered [**26**, **27**] that similar statements also apply to other parts of the curvature tensor. This article presents the most salient aspects of these curvature estimates in a self-contained manner, and shows how they can be applied to the theory of Einstein manifolds. We then probe the issue of whether the known estimates are optimal by relating this question to a certain conjecture in Kähler geometry.

***Résumé* (Courbure et topologie lisse en dimension 4).** — La théorie de Seiberg-Witten révèle des liens étonnants entre la géométrie riemannienne et la topologie lisse en dimension 4. En particulier, sur une variété compacte dont un invariant Seiberg-Witten ne s'annule pas, la norme de la courbure scalaire est minorée, d'une manière uniforme et non triviale, pour toute métrique riemannienne. Cependant, on a récemment démontré [**26**, **27**] des estimées analogues à l'égard de la courbure de Weyl. Dans cet article, nous rendrons compte de ces estimées de courbure, y compris leurs conséquences pour la théorie des variétés d'Einstein. Nous finissons par un examen du problème d'optimalité des estimées actuelles, en reliant cette question à une conjecture en géométrie kählérienne.

1. Introduction

In 1994, Witten [**39**] shocked the mathematical world by announcing that the differential-topological invariants of Donaldson [**9**] are intimately tied to the scalar curvature of Riemannian 4-manifolds. His central discovery was a new family of 4-manifold invariants, now called the *Seiberg-Witten invariants*, obtained by counting

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solutions of a non-linear Dirac equation [**8, 10, 12, 22, 36**]. When a 4-manifold has a non-zero Seiberg-Witten invariant, a Weitzenböck argument shows that it cannot admit metrics of positive scalar curvature; and as a consequence, there are many simply connected, non-spin 4-manifolds which do not admit positive-scalar-curvature metrics. Since this last assertion stands in stark opposition to results concerning manifolds of higher dimension [**14, 34**], one can only conclude that dimension four must be treated as *sui generis*.

In fact, the idiosyncratic nature of four-dimensional geometry largely stems from a single Lie-group-theoretic fluke: *the four-dimensional rotation group $SO(4)$ isn't simple*. Indeed, the decomposition

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

induces an invariant decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{1}$$

of the bundle of 2-forms on any oriented Riemannian 4-manifold (M, g). The rank-3 bundles $\Lambda^\pm$ are in fact exactly the eigenspaces of the Hodge (star) duality operator

$$\star : \Lambda^2 \longrightarrow \Lambda^2,$$

the eigenvalues of which are ± 1; sections of Λ^+ are therefore called *self-dual* 2-forms, whereas sections of Λ^- are called *anti-self-dual* 2-forms. Since $\star$ is unchanged on middle-dimensional forms if g is multiplied by a smooth positive function, the decomposition (1) really only depends on the conformal class $\gamma = [g]$ rather than on the Riemannian metric g itself.

Now this, in turn, has some peculiarly four-dimensional consequences for the Riemann curvature tensor $\mathcal{R}$. Indeed, since $\mathcal{R}$ may be identified with a linear map $\Lambda^2 \to \Lambda^2$, there is an induced decomposition [**32**]

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W_- + \frac{s}{12} \end{array} \right)$$

into simpler pieces. Here the *self-dual* and *anti-self-dual Weyl curvatures* $W_\pm$ are defined to be the trace-free pieces of the appropriate blocks. The scalar curvature s is understood to act by scalar multiplication, and $\overset{\circ}{r}$ can be identified with the trace-free part $r - \frac{s}{4}g$ of the Ricci curvature.

Witten's remarkable discoveries include the fact that the Seiberg-Witten equations (cf. §2 below) give a lower bound for the L^2 norm of the scalar curvature [**23, 24, 39**]. As will be explained in this article, however, they also imply estimates [**26, 27**] which involve the L^2 norms of both s and W_+. The importance of this is enhanced by the fact that the L^2 norms of the four pieces of the curvature tensor $\mathcal{R}$ are interrelated by two formulæ of Gauss-Bonnet type, so that the L^2 norms of s and W_+ actually determine the L^2 norms of W_- and $\mathring{r}$, too.

To clarify this last point, observe that the intersection form

$$\begin{array}{rcl} \smile : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\ [\phi]\ ,\ [\psi]\ ) & \longmapsto & \displaystyle\int_M \phi \wedge \psi \end{array}$$

may be diagonalized[1] as

$$\left[\begin{array}{cc} \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{b_+(M)} & \\ & {\scriptstyle b_-(M)}\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right. \end{array}\right]$$

by choosing a suitable basis for the de Rham cohomology $H^2(M,\mathbb{R})$. The numbers $b_\pm(M)$ are independent of the choice of basis, and so are oriented homotopy invariants of M. Their difference

$$\tau(M) = b_+(M) - b_-(M),$$

is called the *signature* of M. The Hirzebruch signature theorem [**16**] asserts that this invariant is expressible as a curvature integral, which may be put in the explicit form [**32**]

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W^+|^2 - |W^-|^2\right) d\mu. \tag{2}$$

Here the curvatures, norms $|\cdot|$, and volume form $d\mu$ are, of course, those of a particular Riemannian metric g, but the entire point is that the answer is independent of *which* metric we use. Thus the L^2 norms of W_+ and W_- determine one another, once the signature τ is known.

A second such relation is given by the 4-dimensional case of the generalized Gauss-Bonnet theorem [**1**]. This asserts that the Euler characteristic

$$\chi(M) = 2 - 2b_1(M) + b_2(M)$$

[1] over $\mathbb{R}$, of course; the story over $\mathbb{Z}$ is a great deal more complicated!

is also given by a curvature integral, which can be put in the explicit form [**32**]

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(|W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\overset{\circ}{r}|^2}{2}\right) d\mu. \tag{3}$$

In conjunction with (2), this allows one to deduce the L^2 norm of $\overset{\circ}{r}$ from those of s and W_+, assuming that χ and τ are both known.

For this reason, Seiberg-Witten theory is able to shed light on all four parts of the curvature tensor $\mathcal{R}$. In particular, we will see in §3 that these ideas naturally lead to subtle obstructions [**25, 19, 26, 27**] to the existence of Einstein metrics on 4-manifolds. Then, in §4, we will derive some new results regarding the optimality of the estimates of §2. It will turn out that this issue bears decisively on a conjecture regarding the existence of constant-scalar-curvature Kähler metrics on complex surfaces of general type.

2. Seiberg-Witten Theory

Let (M, g) be a compact oriented Riemannian 4-manifold. On any contractible open subset $U \subset M$, one can define Hermitian vector bundles

$$\begin{array}{ccc} \mathbb{C}^2 \to & \mathbb{S}_\pm|_U \\ & \downarrow \\ & U \subset M \end{array}$$

called spin bundles, with two characteristic properties: their determinant line bundles $\wedge^2\mathbb{S}_\pm$ are canonically trivial, and their projectivizations

$$\begin{array}{ccc} \mathbb{CP}_1 \to & \mathbb{P}(\mathbb{S}_\pm) \\ & \downarrow \\ & M \end{array}$$

are exactly the unit 2-sphere bundles $S(\Lambda^\pm)$. As one passes between open subset U and U', however, the corresponding locally-defined spin bundles are not quite canonically isomorphic; instead, there are two equally 'canonical' isomorphisms, differing by a sign. Because of this, one cannot generally define the bundles $\mathbb{S}_\pm$ globally on M; manifolds on which this can be done are called *spin*, and are characterized by the vanishing of the Stiefel-Whitney class $w_2 = w_2(TM) \in H^2(M, \mathbb{Z}_2)$. However, one can always find Hermitian complex line bundles $L \to M$ with first Chern class $c_1 = c_1(L)$ satisfying

$$c_1 \equiv w_2 \mod 2. \tag{4}$$

Given such a line bundle, one can then construct Hermitian vector bundles $\mathbb{V}_\pm$ with

$$\mathbb{P}(\mathbb{V}_\pm) = S(\Lambda^\pm)$$

by formally setting

$$\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2},$$

because the sign problems encountered in consistently defining the transition functions of $\mathbb{S}_\pm$ are exactly canceled by those associated with trying to find consistent square-roots of the transition functions of L.

The isomorphism class $\mathfrak{c}$ of such a choice of $\mathbb{V}_\pm$ is called a *spin^c structure* on M. The cohomology group $H^2(M,\mathbb{Z})$ acts freely and transitively on the spin^c structures by tensoring $\mathbb{V}_\pm$ with complex line bundles. Each spin^c structure has a first Chern class $c_1 := c_1(L) = c_1(\mathbb{V}_\pm) \in H^2(M,\mathbb{Z})$ satisfying (4), and the $H^2(M,\mathbb{Z})$-action on spin^c structures induces the action

$$c_1 \longmapsto c_1 + 2\alpha,$$

$\alpha \in H^2(M,\mathbb{Z})$, on first Chern classes. Thus, if $H^2(M,\mathbb{Z})$ has trivial 2-torsion — as will be true, for example, if M is simply connected — then the spin^c structures are precisely in one-to-one correspondence with the set of cohomology classes $c_1 \in H^2(M,\mathbb{Z})$ satisfying (4).

To make this discussion more concrete, suppose that M admits an almost-complex structure. Any given almost-complex structure can be deformed to an almost complex structure J which is compatible with g in the sense that $J^*g = g$. Choose such a J, and consider the rank-2 complex vector bundles

$$\begin{aligned} \mathbb{V}_+ &= \Lambda^{0,0} \oplus \Lambda^{0,2} \\ \mathbb{V}_- &= \Lambda^{0,1}. \end{aligned} \tag{5}$$

These are precisely the twisted spinor bundles of the spin^c structure obtained by taking L to be the anti-canonical line bundle $\Lambda^{0,2}$ of the almost-complex structure. A spin^c structure $\mathfrak{c}$ arising in this way will be said to be of *almost-complex type.* These are exactly the spin^c structures for which

$$c_1^2 = (2\chi + 3\tau)(M).$$

On a spin manifold, the spin bundles $\mathbb{S}_\pm$ carry natural connections induced by the Levi-Civita connection of the given Riemannian metric g. On a spin^c manifold, however, there is not a natural unique choice of connections on $\mathbb{V}_\pm$. Nonetheless, since we formally have $\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2}$, every Hermitian connection A on L induces associated Hermitian connections ∇_A on $\mathbb{V}_\pm$.

On the other hand, there is a canonical isomorphism $\Lambda^1 \otimes \mathbb{C} = \mathrm{Hom}\,(\mathbb{S}_+, \mathbb{S}_-)$, so that $\Lambda^1 \otimes \mathbb{C} \cong \mathrm{Hom}\,(\mathbb{V}_+, \mathbb{V}_-)$ for any spin^c structure, and this induces a canonical homomorphism

$$\mathit{Cliff}\ : \Lambda^1 \otimes \mathbb{V}_+ \longrightarrow \mathbb{V}_-$$

called *Clifford multiplication.* Composing these operations allows us to define a so-called *twisted Dirac operator*

$$D_A : \Gamma(\mathbb{V}_+) \longrightarrow \Gamma(\mathbb{V}_-)$$

by $D_A\Phi = \mathit{Cliff}\,(\nabla_A\Phi)$.

For any spin^c structure, we have already noted that there is a canonical diffeomorphism $\mathbb{P}(\mathbb{V}_+) \stackrel{\cong}{\to} S(\Lambda^+)$. In polar coordinates, we now use this to define the angular part of a unique continuous map

$$\sigma : \mathbb{V}_+ \longrightarrow \Lambda^+$$

with radial part specified by

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

This map is actually real-quadratic on each fiber of $\mathbb{V}_+$; indeed, assuming our spin^c structure is induced by a complex structure J, then, in terms of (5), σ is explicitly given by

$$\sigma(f,\phi) = (|f|^2 - |\phi|^2)\frac{\omega}{4} + \Im m(\bar{f}\phi),$$

where $f \in \Lambda^{0,0}$, $\phi \in \Lambda^{0,2}$, and where $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ is the associated 2-form of (M,g,J). On a deeper level, σ directly arises from the fact that $\mathbb{V}_+ = \mathbb{S}_+ \otimes L^{1/2}$, while $\Lambda^+ \otimes \mathbb{C} = \odot^2\mathbb{S}_+$. For this reason, σ is invariant under parallel transport.

We are now in a position to introduce the Seiberg-Witten equations

$$D_A\Phi = 0 \tag{6}$$

$$F_A^+ = i\sigma(\Phi), \tag{7}$$

where the unknowns are a Hermitian connection A on L and a section Φ of $\mathbb{V}_+$. Here F_A^+ is the self-dual part of the curvature of A, and so is a purely imaginary 2-form.

For many 4-manifolds, it turns out that there is a solution of the Seiberg-Witten equations for each metric. Let us introduce some convenient terminology [**21**] to describe this situation.

Definition 1. — Let M be a smooth compact oriented 4-manifold with $b_+ \geq 2$, and suppose that M carries a spin^c structure $\mathfrak{c}$ for which the Seiberg-Witten equations (6–7) have a solution for every Riemannian metric g on M. Then the first Chern class $c_1 \in H^2(M,\mathbb{Z})$ of $\mathfrak{c}$ will be called a *monopole class*.

This definition is useful in practice primarily because there are mapping degree arguments which lead to the existence of solutions the Seiberg-Witten equations. For example [**8, 30, 39**], if $\mathfrak{c}$ is a spin^c structure of almost-complex type, then the *Seiberg-Witten invariant* $\mathcal{SW}_\mathfrak{c}(M)$ can be defined as the number of solutions, modulo gauge transformations and counted with orientations, of a generic perturbation

$$\begin{aligned} D_A\Phi &= 0 \\ iF_A^+ + \sigma(\Phi) &= \phi \end{aligned}$$

of (6–7), where ϕ is a smooth self-dual 2-form. If $b_+(M) \geq 2$, this integer is independent of the metric g; and if it is non-zero, the first Chern class c_1 of $\mathfrak{c}$ is then a monopole class. Similar things are also true when $b_+(M) = 1$, although the story [**12**] becomes rather more complicated.

Now, via the Hodge theorem, every Riemannian metric g on M determines a direct sum decomposition

$$H^2(M, \mathbb{R}) = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-,$$

where $\mathcal{H}_g^+$ (respectively, $\mathcal{H}_g^-$) consists of those cohomology classes for which the harmonic representative is self-dual (respectively, anti-self-dual). Because the restriction of the intersection form to $\mathcal{H}_g^+$ (respectively, $\mathcal{H}_g^-$) is positive (respectively, negative) definite, and because these subspaces are mutually orthogonal with respect to the intersection pairing, the dimensions of these spaces are exactly the invariants $b_\pm$ defined in §1. If the first Chern class c_1 of the spin^c structure $\mathfrak{c}$ is now decomposed as

$$c_1 = c_1^+ + c_1^-,$$

where $c_1^\pm \in \mathcal{H}_g^\pm$, we get the important inequality

$$\int_M |\Phi|^4 d\mu \geq 32\pi^2 (c_1^+)^2 \tag{8}$$

because (7) tells us that $2\pi c_1^+$ is the harmonic part of $-\sigma(\Phi)$.

Many of the most remarkable consequences of Seiberg-Witten theory stem [**8, 22, 30**] from the fact that the equations (6–7) imply the Weitzenböck formula

$$0 = 4\nabla^*\nabla\Phi + s\Phi + |\Phi|^2\Phi, \tag{9}$$

where s denotes the scalar curvature of g, and where we have introduced the abbreviation $\nabla_A = \nabla$. Taking the inner product with Φ, it follows that

$$0 = 2\Delta|\Phi|^2 + 4|\nabla\Phi|^2 + s|\Phi|^2 + |\Phi|^4. \tag{10}$$

If we multiply (10) by $|\Phi|^2$ and integrate, we have

$$0 = \int_M \left[2\,\left|\, d|\Phi|^2 \,\right|^2 + 4|\Phi|^2|\nabla\Phi|^2 + s|\Phi|^4 + |\Phi|^6\right] d\mu_g,$$

so that

$$\int (-s)|\Phi|^4 d\mu \geq 4\int |\Phi|^2|\nabla\Phi|^2 d\mu + \int |\Phi|^6 d\mu. \tag{11}$$

This leads [**27**] to the following curvature estimate:

Theorem 2. — *Let M be a smooth compact oriented 4-manifold with monopole class c_1. Then every Riemannian metric g on M satisfies*

$$\int_M \left(\frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+|\right)^2 d\mu \geq 32\pi^2 (c_1^+)^2, \tag{12}$$

where c_1^+ is the self-dual part of c_1 with respect to g.

Proof. — The first step is to prove the inequality

$$V^{1/3}\left(\int_M \left|\frac{2}{3}s_g - 2\sqrt{\frac{2}{3}}|W_+|\right|^3 d\mu\right)^{2/3} \geq 32\pi^2(c_1^+)^2, \tag{13}$$

where $V = \mathrm{Vol}(M,g) = \int_M d\mu_g$ is the total volume of (M,g).

Any self-dual 2-form ψ on any oriented 4-manifold satisfies the Weitzenböck formula **[6]**

$$(d+d^*)^2\psi = \nabla^*\nabla\psi - 2W_+(\psi,\cdot) + \frac{s}{3}\psi.$$

It follows that

$$\int_M (-2W_+)(\psi,\psi)d\mu \geq \int_M (-\frac{s}{3})|\psi|^2\ d\mu - \int_M |\nabla\psi|^2\ d\mu.$$

However,

$$|W_+|_g|\psi|^2 \geq -\sqrt{\frac{3}{2}}W_+(\psi,\psi)$$

simply because W_+ is trace-free. Thus

$$\int_M 2\sqrt{\frac{2}{3}}|W_+||\psi|^2 d\mu \geq \int_M (-\frac{s}{3})|\psi|^2\ d\mu - \int_M |\nabla\psi|^2\ d\mu,$$

and hence

$$-\int_M (\frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+|)|\psi|^2 d\mu \geq \int_M (-s)|\psi|^2\ d\mu - \int_M |\nabla\psi|^2\ d\mu.$$

On the other hand, the particular self-dual 2-form $\varphi = \sigma(\Phi) = -iF_A^+$ satisfies

$$\begin{aligned} |\varphi|^2 &= \frac{1}{8}|\Phi|^4, \\ |\nabla\varphi|^2 &\leq \frac{1}{2}|\Phi|^2|\nabla\Phi|^2. \end{aligned}$$

Setting $\psi = \varphi$, we thus have

$$-\int_M (\frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+|)|\Phi|^4 d\mu \geq \int_M (-s)|\Phi|^4\ d\mu - 4\int_M |\Phi|^2|\nabla\Phi|^2\ d\mu.$$

But (11) tells us that

$$\int_M (-s)|\Phi|^4\ d\mu - 4\int_M |\Phi|^2|\nabla\Phi|^2\ d\mu \geq \int_M |\Phi|^6\ d\mu,$$

so we obtain

$$-\int_M (\frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+|)|\Phi|^4 d\mu \geq \int_M |\Phi|^6\ d\mu. \tag{14}$$

By the Hölder inequality, we thus have

$$\left(\int \left|\frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+|\right|^3 d\mu\right)^{1/3} \left(\int |\Phi|^6 d\mu\right)^{2/3} \geq \int |\Phi|^6\ d\mu,$$

Since the Hölder inequality also tells us that

$$\int |\Phi|^6 \, d\mu \geq V^{-1/2} \left(\int |\Phi|^4 d\mu \right)^{3/2},$$

we thus have

$$V^{1/3} \left(\int_M \left| \frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+| \right|^3 d\mu \right)^{2/3} \geq \int |\Phi|^4 d\mu \geq 32\pi^2 (c_1^+)^2,$$

where the last inequality is exactly (8). This completes the first part of the proof.

Next, we observe that any smooth conformal γ class on any oriented 4-manifold contains a C^2 metric such that $s - \sqrt{6}|W_+|$ is constant. Indeed, as observed by Gursky [**15**], this readily follows from the standard proof of the Yamabe problem. The main point is that the curvature expression

$$\mathfrak{S}_g = s_g - \sqrt{6}|W_+|_g$$

transforms under conformal changes $g \mapsto \hat{g} = u^2 g$ by the rule

$$\mathfrak{S}_{\hat{g}} = u^{-3} \left(6\Delta_g + \mathfrak{S}_g \right) u,$$

just like the ordinary scalar curvature s. We will actually use this only in the negative case, where the proof is technically the simplest, and simply repeats(2) the arguments of Trudinger [**38**].

The conformal class γ of a given metric g thus always contains a metric g_γ for which $\frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+|$ is constant. But since the existence of solutions of the Seiberg-Witten equations precludes the possibility that we might have $s_{g_\gamma} > 0$, this constant is necessarily non-positive. We thus have

$$\int_M \left(\frac{2}{3}s_{g_\gamma} - 2\sqrt{\frac{2}{3}}|W_+|_{g_\gamma} \right)^2 d\mu_{g_\gamma} = V_{g_\gamma}^{1/3} \left(\int_M \left| (\frac{2}{3}s_{g_\gamma} - 2\sqrt{\frac{2}{3}}|W_+|_{g_\gamma} \right|^3 d\mu_{g_\gamma} \right)^{2/3},$$

so that

$$\int_M \left(\frac{2}{3}s_{g_\gamma} - 2\sqrt{\frac{2}{3}}|W_+|_{g_\gamma} \right)^2 d\mu_{g_\gamma} \geq 32\pi^2 (c_1^+)^2.$$

Thus we at least have the desired L^2 estimate for a specific metric g_γ which is conformally related to the given metric g.

Let us now compare the left-hand side with analogous expression for the given metric g. To do so, we express g in the form $g = u^2 g_\gamma$, where u is a positive C^2

(2)However, since $|W_+|$ is generally only Lipschitz continuous, the minimizer generally only has regularity $C^{2,\alpha}$ in the vicinity of a zero of W_+.

function, and observe that

$$\begin{aligned}\int \mathfrak{S}_g u^2 d\mu_{g_\gamma} &= \int u^{-3}\left(6\Delta_{g_\gamma} u + \mathfrak{S}_{g_\gamma} u\right) u^2 d\mu_{g_\gamma} \\ &= \int \left(-6u^{-2}|du|^2_{g_\gamma} + \mathfrak{S}_{g_\gamma}\right) d\mu_{g_\gamma} \\ &\leq \int \mathfrak{S}_{g_\gamma} d\mu_{g_\gamma}.\end{aligned}$$

Applying Cauchy-Schwarz, we thus have

$$\begin{aligned}-V_{g_\gamma}^{1/2}\left(\int \mathfrak{S}_g^2 d\mu_g\right)^{1/2} &= -V_{g_\gamma}^{1/2}\left(\int \left(\mathfrak{S}_g u^2\right)^2 d\mu_{g_\gamma}\right)^{1/2} \\ &\leq \int_M \mathfrak{S}_g u^2 d\mu_{g_\gamma} \\ &\leq \int_M \mathfrak{S}_{g_\gamma} d\mu_{g_\gamma} \\ &= -V_{g_\gamma}^{1/2}\left(\int \mathfrak{S}^2_{g_\gamma} d\mu_{g_\gamma}\right)^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}\int_M \left(\frac{2}{3}s_g - 2\sqrt{\frac{2}{3}}|W_+|_g\right)^2 d\mu_g &= \frac{4}{9}\int \mathfrak{S}_g^2 d\mu_g \\ &\geq \frac{4}{9}\int \mathfrak{S}^2_{g_\gamma} d\mu_{g_\gamma} \\ &= \int_M \left(\frac{2}{3}s_{g_\gamma} - 2\sqrt{\frac{2}{3}}|W_+|_{g_\gamma}\right)^2 d\mu_{g_\gamma} \\ &\geq 32\pi^2 (c_1^+)^2,\end{aligned}$$

exactly as claimed. □

Notice that we can rewrite the inequality (12) as

$$\left\|\frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+|\right\| \geq 4\sqrt{2}\pi|c_1^+|,$$

where $\|\cdot\|$ denotes the L^2 norm with respect to g. Dividing by $\sqrt{24}$ and applying the triangle inequality, we thus have

Corollary 3. — *Let M be a smooth compact oriented 4-manifold with monopole class c_1. Then every Riemannian metric g on M satisfies*

$$\frac{2}{3}\left\|\frac{s}{\sqrt{24}}\right\| + \frac{1}{3}\|W_+\| \geq \frac{2\pi}{\sqrt{3}}|c_1^+|. \tag{15}$$

Inequality (12) actually belongs to a family of related estimates:

Theorem 4. — *Let M be a smooth compact oriented 4-manifold with monopole class c_1, and let $\delta \in [0, \frac{1}{3}]$ be a constant. Then every Riemannian metric g on M satisfies*

$$\int_M \left[(1-\delta)s - \delta\sqrt{24}|W_+|\right]^2 d\mu \geq 32\pi^2(c_1^+)^2, \tag{16}$$

Proof. — Inequality (11) implies

$$\int (-s)|\Phi|^4 d\mu \geq \int |\Phi|^6 d\mu. \tag{17}$$

On the other hand, inequality (14) asserts that

$$-\int_M (\frac{2}{3}s - 2\sqrt{\frac{2}{3}}|W_+|)|\Phi|^4 d\mu \geq \int_M |\Phi|^6 \ d\mu.$$

Now multiply (17) by $1-3\delta$, multiply (14) by 3δ, and add. The result is

$$\int \left[(1-\delta)s - \delta\sqrt{24}|W_+|\right] |\Phi|^4 d\mu \geq \int |\Phi|^6 d\mu. \tag{18}$$

Applying the same Hölder inequalities as before, we now obtain

$$V^{1/3}\left(\int_M \left|(1-\delta)s - \delta\sqrt{24}|W_+|\right|^3 d\mu\right)^{2/3} \geq \int |\Phi|^4 d\mu \geq 32\pi^2(c_1^+)^2.$$

Passage from this L^3 estimate to the desired L^2 estimate is then accomplished by the same means as before: every conformal class contains a metric for which $(1-\delta)s - \delta\sqrt{24}|W_+|$ is constant, and this metric minimizes

$$\int_M \left[(1-\delta)s - \delta\sqrt{24}|W_+|\right]^2 d\mu$$

among metrics in its conformal class. □

Rewriting (16) as

$$\left\|(1-\delta)s - \delta\sqrt{24}|W_+|\right\| \geq 4\sqrt{2}\pi|c_1^+|,$$

dividing by $\sqrt{24}$, and applying the triangle inequality, we thus have

Corollary 5. — *Let M be a smooth compact oriented 4-manifold with monopole class c_1. Then every Riemannian metric g on M satisfies*

$$(1-\delta)\|\frac{s}{\sqrt{24}}\| + \delta\|W_+\| \geq \frac{2\pi}{\sqrt{3}}|c_1^+| \tag{19}$$

for every $\delta \in [0, \frac{1}{3}]$.

The $\delta = 0$ version of (16) is implicit in the work of Witten **[39]**; it was later made explicit in **[24]**, where it was also shown that equality holds for $\delta = 0$ iff g is a Kähler metric of constant, non-positive scalar curvature. But indeed, since $\sqrt{24}|W_+| \equiv |s|$ for any Kähler manifold of real dimension 4, metrics of this kind saturate (16) for each value of δ. Conversely:

Proposition 6. — *Let $\delta \in [0, \frac{1}{3})$ be a fixed constant. If g is a metric such that equality holds in (16), then g is Kähler, and has constant scalar curvature.*

Proof. — Equality in (16) implies equality in (18). However, $(1-3\delta)$ times inequality (11) plus 3δ times inequality (14) reads

$$\int \left[(1-\delta)s - \delta\sqrt{24}|W_+|\right] |\Phi|^4 d\mu \geq \int |\Phi|^6 d\mu + 4(1-3\delta) \int |\Phi|^2 |\nabla\Phi|^2 d\mu.$$

Equality in (16) therefore implies that

$$0 = \frac{1}{2} \int |\Phi|^2 |\nabla\Phi|^2 d\mu \geq \int |\nabla\varphi|^2 d\mu,$$

forcing the 2-form φ to be parallel. If $\varphi \not\equiv 0$, we conclude that the metric is Kähler, and the constancy of s then follows from the Yamabe portion of the argument.

On the other hand, since $b_+(M) \geq 2$ and c_1 is a monopole class, M does not admit metrics of positive scalar curvature. If $\varphi \equiv 0$ and (16) is saturated, one can therefore show that (M, g) is $K3$ or T^4 with a Ricci-flat Kähler metric. The details are left as an exercise for the interested reader. □

When $\delta = \frac{1}{3}$, the above argument breaks down. However, a metric g can saturate (12) only if equality holds in (8), and this forces the self-dual 2-form $\varphi = \sigma(\Phi)$ to be harmonic. Moreover, the relevant Hölder inequalities would also have to be saturated, forcing φ to have constant length. This forces g to be *almost-Kähler*, in the sense that there is an orientation-compatible orthogonal almost-complex structure for which the associated 2-form is closed. For details, see **[27]**.

It is reasonable to ask whether the inequalities (16) and (19) continue to hold when $\delta > 1/3$. This issue will be addressed in §4.

3. Einstein Metrics

Recall that a smooth Riemannian metric g is said to be *Einstein* if its Ricci curvature r is a constant multiple of the metric:

$$r = \lambda g.$$

Not every 4-manifold admits such metrics. A necessary condition for the existence of an Einstein metric on a compact oriented 4-manifold is that the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$ must hold **[37, 17, 5]**. Indeed, (2) and (3) tell us that

$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left(\frac{s^2}{24} + 2|W_\pm|^2 - \frac{|\overset{\circ}{r}|^2}{2} \right) d\mu.$$

The Hitchin-Thorpe inequality follows, since the integrand is non-negative when $\overset{\circ}{r} = 0$. This argument, however, treats the scalar and Weyl contributions as 'junk' terms, about which one knows nothing except that they are non-negative. We now remedy this by invoking the estimates of §2.

Proposition 7. — *Let M be a smooth compact oriented 4-manifold with monopole class c_1. Then every metric g on M satisfies*

$$\frac{1}{4\pi^2}\int_M \left(\frac{s_g^2}{24}+2|W_+|_g^2\right)d\mu_g \geq \frac{2}{3}(c_1^+)^2.$$

If $c_1^+ \neq 0$, moreover, equality can only hold if g is almost-Kähler, with almost-Kähler class proportional to c_1^+.

Proof. — We begin with inequality (15)

$$\frac{2}{3}\|\frac{s}{\sqrt{24}}\|+\frac{1}{3}\|W_+\| \geq \frac{2\pi}{\sqrt{3}}|c_1^+|,$$

and elect to interpret the left-hand side as the dot product

$$(\frac{2}{3},\frac{1}{3\sqrt{2}})\cdot\left(\|\frac{s}{\sqrt{24}}\|,\sqrt{2}\|W_+\|\right)$$

in $\mathbb{R}^2$. Applying Cauchy-Schwarz, we thus have

$$\left((\frac{2}{3})^2+(\frac{1}{3\sqrt{2}})^2\right)^{1/2}\left(\|\frac{s}{\sqrt{24}}\|^2+2\|W_+\|^2\right)^{1/2} \geq \frac{2}{3}\|\frac{s}{\sqrt{24}}\|+\frac{1}{3}\|W_+\|.$$

Thus

$$\frac{1}{2}\int_M\left(\frac{s^2}{24}+2|W_+|^2\right)d\mu \geq \left(\frac{2}{3}\|\frac{s}{\sqrt{24}}\|+\frac{1}{3}\|W_+\|\right)^2 \geq \frac{4\pi^2}{3}(c_1^+)^2,$$

and hence

$$\frac{1}{4\pi^2}\int_M \left(\frac{s_g^2}{24}+2|W_+|_g^2\right)d\mu_g \geq \frac{2}{3}(c_1^+)^2,$$

as claimed.

In the equality case, φ would be a closed self-dual form of constant norm, so g would be almost-Kähler unless $\varphi \equiv 0$. $\square$

To give some concrete applications, we now focus on the case of complex surfaces.

Proposition 8. — *Let (X, J_X) be a compact complex surface with $b_+ > 1$, and let (M, J_M) be the complex surface obtained from X by blowing up $k > 0$ points. Then any Riemannian metric g on the 4-manifold*

$$M = X\#k\overline{\mathbb{CP}}_2$$

satisfies

$$\frac{1}{4\pi^2}\int_M \left(\frac{s_g^2}{24}+2|W_+|_g^2\right)d\mu_g > \frac{2}{3}(2\chi+3\tau)(X).$$

Proof. — Let $c_1(X)$ denote the first Chern class of the given complex structure J_X, and, by a standard abuse of notation, let $c_1(X)$ also denote the pull-back class of this class to M. If $E_1, \ldots, E_k$ are the Poincaré duals of the exceptional divisors in M introduced by blowing up, the complex structure J_M has Chern class

$$c_1(M) = c_1(X) - \sum_{j=1}^{k} E_j.$$

By a result of Witten [**39**], this is a monopole class of M. However, there are self-diffeomorphisms of M which act on $H^2(M)$ in a manner such that

$$\begin{array}{ccc} c_1(X) & \longmapsto & c_1(X) \\ E_j & \longmapsto & \pm E_j \end{array}$$

for any choice of signs we like. Thus

$$c_1 = c_1(X) + \sum_{j=1}^{k} (\pm E_j)$$

is a monopole class on M for each choice of signs. We now fix our choice of signs so that

$$[c_1(X)]^+ \cdot (\pm E_j) \geq 0,$$

for each j, with respect to the decomposition induced by the given metric g. We then have

$$\begin{aligned} (c_1^+)^2 &= \left([c_1(X)]^+ + \sum_{j=1}^{k} (\pm E_j^+) \right)^2 \\ &= ([c_1(X)]^+)^2 + 2\sum_{j=1}^{k} [c_1(X)]^+ \cdot (\pm E_j) + (\sum_{j=1}^{k} (\pm E_j^+))^2 \\ &\geq ([c_1(X)]^+)^2 \\ &\geq (2\chi + 3\tau)(X). \end{aligned}$$

This shows that

$$\frac{1}{4\pi^2} \int_M \left(\frac{s_g^2}{24} + 2|W_+|_g^2 \right) d\mu_g \geq \frac{2}{3}(2\chi + 3\tau)(X).$$

If equality held, g would be almost-Kähler, with almost-Kähler class $[\omega]$ proportional to c_1^+. On the other hand, we would also have $[c_1(X)]^+ \cdot E_j = 0$, so it would then follow that $[\omega] \cdot E_j = 0$ for all j. However, the Seiberg-Witten invariant would be non-trivial for a spin^c structure with $c_1(\tilde{L}) = c_1(L) - 2(\pm E_1)$, and a celebrated theorem of Taubes [**36**] would then force the homology class E_j to be represented by a pseudo-holomorphic 2-sphere in the symplectic manifold (M, ω). But the (positive!)

area of this sphere with respect to g would then be exactly $[\omega]\cdot E_j$, contradicting the observation that $[\omega]\cdot E_j = 0$. □

Theorem 9. — *Let (X, J_X) be a compact complex surface with $b_+ > 1$, and let (M, J_M) be obtained from X by blowing up k points. Then the smooth compact 4-manifold M does not admit any Einstein metrics if $k \geq \frac{1}{3}c_1^2(X)$.*

Proof. — We may assume that $(2\chi + 3\tau)(X) > 0$, since otherwise the result follows from the Hitchin-Thorpe inequality.

Now

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s_g^2}{24} + 2|W_+|_g^2 - \frac{|\overset{\circ}{r}|^2}{2}\right) d\mu_g$$

for any metric on g on M. If g is an Einstein metric, the trace-free part $\overset{\circ}{r}$ of the Ricci curvature vanishes, and we then have

$$\begin{aligned}(2\chi + 3\tau)(X) - k &= (2\chi + 3\tau)(M)\\ &= \frac{1}{4\pi^2}\int_M \left(\frac{s_g^2}{24} + 2|W_+|_g^2\right) d\mu_g\\ &> \frac{2}{3}(2\chi + 3\tau)(X)\end{aligned}$$

by Proposition 8. If M carries an Einstein metric, it therefore follows that

$$\frac{1}{3}(2\chi + 3\tau)(X) > k.$$

The claim thus follows by contraposition. □

Example. — Let $X \subset \mathbb{CP}_4$ be the intersection of two cubic hypersurfaces in general position. Since the canonical class on X is exactly the hyperplane class, $c_1^2(X) = 1^2 \cdot 3 \cdot 3 = 9$. Theorem 9 therefore tells us that if we blow up X at 3 points, the resulting 4-manifold

$$M = X \# 3\overline{\mathbb{CP}}_2$$

does not admit Einstein metrics.

But now consider the *Horikawa surface* N obtained as a ramified double cover of the blown-up projective plane $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$ branched over the (smooth) proper transform $\hat{C}$ of the singular curve C given by

$$x^{10} + y^{10} + z^6(x^4 + y^4) = 0$$

in the complex projective plane, where the singular point $[0 : 0 : 1]$ of C is the point at which we blow up $\mathbb{CP}_2$. By the Freedman classification of 4-manifolds [**11**], both of these complex surfaces are homeomorphic to

$$11\mathbb{CP}_2\#53\overline{\mathbb{CP}}_2.$$

However, N has $c_1 < 0$, and so admits a Kähler-Einstein metric by the Aubin/Yau theorem [**3, 40**]. Thus, although M and N are homeomorphic, one admits Einstein metrics, while the other doesn't. $\diamondsuit$

Example. — Let $X \subset \mathbb{CP}_3$ be a hypersurface of degree 6. Since the canonical class on X is twice the hyperplane class, $c_1^2(X) = 2^2 \cdot 6 = 24$. Theorem 9 therefore tells us that if we blow up X at 8 points, the resulting 4-manifold

$$M = X \# 8\overline{\mathbb{CP}}_2$$

does not admit Einstein metrics.

However, the Freedman classification can be used to show that M is homeomorphic to the Horikawa surface N obtained as a ramified double cover of $\mathbb{CP}_1 \times \mathbb{CP}_1$ branched at a generic curve of bidegree $(6, 12)$; indeed, both of these complex surfaces are homeomorphic to

$$21\mathbb{CP}_2 \# 93\overline{\mathbb{CP}}_2.$$

However, this N also admits a Kähler-Einstein metric, even though the existence of Einstein metric is obstructed on M. $\diamondsuit$

Example. — Let $X \subset \mathbb{CP}_3$ be a hypersurface of degree 10. Since the canonical class on X is six times the hyperplane class, $c_1^2(X) = 6^2 \cdot 10 = 360$. Theorem 9 therefore tells us that if we blow up X at 120 or more points, the resulting 4-manifold does not admit Einstein metrics. In particular, this assertion applies to

$$M = X \# 144\overline{\mathbb{CP}}_2.$$

Now let N be obtained from $\mathbb{CP}_1 \times \mathbb{CP}_1$ as a ramified double cover branched at a generic curve of bidegree $(8, 58)$. Both M and N are then simply connected, and have $c_1^2 = 216$ and $p_g = 84$; and both are therefore homeomorphic to

$$129\mathbb{CP}_2 \# 633\overline{\mathbb{CP}}_2.$$

But again, N has $c_1 < 0$, and so admits a Kähler-Einstein metric, even though M does not admit an Einstein metric of any kind whatsoever.

In most respects, this example is much like the previous ones. However, this last choice of N is not a Horikawa surface, but instead sits well away from the Noether line [**4**] of complex-surface geography. $\diamondsuit$

Infinitely many such examples can be constructed using the above techniques, and the interested reader might wish to explore their geography.

It should be noted that Theorem 9 is the direct descendant of an analogous result in [**25**], where scalar curvature estimates alone were used to obtain an obstruction when $k \geq \frac{2}{3}c_1^2(X)$. It was later pointed out by Kotschick [**19**] that this suffices to imply the existence of homeomorphic pairs consisting of an Einstein manifold and a 4-manifold which does not admit Einstein metrics. An intermediate step between

[**25**] and Theorem 9 may be found in [**26**], where cruder Seiberg-Witten estimates of Weyl curvature were used to obtain an obstruction for $k \geq \frac{25}{57}c_1^2(X)$.

4. How Sharp are the Estimates?

The estimates we have described in §2 are optimal in the sense that equality is achieved for Kähler metrics of constant negative scalar curvature, with the standard orientation and spin^c structure. In this section, we will attempt to probe the limits of these estimates by considering metrics of precisely this type, but with *non-standard* choices of orientation and spin^c structure.

One interesting class of 4-manifolds which admit constant-scalar-curvature Kähler metrics are the complex surfaces with ample canonical line bundle. In terms of complex-surface classification [**4**], these are precisely those minimal surfaces of general type which do not contain $\mathbb{CP}_1$'s of self-intersection -2. The ampleness of the canonical line bundle is often written as $c_1 < 0$, meaning that $-c_1$ is a Kähler class. A celebrated result of Aubin/Yau [**3, 40**] guarantees that there is a unique Kähler-Einstein metric on M, compatible with the given complex structure, and with Kähler class $[\omega] = -c_1 = H^{1,1}(M, \mathbb{R})$. The scalar curvature of such a metric is, of course, a negative constant; indeed, $s = -\dim_{\mathbb{R}} M = -4$.

Now if M is a compact complex manifold without holomorphic vector fields, the set of Kähler classes which are representable by metrics of constant scalar curvature is open [**13, 28**] in $H^{1,1}(M, \mathbb{R})$. On the other hand, a manifold with $c_1 < 0$ never carries a non-zero holomorphic vector field, so it follows that a complex surface with ample canonical line bundle will carry lots of constant-scalar-curvature Kähler metrics which are non-Einstein if $b_- = h^{1,1} - 1$ is non-zero. However, one might actually hope to find such metrics even in those Kähler classes which are far from the anti-canonical class. This expectation may be codified as follows:

Conjecture 10. — *Let M be any compact complex surface with $c_1 < 0$. Then every Kähler class $[\omega] \in H^{1,1}(M, \mathbb{R})$ contains a unique Kähler metric of constant scalar curvature.*

The uniqueness clause was recently proved by X.-X. Chen [**7**], using ideas due to Donaldson and Semmes. A direct continuity-method attack on conjecture has also been explored by S.-R. Simanca.

Let us now narrow our discussion to a very special class of complex surfaces.

Definition 11. — A *Kodaira fibration* is a holomorphic submersion $\varpi : M \to B$ from a compact complex surface to a compact complex curve, such that the base B and fiber $F_z = \varpi^{-1}(z)$ both have genus ≥ 2. If M admits such a fibration ϖ, we will say that is a *Kodaira-fibered surface*.

The underlying 4-manifold M of a Kodaira-fibered surface is a fiber bundle over B, with fiber F. We thus have a long exact sequence [**33**]

$$\cdots \longrightarrow \pi_k(F) \longrightarrow \pi_k(M) \longrightarrow \pi_k(B) \longrightarrow \pi_{k-1}(F) \longrightarrow \cdots$$

of homotopy groups, and M is therefore a $K(\pi, 1)$. In particular, any 2-sphere in M is homologically trivial, and so has self-intersection 0; in particular, the complex surface M cannot contain any $\mathbb{CP}_1$'s of self-intersection -1 or -2. On the other hand, M is of general type, so the above implies that $c_1(M) < 0$. Kodaira-fibered surfaces thus provide us with an interesting testing-ground for Conjecture 10.

Now the product $B \times F$ of two complex curves of genus ≥ 2 is certainly Kodaira fibered, but such a product also admits orientation-reversing diffeomorphisms, and so has signature $\tau = 0$. However, as was first observed by Kodaira [**18**], one can construct examples with $\tau > 0$ by taking *branched covers* of products; cf. [**2, 4**]. For example, let B be a curve of genus 3 with a holomorphic involution $\iota : B \to B$ without fixed points; one may visualize such an involution as a 180° rotation of a 3-holed doughnut about an axis which passes though the middle hole, without meeting the doughnut. Let $f : C \to B$ be the unique 64-fold unbranched cover with $f_*[\pi_1(C)] = \ker[\pi_1(B) \to H_1(B, \mathbb{Z}_2)]$; thus C is a complex curve of genus 129. Let $\Sigma \subset C \times B$ be the union of the graphs of f and $\iota \circ f$. Then the homology class of Σ is divisible by 2. We may therefore construct a ramified double cover $M \to B \times C$ branched over Σ. The projection $M \to B$ is then a Kodaira fibration, with fiber F of genus 321. The projection $M \to C$ is also a Kodaira fibration, with fiber of genus 6. The signature of this example is $\tau(M) = 256$, and so coincidentally equals one-tenth of its Euler characteristic $\chi(M) = 2560$.

Now, more generally, let M be any Kodaira-fibered surface with $\tau > 0$, and let $\varpi : M \to B$ be a Kodaira fibration. Let p denote the genus of B, and let q denote the genus of a fiber F of ϖ. Indulging in a standard notational abuse, let us also use F to denote the Poincaré dual of the homology class of the fiber. Since F can be represented in de Rham cohomology by the pull-back of an area form on B, this $(1,1)$-class is positive semi-definite. On the other hand, $-c_1$ is a Kähler class on M, and so it follows that

$$[\omega_\varepsilon] = 2(p-1)F - \varepsilon c_1$$

is a Kähler class on M for any $\varepsilon > 0$. If Conjecture 10 is true, there must therefore exist a Kähler metric g_ε on M of constant scalar curvature with Kähler class $[\omega_\varepsilon]$. Let us explore the global geometric invariants of this putative metric.

The metric in question, being Kähler, would have total scalar curvature

$$\int s_{g_\varepsilon} d\mu_{g_\varepsilon} = 4\pi c_1 \cdot [\omega_\varepsilon] = -4\pi(\chi + \varepsilon c_1^2)(M)$$

and total volume

$$\int d\mu_{g_\varepsilon} = \frac{[\omega_\varepsilon]^2}{2} = \frac{\varepsilon}{2}(2\chi + \varepsilon c_1^2)(M).$$

The assumption that $s_{g_\varepsilon} = \text{const}$ would thus imply that

$$\begin{aligned}\|s\|^2 = \int s_{g_\varepsilon}^2 d\mu_{g_\varepsilon} &= \frac{32\pi^2}{\varepsilon}\frac{(\chi+\varepsilon c_1^2)^2}{2\chi+\varepsilon c_1^2} \\ &= 16\pi^2\frac{\chi}{\varepsilon}\left[1+(3+\frac{9}{2}\varrho)\varepsilon+O(\varepsilon^2)\right],\end{aligned}$$

where we have set

$$\varrho = \frac{\tau(M)}{\chi(M)} \quad .$$

Since a Kähler metric on a complex surface satisfies $|W_+|^2 \equiv s^2/24$, we would also consequently have

$$\begin{aligned}\int |W_+|_{g_\varepsilon}^2 d\mu_{g_\varepsilon} &= \frac{1}{24}\int s_{g_\varepsilon}^2 d\mu_{g_\varepsilon} \\ &= \frac{2}{3}\pi^2\frac{\chi}{\varepsilon}\left[1+(3+\frac{9}{2}\varrho)\varepsilon+O(\varepsilon^2)\right].\end{aligned}$$

It would thus follow that

$$\begin{aligned}\|W_-\|^2 = \int |W_-|_{g_\varepsilon}^2 d\mu_{g_\varepsilon} &= -12\pi^2\tau(M) + \int |W_+|_{g_\varepsilon}^2 d\mu_{g_\varepsilon} \\ &= \frac{2}{3}\pi^2\frac{\chi}{\varepsilon}\left[1+(3-\frac{27}{2}\varrho)\varepsilon+O(\varepsilon^2)\right].\end{aligned}$$

On the other hand, there are symplectic forms on M which are compatible with the *non-standard* orientation of M; for example, the cohomology class $F + \varepsilon c_1$ is represented by such forms if ε is sufficiently small. A celebrated theorem of Taubes [**35**] therefore tells us that the reverse-oriented version $\overline{M}$ of M has a non-trivial Seiberg-Witten invariant [**29, 31, 20**]. The relevant spin^c structure on $\overline{M}$ is of almost-complex type, and its first Chern class, which we will denote by $\bar{c}_1$, is given by

$$\bar{c}_1 = c_1 + 4(p-1)F.$$

Of course, the conjugate almost-complex structure, with first Chern class $-\bar{c}_1$, is also a monopole class of $\overline{M}$, and $\overline{M}$ will have yet other monopole classes if, for example, M admits more than one Kodaira fibration and $\tau(M) \neq 0$.

Now recall that (19) asserts that

$$(1-\delta)\|\frac{s}{\sqrt{24}}\| + \delta\|W_+\| \geq \frac{2\pi}{\sqrt{3}}|c_1^+|$$

for all $\delta \in [0, \frac{1}{3}]$. One would like to know whether this inequality might also hold, quite generally, for some value of $\delta > \frac{1}{3}$. In order to find out, we apply this inequality to $\overline{M}$ with the above monopole class. Rewriting the inequality with respect to the *complex* orientation of M, we then get

$$(1-\delta)\|\frac{s}{\sqrt{24}}\| + \delta\|W_-\| \geq \frac{2\pi}{\sqrt{3}}|\bar{c}_1^-|, \tag{20}$$

and it is this inequality we shall now use to probe the limits of the theory.

Relative to any Kähler metric with Kähler class $[\omega_\varepsilon]$, one has

$$\begin{aligned} \bar{c}_1^+ &= \frac{\bar{c}_1\cdot[\omega_\varepsilon]}{[\omega_\varepsilon]^2}[\omega_\varepsilon] \\ &= \frac{[c_1+4(p-1)F]\cdot[2(p-1)F-\varepsilon c_1]}{[\omega_\varepsilon]^2}[\omega_\varepsilon] \\ &= -\frac{(\chi+3\varepsilon\tau)}{[\omega_\varepsilon]^2}[\omega_\varepsilon], \end{aligned}$$

so that

$$\begin{aligned} |\bar{c}_1^+|^2 &= \frac{(\chi+3\varepsilon\tau)^2}{[\omega_\varepsilon]^2} \\ &= \frac{1}{\varepsilon}\frac{(\chi+3\varepsilon\tau)^2}{2\chi+\varepsilon c_1^2} \\ &= \frac{\chi}{2\varepsilon}\left[1-(1-\frac{9}{2}\varrho)\varepsilon+O(\varepsilon^2)\right]. \end{aligned}$$

Now since $\bar{c}_1$ is the first Chern class of an almost-complex structure on $\overline{M}$, we have

$$|\bar{c}_1^-|^2-|\bar{c}_1^+|^2=2\chi-3\tau,$$

and it follows that

$$\begin{aligned} |\bar{c}_1^-|^2 &= (2\chi-3\tau)+\frac{\chi}{2\varepsilon}\left[1-(1-\frac{9}{2}\varrho)\varepsilon+O(\varepsilon^2)\right] \\ &= \frac{\chi}{2}(4-6\varrho)+\frac{\chi}{2\varepsilon}\left[1-(1-\frac{9}{2}\varrho)\varepsilon+O(\varepsilon^2)\right] \\ &= \frac{\chi}{2\varepsilon}\left[1+(3-\frac{3}{2}\varrho)\varepsilon+O(\varepsilon^2)\right]. \end{aligned}$$

After dividing by $\pi\sqrt{2\chi/3\varepsilon}$, the inequality (20) would thus read

$$\begin{aligned} (1-\delta)\sqrt{1+(3+\frac{9}{2}\varrho)\varepsilon+O(\varepsilon^2)} &+ \delta\sqrt{1+(3-\frac{27}{2}\varrho)\varepsilon+O(\varepsilon^2)} \\ &\geq \sqrt{1+(3-\frac{3}{2}\varrho)\varepsilon+O(\varepsilon^2)}. \end{aligned}$$

Dropping the terms of order ε^2, we would thus have

$$(1-\delta)\left[1+(\frac{3}{2}+\frac{9}{4}\varrho)\varepsilon\right]+\delta\left[1+(\frac{3}{2}-\frac{27}{4}\varrho)\varepsilon\right]\geq 1+(\frac{3}{2}-\frac{3}{4}\varrho)\varepsilon,$$

so that, upon collecting terms, we would obtain

$$3\varrho\varepsilon\geq 9\varrho\varepsilon\delta.$$

Taking $\varrho=\tau/\chi$ to be positive, and noting that ε is positive by construction, this shows that Conjecture 10 would imply that

$$\frac{1}{3}\geq\delta,$$

or in other words that (15) is optimal. We have thus proved the following result:

Theorem 12. — *Either*

- *the estimate (15) is optimal; or else*
- *Conjecture 10 is false.*

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